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BOUNDS FOR THE SMALLEST NORM IN AN IDEAL CLASS

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Abstract. We develop a method for obtaining upper bounds for the smallest norm among all norms of integral ideals in an ideal class. Applying this to number fields of small degree, we are able to substantially improve on the best previously known bounds.

1. Introduction

Let K be a number field with $[K:\mathbb{Q}] = r_1 + 2r_2$, where K has r_1 real embeddings and $2r_2$ complex embeddings. Minkowski proved that there exists a constant $C(r_1, r_2)$, which depends only on r_1 and r_2, such that for any ideal class $\mathcal{C}$ of K, there exists an integral ideal $\mathfrak{a}_{\mathcal{C}} \in \mathcal{C}$ satisfying $N(\mathfrak{a}_{\mathcal{C}}) \leq (C(r_1, r_2))^{-1}\sqrt{|d_K|}$. Here N denotes the absolute norm and d_K is the discriminant of the field K.

By results of C. A. Rogers [R] and H. P. Mulholland [M], one has that for $[K:\mathbb{Q}]$ large

$$N(\mathfrak{a}_{\mathcal{C}}) \leq \left((32.5)^{\frac{r_1}{2}}(15.7)^{r_2}\right)^{-1}\sqrt{|d_K|}.$$

The best bound so far for the constant $C(r_1, r_2)$ was given by Zimmert [Zi] in 1981, who found that

$$N(\mathfrak{a}_{\mathcal{C}}) \leq ((50.7)^{\frac{r_1}{2}}(19.9)^{r_2})^{-1}\sqrt{|d_K|}$$

(for $[K:\mathbb{Q}]$ large). He also obtained the best known bounds when the degree of K is small.

Before Zimmert, the bound was always obtained using methods from the geometry of numbers [N, p. 129]. The paper [Zi] in contrast introduces a new analytic method for deriving the bound. We will modify this method to obtain, for fields of small degree, a bound which improves on Zimmert's. In Table 1 at the end of the introduction we give both Zimmert's bound and the new bound found for each case.

The main technique for obtaining the new bounds is contained in Theorem 1 and its corollary below. To formulate the result, we need some definitions.

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For parameters $r_1,\ r_2$ and $\gamma > 0$, let the functions $P(s)$ and $T(s)$ be defined as follows:

$$P(s) = \frac{\Gamma_{r_1,r_2}(s)}{\Gamma_{r_1,r_2}(s+2\gamma+1)}, \tag{1}$$

$$T(s) = \frac{\Gamma_{r_1,r_2}(1-s)}{\Gamma_{r_1,r_2}(s+2\gamma+1)}, \tag{2}$$

where Γ_{r_1,r_2} is given by

$$\Gamma_{r_1,r_2}(s) = \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2} \tag{3}$$

(and $\Gamma(\cdot)$ denotes the gamma function). For given values $r_1,\ r_2$, and parameter $\gamma\ (> 0)$, a rational function $R_\gamma(s)$ is called an *admissible rational function* if it can be written as

$$R_\gamma(s) = \left(1 + \frac{1+2\gamma}{s}\right)^{-r_1-r_2} \left(1 + \frac{1+2\gamma}{s+1}\right)^{-r_2} \sum_{i=0}^{l} e_i \prod_{j=0}^{n_i} (s+a_{ij})^{-1}, \tag{4}$$

where $l \geq 0$ and all $e_i \geq 0,\ a_{ij} \geq 0$ and $n_i > 0$. To an admissible rational function R_γ we associate a *weight function* $F_\gamma(y) : \mathbb{R} \to \mathbb{R}$, via the contour integral

$$F_\gamma(y) = \frac{1}{2\pi i} \int_{-\delta_1 - i\infty}^{-\delta_1 + i\infty} (e^y)^{1-s} T(s) R_\gamma(s) ds, \tag{5}$$

where $T(s)$ is as in (2) (with the same γ) and $\delta_1 > 0$ such that $R_\gamma(s)$ has no pole in the strip $-\delta_1 \leq \mathrm{Re}\ s \leq 0$.

The partial zeta function of an ideal class $\mathcal{C}$ is defined as

$$\zeta_{\mathcal{C}}(s) = \sum_{\mathfrak{a} \in \mathcal{C}} (N(\mathfrak{a}))^{-s},$$

where the sum runs over all the integral ideals in $\mathcal{C}$. It can be alternatively written as

$$\zeta_{\mathcal{C}}(s) = \sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m m^{-s}, \tag{6}$$

where $\mathfrak{a}_{\mathcal{C}}$ is an integral ideal in $\mathcal{C}$ with minimal norm and a_m denotes the number of integral ideals in $\mathcal{C}$ with norm equal to m.

Theorem 1. *Let $\mathcal{C}$ be an ideal class for a field K, where K has r_1 real embeddings and $2r_2$ complex embeddings. Then for any parameter γ and any weight function F_γ, we have that*

$$B \sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m F_\gamma\left(y - \log\left(\frac{m}{N(\mathfrak{a}_{\mathcal{C}})}\right)\right) \geq t_0 e^y - \frac{\sqrt{d_K}}{N(\mathfrak{a}_{\mathcal{C}})} \text{ for } y \in \mathbb{R},$$

where a_m is as in (6)*, and B, t_0 are positive numbers given by*

$$B = \frac{\sqrt{\pi^n}}{\kappa R_\gamma(1) P(1) N(\mathfrak{a}_{\mathcal{C}})} \quad \textit{and} \quad t_0 = \frac{R_\gamma(0) T(0)}{R_\gamma(1) P(1)} \sqrt{\pi^n},$$

with $n = [K : \mathbb{Q}]$.

The next result is an immediate consequence.

Table 1.

n	r_1	r_2	$Z_0(r_1, r_2)$	$Z(r_1, r_2)$	minimal value for $\sqrt{\lvert d\rvert}$ known
2	2	0	1.760	2.137	2.236
2	0	1	1.400	1.651	1.732
3	3	0	4.636	6.235	7.0
3	1	1	3.355	4.340	4.795
4	4	0	14.45	21.21	26.92
4	2	1	9.749	13.76	16.58
4	0	2	6.792	9.250	10.81
5	5	0	50.21	79.19	121.0
5	3	1	32.12	49.57	67.16
5	1	2	21.11	31.02	40.11
6	6	0	188.1	315.0	547.8
6	0	3	46.74	70.98	98.72
8	8	0	3088	5644	16801
8	0	4	385.5	635.5	1121
10	10	0	58540	121120	716099
10	0	5	3560	6443	14464

Corollary. *Suppose that for a given weight function F_γ there exists a $y_1 \in \mathbb{R}$ such that*

$$F_\gamma(y) \ \le 0 \quad \textit{for} \quad -\infty < y \le y_1. \tag{7}$$

Then

$$N(\mathfrak{a}_{\mathcal{C}}) \le \ (t_0 e^{y_1})^{-1} \ \sqrt{|d_K|}.$$

Thus to obtain a bound for the smallest norm of an ideal using the above corollary, we need to find a suitable y_1. Unfortunately, very little is known in general about a weight function F_γ as in Theorem 1. Analyzing Zimmert's technique, we are able to show that indeed y_1 exists. However, to obtain new bounds we need a far larger value of y_1 than the one given by Zimmert's proof. To do this we must numerically calculate $F_\gamma(y)$ (see Theorem 2 below) and also develop an algorithm to ensure that for all $y \le y_1$ we have $F_\gamma(y) \le 0$.

In Table 1, we give Zimmert's lower bound $\sqrt{|d_K|}/N(\mathfrak{a}_{\mathcal{C}}) \ge Z_0(r_1, r_2)$ and our new lower bound $Z(r_1, r_2)$. In the last column we give the smallest $\sqrt{|d_K|}$ known for K with the given signature (r_1, r_2) [O, p. 133]. Taking $\mathcal{C}$ to be the trivial class, for which $N(\mathfrak{a}_{\mathcal{C}}) = 1$, we see that no general lower bound for $\sqrt{|d_K|}/N(\mathfrak{a}_{\mathcal{C}})$ could exceed the last column.

The numerical approximation of $F_\gamma(y)$ is based on the following theorem.

Theorem 2. *The function $F_\gamma(y)$ admits an expansion of the form*

$$F_\gamma(y) = \sum_{j=1}^{m} (e^y)^{1-j} P_j(y) + \epsilon(m, y) \qquad (m \ge 1).$$

Here the error term is given by

$$\epsilon(m,y) = \frac{1}{2\pi i}\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} (e^y)^{1-s} R_\gamma(s)T(s)ds,$$

which tends to zero as $m \to \infty$ *and* $P_j(y)$ *is a polynomial in* y *of degree at most* $r_1 + r_2$.

In fact, the above result allows us to quickly calculate $F_\gamma(y)$ numerically for any given y, since $|\epsilon(m,y)|$ can be bounded explicitly (see Proposition 3) and the polynomials P_j can be determined recursively. Specifically, if $P_j(y) = \sum_{k=0}^{t_j} a_{k,j} y^k$, then the coefficients of $P_{j+1}(y)$ can be found by a recursion of the form $a_{k,j+1} = f(a_{0,j}, \cdots, a_{t_j,j})$. The exact form of the function f is obtained with the help of a formula analogous to the gamma function formula $x\Gamma(x) = \Gamma(x+1)$ (see Section 3).

It seems to be difficult to prove that a given point y_1 satisfies the condition (7), i.e., $F_\gamma(y) < 0$ for all $-\infty < y \le y_1$. We need to work carefully with numerical estimates. We used ***PARI*** [C] to calculate $F_\gamma(y)$ numerically and then to obtain a bound y_1. To assure their reliability, we have done an independent check of the numerical computation of the function $F_\gamma(s)$ based on a numerical integration through Simpson's rule with a variable cut off. Both numerical methods coincided in at least twice as many digits as those displayed in Table 2 (i.e., they coincided at least in 10 significant digits.)

The paper is organized as follows. In Section 2 we present the basic idea of Zimmert's method, the proof of Theorem 1 and we obtain a point y_0 that satisfies the inequalities of the corollary. In Section 3 we give the proof of Theorem 2 and different expressions for $T(s)$ that permit computing the polynomials P_j. Finally, in Section 4 we give an algorithm to find a largest possible point y_1 satisfying the inequalities of the corollary.

2. Zimmert's method

Zimmert's method uses the functional equation of the zeta function of an ideal class. We present his method, slightly reformulated.

Lemma 1 (Zimmert)**.** *Let* R_γ *be an admissible rational function as in* (4)*. Let* $f(s)$ *be a Dirichlet series with nonnegative coefficients, convergent in the half-plane* $\mathrm{Re}(s) > 1$*. Then for any* $x > 0$ *and* $\tau > 1$,

$$\frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} x^s R_\gamma(s)P(s)f(s)ds \ge 0, \tag{8}$$

where $P(s)$ *is as in* (1) *(with the same parameter* γ*).*

Proof. Zimmert [Zi, p. 369] proved this for a certain $R_\gamma(s)$, but his proof is actually valid for all admissible rational functions, as was pointed out by E. Friedman [F1, p. 618]. □

Given an ideal class $\mathcal{C}$ of K, denote by $\mathcal{C}' = \partial_K \mathcal{C}^{-1}$ the conjugate class of $\mathcal{C}$, where ∂_K is the ideal class of the different of K. We can write the functional equation of $\zeta_{\mathcal{C}}(s)$ as $\Delta(s,\mathcal{C}') = \Delta(1-s,\mathcal{C})$, where

$$\Delta(s,\mathcal{C}) = \left(\sqrt{\frac{|d_K|}{\pi^n}}\right)^s \Gamma_{r_1,r_2}(s)\ \zeta_{\mathcal{C}}(s),$$

with $n = [K:\mathbb{Q}]$ and $\Gamma_{r_1,r_2}(s)$ as in (3).

Theorem 1. *Let $\mathcal{C}$ be an ideal class for a field K, where K has r_1 real embeddings and $2r_2$ complex embeddings. Then for any parameter γ and any weight function F_γ, we have that*

$$B\sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m F_\gamma\left(y - \log\left(\frac{m}{N(\mathfrak{a}_{\mathcal{C}})}\right)\right) \geq t_0 e^y - \frac{\sqrt{d_K}}{N(\mathfrak{a}_{\mathcal{C}})} \quad \text{for} \quad y \in \mathbb{R},$$

where a_m is as in (6), and B, t_0 are positive numbers given by

$$B = \frac{\sqrt{\pi^n}}{\kappa R_\gamma(1)P(1)N(\mathfrak{a}_{\mathcal{C}})} \quad \text{and} \quad t_0 = \frac{R_\gamma(0)T(0)}{R_\gamma(1)P(1)}\sqrt{\pi^n},$$

with $n = [K:\mathbb{Q}]$, P, T, and R_γ are as in (1), (2) and (4) .

Proof. In Lemma 1 take $f(s) = \zeta_{\mathcal{C}'}(s)$, any $x > 0$ and $\tau > 1$. By the functional equation, a convexity theorem [L, p. 266] and the asymptotic formula $|\Gamma(\sigma+it)| \sim e^{-\frac{\pi}{2}|t|}|t|^{\sigma-\frac{1}{2}}$ (uniformly for real σ in an interval and real t with $|t| \gg 0$ [G-R, p. 945]), we can shift the line of integration in (8), from Re $s = \tau$ to Re $s = -\delta_1$. Thus we pick up the residue at $s = 0$ and $s = 1$ corresponding to the (simple) poles of $\Delta(s,\mathcal{C})$. By using the functional equation for the gamma function we get:

$$\begin{aligned} 0 \quad &\leq \quad R_\gamma(1)P(1) - AR_\gamma(0)T(0) \\ &\quad + \frac{x}{A\kappa 2\pi i}\int_{-\delta_1-i\infty}^{-\delta_1+i\infty}\left(\frac{A^2}{x}\right)^{1-s} R_\gamma(s)T(s)\zeta_{\mathcal{C}}(1-s)ds, \end{aligned} \tag{9}$$

where $\kappa = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{w_K\sqrt{d_K}}$, $A = \sqrt{\frac{|d_K|}{\pi^n}}$, R_K is the regulator of K, and w_K is the number of roots of unity in K. Hence

$$\begin{aligned} &\frac{AR_\gamma(0)T(0)}{xR_\gamma(1)P(1)} - 1 \leq \\ &\qquad \frac{1}{\kappa AR_\gamma(1)P(1)}\sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m \frac{1}{2\pi i}\int_{-\delta_1-i\infty}^{-\delta_1+i\infty}\left(\frac{A^2}{xm}\right)^{1-s} R_\gamma(s)T(s)ds. \end{aligned} \tag{10}$$

Let

$$t_0 = \frac{R_\gamma(0)T(0)}{R_\gamma(1)P(1)}\sqrt{\pi}^n = \frac{R_\gamma(0)}{R_\gamma(1)}\left(\frac{\Gamma(1+\gamma)}{\Gamma(\frac{1}{2}+\gamma)}\right)^{r_1}\left(\frac{1}{2}+\gamma\right)^{r_2}\sqrt{\pi}^n, \tag{11}$$

and choose x so that $y = \log(\frac{A^2}{xN(\mathfrak{a}_{\mathcal{C}})})$. We can rewrite (10) as

$$t_0 e^y \ - \frac{\sqrt{|d_K|}}{N(\mathfrak{a}_{\mathcal{C}})} \leq \frac{\sqrt{\pi^n}}{\kappa\ R_\gamma(1)P(1)N(\mathfrak{a}_{\mathcal{C}})}\sum_{m=N(\mathfrak{a}_{\mathcal{C}})}^{\infty} a_m F\left(y - \log\left(\frac{m}{N(\mathfrak{a}_{\mathcal{C}})}\right)\right).$$

We note that hypothesis (4) on $R_\gamma(s)$ implies that $R_\gamma(t) > 0$ for $t > 0$. Hence, t_0 as in (11) is positive and letting $B = \frac{\sqrt{\pi^n}}{\kappa R_\gamma(1)P(1)N(\mathfrak{a}_{\mathcal{C}})} > 0$, we are done. $\square$

Zimmert used the admissible rational function $R_\gamma(s) = \frac{(s+\alpha)}{(s+\beta)\ (s+2\gamma-\beta)\ (s+2\gamma-\alpha)}$, with $0 \le \alpha < \beta < \gamma$. By estimating the integral in (9) and taking the limit $\beta \to \gamma$, he obtained the bound

$$Z_0(r_1, r_2) \le \frac{\sqrt{|d_k|}}{N(\mathfrak{a}_{\mathcal{C}})}, \quad \text{where} \quad Z_0(r_1, r_2) = t_0 e^{Y(r_1, r_2, \gamma)},$$

$$Y(r_1, r_2, \gamma) = -r_1 \frac{\Gamma'}{\Gamma}\left(\frac{1+\gamma}{2}\right) - 2r_2\left(\frac{\Gamma'}{\Gamma}(1+\gamma) - \log(2)\right) - \frac{2}{\gamma - \alpha},$$

and t_0 as in (11). For each signature (r_1, r_2), Zimmert chose appropiate γ and α in order to obtain his bound [Zi, p. 368]. By modifiyng the admisible rational function $R_\gamma(s)$, we will now improve the bound of Zimmert.

Proposition 1. *For all $s \in \mathbb{C}$ with $\operatorname{Re} s \ge -\gamma$ and $\operatorname{Re} s \ne 1, 2, 3, \cdots$, the function $T(s)$ introduced in (2) satisfies the inequality*

$$|T(s)| \le |T(\operatorname{Re} s)|. \tag{12}$$

Proof. Setting

$$G(s, \gamma) \ = \ \frac{\Gamma(s)}{\Gamma(1 + \gamma - s)}, \tag{13}$$

we can write $T(s)$ in the following way:

$$T(s) \ = \ G\left(\frac{1-s}{2}, \gamma\right)^{r_1 + r_2} \ G\left(\frac{1}{2} + \frac{1-s}{2},\ 1 + \gamma\right)^{r_2}. \tag{14}$$

We claim now:

$(*)$ *For all $\gamma > 0$ and $s \in \mathbb{C}$ with $\operatorname{Re} s \le \frac{1+\gamma}{2}$ and $\operatorname{Re} s \ne 0, -1, -2 \cdots$, we have $|G(s, \gamma)| \le |G(\operatorname{Re} s, \gamma)|$.*

For $s = \sigma + it$, with σ and t real, and $\sigma \ne 0, -1, -2, \cdots$, we have [G-R, 8.326]

$$\left|\frac{\Gamma(\sigma + it)}{\Gamma(\sigma)}\right|^2 = \prod_{n=0}^{\infty}\left(1 + \left(\frac{t}{\sigma + n}\right)^2\right)^{-1}.$$

Hence

$$\left|\frac{G(\sigma + it, \gamma)}{G(\sigma, \gamma)}\right|^2 = \frac{\prod_{n=0}^{\infty}\left(1 + \left(\frac{t}{1+\gamma-\sigma+n}\right)^2\right)}{\prod_{n=0}^{\infty}\left(1 + \left(\frac{t}{\sigma+n}\right)^2\right)} \ \le \ 1, \quad \text{for} \quad \sigma \ \le \ \frac{1+\gamma}{2}.$$

This proves the claim. Using the claim $(*)$ and (14) we obtain the proposition. $\square$

Using an analogous method to Zimmert's, we obtain in the next lemma a point $y_0 = y_0(\delta_2)$ satisfying (7). This value of y_0 is in general a bad bound for the minimal norm of ideals, but we use it as a starting point in the algorithm to obtain better bounds (cf. Section 4).

Lemma 2. *Let $R_\gamma(s)$ be an admissible rational function and let δ_2 be such that $\delta_2 \le \gamma$ and the function $R_\gamma(s)$ has the unique simple pole $-\beta$ in the strip $-\delta_2 \le \operatorname{Re} s \le 0$. Suppose furthermore that the residue $\rho = \operatorname{Res}_{s=-\beta}(R_\gamma(s)T(s))$ is negative. Let $y_0 = y_0(\delta_2)$ be defined by*

$$y_0 = \frac{1}{\delta_2 - \beta}\left(\log(\ -\rho\ ) - \log\left(\frac{|T(-\delta_2)|}{2\pi}\int_{-\delta_2 - i\infty}^{-\delta_2 + i\infty} |R_\gamma(s) ds|\right)\right). \tag{15}$$

Then the weight function $F_\gamma(y)$ specified by $R_\gamma(s)$ satisfies $F_\gamma(y) < 0$ for all y in the interval $-\infty < y \le y_0$.

Proof. We shift the line of integration in (5) (the integral that defines $F_\gamma(y)$) from Re $s = -\delta_1$ to Re $s = -\delta_2$. Then

$$F_\gamma(y) = (e^y)^{1+\beta}\ \rho + \frac{1}{2\pi i}\int_{-\delta_2 - i\infty}^{-\delta_2 + i\infty} (e^y)^{1-s}\, R_\gamma(s) T(s) ds.$$

Using $\delta_2 \le \gamma$ in Proposition 1, we have $F_\gamma(y) \le 0$ if $y \le y_0$. □

The following proposition provides an explicit admissible rational function satisfying the conditions of Lemma 2 together with a bound for y_0 (15).

Proposition 2. *The rational function $R_\gamma(s)$ given by*

$$R_\gamma(s) = \frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)(s+\alpha_2)},$$

with

$$0 \le \alpha < \beta < \gamma < \alpha_1 \le \alpha_2, \tag{16}$$

is an admissible rational function. Furthermore, let δ_2 be such that $\gamma \ge \delta_2 \ge \beta$. The point $y_0(\delta_2)$ given in Lemma 2 satisfies

$$y_0 \ge \frac{1}{\delta_2 - \beta}\log\left(\frac{(\beta-\alpha)T(-\beta)}{(\alpha_1-\beta)(\alpha_2-\beta)}\right) - \log\left(\frac{(\delta_2-\alpha)|T(-\delta_2)|}{2(\delta_2-\beta)(\alpha_1-\delta_2)}\right). \tag{17}$$

Proof. First, note that

$$\frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)}\left(1 + \frac{1+2\gamma}{s}\right) = \left(1+\frac{\alpha}{s}\right)\left(1 + \frac{1+2\gamma-\beta}{s+\beta}\right)\left(\frac{1}{s+\alpha_1}\right).$$

Hence by (16), $R_\gamma(s)$ is of the form (4), i.e., $R_\gamma(s)$ is an admissible rational function; furthermore, by this inequalities we obtain that $\mathrm{Res}_{s=-\beta}(R_\gamma(s)T(s))$ is negative. By the condition on δ_2, $R_\gamma(s)$ has a unique pole $-\beta$ in the strip $-\delta_2 \le \mathrm{Re}\, s \le 0$; furthermore we see that $|\frac{s+\alpha}{s+\beta}|$ in $s = -\delta_2 + it$ has a maximum at $t = 0$.

Hence, one has

$$\int_{-\delta_2 - i\infty}^{-\delta_2+i\infty} |R_\gamma(s) ds| \le \int_{-\infty}^{\infty} \frac{\delta_2 - \alpha}{(\delta_2-\beta)((\alpha_1-\delta_2)^2 + t^2)} dt \le \frac{(\delta_2-\alpha)\pi}{(\delta_2-\beta)(\alpha_1-\delta_2)}$$

(using also the condition $\delta_2 \le \alpha_1 \le \alpha_2$). □

3. An approximation of F_γ

We will now give an approximation of the weight function F_γ ($\gamma > 0$), which is given by

$$F_\gamma(y) = \frac{1}{2\pi i}\int_{-\delta_1 - i\infty}^{-\delta_1 + i\infty} (e^y)^{1-s} R_\gamma(s) T(s) ds,$$

where

$$T(s) = \left(\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s+1}{2}+\gamma\right)}\right)^{r_1+r_2} \left(\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{s}{2}+1+\gamma\right)}\right)^{r_2},$$

and R_γ is an admissible rational function without poles in the strip $-\delta_1 < \mathrm{Re}\ s \le 0$.

Theorem 2. *The function $F_\gamma(y)$ admits an expansion of the form*

$$F_\gamma(y) = \sum_{j=1}^{m} (e^y)^{1-j} P_j(y) + \epsilon(m,y) \qquad (m \geq 1). \tag{18}$$

Here the error term is given by

$$\epsilon(m,y) = \frac{1}{2\pi i} \int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} (e^y)^{1-s} R_\gamma(s) T(s) ds, \tag{19}$$

which tends to zero as $m \to \infty$, and

$$P_j(y) = \sum_{k=0}^{t_j-1} \left(\frac{1}{k!} \sum_{i=0}^{t_j-1} c_{i,j}\ d_{-(i+k+1),j} \right) y^k, \tag{20}$$

with $t_j = r_1 + r_2$ for j odd and $t_j = r_2$ for j even, and with $c_{k,j}$ and $d_{k,j}$ given by the Laurent expansions of $R_\gamma(s)$ and $T(s)$ near $s = j$:

$$R_\gamma(s) = \sum_{k=0}^{\infty} c_{k,j}\ (s-j)^k \quad \textit{and} \quad T(s) = \sum_{k=-t_j}^{\infty} d_{k,j}\ (s-j)^k.$$

Proof. The function $(e^y)^{1-s} R_\gamma(s)$ is analytic in the half-plane $\mathrm{Re}\, s \geq -\delta_1$ and $T(s)$ has poles of order $r_1 + r_2$ at $s = 1, 3, 5, \cdots$ and of order r_2 at $s = 2, 4, 6, \cdots$. Hence, if we shift the line of integration in (5) (the integral that defines $F_\gamma(y)$) from $\mathrm{Re}(s) = -\delta_1$ to $\mathrm{Re}(s) = m + \frac{1}{2}$, we pick up the residues at these poles. Given a pole at $s = j$ of order t_j, we have $(e^y)^{1-s} R_\gamma(s) = (e^y)^{1-j+(j-s)} R_\gamma(s) = (e^y)^{1-j} \sum_{k=0}^{\infty} e_{k,j}(y)(s-j)^k$ with $e_{k,j}(y) = \sum_{i=0}^{k} \frac{(-1)^i c_{k-i,j}}{i!} y^i$. Hence

$$\mathrm{Res}_{s=j} \left((e^y)^{1-s} R_\gamma(s) T(s) \right) \ = \ (e^y)^{1-j} \sum_{k=0}^{t_j-1} e_{k,j}(y)\ d_{-k-1,j}.$$

Inserting the explicit expressions of the $e_{k,j}$ in this equalities and collecting powers of y, we are led to the polynomials $P_j(y)$.

The fact that the error term $|\epsilon(m,y)|$ tends to 0 as $m \to \infty$ is an immediate consequence of Proposition 3 below. □

Proposition 3. a) *For $m \in \mathbb{N}$ and $y \in \mathbb{R}$, we have*

$$|\epsilon(m,y)| \ \leq \ (e^y)^{\frac{1}{2}-m} \frac{|T(m+\frac{1}{2})|}{2\pi} \int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R_\gamma(s)| ds.$$

b) *If we take $R_\gamma(s) = \frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)(s+\alpha_2)}$ with $0 \leq \alpha < \beta < \gamma < \alpha_1 \leq \alpha_2$, we have*

$$|\epsilon(m,y)| \leq (e^y)^{\frac{1}{2}-m} \frac{|T(m+\frac{1}{2})|}{2(\alpha_1 + m + \frac{1}{2})}.$$

c) *For each $m \in \mathbb{N}$, we have*

$$\left| T\left(m + \frac{1}{2}\right) \right| = \frac{(\sqrt{2}\pi)^{a+b}}{\left(\Gamma(\frac{m}{2}+\frac{3}{4})\Gamma(\frac{m}{2}+\frac{3}{4}+\gamma)\right)^a \left(\Gamma(\frac{m}{2}+1+\frac{1}{4})\Gamma(\frac{m}{2}+\frac{1}{4}+\gamma)\right)^b},$$

where $a = r_1 + r_2$ and $b = r_2$.

Proof. a) Follows from inequality for $T(s)$ in (12) and the expression for $\epsilon(m, y)$ in (19).

b) When

$$R_\gamma(s) \;=\; \frac{(s+\alpha)}{(s+\beta)(s+\alpha_1)(s+\alpha_2)},$$

we construct the bound

$$\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R_\gamma(s)|ds$$

just as in the proof of Proposition 2, upon bounding $|\frac{s+\alpha}{s+\beta}|$ at $s = (m+\frac{1}{2})+it$ by 1. Hence, one has

$$\int_{m+\frac{1}{2}-i\infty}^{m+\frac{1}{2}+i\infty} |R_\gamma(s)ds| \leq \frac{\pi}{\alpha_1 + m + \frac{1}{2}}.$$

c) Using the definition of $T(s)$ and the reflection-relation $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$ to rewrite the numerators, we see that

$$\begin{aligned} \left|T\left(m+\frac{1}{2}\right)\right| &= \left|\frac{\Gamma(\frac{1-(m+\frac{1}{2})}{2})}{\Gamma(\frac{m}{2}+\frac{3}{4}+\gamma)}\right|^a \left|\frac{\Gamma(1-(\frac{m}{2}+\frac{1}{4}))}{\Gamma(1+\frac{m}{2}+\frac{1}{4}+\gamma)}\right|^b \\ &= \left(\frac{\sqrt{2}\pi}{\Gamma(\frac{m}{2}+\frac{3}{4})\Gamma(\frac{m}{2}+\frac{3}{4}+\gamma)}\right)^a \left(\frac{\sqrt{2}\pi}{\Gamma(\frac{m}{2}+\frac{1}{4})\Gamma(\frac{m}{2}+1+\frac{1}{4}+\gamma)}\right)^b. \end{aligned}$$

□

Notice that it is immediate from the proposition that $|T(m+\frac{1}{2})|$, and hence the error term $\epsilon(m, y)$ tends to zero rapidly as $m \to \infty$.

Let us now demonstrate that the coefficients $c_{k,j}$ and $d_{k,j}$ appearing in the polynomial P_j (see (20)) can be found recursively (in the variable j). The proof uses several steps and culminates in a method for computing P_j given at the end of this section.

The following proposition lies at the basis of the recursive computation of the coefficients $c_{k,j}$.

Proposition 4. *Let $R_\gamma(s)$ be an admissible rational function with $-a_1, \cdots, -a_r$ and $-b_1, \cdots, -b_t$ the zeros and poles (i.e., $R_\gamma(s) = \alpha\frac{(s+a_1)\cdots(s+a_r)}{(s+b_1)\cdots(s+b_t)}$).*

a) Then we have that

$$R_\gamma(s) = \frac{\alpha}{b_1 \cdots b_t}(s+a_1)\cdots(s+a_r)\prod_{i=1}^{t}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{b_i^k}s^k\right).$$

b) If $R_\gamma(s) = \sum_{k=0}^{\infty} c_{k,j}(a_1, \cdots, a_s, b_1, \cdots, b_t)(s-j)^k$ for s near j ($j = 0, 1, 2\cdots$), then the expansion coefficients $c_{k,j}$ satisfy the recurrence relation

$$c_{k,j+1}(a_1, \cdots, a_s, b_1, \cdots, b_t) = c_{k,j}(a_1+1, \cdots, a_r+1, b_1+1, \cdots, b_t+1).$$

Proof. We get the expansion formula of part a) upon expanding the factors of the denominator by means of the geometric series $\frac{1}{s+b} = \frac{1}{b}\sum_{k=0}^{\infty}\frac{(-1)^k}{(b)^k}(s)^k$. The recurrence relation for the coefficients $c_{k,j}$ of part b) is then obtained by writing

$s+a$ as $a+j+(s-j)$ in the numerator and $s+b$ as $b+j+(s-j)$ in the denominator and invoking of the geometric series. $\square$

Let us recall that (cf. eqs. (13), (14))

$$T(s) = G\left(\frac{1-s}{2},\gamma\right)^{r_1+r_2} \; G\left(\frac{1}{2}+\frac{1-s}{2},\ 1+\gamma\right)^{r_2},$$

with

$$G(s,\gamma) = \frac{\Gamma(s)}{\Gamma(1+\gamma-s)}.$$

To find the recurrence relations for the expansion coefficients $d_{k,j}$ of the Laurent series of $T(s)$ around $s=j$, we will employ functional equations for $G(s,\gamma)$ analogous to the difference equation $s\Gamma(s)=\Gamma(s+1)$ and duplication formula for the gamma function.

Lemma 3. a) $G(s+1,\gamma)=(\gamma-s)sG(s,\gamma)$.
b) $G(s,\gamma)\; G\left(\frac{1}{2}+s,\gamma\right) = 2^{1+2\gamma-4s}G(2s,2\gamma) = \frac{2^{1+2\gamma-4s}}{4(\gamma-s)s}G\left(2\left(s+\frac{1}{2}\right),2\gamma\right).$

Proof. These formulas readily follow from the difference equation $\Gamma(s+1)=s\Gamma(s)$ and the duplication formula $2^{2s-1}\Gamma(s)\Gamma(\frac{1}{2}+s)=\sqrt{\pi}\Gamma(2s)$ [G-R, p. 946]. $\square$

The next lemma (together with Proposition 5 below) is a key step for obtaining the recurrence for $d_{k,j}$ in terms of $d_{k,j-2}$.

Lemma 4. *We have that*

$$T(s) \;=\; 2^{2(1+\gamma-2w)b}\,\frac{G(w,\gamma)^{a-b}\;G(2w,2\gamma)^b}{(1+2\gamma-2w)^b},\quad w=\frac{1-s}{2}, \tag{21}$$

and

$$T(s) \;=\; \frac{2^{(1+2\gamma-4w)a}\;G(w,\gamma)^{b-a}G(2w,2\gamma)^a}{\left((\frac{1}{2}+\gamma-w)(w-\frac{1}{2})\right)^a\left(1+\gamma-w\right)^b},\quad w=1-\frac{s}{2}, \tag{22}$$

where $a=r_1+r_2$ and $b=r_2$.

Proof. By (14) and using $\Gamma(1+x)=x\Gamma(x)$ in (13) (with $s=\frac{1}{2}+\gamma-w$), we have

$$T(s) \;=\; 2^b\,\frac{G(w,\gamma)^a\;G(\frac{1}{2}+w,\gamma)^b}{(1+2\gamma-2w)^b}.$$

Using b) above, we have the lemma. $\square$

The following proposition encodes a recurrence for the coefficients of the Laurent series of $G(s,\gamma)$ (and $G(2s,2\gamma)$) near near $s=-j-1$ in terms of the coefficients near $s=-j$.

Proposition 5. a) *If $\gamma>0$ and $G(s,\gamma)=\sum_{k=-1}^{\infty}a_k(s+j)^k$ for s near $-j$ $(j=0,1,2\cdots)$, then near $s=-(j+1)$*

$$G(s,\gamma) = -\frac{\sum_{k=0}^{\infty}\left(\frac{s+j+1}{j+1}\right)^k\sum_{k=0}^{\infty}\left(\frac{s+j+1}{\gamma+j+1}\right)^k}{(j+1)(\gamma+j+1)}\sum_{k=-1}^{\infty}a_k(s+j+1)^k.$$

b) *If* $\gamma > 0$ *and* $G(2s, 2\gamma) = \sum_{k=-1}^{\infty} b_k (s+j)^k$ *near* $s = -j$, *then near* $s = -(j+1)$

$$G(2s, 2\gamma) = \frac{\sum_{k=0}^{\infty} \left(\frac{s+j+1}{j+1}\right)^k \cdot \sum_{k=0}^{\infty} \left(\frac{2(s+j+1)}{2j+1}\right)^k \cdot \sum_{k=0}^{\infty} \left(\frac{s+j+1}{\gamma+j+1}\right)^k}{2(j+1)(2j+1)2(\gamma+j+1)}$$

$$\cdot \frac{\sum_{k=0}^{\infty} \left(\frac{2(s+j+1)}{2\gamma+2j+1}\right)^k}{(2\gamma+2j+1)} \sum_{k=-1}^{\infty} b_k (s+j+1)^k.$$

Proof. a) For each $j \geq 0$, we only need to rewrite a) in Lemma 3 as

$$G(s, \gamma) = \frac{-G(s+1, \gamma)}{(\gamma + j + 1 - (s+j+1))(1 + j - (s+j+1))}.$$

We note that when s is near $-(j+1)$, then $s+1$ is near $-j$. Hence we obtain a) by using the geometric series.

To prove b), we write part b) of Lemma 3 as

$$G(2s, 2\gamma) = \frac{G(2(s+1), 2\gamma)}{(2\gamma - 2s)\ 2s\ (2\gamma - (2s+1))\ (2s-1)},$$

replace $-2s$ by $2(j+1) - 2(s+j+1)$, and proceed as in the proof of a). $\square$

The explicit form of the recurrence relations for the coefficients $d_{k,j}$ is rather complicated and will be omitted here (as we do not need it). For our purposes it suffices to combine the above results into an effective method for computing the coefficients of the polynomials P_j quickly by means of a computer. We will now describe this method.

Method for computing the polynomials P_j. We compute the polynomials $P_j(y) = \sum_{k=0}^{t_j-1} a_{k,j} y^k$ recursively. Let us recall that

$$a_{k,j} = \frac{1}{k!} \sum_{i=0}^{t_j-1} c_{i,j} d_{-(i+k+1),j},$$

where $c_{k,j}$ and $d_{k,j}$ are the coefficients of the Laurent expansions of $R_\gamma(s)$ and $T(s)$ around $s = j$:

$$R_\gamma(s) = \sum_{k=0}^{\infty} c_{k,j}\ (s-j)^k, \quad T(s) = \sum_{k=-t_j}^{\infty} d_{k,j}\ (s-j)^k.$$

The coefficients $c_{k,j}$ are determined from Proposition 4. First part a) of the proposition is used to compute $c_{k,0}$, and next one uses the recursion of part b) to obtain $c_{k,j}$ for $j > 0$.

The coefficients $d_{k,j}$ are determined from Lemma 4 and Proposition 5. For j odd we use formula (21) for $T(s)$ and for j even we use formula (22). Expanding $G(w, \gamma)$ and $G(2w, 2\gamma)$ (with $w = (1-s)/2$ and $w = 1 - s/2$ respectively) by means of Proposition 5 yields a recurrence relation for $d_{k,j}$ in terms of $d_{k,j-2}$. To start the recursion we must compute $d_{k,j}$ for $j = 1$ and $j = 0$. To this end we expand formula (21) around $s = 1$ and formula (22) around $s = 0$, respectively. This involves the expansion of exponential factors and geometric series, and the expansion of the gamma factors $G(w, \gamma)$ and $G(2w, 2\gamma)$ (with $w = (1-s)/2$ and

$w = 1 - s/2$, respectively). The latter expansions depends on the standard Laurent series [G-R, p. 944] (for $|w| < 1$)

$$\Gamma(w) = \frac{1}{w}\Gamma(w+1) = \sum_{n=0}^{\infty} c_n w^{n-1}, \tag{23}$$

with $c_{n+1} = (n+1)^{-1} \sum_{k=0}^{n} (-1)^{k+1} s_{k+1} c_{n-k}$, $c_0 = 1$, and

$$\frac{1}{\Gamma(w)} = \frac{w}{\Gamma(w+1)} = \sum_{n=0}^{\infty} d_n w^{n+1}, \tag{24}$$

with $d_{n+1} = (n+1)^{-1} \sum_{k=0}^{n} (-1)^{k} s_{k+1} d_{n-k}$, $d_0 = 1$, where $s_1 = \mathbf{C} = 0,577215\ldots$ denotes Euler's constant and $s_n = \zeta(n)$ for $n > 1$. (Here $\zeta(\cdot)$ refers to the Riemann zeta function.) Indeed, the gamma factors of the numerator and denominator are expanded by means of (23) and (24), respectively, after translating the arguments of the gamma functions to a neighborhood of the origin by means of the functional equation $\Gamma(w+1) = w\Gamma(w)$.

4. The algorithm

In this section, we will describe the algorithm to approximate (from below) the largest point y^* satisfying property (7):

$$F_\gamma(y) \ \leq 0 \quad \text{ for } -\infty < y \leq y^*.$$

We implemented this algorithm using ***PARI*** [C]. From (the lower estimate of) y^* we then get a bound on the Minkowski constant via the corollary of Theorem 1 stated in the introduction.

The following proposition describes a procedure to augment lower estimates of y^* (thus improving the bound).

Proposition 6. *Let $\epsilon(m,y)$ be given by (19) and let $\varepsilon > 0$. Suppose that $|\epsilon(m,y)| \leq \varepsilon$ for all y in an interval $[x_1, x_2]$. Furthermore, let $a_1 \in [x_1, x_2]$ be such that there exists a $\delta = \delta(a_1) > 0$ satisfying $[a_1, a_1+\delta] \subseteq [x_1, x_2]$ and*

$$\delta c M + \sum_{j \in A} g_j(a_1+\delta) + \sum_{j \in B \cup C} g_j(a_1) \leq -\varepsilon. \tag{25}$$

Here $g_j(y) := e^{y(1-j)} P_j(y)$ for $1 \leq j \leq m$ with P_j given by (20),

$$\begin{aligned} A &= \{1 \leq j \leq m |\ g_j \textit{ is increasing on } [x_1, x_2]\}, \\ B &= \{1 \leq j \leq m |\ g_j \textit{ is decreasing on } [x_1, x_2]\}, \\ C &= \{1 \leq j \leq m |\ g_j \textit{ is not monotone on } [x_1, x_2]\}, \end{aligned}$$

$c = card(C)$, and for each $j \in C$ we have that $|g_j'(y)| < M$ for $x_1 \leq y \leq x_2$.

Then all $y \in [a_1, a_1+\delta]$ satisfy property (7), provided a_1 satisfies property (7).

Proof. Note that by (18), we have that $F_\gamma(y) = \sum_{j=1}^{m} g_j(y) + \epsilon(m,y)$. Moreover, if $y \in [a_1, a_1+\delta]$, we have by the mean value theorem and the definition of A, B and C and (25) that

$$\begin{aligned} F_\gamma(y) &\leq \sum_{j \in A} g_j(a_1+\delta) + \sum_{j \in B} g_j(a_1) + \sum_{j \in C} g_j(y) + \epsilon(m,y) \\ &\leq \sum_{j \in A} g_j(a_1+\delta) + \sum_{j \in B \cup C} g_j(a_1) + \delta c M + \epsilon(m,y) \leq 0. \quad \square \end{aligned}$$

TABLE 2.

n	r_1	r_2	γ	$z_0 = t_0 e^{y_0}$	upper bound for $F_\gamma(y_0)$	m	new bound $Z(r_1, r_2)$
2	2	0	2.48	0.2036	$-5.9027 \cdot 10^{-7}$	30	2.1379
2	0	1	3.09	0.1500	$-4.7945 \cdot 10^{-9}$	40	1.6518
3	3	0	1.63	0.6069	$-7.0926 \cdot 10^{-5}$	12	6.2350
3	1	1	1.92	0.4199	$-4.327 \cdot 10^{-6}$	16	4.3407
4	4	0	1.25	2.0029	$-1.781 \cdot 10^{-3}$	8	21.219
4	2	1	1.41	1.3195	$-1.651 \cdot 10^{-4}$	9	13.768
4	0	2	1.61	0.8911	$-1.146 \cdot 10^{-5}$	11	9.2504
5	5	0	1.04	7.1184	$-2.062 \cdot 10^{-2}$	6	79.190
5	3	1	1.15	4.5012	$-2.821 \cdot 10^{-3}$	7	49.572
5	1	2	1.27	2.9145	$-3.284 \cdot 10^{-4}$	8	31.025
6	6	0	0.91	26.716	$-1.543 \cdot 10^{-1}$	5	315.00
6	0	3	1.16	6.5421	$-6.095 \cdot 10^{-4}$	7	70.987
8	8	0	0.74	424.17	-6.005	4	5644.0
8	0	4	0.94	54.767	$-1.032 \cdot 10^{-2}$	5	635.5
10	10	0	0.47	7452.2	-182.064	4	112120
10	0	5	0.82	98.560	$-6.604 \cdot 10^{-3}$	5	6443.8

To obtain the numerical bounds on the Minkowski constants we work with the admissible rational function

$$R_\gamma(s) \;=\; \frac{(s+\alpha)}{(s+\beta)(s+2(2\gamma-\beta))(s+2(2\gamma-\alpha))},$$

where $\alpha = \gamma - \frac{\gamma(\gamma+1)}{\sqrt{1+3\gamma(\gamma+1)}}$ as in [Zi, p. 373], and β is near to γ such that $0 \le \alpha < \beta < \gamma$. Notice that $R_\gamma(s)$ is of the form given in Proposition 2 with $\alpha_1 = 2(2\gamma - \beta)$ and $\alpha_2 = 2(2\gamma - \alpha)$. (In particular, the parameters of $R_\gamma(s)$ satisfy the constraints in (16).) As a starting value for the lower estimate of y^* we take the lower bound for y_0 (15) given by (17). We pick $0 < \varepsilon < -\frac{F_\gamma(y_0)}{2}$ and use Proposition 3 to select an m such that $|\epsilon(m, y)| < \varepsilon$ for each $y \ge y_0$. Then, by means of Proposition 6, we move from (the lower estimate of) y_0 to a larger value y_1 such that $F_\gamma(y)$ remains negative on the interval $[y_0, y_1]$. By iterating this process one constructs a sequence $\{y_l\}$, $l = 0, 1, 2, \ldots$, converging from below to y^*. In principle the value $Z^*(r_1, r_2) := t_0 e^{y^*}$ (with t_0 from (11)) now provides a new lower bound such that $\sqrt{|d_K|}/N(\mathfrak{a}) \ge Z^*(r_1, r_2)$ (cf. the corollary of Theorem 1 in the introduction). In practice, however, the iteration ends after a finite (but large) number of steps producing a value Y very close but smaller than y^*. This leads to a numerical approximation (from below) $Z(r_1, r_2) = t_0 e^Y$ of the bound $Z^*(r_1, r_2)$. In Table 2, we give the parameter γ, the starting bound $z_0 := t_0 e^{y_0}$, an upper bound for $F_\gamma(y_0)$, m (the number of polynomials P_j used in the computation), and the new numerical bound $Z(r_1, r_2)$ produced by the algorithm.

It is illuminating to illustrate the state of affairs by means of a plot of the function $F_\gamma(\cdot)$. In Figure 1 we have plotted $F_\gamma(\log(\frac{z}{t_0}))$ as a function of z for the case $r_1 = 1$, $r_2 = 2$ (corresponding to the 10th line of Table 2). It is an empirical observation that for the other cases the graph of $F_\gamma(\log(\frac{z}{t_0}))$ is qualitatively of the

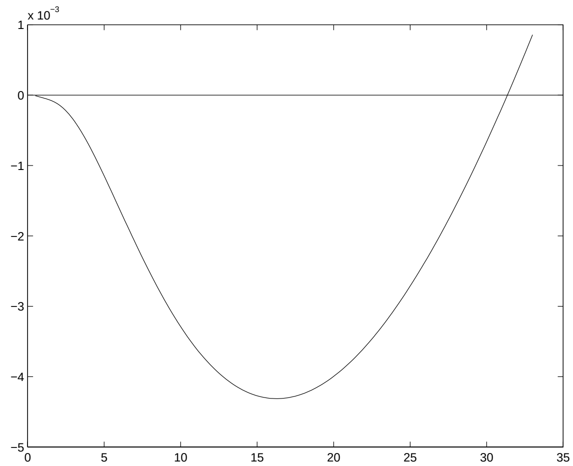

FIGURE 1. Graph of $F_\gamma(\log(\frac{z}{t_0}))$ for $r_1 = 1,\ r_2 = 2$ and $\gamma = 1.27$

same shape. Notice that in the case under consideration we have at the starting point y_0 that $z_0 = 2.9145$ (see Table 2). This point is close to the zero on the left. For the numerical approximation (from below) Y of y^* we have on the other hand that $z = 31.025$, which is much bigger and close to the point where the function changes sign.

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References

[C] Cohen, H. et al., ***PARI***, *Freeware available by anonymous FTP from* `megrez.math.-u-bordeaux.fr, directory pub/pari`.

[F1] Friedman, E.: *Analytic formulas for the regulator of a number field.* Invent. Math. **98** (1989), 599–622. MR **91c:**11061

[F2] Friedman, E.: *Hecke's integral formula.* Sém. Théorie des Nombres de Bordeaux 1987–1988, exposé 5. MR **90i:**11136

[G-R] Gradshteyn, I. S. and Ryzhik, I. M.: *Table of integrals, series, and products.* New York: Academic Press, 1994. MR **94g:**00008

[L] Lang, S.: *Algebraic Number Theory.* Addison-Wesley, Reading. Mass., 1970. MR **44:**181

[M] Mulholland, H.P.: *On the product of n complex homogeneous linear forms.* J. London Math. Soc. **35**, 241-250 (1960). MR **22:**4703

[N] Narkiewicz, W.: *Elementary and Analytic Theory of Algebraic Numbers.* Springer- Verlag, Berlin, 1990. MR **91h:**11107

[O] Odlyzko, A.M.: *Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta function: a survey of recent results.* Sém. Théorie des Nombres de Bordeaux **2**, 119-141 (1990). MR **91i:**11154

[R] Rogers, C.A.: *The product of n real homogeneous linear forms.* Acta Math. **83**, 185-208 (1950). MR **11:**501e

[Zi] Zimmert, R.: *Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung.* Invent. Math. **62**, 367-380 (1981). MR **83g:**12008

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