EACH AVERAGING TECHNIQUE YIELDS RELIABLE
A POSTERIORI ERROR CONTROL IN FEM
ON UNSTRUCTURED GRIDS.
PART II: HIGHER ORDER FEM

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ABSTRACT. Averaging techniques are popular tools in adaptive finite element methods since they provide efficient a posteriori error estimates by a simple postprocessing. In the second paper of our analysis of their reliability, we consider conforming h-FEM of higher (i.e., not of lowest) order in two or three space dimensions. In this paper, reliablility is shown for conforming higher order finite element methods in a model situation, the Laplace equation with mixed boundary conditions. Emphasis is on possibly unstructured grids, non-smoothness of exact solutions, and a wide class of local averaging techniques. Theoretical and numerical evidence supports that the reliability is up to the smoothness of given right-hand sides.

1. INTRODUCTION

Given the exact solution u and an approximate solution u_h of a second order elliptic partial differential equation, an averaging technique for a posteriori error control is a postprocessing algorithm that provides q_h from the input ∇u_h. The true energy error $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ is then estimated by $\|q_h - \nabla u_h\|_{L^2(\Omega)}$. The underlying motivation is that q_h is a "smoother" approximation of ∇u or/and of "higher order". So far, mathematical justifications for regarding $\|q_h - \nabla u_h\|_{L^2(\Omega)}$ as an approximation to $\|\nabla(u - u_h)\|_{L^2(\Omega)}$ have been based on superconvergence phenomena or given for ∇u_h piecewise constant only. This paper aims to establish the reliability and efficiency of local averaging: For each edge E, written $E \in \mathcal{E}$, with neighbourhood ω_E (i.e., the union of elements with face E) we suggest to compute (in case of pure Dirichlet problems)

$$\eta_E := \min\{\|\nabla u_h - q_E\|_{L^2(\omega_E)} : q_E \in \mathcal{P}_{d_E}(\omega_E)^d\} \qquad \text{and} \qquad \eta_Z^2 := \sum_{E \in \mathcal{E}} \eta_E^2.$$

The polynomial degree d_E on ω_E is chosen according to the elementwise degrees of u_h on ω_E, and $\mathcal{P}_{d_E}(\omega_E)$ denotes algebraic polynomials of degree $\leq d_E$ regarded as functions on ω_E. We analyse estimates of the form

$$(1.1) \qquad c_1 \eta_Z + \text{h.o.t.} \leq \|\nabla(u - u_h)\|_{L^2(\Omega)} \leq c_2 \eta_Z + \text{h.o.t.}$$

for the error estimator $\eta_Z = \left(\sum_{E \in \mathcal{E}} \eta_E^2\right)^{1/2}$ and study the qualitative behaviour of the constants $c_1, c_2 > 0$ and the higher order terms (h.o.t.) in (1.1). It turns out that the higher order terms in the efficiency estimate, i.e., the first inequality in (1.1), depend on the smoothness of the exact solution u while the higher order terms in the reliability estimate, i.e., the second inequality in (1.1), merely depend on the smoothness of given right-hand sides. The constants c_1 and c_2 are independent of the mesh-size but depend on the polynomial degrees.

We stress that the upper bound of (1.1) shows reliability of *any* local averaging scheme in the sense that, whatever choice of $q_E \in \mathcal{P}_{d_E}(\omega_E)^d$ we have, up to higher order terms (which are easily computed from the given right-hand sides f, g, and u_D),

$$(1.2) \qquad \|\nabla(u - u_h)\|_{L^2(\Omega)} \le c_3 \left(\sum_{E \in \mathcal{E}} \|\nabla u_h - q_E\|_{L^2(\omega_E)}^2\right)^{1/2}.$$

The error term η_E measures the flux difference of two neighbouring elements and so might be regarded as an edge contribution generalising $h_E \int_E [\partial u_h / \partial n]^2 \, ds$ from standard residual-based error estimations to higher order finite elements. Hence, (1.1) indicates the dominance of edge-contributions, which is true for lowest order finite elements [CV], [R] but known to be false for general higher-order polynomials [Y1], [Y2]. From this point of view, (1.2) appears surprising: the volume residual $f - \Delta_{\mathcal{T}} u_h$ does not enter the upper bound directly.

In case of lowest order finite elements, local and global averaging is indeed equivalent [CB], and so q_E can be chosen as a $\mathcal{T}$-piecewise polynomial that is globally continuous, as suggested in [ZZ]. However, for higher order finite element methods it is conjectured that global averaging is *not* equivalent to local averaging.

The outline of the remaining part of this paper is as follows. Approximation and stability properties of a weak approximation operator of [Ca], [CV] are generalised to higher-order approximation in Section 2. Some preliminary results are given in Section 3 in the spirit of [CB]. The main results on (1.1) are stated and proved in Section 4. Three examples with uniform, adapted, and perturbed meshes and a variety of polynomial order finite element methods conclude the paper in Section 5.

2. Approximation in higher order finite element spaces

The Lipschitz boundary $\Gamma = \partial\Omega$ of the bounded domain Ω is split into a closed Dirichlet part Γ_D with positive surface measure and a remaining, relatively open and possibly empty, Neumann part $\Gamma_N := \Gamma \setminus \Gamma_D$. Suppose $\mathcal{T}$ be a regular triangulation of the domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, in the sense of Ciarlet [BS], [Ci] (no hanging node, domain is matched exactly) with piecewise affine Lipschitz boundary $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$, i.e., $\mathcal{T}$ consists of a finite number of closed subsets of $\overline{\Omega}$, that cover $\overline{\Omega} = \bigcup \mathcal{T}$. Each element $T \in \mathcal{T}$ is either an interval $T = \text{conv}\{a, b\}$ if $d = 1$, a triangle $T = \text{conv}\{a, b, c\}$ or a parallelogram $T = \text{conv}\{a, b, c, d\}$ if $d = 2$, and a tetrahedron $T = \text{conv}\{a, b, c, d\}$ or a parallelepiped $T = \text{conv}\{a, ..., h\}$ if $d = 3$. The extremal points a, b, c, d are called vertices; the faces $E \subseteq \partial T$, e.g. $E = \text{conv}\{a, b\}$ in $d = 2$ or $E = \text{conv}\{a, b, c\}$ in $d = 3$, are called edges. The set of all vertices and all edges appearing for some T in $\mathcal{T}$ are denoted as $\mathcal{N}$ and $\mathcal{E}$. Two distinct and intersecting T_1 and T_2 share either an entire edge or a vertex. Each edge $E \in \mathcal{E}$ on the boundary Γ belongs either to Γ_D, written $E \in \mathcal{E}_D$, or to $\overline{\Gamma}_N$, written

$E \in \mathcal{E}_N$. Therefore the set of edges is partitioned into $\mathcal{E}_\Omega := \{E \in \mathcal{E} : E \not\subset \Gamma\}$, $\mathcal{E}_D$, and $\mathcal{E}_N$. We stress that the union of all edges $\bigcup \mathcal{E}$ denotes the skeleton of edges in $\mathcal{T}$, i.e., the set of all points x that belong to some boundary $x \in \partial T$ of some element $T \in \mathcal{T}$. Finally, $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ denotes the set of free nodes.

For $T \in \mathcal{T}$, let $P_T^k := \mathcal{P}_k(T)$ if T is a triangle or tetrahedron, or $P_T^k := \mathcal{Q}_k(T)$ if T is a parallelogram or parallelepiped. Here, $\mathcal{P}_k(K)$ resp. $\mathcal{Q}_k(K)$ denotes the set of algebraic polynomials in d variables on K of total resp. partial degree $\leq k$. The space $\mathcal{L}^k(\mathcal{T})$ of (possibly discontinuous) $\mathcal{T}$–piecewise polynomials of degree $\leq k$ is the set of all $U \in L^\infty(\Omega)$ with $U|_T \in P_T^k$ for all T in $\mathcal{T}$. Set

$$\mathcal{S}^k(\mathcal{T}) := \mathcal{L}^k(\mathcal{T}) \cap C(\Omega) \quad \text{and} \quad \mathcal{S}_D^1(\mathcal{T}) := \{u_h \in \mathcal{S}^1(\mathcal{T}) : u_h|_{\Gamma_D} = 0\}.$$

Let $(\varphi_z : z \in \mathcal{N})$ denote the nodal basis of $\mathcal{S}^1(\mathcal{T})$, i.e., $\varphi_z \in \mathcal{S}^1(\mathcal{T})$ satisfies $\varphi_z(x) = 0$ if $x \in \mathcal{N} \setminus \{z\}$, and $\varphi_z(z) = 1$. Note that $(\varphi_z | z \in \mathcal{N})$ is a partition of unity and the open patches

$$(2.1) \qquad\qquad\qquad \omega_z := \{x \in \Omega : 0 < \varphi_z(x)\}$$

form an open cover $(\omega_z : z \in \mathcal{N})$ of Ω with finite overlap.

In order to define an approximation-operator $\mathcal{J}$, we choose for each fixed node $z \in \mathcal{N} \setminus \mathcal{K}$ a neighbouring free node $\zeta \in \mathcal{K}$ and thereby define a relation R on $\mathcal{N}$ where zRz if $z \in \mathcal{K}$. Then, let

$$(2.2) \qquad\qquad \psi_z := \sum_{\zeta \in \mathcal{N}, \zeta R z} \varphi_\zeta \quad \text{and} \quad \Omega_z := \text{interior}(\text{supp}\,\psi_z).$$

We require that for each $z \in \mathcal{K}$, Ω_z is connected and $\varphi_z \neq \psi_z$ implies that $(\partial\Omega_z) \cap \Gamma_D$ has a positive surface measure. As $(\{\zeta \in \mathcal{N} : \zeta R z\} : z \in \mathcal{K})$ is a partition of $\mathcal{N}$, $(\psi_z : z \in \mathcal{K})$ is a partition of unity. For each $z \in \mathcal{K}$, we define the degree (minimal degree allowed on Ω_z minus one)

$$(2.3) \qquad\qquad\qquad d(z) := \max\{k \in \mathbb{N}_0 : \mathcal{P}_k(\Omega_z)\,\varphi_z \subseteq \mathcal{S}\},$$

where $\mathcal{P}_k(\Omega_z)$ denotes the set of all polynomials on $\mathbb{R}^d$ of total degree at most k restricted to Ω_z. The set $\mathcal{S} \subseteq H^1(\Omega)$ is some finite element space consisting of functions that are $\mathcal{T}$-elementwise polynomials and globally continuous. Moreover, we require that $\mathcal{S}_D^1(\mathcal{T}) \subseteq \mathcal{S}$, which implies that $d(z)$ is well defined and greater than or equal than zero.

For $g \in L^1(\Omega)$ and $z \in \mathcal{K}$, let $g_z \in \mathcal{P}_{d(z)}(\Omega_z)$ be defined by

$$(2.4) \qquad\qquad \int_{\Omega_z} (g_z\varphi_z - g\psi_z)q_z \, dx = 0 \quad \text{for all} \quad q_z \in \mathcal{P}_{d(z)}(\Omega_z),$$

and then let

$$(2.5) \qquad\qquad\qquad \mathcal{J}g := \sum_{z \in \mathcal{K}} g_z\varphi_z \in \mathcal{S} \cap H_D^1(\Omega).$$

Remark 2.1. Notice that $g_z\varphi_z \in \mathcal{S}$, because of (2.3).

Remark 2.2. Condition (2.4) characterises the unique minimiser g_z of

$$(2.6) \qquad\qquad\qquad \frac{1}{2}\int_{\Omega_z} \varphi_z q_z^2 \, dx - \int_{\Omega_z} g\psi_z q_z \, dx$$

amongst all q_z in $\mathcal{P}_{d(z)}(\Omega_z)$. In particular, there exists a unique solution to (2.4), i.e., $\mathcal{J}g$ is well defined and belongs to $\mathcal{S}$ even for $g \in L^1(\Omega)$.

The local mesh-sizes are denoted by $h_{\mathcal{T}}$ and $h_{\mathcal{E}}$, where $h_{\mathcal{T}} \in \mathcal{L}^0(\mathcal{T})$ is the element-size, $h_{\mathcal{T}}|_T := h_T := \operatorname{diam}(T)$ for $T \in \mathcal{T}$, and the edge-size $h_{\mathcal{E}} \in L^\infty(\bigcup \mathcal{E})$ is defined on the union or skeleton $\bigcup \mathcal{E}$ of all edges E in $\mathcal{E}$ by $h_{\mathcal{E}}|_E := h_E := \operatorname{diam}(E)$. The patch-size $h_z := \operatorname{diam}(\Omega_z)$ is defined for each node $z \in \mathcal{K}$ separately.

Theorem 2.1. *There exist $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constants $c_4, c_5, c_6, c_7 > 0$ such that, for all $g \in H_D^1(\Omega)$ and $f \in L^2(\Omega)$,*

$$(2.7) \qquad \|\nabla(g - \mathcal{J}g)\|_{L^2(\Omega)} \le c_4 \|\nabla g\|_{L^2(\Omega)},$$

$$\int_\Omega f(g - \mathcal{J}g)\, dx \le c_5 \|\nabla g\|_{L^2(\Omega)}$$

$$(2.8) \qquad \times \left(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f - f_z\|_{L^2(\Omega_z)}^2 \right)^{1/2},$$

$$(2.9) \qquad \|h_{\mathcal{T}}^{-1}(g - \mathcal{J}g)\|_{L^2(\Omega)} \le c_6 \|\nabla g\|_{L^2(\Omega)},$$

$$(2.10) \qquad \|h_{\mathcal{E}}^{-1/2}(g - \mathcal{J}g)\|_{L^2(\Gamma_N)} \le c_7 \|\nabla g\|_{L^2(\Omega)}.$$

The constants c_4, c_5, c_6, c_7 only depend on Ω, Γ_D, Γ_N, the degrees $d(z)$, $z \in \mathcal{K}$, and the shapes of the elements $T \in \mathcal{T}$ and the patches Ω_z, $z \in \mathcal{K}$.

Proof. In this proof and at similar occasions, $\lesssim$ abbreviates an inequality $\le$ up to a constant $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent factor. Also, $\|\cdot\|_{p,K}$ abbreviates $\|\cdot\|_{L^p(K)}$ and we neglect K if Ω is meant, i.e., $\|\cdot\|_2 := \|\cdot\|_{2,\Omega}$. Hence, e.g., (2.7) could be phrased as $\|\nabla \mathcal{J}g - \nabla g\|_2 \lesssim \|\nabla g\|_2$.

The local key estimate for the stability and the approximation property of $\mathcal{J}$ will be

$$(2.11) \qquad \|g_z \varphi_z - g \psi_z\|_{2,\Omega_z} \lesssim h_z \|\nabla g\|_{2,\Omega_z}.$$

(The constant in (2.11) is h_z–independent but depends on $d(z)$ and the shape of Ω_z.) For the proof of (2.11) let $\overline{g}_z$ denote the integral mean of g on Ω_z. Then, $g_z - \overline{g}_z \in \mathcal{P}_{d(z)}(\Omega_z)$ and

$$(2.12) \qquad \|g_z - \overline{g}_z\|_{2,\Omega_z} \le c_8 \|\varphi_z(g_z - \overline{g}_z)\|_{2,\Omega_z}.$$

For a proof of (2.12), notice that $\|\cdot\|_{2,\Omega_z}$ and $\|\varphi_z \cdot\|_{2,\Omega_z}$ are norms on $\mathcal{P}_{d(z)}(\Omega_z)$ and so are equivalent. A scaling argument shows that the constant c_8 is h_z–independent (but of course depends on the shape of Ω_z and the degree $d(z)$). Since $\overline{g}_z \in \mathcal{P}_{d(z)}(\Omega_z)$ is allowed in (2.6), we have

$$(2.13) \qquad \int_{\Omega_z} \varphi_z g_z^2\, dx \le \int_{\Omega_z} \varphi_z \overline{g}_z^2\, dx - 2 \int_{\Omega_z} \psi_z g \overline{g}_z\, dx + 2 \int_{\Omega_z} \psi_z g g_z\, dx$$

and infer, with Cauchy's and Young's inequality,

$$\|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z}^2 \le \|(g_z - \overline{g}_z)\varphi_z^{1/2}\|_{2,\Omega_z}^2$$

$$= 2\int_{\Omega_z} \varphi_z \overline{g}_z(\overline{g}_z - g_z)\,dx + 2\int_{\Omega_z} \psi_z g(g_z - \overline{g}_z)\,dx$$

$$(2.14) \qquad = 2\int_{\Omega_z} \varphi_z(\overline{g}_z - g)(\overline{g}_z - g_z)\,dx + 2\int_{\Omega_z}(\psi_z - \varphi_z)g(g_z - \overline{g}_z)\,dx$$

$$\le \frac{1}{4}\|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z}^2 + 4\|g - \overline{g}_z\|_{2,\Omega_z}^2$$

$$+ \frac{1}{4c_8^2}\|g_z - \overline{g}_z\|_{2,\Omega_z}^2 + 4c_8^2\|(\psi_z - \varphi_z)g\|_{2,\Omega_z}^2.$$

Utilising (2.12) and absorbing $\|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z}^2$, we deduce from (2.14) that

$$(2.15) \qquad \|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z}^2 \le 8\|g - \overline{g}_z\|_{2,\Omega_z}^2 + 8c_8^2\|(\psi_z - \varphi_z)g\|_{2,\Omega_z}^2.$$

A Poincaré inequality yields

$$(2.16) \qquad \|g - \overline{g}_z\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}$$

with a constant factor that only depends on the shape of Ω_z; the weight-function $(\psi_z - \varphi_z)$ is non-zero only if $\Gamma_D \cap (\partial\Omega_z)$ has positive surface measure. Since $g = 0$ there, a Friedrichs' inequality yields

$$(2.17) \qquad \|(\psi_z - \varphi_z)g\|_{2,\Omega_z} \le \|g\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}$$

for an h_z–independent constant that depends on the shape of Ω_z and $(\partial\Omega_z) \cap \Gamma_D$ only. Therefore, (2.15) yields

$$(2.18) \qquad \|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}.$$

To prove (2.11), we use the triangle inequality and (2.15)-(2.18) to verify

$$(2.19) \qquad \begin{aligned} \|g_z\varphi_z - g\psi_z\|_{2,\Omega_z} &\le \|(g_z - \overline{g}_z)\varphi_z\|_{2,\Omega_z} + \|(g - \overline{g}_z)\varphi_z\|_{2,\Omega_z} \\ &\quad + \|(\psi_z - \varphi_z)g\|_{2,\Omega_z} \lesssim h_z\|\nabla g\|_{2,\Omega_z}, \end{aligned}$$

which is (2.11). To verify (2.8), we use that $(\psi_z : z \in \mathcal{K})$ is a partition of unity and obtain with (2.11), (2.4) for any $f_z \in \mathcal{P}_{d(z)}(\Omega_z)$ that

$$(2.20)$$

$$\int_\Omega f(g - \mathcal{J}g)\,dx = \sum_{z\in\mathcal{K}}\int_{\Omega_z} f(g\psi_z - g_z\varphi_z)\,dx = \sum_{z\in\mathcal{K}}\int_{\Omega_z}(f - f_z)(g\psi_z - g_z\varphi_z)\,dx$$

$$\lesssim \sum_{z\in\mathcal{K}}\|f - f_z\|_{2,\Omega_z}h_z\|\nabla g\|_{2,\Omega_z} \lesssim \left(\sum_{z\in\mathcal{K}}h_z^2\|f - f_z\|_{2,\Omega_z}^2\right)^{1/2}\|\nabla g\|_2.$$

In the last step we used that $(\psi_z : z \in \mathcal{K})$ has a finite overlap that depends on the shape of the elements only. The proof of (2.8) is finished. Notice that $h_z \lesssim h_T$ for all $z \in \mathcal{K}$ and $T \in \mathcal{T}$ with $T \subseteq \overline{\Omega}_z$. Letting $f := h_T^{-2}(g - \mathcal{J}g)$ and $f_z = 0$, $z \in \mathcal{K}$, we deduce from (2.8) that

$$(2.21) \qquad \begin{aligned} \|h_T^{-1}(g - \mathcal{J}g)\|_2^2 &\lesssim \|\nabla g\|_2\left(\sum_{z\in\mathcal{K}}\|h_T^{-1}(g - \mathcal{J}g)\|_{2,\Omega_z}^2\right)^{1/2} \\ &\lesssim \|\nabla g\|_2\|h_T^{-1}(g - \mathcal{J}g)\|_2, \end{aligned}$$

which implies (2.9). To verify (2.7) we argue as above and additionally utilise $\sum_{z \in \mathcal{K}} \nabla \psi_z = 0$. Repeating the triangle inequality only a limited number of times (according to the finite overlap of the patches Ω_z), we infer

$$(2.22) \qquad \|\nabla g - \nabla \mathcal{J} g\|_2^2 \lesssim \sum_{z \in \mathcal{K}} \|\nabla(\psi_z g - \varphi_z g_z)\|_2^2.$$

Since $\overline{g}_z$ is constant, and recalling (2.16), we have

$$(2.23) \qquad h_z^{-1} \|g - \overline{g}_z\|_{2,\Omega_z} + \|\nabla(g - \overline{g}_z)\|_{2,\Omega_z} \lesssim \|\nabla g\|_{2,\Omega_z}.$$

Then, with the triangle inequality, Friedrichs' inequality, $\|\nabla \phi_\zeta\|_{\infty,\Omega_z} \leq 1/\rho_z$, and (2.23) we conclude that

$$(2.24) \qquad \begin{aligned} &\|\nabla(g\psi_z - g_z \varphi_z)\|_{2,\Omega_z} \\ &\quad \leq \|(\psi_z - \varphi_z)\nabla g\|_{2,\Omega_z} + \|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z} \\ &\qquad + \|\nabla(\varphi_z(\overline{g}_z - g))\|_{2,\Omega_z} + \|g\nabla(\psi_z - \varphi_z)\|_{2,\Omega_z} \\ &\quad \lesssim \|\nabla g\|_{2,\Omega_z} + h_z/\rho_z \|\nabla g\|_{2,\Omega_z} + \|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z}. \end{aligned}$$

Note that $h_z \lesssim \rho_z$. To estimate $\|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z}$ we observe that $\|\nabla(\varphi_z \cdot)\|_{2,\Omega_z}$ is a norm on $\mathcal{P}_{d(z)}(\Omega_z)$ and so is equivalent to $\|\varphi_z \cdot\|_{2,\Omega_z}$. A scaling argument shows that

$$(2.25) \qquad h_z \|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z} \lesssim \|\varphi_z(g_z - \overline{g}_z)\|_{2,\Omega_z},$$

where the h_z–independent constant depends on the shape of Ω_z and the degree $d(z)$ only. Utilising (2.11), (2.23), and Friedrichs' inequality, we obtain from (2.25) that

$$\begin{aligned} \frac{h_z}{c_{12}} \|\nabla(\varphi_z(g_z - \overline{g}_z))\|_{2,\Omega_z} &\leq \|\varphi_z g_z - \psi_z g\|_{2,\Omega_z} \\ &\qquad + \|(\psi_z - \varphi_z)g\|_{2,\Omega_z} + \|\varphi_z(g - \overline{g}_z)\|_{2,\Omega_z} \\ &\lesssim h_z \|\nabla g\|_{2,\Omega_z}. \end{aligned}$$

Employing this estimate in (2.24) and the resulting estimate in (2.22), we verify (2.7).

A trace inequality [BS], [Cl], [CF] is required for the proof of (2.10). For $E \in \mathcal{E}_N$ and a neighbouring element $T \in \mathcal{T}$ with $E \subset \partial T \cap \overline{\Gamma}_N$ we have, for all $w \in H^1(T)$,

$$(2.26) \qquad \|w\|_{2,E} \lesssim h_E^{-1/2} \|w\|_{2,T} + h_E^{1/2} \|\nabla w\|_{2,T}.$$

We denote $\omega_E := T$ for $E \in \mathcal{E}_N$ and $E \subseteq \partial T$. Since $h_T \lesssim h_E$, we deduce from (2.26) with $w = g - \mathcal{J}g$ by summing all edges E on Γ_N that

$$\begin{aligned} \sum_{E \in \mathcal{E}_N} h_E^{-1} \|g - \mathcal{J}g\|_{2,E}^2 &\lesssim \sum_{E \in \mathcal{E}_N} h_T^{-2} \|g - \mathcal{J}g\|_{2,\omega_E}^2 + \sum_{E \in \mathcal{E}_N} \|\nabla(g - \mathcal{J}g)\|_{2,\omega_E}^2 \\ &\lesssim \|\nabla g\|_{2,\Omega}^2, \end{aligned}$$

according to (2.7) and (2.9). This concludes the proof. $\qquad\qquad\qquad\square$

Remark 2.3. The constants in the theorem depend on the polynomial degrees in the finite element method and are expected to increase to infinity with the degree.

3. Basic estimates

In this section we first derive with the approximation operator $\mathcal{J}$ a global error estimate for a posteriori error control by averaging processes in an abstract setting. The estimate of this section is specified in the subsequent section to higher order conforming finite element methods.

Theorem 3.1. *Suppose* $p, q \in H(\mathrm{div}\,;\Omega)$ *and* $p_h \in \mathcal{L}^k(\mathcal{T})^d$ *with* $p \cdot n, q \cdot n \in L^2(\Gamma_N)$ *and*

$$(3.1) \qquad \int_\Omega (p - p_h) \cdot \nabla w_h \, dx = 0 \quad \text{for all } w_h \in \mathcal{S} \cap H_D^1(\Omega).$$

Then

$$
(3.2) \quad
\begin{aligned}
\sup_{\substack{w \in H_D^1(\Omega) \\ \|\nabla w\|_{L^2(\Omega)} = 1}} & \int_\Omega (p - p_h) \cdot \nabla w \, dx \leq c_4 \|p_h - q\|_{L^2(\Omega)} \\
& + c_5 \Big(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|\mathrm{div}\,(p - q) - f_z\|_{L^2(\Omega_z)}^2 \Big)^{1/2} \\
& + c_7 \|h_{\mathcal{E}}^{1/2} (p - q) \cdot n\|_{L^2(\Gamma_N)}.
\end{aligned}
$$

Proof. According to (3.1), (2.7), Cauchy's inequality, and an integration by parts we have, for each $w \in H_D^1(\Omega)$ with $\|\nabla w\|_2 = 1$, that

$$
(3.3) \quad
\begin{aligned}
\int_\Omega (p - p_h) \cdot \nabla w \, dx &= \int_\Omega (p - p_h) \cdot \nabla (w - \mathcal{J}w) \, dx \\
&= \int_\Omega (p - q) \cdot \nabla (w - \mathcal{J}w) \, dx + \int_\Omega (q - p_h) \cdot \nabla (w - \mathcal{J}w) \, dx \\
&\leq \int_{\Gamma_N} (w - \mathcal{J}w)\,(p - q) \cdot n \, dx \\
&\quad - \int_\Omega (w - \mathcal{J}w)\,\mathrm{div}\,(p - q) \, dx + c_4 \|p_h - q\|_{2,\Omega},
\end{aligned}
$$

since w and $\mathcal{J}w$ vanish on $\partial\Omega \setminus \Gamma_N$. Owing to (2.8) and (2.10) in Theorem 2.1, we conclude (3.2) from (3.3) and Cauchy's inequality. $\qquad\square$

The next lemma states that averaging over the patches Ω_z is dominated by averaging over smaller domains. Let $\omega_E := T_1 \cup T_2$ for an interior edge $E = T_1 \cap T_2 \in \mathcal{E}_\Omega$ and $\omega_E := T$ for an edge $E = T \cap \partial\Omega \in \mathcal{E}_D \cup \mathcal{E}_N$ on the boundary. Moreover, for $z \in \mathcal{K}$, let $\mathcal{E}_{\Omega_z}$ denote the set of all edges $E \subset \overline{\Omega}_z$ with $E \not\subset \partial\Omega_z$.

Lemma 3.1. *Suppose* $\mathcal{S} = \{v_h \in C(\overline{\Omega}) : \forall T \in \mathcal{T}, v_h|_T \in \mathcal{P}_{d(T)}(T)\}$ *for positive integers* $d(T)$, $T \in \mathcal{T}$, *and let* d_E, $E \in \mathcal{E}$, *be nonnegative integers. Then there exists a constant* $c_9 > 0$ *such that, for all* $u_h \in \mathcal{S}$ *and each* $z \in \mathcal{K}$, *we have*

$$(3.4) \quad \min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|\nabla u_h - q_z\|_{L^2(\Omega_z)}^2 \leq c_9 \sum_{E \in \mathcal{E}_{\Omega_z}} \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \|\nabla u_h - q_E\|_{L^2(\omega_E)}^2.$$

The constant c_9 *depends on the degrees* d_z *and* d_E *as well as on the shapes of the elements and patches, but not on their diameters.*

Proof. Set $p_h := \nabla u_h$ and let $\|\|p_h\|\|_{1,z}^2$ and $\|\|p_h\|\|_{2,z}^2$ denote the expressions on the left- and right-hand side in (3.4), respectively. Assume $u_h \in \mathcal{S}$ with $\|\|p_h\|\|_{2,z} = 0$.

We claim that $\||p_h\||_{1,z} = 0$. (The statement of the lemma then follows from compactness and scaling argument.) Since Ω_z is connected, there exists a sequence of edges $E_1, ..., E_J \in \mathcal{E}_{\Omega_z}$ such that $\omega_{E_j} \cap \omega_{E_{j+1}} \neq \emptyset$ and the open patches ω_{E_j} cover Ω_z, i.e., $\omega_{E_1} \cup ... \cup \omega_{E_J} = \Omega_z$. Since $\||p_h\||_{2,z} = 0$ we deduce that $p_h|_{\omega_{E_j}} \in \mathcal{P}_{d_{E_j}}(\omega_{E_j})^d$, $j = 1, ..., J$, and so $p_h|_{\Omega_z} \in \mathcal{P}_m(\Omega_z)^d$, where m is the smallest polynomial degree of p_h restricted to elements $T \subseteq \overline{\Omega}_z$, i.e., $m = \min\{m_T : T \subseteq \overline{\Omega}_z\}$ for $m_T := \min\{\ell \in \mathbb{N}_0 : p_h|_T \in \mathcal{P}_\ell(T)^d\}$. Since $m \leq d(T) - 1$ for all $T \subseteq \overline{\Omega}_z$ and $d(z) \geq \min_{T \in \mathcal{T}, T \subseteq \overline{\Omega}_z} d(T) - 2$, we have $m \leq d(z) + 1$. This implies $\||p_h\||_{1,z} = 0$. $\square$

Remark 3.1. Note that a hat function φ_z is not elementwise affine when z belongs to a parallelogram if $d = 2$ or a parallelepiped if $d = 3$. Then, $\mathcal{P}_{d(z)+1}(\Omega_z)^d$ cannot be reduced to $\mathcal{P}_{d(z)}(\Omega_z)^d$ in (3.4).

The following lemma includes the approximation of given boundary data by discrete functions. We denote by $\mathcal{L}^k(\mathcal{E}_N)$ the space of all (possibly discontinuous) functions on Γ_N which equal a polynomial of degree at most k on each edge $E \in \mathcal{E}_N$.

Lemma 3.2. *Let k and d_E, $E \in \mathcal{E}$, be nonnegative integers and let $p_h \in \mathcal{L}^k(\mathcal{T})^d$ and $g_h \in \mathcal{L}^{k+1}(\mathcal{E}_N)$. Then,*

(3.5)
$$\min_{q_h \in \mathcal{S}^{k+1}(\mathcal{T})^d} \left(\|p_h - q_h\|_{L^2(\Omega)}^2 + \|h_{\mathcal{E}}^{1/2}(g_h - q_h \cdot n)\|_{L^2(\Gamma_N)}^2 \right)$$
$$\leq c_{10} \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \left(\|p_h - q_E\|_{L^2(\omega_E)}^2 + h_E\|g_h - q_E \cdot n\|_{L^2(E \cap \Gamma_N)}^2 \right)$$

with a constant $c_{10} > 0$ that depends on the degrees k and d_E as well as on the shapes of the elements and patches but not on their diameters.

Proof. Let $\mathcal{T}_E := \{T \in \mathcal{T} : T \subseteq \overline{\omega}_E\}$ denote the restriction of $\mathcal{T}$ to ω_E, $E \in \mathcal{E}$. It is shown in [CB] that

(3.6)
$$\min_{q_h \in \mathcal{S}^{k+1}(\mathcal{T})^d} \left(\|p_h - q_h\|_{2,\Omega}^2 + \|h_{\mathcal{E}}^{1/2}(g_h - q_h \cdot n)\|_{2,\Gamma_N}^2 \right)$$
$$\lesssim \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{q_E \in \mathcal{S}^{k+1}(\mathcal{T}_E)^d} \left(\|p_h - q_E\|_{2,\omega_E}^2 + h_E\|g_h - q_E \cdot n\|_{2,E \cap \Gamma_N}^2 \right).$$

For $E \in \mathcal{E}_\Omega$ we have $\min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \|p_h - q_E\|_{2,\omega_E}^2 = 0$ if and only if $p_h \in \mathcal{P}_{d_E}(\omega_E)^d \cap \mathcal{L}^k(\mathcal{T}_E)^d \subset C(\omega_E)^d$, and $\min_{q_E \in \mathcal{S}^{k+1}(\mathcal{T}_E)^d} \|p_h - q_E\|_{2,\omega_E}^2 = 0$ if and only if $p_h \in C(\omega_E)^d \cap \mathcal{L}^k(\mathcal{T}_E)^d$. Hence a compactness and a scaling argument (from the context of equivalence of norms) show (with an $(h_{\mathcal{T}}, h_{\mathcal{E}})$-independent constant factor)

(3.7)
$$\min_{q_E \in \mathcal{S}^{k+1}(\mathcal{T}_E)^d} \|p_h - q_E\|_{2,\omega_E}^2 \lesssim \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \|p_h - q_E\|_{2,\omega_E}^2.$$

For an edge $E \in \mathcal{E}_N$ the corresponding minimum in the right-hand-side of (3.6) is zero if and only if $g_h = p_h \cdot n$, while the corresponding minimum over $\mathcal{P}_{d_E}(\omega_E)^d$ is zero if and only if $g_h = p_h \cdot n$ and $p_h|_{\omega_E} \in \mathcal{P}_{d_E}(\omega_E)^d$. By a compactness and a

scaling argument we deduce the inequality

$$(3.8) \quad \min_{q_E \in \mathcal{S}^{k+1}(\mathcal{T}_E)^d} \Big(\|p_h - q_E\|^2_{2,\omega_E} + h_E \|g_h - q_E \cdot n\|^2_{2,E \cap \Gamma_N} \Big)$$

$$\lesssim \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \Big(\|p_h - q_E\|^2_{2,\omega_E} + h_E \|g_h - q_E \cdot n\|^2_{2,E \cap \Gamma_N} \Big).$$

Combining (3.6), (3.7), and (3.8), we obtain (3.5). □

4. Higher order finite element methods

Given right-hand sides $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, and $u_D \in H^1(\Gamma_D) \cap C(\Gamma_D)$, let $u \in H^1(\Omega)$ denote the unique weak solution to

$$(4.1) \qquad\qquad -\Delta u \;=\; f \qquad \text{in } \Omega,$$

$$(4.2) \qquad\qquad u \;=\; u_D \qquad \text{on } \Gamma_D,$$

$$(4.3) \qquad\qquad \partial u / \partial n \;=\; g \qquad \text{on } \Gamma_N.$$

Suppose a finite element scheme, based on a regular triangulation $\mathcal{T}$, provided a discrete flux $p_h := \nabla u_h$ to the exact flux $p := \nabla u \in H(\text{div}; \Omega)$ such that $u_h \in \mathcal{S}$ and

$$(4.4) \quad \int_\Omega \nabla u_h \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx + \int_{\Gamma_N} g w_h \, ds \quad \text{for all } w_h \in \mathcal{S} \cap H^1_D(\Omega).$$

We will assume in Lemma 4.3 below that $u_h|_{\Gamma_D}$ interpolates u_D in nodes on Γ_D.

Theorem 4.1. *Assume $\nabla u_h \in \mathcal{L}^k(\mathcal{T})^d$ and (4.4). There exist $(h_\mathcal{T}, h_\mathcal{E})$-independent constants $c_{11}, c_{12} > 0$ such that*

$$(4.5) \quad
\begin{aligned}
\|\nabla(u &- u_h)\|_{L^2(\Omega)} \\
&\leq \min_{q_h \in \mathcal{S}^{k+1}(\mathcal{T})^d} \Big(c_{11} \|\nabla u_h - q_h\|_{L^2(\Omega)} + 2c_7 \|h_\mathcal{E}^{1/2}(g - q_h \cdot n)\|_{L^2(\Gamma_N)} \Big) \\
&\quad + c_{12} \Big(\sum_{z \in \mathcal{K}} \min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|\nabla u_h - q_z\|^2_{L^2(\Omega_z)} \Big)^{1/2} \\
&\quad + \inf_{v|_{\Gamma_D} = u_D} \|\nabla(u_h - v)\|_{L^2(\Omega)} \\
&\quad + 2\sqrt{3} c_5 \Big(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f - f_z\|^2_{L^2(\Omega_z)} \Big)^{1/2}.
\end{aligned}$$

In the infimum, "$v|_{\Gamma_D} = u_D$" stands for all $v \in H^1(\Omega)$ with $v = u_D$ on Γ_D.

Proof. Abbreviate $e := u - u_h$ and let $q_h \in \mathcal{S}^{k+1}(\mathcal{T})^d$. Assume that $v \in H^1(\Omega)$ satisfies $v = u_D$ on Γ_D and $\|\nabla(u_h - v)\|_2 \leq \|\nabla e\|_2$. Then (4.1)-(4.4) imply (3.1). Hence, we may choose $q = q_h$ and $w = u - v$ in Theorem 3.1 to obtain (with Cauchy's inequality for the second term in the right-hand side of the subsequent

equality) that

$$\|\nabla e\|_2^2 = \int_\Omega \nabla e \cdot \nabla w \, dx + \int_\Omega \nabla e \cdot \nabla(v - u_h) \, dx$$

$$\leq \|\nabla w\|_2 \left(c_4 \|\nabla u_h - q_h\|_2 + c_7 \|h_{\mathcal{E}}^{1/2} (g - q_h \cdot n)\|_{2,\Gamma_N} \right.$$

(4.6)

$$\left. + c_5 \left(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f + \operatorname{div} q_h - f_z\|_{2,\Omega_z}^2 \right)^{1/2} \right)$$

$$+ \|\nabla(u_h - v)\|_2 \|\nabla e\|_2.$$

Since $\|\nabla w\|_2 \leq \|\nabla e\|_2 + \|\nabla(u_h - v)\|_2 \leq 2\|\nabla e\|_2$, we can divide (4.6) by $\|\nabla e\|_2$ to verify

(4.7) $\|\nabla e\|_2 \leq 2c_4 \|\nabla u_h - q_h\|_2 + 2c_7 \|h_{\mathcal{E}}^{1/2} (g - q_h \cdot n)\|_{2,\Gamma_N} + \|\nabla(u_h - v)\|_2$

$$+ 2c_5 \left(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f + \operatorname{div} q_h - f_z\|_{2,\Omega_z}^2 \right)^{1/2}.$$

Let $\operatorname{div}_{\mathcal{T}}$ denote the $\mathcal{T}$-piecewise action of the div-operator. The triangle inequality in the last summand in (4.7) and $h_z \lesssim h_T$ for $z \in T \cap \mathcal{N}$ and $T \in \mathcal{T}$, a summation over elements, and $\operatorname{div}\left(\mathcal{P}_{d(z)+1}(\Omega_z)^d\right) \subseteq \mathcal{P}_{d(z)}(\Omega_z)$ show that

$$\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f + \operatorname{div} q_h - f_z\|_{2,\Omega_z}^2$$

$$\leq 3 \|h_{\mathcal{T}} \operatorname{div}_{\mathcal{T}}(\nabla u_h - q_h)\|_2^2$$

(4.8)

$$+ 3 \sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f - f_z\|_{2,\Omega_z}^2$$

$$+ 3 \sum_{z \in \mathcal{K}} \min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|h_z \operatorname{div}_{\mathcal{T}}(\nabla u_h - q_z)\|_{2,\Omega_z}^2.$$

A $\mathcal{T}$-elementwise inverse estimate shows

$$\|h_{\mathcal{T}} \operatorname{div}_{\mathcal{T}}(\nabla u_h - q_h)\|_2 \lesssim \|\nabla u_h - q_h\|_2$$

and

$$\|h_z \operatorname{div}_{\mathcal{T}}(\nabla u_h - q_z)\|_{2,\Omega_z} \lesssim \|\nabla u_h - q_z\|_{2,\Omega_z}$$

(with constants that depend on the polynomial degrees and on the shape of the finite elements but not on their diameters). Utilising this in (4.7)-(4.8), we deduce (4.5). □

The following lemmas show that the terms concerning the right-hand-sides f, g, and u_D in (4.5) are of higher order, provided the given data functions are smooth enough.

Lemma 4.1. *For all $z \in \mathcal{K}$, there exists an h_z-independent constant $c_{13} > 0$ such that, if $f|_{\Omega_z} \in H^{d(z)+1}(\Omega_z)$, we have $(D^{d(z)+1} f = (\partial^\alpha f)_{|\alpha|=d(z)+1}$ denotes the vector of all partial derivatives of order $d(z) + 1$)*

(4.9) $$\min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f - f_z\|_{L^2(\Omega_z)} \leq c_{13} \, h_z^{d(z)+1} \|D^{d(z)+1} f\|_{L^2(\Omega_z)}.$$

Proof. The proof of the lemma can be found, e.g., in [BS]. □

For a function $w \in H^1(\gamma)$ we denote by $\partial w/\partial t$ and $\partial w/\partial s$ its gradient with respect to the arc-length along the $(d-2)$– respectively $(d-1)$–dimensional sub-manifold γ.

Lemma 4.2. *Let $E \in \mathcal{E}_N$ and assume $g|_E \in H^{d_E}(E)$ for some $d_E \geq 1$. Then there exist $q_h \in \mathcal{L}^{d_E}(\omega_E)^d$ and a constant $c_{14} > 0$ that is independent of h_E and g, such that, for $g_h := q_h \cdot n$, we have*

$$(4.10) \qquad \|g - g_h\|_{L^2(E)} \leq c_{14}\, h_E^{d_E} \|\partial^{d_E} g/\partial s^{d_E}\|_{L^2(E)}.$$

Proof. The statement is the $(d-1)$-dimensional version of the previous lemma. □

Lemma 4.3. *Let $u_D \in H^1(\Gamma_D) \cap C(\Gamma_D)$ satisfy $u_D|_E \in H^{d_E+1}(E)$ for some $d_E \geq 1$, $E \in \mathcal{E}_D$, and $u_h(z) = u_D(z)$ for all $z \in \mathcal{N} \cap \Gamma_D$. If u_h also satisfies*

$$(4.11) \qquad \|\partial^\ell(u_h - u_D)/\partial s^\ell\|_{L^2(E)} \leq c_{15} h_E^{d_E+1-\ell} \|\partial^{d_E+1} u_D/\partial s^{d_E+1}\|_{L^2(E)}$$

for all $E \in \mathcal{E}_D$ and $\ell = 0, 1, 2$ with an $h_\mathcal{E}$-independent constant $c_{15} > 0$, then there exists an $h_\mathcal{E}$-independent constant $c_{16} > 0$ such that

$$(4.12)$$
$$\inf_{v|_{\Gamma_D} = u_D} \|\nabla(v - u_h)\|_{L^2(\Omega)} \leq c_{16}\Big(\sum_{E \in \mathcal{E}_D} h_E^{2d_E+1} \|\partial^{d_E+1} u_D/\partial s^{d_E+1}\|_{L^2(E)}^2\Big)^{1/2}.$$

Proof. The trace theorems yield (both infima are attained)

$$(4.13) \qquad \inf_{\substack{v \in H^1(\Omega) \\ v|_{\Gamma_D} = u_D}} \|\nabla(u_h - v)\|_2 = \inf_{\substack{\tilde{v} \in H^1(\Omega) \\ \tilde{v}|_{\Gamma_D} = u_h|_{\Gamma_D} - u_D}} \|\nabla\tilde{v}\|_2 \lesssim \|u_h - u_D\|_{H^{1/2}(\Gamma_D)}.$$

The norm in $H^{1/2}(\Gamma_D)$ is equivalent to the norm of minimal extension to Γ, and so (4.13) leads to

$$\inf_{\substack{v \in H^1(\Omega) \\ v|_{\Gamma_D} = u_D}} \|\nabla(u_h - v)\|_2 \lesssim \inf_{\substack{w \in H^{1/2}(\Gamma) \\ w|_{\Gamma_D} = u_h|_{\Gamma_D} - u_D}} \|w\|_{H^{1/2}(\Gamma)}.$$

We construct an extension w of $u_h|_{\Gamma_D} - u_D$ to Γ. For $d = 2$ we extend $u_h|_{\Gamma_D} - u_D$ by zero to Γ and obtain a continuous extension w. For $d = 3$ this extension might be discontinuous, which forces a modification on edges near Γ_D. For $E \in \mathcal{E} \setminus \mathcal{E}_D$ with $E \cap \Gamma_D = \emptyset$ or $E \cap \Gamma_D = \{z\}$ for some $z \in \mathcal{N}$, set $w|_E = 0$. In the remaining cases $E \in \mathcal{E} \setminus \mathcal{E}_D$ with $E \cap \Gamma_D = \bigcup_{j=1}^J E \cap F_j = \bigcup_{j=1}^J \text{conv}\{a_j, b_j\}$ with $J \in \{1, 2, 3\}$, $F_j \in \mathcal{E}_D$, and $a_j, b_j \in \mathcal{N} \cap F_j$, $a_j \neq b_j$, $j = 1, ..., J$, we proceed as follows. Let s_E denote the center of inertia of E. On each triangle $G_j := \text{conv}\{a_j, b_j, s_E\}$ we let w be an harmonic extension of $u_h|_{\Gamma_D} - u_D$ from $S_j := \text{conv}\{a_j, b_j\}$ to G_j such that $w|_{\partial G_j \setminus S_j} = 0$. Note that w is continuous on G_j. An interpolation argument and $\|w\|_{2, \partial G_j} \leq h_E \|\partial w/\partial t\|_{2, \partial G_j}$ reveal that

$$(4.14) \qquad \begin{aligned} \|\partial w/\partial s\|_{2, G_j}^2 &\lesssim \|w\|_{H^{1/2}(\partial G_j)}^2 \lesssim \|w\|_{2, \partial G_j} \|w\|_{H^1(\partial G_j)} \leq h_E \|\partial w/\partial t\|_{2, \partial G_j}^2 \\ &= h_E \|\partial(u_h - u_D)/\partial t\|_{2, S_j}^2. \end{aligned}$$

By a trace theorem [CF], [BS] for the tangential gradient $\partial(u_h - u_D)/\partial s$ from F_j to $G_j \cap F_j = S_j$ we have

$$\begin{aligned} h_E &\|\partial(u_h - u_D)/\partial t\|_{2, S_j}^2 \\ &\lesssim \|\partial(u_h - u_D)/\partial s \cdot t\|_{2, F_j}^2 + h_E^2 \|\partial(\partial(u_h - u_D)/\partial s \cdot t)/\partial s\|_{2, F_j}^2, \end{aligned}$$

where t denotes the unit tangent vector along S_j. Since t is constant, this shows that

$$(4.15) \qquad h_E \|\partial(u_h - u_D)/\partial t\|_{2,S_j}^2 \lesssim \|u_h - u_D\|_{H^1(F_j)}^2 + h_E^2 \|u_h - u_D\|_{H^2(F_j)}^2.$$

Applying (4.15) in (4.14), we find that

$$(4.16) \qquad \|\partial w/\partial s\|_{2,G_j}^2 \lesssim \|u_h - u_D\|_{H^1(F_j)}^2 + h_E^2 \|u_h - u_D\|_{H^2(F_j)}^2$$

and, using Friedrichs' inequality $\|w\|_{2,G_j} \lesssim h_E \|\partial w/\partial s\|_{2,G_j}$,

$$(4.17) \qquad \|w\|_{2,G_j}^2 \lesssim h_E^2 \|(u_h - u_D)\|_{H^1(F_j)}^2 + h_E^4 \|(u_h - u_D)\|_{H^2(F_j)}^2.$$

Note that the extension is continuous and affects neighbouring edges of Γ_D only.

Let $w_1, ..., w_n$ denote the nonvanishing functions amongst $(w\,\varphi_z|_\Gamma : z \in \mathcal{N} \cap \Gamma)$. By an interpolation estimate and Friedrichs' inequality we infer that, for each $j = 1, ..., n$,

$$(4.18) \qquad \begin{aligned} \|w_j\|_{H^{1/2}(\Gamma)}^2 &\lesssim \|w_j\|_{2,\Gamma} \|w_j\|_{H^1(\Gamma)} \\ &\leq \|w_j\|_{2,\Gamma}(\|w_j\|_{2,\Gamma} + \|\partial w_j/\partial s\|_{2,\Gamma}) \lesssim \|h_{\mathcal{E}}^{1/2} \partial w_j/\partial s\|_{2,\Gamma}^2. \end{aligned}$$

To localise the $H^{1/2}$-norm on Γ_D, we employ the arguments from [CMS]. It was shown therein that there exists a partition $(I_\ell : \ell = 1, ..., L)$ of $\{1, ..., n\}$ such that $w_j w_k = 0$ for any distinct $j, k \in I_\ell$. L is independent of $h_\mathcal{T}$, and for each $\ell = 1, ..., L$ we have

$$(4.19) \qquad \|\sum_{j \in I_\ell} w_j\|_{H^{1/2}(\Gamma)}^2 \lesssim \sum_{j \in I_\ell} \|w_j\|_{H^{1/2}(\Gamma)}^2.$$

Using the inequality $(\sum_{\ell=1}^L a_\ell)^2 \leq L \sum_{\ell=1}^L a_\ell^2$ and (4.19), we deduce that

$$(4.20) \qquad \begin{aligned} \|w\|_{H^{1/2}(\Gamma)}^2 &\leq L \sum_{\ell=1}^L \|\sum_{j \in I_\ell} w_j\|_{H^{1/2}(\Gamma)}^2 \lesssim L \sum_{\ell=1}^L \sum_{j \in I_\ell} \|w_j\|_{H^{1/2}(\Gamma)}^2 \\ &\lesssim L \sum_{\ell=1}^L \sum_{j \in I_\ell} \|h_{\mathcal{E}}^{1/2} \partial w_j/\partial s\|_{2,\Gamma}^2. \end{aligned}$$

With $w_j = w\,\varphi_z$, $|\partial w_j/\partial s| \leq |\partial w/\partial s| + |w||\partial \varphi_z/\partial s| \lesssim |\partial w/\partial s| + |w|/h_E$, and $L \lesssim 1$ this shows that

$$(4.21) \qquad \begin{aligned} \|w\|_{H^{1/2}(\Gamma)} &\lesssim \sum_{\ell=1}^J \sum_{j \in I_\ell} \left(\|h_{\mathcal{E}}^{1/2} \partial w/\partial s\|_{2,\mathrm{supp}\,w_j}^2 + \|h_{\mathcal{E}}^{-1/2} w\|_{2,\mathrm{supp}\,w_j}^2 \right) \\ &\lesssim \|h_{\mathcal{E}}^{1/2} \partial w/\partial s\|_{2,\Gamma}^2 + \|h_{\mathcal{E}}^{-1/2} w\|_{2,\Gamma}^2. \end{aligned}$$

Utilising $w|_E = (u_h|_{\Gamma_D} - u_D)|_E$ for all $E \in \mathcal{E}_D$, and the estimates (4.16) and (4.17), we infer

$$(4.22) \qquad \begin{aligned} \|w\|_{H^{1/2}(\Gamma)}^2 &\lesssim \|h_{\mathcal{E}}^{-1/2}(u_h - u_D)\|_{2,\Gamma_D}^2 + \|h_{\mathcal{E}}^{1/2} \partial(u_h - u_D)/\partial s\|_{2,\Gamma_D}^2 \\ &\quad + \|h_{\mathcal{E}}^{3/2} \partial_{\mathcal{E}}^2(u_h - u_D)/\partial s^2\|_{2,\Gamma_D}^2. \end{aligned}$$

Estimating the summands in the right-hand side of (4.22) on each $E \in \mathcal{E}_D$ by (4.11), we conclude the proof. $\qquad \square$

Theorem 4.2. *Suppose that $\mathcal{S}$ satisfies the hypothesis of Lemma 3.1 and let d_E, $E \in \mathcal{E}$, be nonnegative integers with $d_E \geq 1$ for $E \in \mathcal{E}_N \cup \mathcal{E}_D$. Assume $f|_{\Omega_z} \in H^{d(z)+1}(\Omega_z)$ for all $z \in \mathcal{K}$, $g|_E \in H^{d_E}(E)$ for all $E \in \mathcal{E}_N$, and $u_D|_E \in H^{d_E+1}(E)$ for all $E \in \mathcal{E}_D$. Then, if u_h satisfies the hypothesis of Lemma 4.3, there exists $g_h \in L^\infty(\Gamma_N)$ with $g_h|_E \in \mathcal{P}_{d_E}(E)$ for all $E \in \mathcal{E}_N$, such that*

$$
(4.23) \quad
\begin{aligned}
\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq{}& c_{17} \left(\sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \, on \, E \cap \Gamma_N}} \|\nabla u_h - q_E\|^2_{L^2(\omega_E)} \right)^{1/2} \\
&+ 2c_7 c_{14} \left(\sum_{E \in \mathcal{E}_N} h_E^{2d_E+1} \|\partial^{d_E} g / \partial s^{d_E}\|^2_{L^2(E)} \right)^{1/2} \\
&+ c_{16} \left(\sum_{E \in \mathcal{E}_D} h_E^{2d_E+1} \|\partial^{d_E+1} u_D / \partial s^{d_E+1}\|^2_{L^2(E)} \right)^{1/2} \\
&+ 2c_5 c_{13} \left(\sum_{z \in \mathcal{K}} h_z^{2(d(z)+2)} \|D^{d(z)+1} f\|^2_{L^2(\Omega_z)} \right)^{1/2}.
\end{aligned}
$$

The constant $c_{17} > 0$ depends only on the shapes of the elements and the polynomial degrees d_E and d_z.

Proof. Let $g_h \in L^\infty(\Gamma_N)$ be such that, for all $E \in \mathcal{E}_N$, $g_h|_E \in \mathcal{P}_{d_E}(E)$ is an approximation of $g|_E$ as in Lemma 4.2. Choose $k \in \mathbb{N}_0$ such that $\nabla u_h \in \mathcal{L}^k(\mathcal{T})$ and $g_h \in \mathcal{L}^{k+1}(\mathcal{E}_N)$. Utilising Theorem 4.1, we verify

$$
(4.24)
$$
$$
\begin{aligned}
\|\nabla&(u - u_h)\|_2 \\
&\leq \min_{q_h \in \mathcal{S}^{k+1}(\mathcal{T})^d} \left(c_{11} \|\nabla u_h - q_h\|_2 + 2c_7 \|h_{\mathcal{E}}^{1/2}(g_h - q_h \cdot n)\|_{2,\Gamma_N} \right) \\
&\quad + \|h_{\mathcal{E}}^{1/2}(g - g_h)\|_{2,\Gamma_N} + c_{12} \left(\sum_{z \in \mathcal{K}} \min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|\nabla u_h - q_z\|^2_{2,\Omega_z} \right)^{1/2} \\
&\quad + \inf_{v|_{\Gamma_D} = u_D} \|\nabla(u_h - v)\|_2 + 2c_5 \left(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f - f_z\|^2_{2,\Omega_z} \right)^{1/2}.
\end{aligned}
$$

By Lemma 3.1 and since each inner edge belongs to a finite number of patches, we have

$$
(4.25)
$$
$$
\begin{aligned}
\sum_{z \in \mathcal{K}} &\min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|\nabla u_h - q_z\|^2_{2,\Omega_z} \\
&\leq c_9 \sum_{z \in \mathcal{K}} \sum_{E \in \mathcal{E}_{\Omega_z}} \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \|\nabla u_h - q_E\|^2_{2,\omega_E} \\
&\lesssim \sum_{E \in \mathcal{E}_\Omega} \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \|\nabla u_h - q_E\|^2_{2,\omega_E}.
\end{aligned}
$$

With Lemma 3.2 and noting that for all $E \in \mathcal{E}_N$ there exists $q_E \in \mathcal{P}_{d_E}(\omega_E)^d = \mathcal{L}^{d_E}(\omega_E)^d$ with $q_E|_E \cdot n = g_h|_E$, we can estimate

(4.26)

$$\min_{q_h \in \mathcal{S}^{k+1}(\mathcal{T})^d} \left(\|\nabla u_h - q_h\|_2^2 + \|h_{\mathcal{E}}^{1/2}(g_h - q_h \cdot n)\|_{2,\Gamma_N}^2 \right)$$

$$\leq c_{10} \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{q_E \in \mathcal{P}_{d_E}(\omega_E)^d} \left(\|\nabla u_h - q_E\|_{2,\omega_E}^2 + h_E \|g_h - q_E \cdot n\|_{2,E \cap \Gamma_N}^2 \right)$$

$$\leq c_{10} \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u_h - q_E\|_{2,\omega_E}^2.$$

Estimating the first and the third term in (4.24) by (4.25) resp. (4.26) and utilising Lemmas 4.1-4.3 to bound the remaining terms in (4.24) yields (4.23). $\square$

The a posteriori error estimate given in the previous theorem is efficient up to higher order terms which depend on the smoothness of ∇u, as the following theorem shows.

Theorem 4.3. *Let $g_h \in \mathcal{L}^k(\mathcal{E}_N)$ with $g_h|_E \in \mathcal{P}_{d_E}(E)$ for $d_E \in \{0, 1, ..., k\}$ on each $E \in \mathcal{E}_N$. Then we have*

$$\left(\sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u_h - q_E\|_{L^2(\omega_E)}^2 \right)^{1/2}$$

(4.27)
$$\leq \sqrt{2d}\, \|\nabla(u - u_h)\|_{L^2(\Omega)}$$

$$+ \left(\sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u - q_E\|_{L^2(\omega_E)}^2 \right)^{1/2}.$$

Proof. Since $g_h \in \mathcal{P}_{d_E}(E)$ for each $E \in \mathcal{E}_N$, the minima in the left-hand side of (4.27) are well defined. We infer, with Young's inequality and an arbitrary $\gamma > 0$,

$$\sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u_h - q_E\|_{2,\omega_E}^2$$

$$\leq \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \left(\|\nabla(u_h - u)\|_{2,\omega_E} + \|\nabla u - q_E\|_{2,\omega_E} \right)^2$$

(4.28)
$$\leq (1 + 1/\gamma) \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \|\nabla(u - u_h)\|_{2,\omega_E}^2$$

$$+ (1 + \gamma) \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u - q_E\|_{2,\omega_E}^2.$$

Rearranging the sum over the edges $E \in \mathcal{E}_\Omega \cup \mathcal{E}_N$ and using the fact that $T \in \mathcal{T}$ has at most $2d$ edges, the first term in the right-hand side of (4.28) can be estimated by

$$\sum_{T \in \mathcal{T}} \sum_{\substack{E \subset \partial T, \\ E \notin \mathcal{E}_D}} \|\nabla(u - u_h)\|_{2,T}^2 \leq 2d\, \|\nabla(u - u_h)\|_{2,\Omega}^2.$$

A direct calculation shows that $(1+1/\gamma)a^2+(1+\gamma)b^2$, where $a, b > 0$, is minimised for $\gamma = b/a$ with minimal value $a^2 + 2ab + b^2 = (a+b)^2$. The optimal choice for γ in the right-hand side of (4.28) yields (4.27). $\qquad\square$

Remark 4.1. The local degrees d_E have to be chosen large enough to obtain higher order terms in the reliability estimate (4.23) and in the efficiency estimate (4.27) at the same time.

Remark 4.2. The definition of the approximation operator $\mathcal{J}$ is related to "partition of unity finite elements" [MBa], where for a fixed nonnegative integer ℓ,

$$(4.29) \qquad \mathcal{S} = \left\{ \sum_{z \in \mathcal{N}} q_z \varphi_z : q_z \in \mathcal{P}_\ell(\omega_z) \right\}$$

(so that $d(z) = \ell$ for all $z \in \mathcal{K}$ by definition). When $\mathcal{S}$ is given by (4.29), a reasonable choice for d_E is $d_E = \ell + 1$, $E \in \mathcal{E}$, so that inequality (4.23) reads

(4.30)

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le c_{17} \left(\sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \min_{\substack{q_E \in \mathcal{P}_{\ell+1}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u_h - q_E\|^2_{L^2(\omega_E)} \right)^{1/2}$$
$$+ c_{18} \Big(\||h_\mathcal{E}^{\ell+3/2} \partial_\mathcal{E}^{\ell+2} u_D / \partial s^{\ell+2}\|_{L^2(\Gamma_D)}$$
$$+ \||h_\mathcal{E}^{\ell+3/2} \partial_\mathcal{E}^{\ell+1} g / \partial s^{\ell+1}\|_{L^2(\Gamma_N)} + \||h_\mathcal{T}^{\ell+2} D^{\ell+1} f\|_{L^2(\Omega)} \Big)$$

with an $(h_\mathcal{T}, h_\mathcal{E})$-independent constant $c_{18} > 0$. Here, $\partial_\mathcal{E} \cdot /\partial s$ denotes the edgewise differentiation along Γ_D and Γ_N.

Remark 4.3. An a posteriori error estimate based on a global averaging technique could be formulated as follows: *There exists an $(h_\mathcal{T}, h_\mathcal{E})$-independent constant $c_{19} > 0$ such that, if $\nabla u_h \in \mathcal{L}^k(\mathcal{T})^d$, we have*

(4.31)

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \le c_{19} \min_{q_h \in \mathcal{P}_m(\Omega)^d} \left(\|\nabla u_h - q_h\|_{L^2(\Omega)} + \|h_\mathcal{E}^{1/2}(q_h \cdot n - g)\|_{L^2(\Gamma_N)} \right)$$
$$+ \inf_{v|_{\Gamma_D} = u_D} \|\nabla(v - u_h)\|_{L^2(\Omega)}$$
$$+ 2c_5 \left(\sum_{z \in \mathcal{K}} h_z^2 \min_{f_z \in \mathcal{P}_{d(z)}(\Omega_z)} \|f - f_z\|^2_{L^2(\Omega_z)} \right)^{1/2},$$

with $m := \min\{k\} \cup \{d(z) : z \in \mathcal{K}\} + 1$. The authors failed to replace $\mathcal{P}_m(\Omega)^d$ in the minimisation over q_h in (4.31) by a larger discrete space (such as certain piecewise polynomials) to make it efficient.

Proof of (4.31). By Theorem 4.1 we only need to estimate the term

$$\sum_{z \in \mathcal{K}} \min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|\nabla u_h - q_z\|^2_{2,\Omega_z}$$

by the minimum in (4.31). For this, let $\tilde{q}_h \in \mathcal{P}_m(\Omega)^d$ denote the minimiser in (4.31). Since $\tilde{q}_z := \tilde{q}_h|_{\Omega_z} \in \mathcal{P}_{d(z)+1}(\Omega_z)^d$, we can conclude that

$$
(4.32)
$$

$$
\sum_{z \in \mathcal{K}} \min_{q_z \in \mathcal{P}_{d(z)+1}(\Omega_z)^d} \|\nabla u_h - q_z\|_{2,\Omega_z}^2 \leq \sum_{z \in \mathcal{K}} \|\nabla u_h - \tilde{q}_z\|_{2,\Omega_z}^2 \leq c_{19}\|\nabla u_h - \tilde{q}_h\|_{2,\Omega}^2. \quad \square
$$

Remark 4.4. For the restriction $\mathcal{T}_E = \{T \in \mathcal{T} : T \subseteq \overline{\omega}_E\}$ of the triangulation $\mathcal{T}$ to ω_E, the equivalence

$$
(4.33) \qquad \min_{q_h \in \mathcal{S}^k(\mathcal{T})^d} \|p_h - q_h\|_{L^2(\Omega)}^2 \approx \sum_{E \in \mathcal{E}_\Omega} \min_{q_E \in \mathcal{S}^k(\mathcal{T}_E)^d} \|p_h - q_E\|_{L^2(\omega_E)}^2
$$

holds for $p_h \in \mathcal{L}^k(\mathcal{T})^d$. For $p_h \in \mathcal{L}^0(\mathcal{T})^d$ we also have (by a compactness argument)

$$
(4.34) \qquad \min_{q_E \in \mathcal{S}^k(\mathcal{T}_E)^d} \|p_h - q_E\|_{L^2(\omega_E)}^2 \approx \min_{q_E \in \mathcal{P}_k(\omega_E)^d} \|p_h - q_E\|_{L^2(\omega_E)}^2.
$$

The left-hand side of (4.33) thus defines an efficient (provided $k \geq 1$) and reliable error estimate for lowest order finite element methods and $p_h = \nabla u_h$. However, there is no analogon for higher order finite element methods, since (4.34) is not valid if p_h is not elementwise constant.

5. NUMERICAL EXPERIMENTS

The theoretical results of this paper are supported by numerical experiments for $d = 2$. Here, we report on three examples of the problem (4.1)-(4.3) on uniform, h-, p-, hp-adapted, and perturbed meshes. Two of these examples were considered in [CB] for lowest order schemes.

Example 5.1. Let $f(x,y) := -8\pi^2 \sin(2\pi x)\sin(2\pi y)$ on the unit square $\Omega := (0,1)^2$ and set $u_D := 0$ on $\Gamma_D := \partial\Omega$. The exact solution is then given by $u(x,y) = \sin(2\pi x)\sin(2\pi y)$. In this example the right-hand sides as well as the exact solution are smooth. The coarsest triangulation $\mathcal{T}_0$ consists of one square halved by the diagonal parallel to $(1,1)$. The initial polynomial degrees $p^0 = (p_T^0)_{T \in \mathcal{T}_0}$ are chosen to be equal to p for all $T \in \mathcal{T}_0$ and various values $p = 1, 2, ..., 6$.

Example 5.2. Let $f := -\Delta u$ for the function

$$
u(x,y) := x(1-x)y(1-y)\arctan(60(r-1))
$$

$r^2 := (x - 1.25)^2 + (y + 0.25)^2$ on the unit square $\Omega := (0,1)^2$, and set $u_D := 0$ on the entire boundary $\Gamma_D := \partial\Omega$. The solution u to (4.1)-(4.3) is H^2-regular but f (although theoretically smooth) has huge gradients on the circle with radius 1 around $(1.25, -0.25)$. The coarsest triangulation $\mathcal{T}_0$ consists of four squares halved by diagonals parallel to $(1,1)$. The initial polynomial degrees $p^0 = (p_T^0)_{T \in \mathcal{T}_0}$ are chosen to be equal to p for all $T \in \mathcal{T}_0$ and various values $p = 1, 2, ..., 6$.

Example 5.3. Let $f := 0$ on the L–shaped domain $\Omega := (-1,1)^2 \setminus [0,1] \times [-1,0]$, $u_D := 0$ on the Dirichlet boundary $\Gamma_D := \{0\} \times [-1,0] \cup [0,1] \times \{0\}$, and, on the Neumann boundary $\Gamma_N := \partial\Omega \setminus \Gamma_D$,

$$
g(r,\varphi) := 2/3\, r^{-1/3}(-\sin(\varphi/3), \cos(\varphi/3)) \cdot n
$$

using polar coordinates (r, φ). The exact solution $u(r,\varphi) := r^{2/3}\sin(2\varphi/3)$ of (4.1)-(4.3) has a typical corner singularity at the origin. In this example, the right-hand sides are smooth, but the solution is not. The coarsest triangulation $\mathcal{T}_0$ consists of
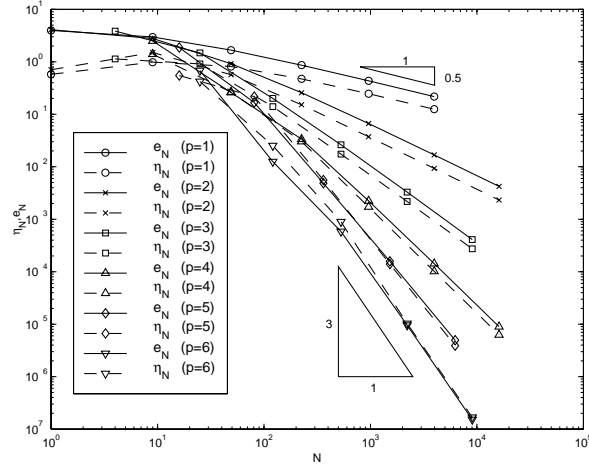
FIGURE 1.   Error and error estimator for uniform mesh-refinement
in Example 5.1 for different fixed polynomial degrees.

three squares halved by diagonals parallel to $(1,1)$. The initial polynomial degrees
$p^0 = (p_T^0)_{T \in \mathcal{T}_0}$ are chosen to be equal to p for all $T \in \mathcal{T}_0$ and various values
$p = 1, 2, ..., 6$.

The following adaptive algorithm generates all the sequences of meshes $\mathcal{T}_0, \mathcal{T}_1,$
$\mathcal{T}_2, ...$ in this paper which are h-uniform for $\Theta = 0$ or h-adapted for $\Theta = 1/2$ in (5.2).
No element is refined if $\Theta = 2$. We also allow raising polynomial degrees steered
by the parameter $\delta \in \{(0,0), (1,1), (0,1)\}$. If $\delta = (0,0)$, the initial polynomial
degrees are kept on all triangles and their subtriangles. For $\delta = (1,1)$, we increase
the degree on each element by one during each iteration of the algorithm while for
$\delta = (0,1)$ the polynomial degree is increased only on those triangles that are not
h-refined during the respective iteration of the algorithm.

Since the resulting meshes might show local symmetries, we considered meshes
that are either unperturbed (relative to $\mathcal{T}_0$) for $\vartheta = 0$ or randomly perturbed for
$\vartheta = 1$ in step (e).

The implementation was performed in Matlab in the spirit of [ACF] with a
direct solution of linear systems of equations. For details on the red-blue-green-
refinements we refer to [V].

Algorithm $(A_\Theta^{\vartheta, \delta})$. (a) Start with a coarse mesh $\mathcal{T}_0$ and polynomial degrees $p^0 =$
$(p_T^0)_{T \in \mathcal{T}_0}$, $k = 0$.

(b) Compute the discrete solution u_h on the actual mesh $\mathcal{T}_k$ in the space

$$\mathcal{S}^{p^k}(\mathcal{T}_k) := \{v_h \in C(\overline{\Omega}) : \forall T \in \mathcal{T}, \, v_h|_T \in P_T^{p_T^k}\}.$$

(c) Set $d_E := \max(p_{T_1}^k, p_{T_2}^k)$ if $E = T_1 \cap T_2 \in \mathcal{E}_\Omega$ and $d_E := p_T^k$ if $E = T \cap \Gamma_N \in \mathcal{E}_N$.
Compute, for all $E \in \mathcal{E}_\Omega \cup \mathcal{E}_N$, error indicators

(5.1) $$\eta_{Z,E} := \frac{1}{\sqrt{3}} \min_{\substack{q_E \in \mathcal{P}_{d_E}(\omega_E)^d, \\ q_E \cdot n = g_h \text{ on } E \cap \Gamma_N}} \|\nabla u_h - q_E\|_{L^2(\omega_E)}$$
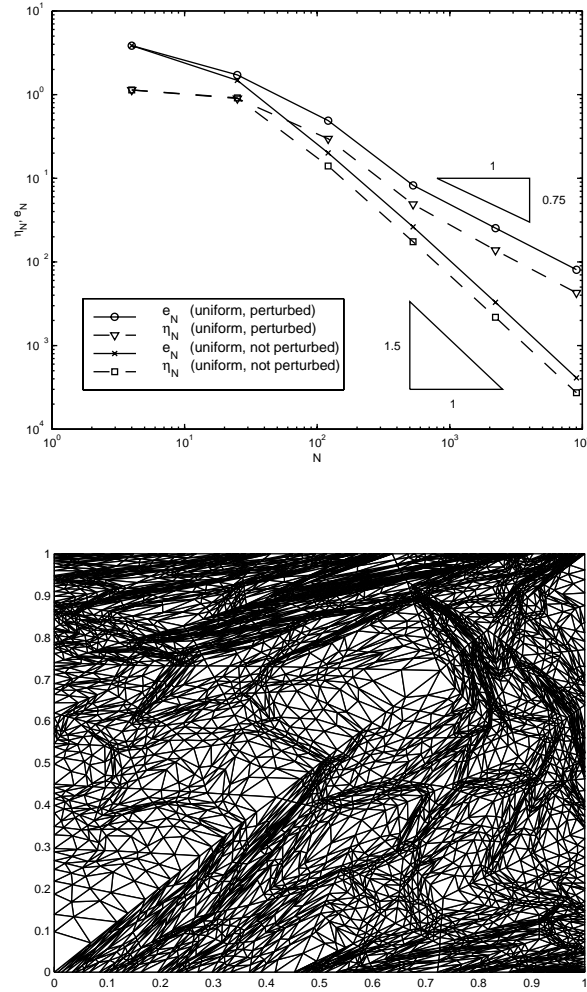
FIGURE 2.   Error and error estimator on perturbed meshes for uniform h-refinement and fixed polynomial degrees $p = 3$ in Example 5.1 (top). Perturbed mesh $\mathcal{T}_6$ with 4225 nodes (bottom).

and plot energy error $e_N := \|\nabla(u - u_h)\|_{L^2(\Omega)}$ and its estimator

$$\eta_N^2 := \sum_{E \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \eta_{Z,E}^2$$

versus the degree of freedom N of the triangulation $\mathcal{T}_k$.

(d) Mark the edge E for *red*-refinement provided

(5.2) $$\eta_{Z,E} \geq \Theta \max_{E' \in \mathcal{E}_\Omega \cup \mathcal{E}_N} \eta_{Z,E'}.$$

(e) Mark further edges (*red–blue–green*-refinement) to avoid hanging nodes. Generate a new triangulation $\tilde{\mathcal{T}}_{k+1}$ using edge–midpoints if $\vartheta = 0$ and points on the edges at a random distance at most $0.1\,h_E$ from the edge–midpoints if $\vartheta = 1$. For $T \in \tilde{\mathcal{T}}_{k+1}$ set $p_T^{k+1} := p_{T'}^k + \delta_1$ if $T \subsetneq T'$ for $T' \in \mathcal{T}^k$ and define $p_T^{k+1} := p_T^k + \delta_2$
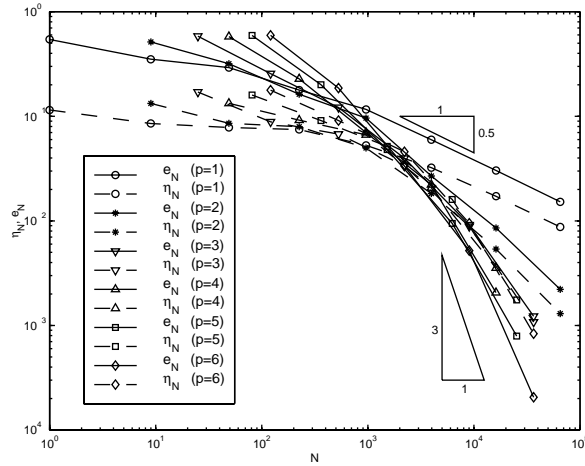
FIGURE 3.   Error and error estimator for uniform mesh-refinement in Example 5.2 for different fixed polynomial degrees.

if $T \in \mathcal{T}_k \cap \tilde{\mathcal{T}}_{k+1}$. Perturb the nodes $z \in \mathcal{N}_{k+1}$ of the mesh $\tilde{\mathcal{T}}_{k+1}$ at random with values taken uniformly from a ball around z of radius $\vartheta \, 2^{-k}/15$. Correct boundary nodes by orthogonal projection onto that boundary piece they are expected such that $\Omega, \Gamma_D, \Gamma_N$ are matched by the resulting mesh $\mathcal{T}_{k+1}$ exactly. Update k and go to (b).

Remark 5.1. The proof of Theorem 4.3 shows that we may choose the factor $1/\sqrt{3}$ in (5.1) to obtain an efficient error estimate with constant 1 for our two-dimensional examples with triangles.

Remark 5.2. In our numerical experiments the minimiser in (5.1) is calculated explicitly. The resulting linear system of equations is badly scaled for large polynomial degrees p (we employed transformed Legendre polynomials in such cases). The constraint $q_E \cdot n = g_h$ is satisfied by matching $q_E \cdot n$ and g in nodes on E.

We used Algorithm $(A_0^{0,(0,0)})$ in Example 5.1 with uniform initial polynomial degrees $p_T = p$ for all $T \in \mathcal{T}_0$, where $p = 1, 2, ..., 6$. Experimental convergence rates are indicted in Figure 1, where we plotted the entries (N, e_N) and (N, η_N). A log-scaling on both axes allows a slope $-\alpha$ of a straight line that connects subsequent entries to be interpreted as an experimental convergence rate 2α (owing to $N \propto h^{-2}$ in two dimensions). Moreover, we see from Figure 1 that the error estimator η_N serves as a good approximation for the error e_N in this example. When a perturbation of the meshes is introduced we obtain with Algorithms $(A_0^{0,(0,0)})$ and $(A_0^{1,(0,0)})$ for polynomial degrees equal to three the results displayed in the top plot of Figure 2. The results become worse for the perturbed meshes but still η_N approximates e_N reasonably. The perturbed mesh $\mathcal{T}_6$ with $N = 9025$ is shown in the bottom plot of Figure 2 and appears quite degenerate.

The lack of smoothness of the exact solution in Example 5.2 results in a poorer quality of the numerical solution and the estimates than in the previous numerical experiment. Figure 3 shows the results for uniform mesh-refinement and fixed polynomial degrees obtained from Algorithm $(A_0^{0,(0,0)})$. For polynomial degrees
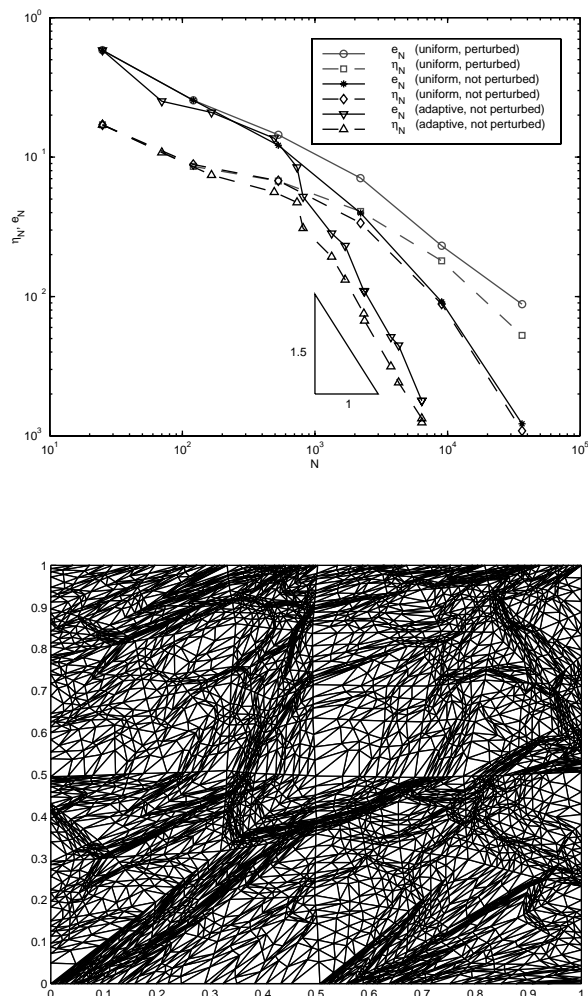
FIGURE 4. Error and error estimator on perturbed meshes for uniform h-refinement and fixed polynomial degree $p = 3$ in Example 5.2 compared to uniform and h-adaptive mesh-refinement without perturbation (top). Perturbed mesh with 4225 nodes (bottom).

$p = 1, 2, 3$ we obtain the expected convergence rates, though the preasymptotic range is very large, while for $p = 4, 5, 6$ the values (N, e_N) and (N, η_N) do not result in a straight line in the plot, but the quotient η_N/e_N remains in a small intervall containing one. This is also true for the perturbed meshes in the top and bottom plots of Figure 4, where we chose $p = 3$ and used Algorithm $(A_0^{1,(0,0)})$ to generate the triangulations. The top plot of Figure 4 also shows that an adaptive mesh refinement, realised by Algorithm $(A_{1/2}^{0,(0,0)})$, yields smaller errors than the uniform mesh-refinement from Algorithm $(A_0^{0,(0,0)})$ at comparable numbers of degrees of freedom.
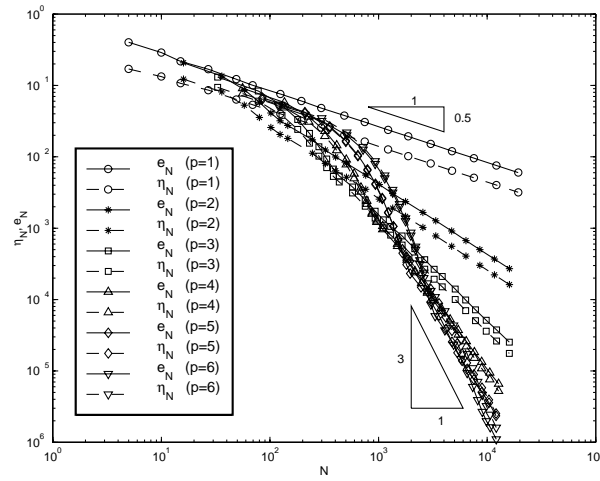
FIGURE 5.   Error and error estimator for adaptive mesh-refinement in Example 5.3 for different fixed polynomial degrees.

The numerical results for (unperturbed) h-adaptive mesh-refinement, i.e., the results obtained with Algorithm $(A_{1/2}^{0,(0,0)})$, in Example 5.3 for fixed polynomial degrees $p = 1, 2, ..., 6$, are displayed in Figure 5. We obtain optimal experimental convergence rates for $p = 1, 2, ..., 6$. Algorithm $(A_2^{0,(1,1)})$ ran a p-version on a fixed mesh $\mathcal{T}$ with 96 elements obtained by two uniform red-refinements of $\mathcal{T}_0$. The top plot of Figure 6 displays the results for the p-refinements and $p = 1, 2, ..., 10$. For small p we observe efficiency (i.e., $\eta_N \leq e_N$) which decreases for larger p (e.g., $e_N < \eta_N$ for $p \geq 6$). This is in agreement with our theoretical predictions, as efficiency is linked to the smoothness of the exact solution, which is limited in this example, while the given data f, g, and u_D are smooth.

For comparison we considered an hp-like refinement strategy by running Algorithm $(A_{1/2}^{0,(0,1)})$. The obtained mesh-refinement and the degree distribution (via different shadings) is given in Figure 5. We observe an automatic geometric refinement towards the origin (where u is singular) and a reverse distribution of the polynomial degrees. This appears as a good strategy: Large elements have a high polynomial degree where the exact solution u is smooth, and small elements of lower degree appear near the singular points of u. The corresponding estimate η_N for e_N is satisfactory and even improves for increasing N. As the constants in our theoretical results may depend on the polynomial degrees, this good behaviour in practice could not be predicted, but suggests that those constants depend only moderately on p.

An h-adaptive algorithm was compared to a uniform mesh-refinement strategy in our last experiment. Algorithms $(A_0^{0,(0,0)})$, $(A_0^{1,(0,0)})$, $(A_{1/2}^{0,(0,0)})$, and $(A_{1/2}^{1,(0,0)})$ generate results displayed in Figure 7. The adaptive meshes refine towards the singular point seen on the bottom plot of Figure 7. This improved the experimental convergence rate from 2/3 to 3.

Remark 5.3. The numerical experiments allow the subsequent conclusions:

(i) Our overall experience with the error estimator η_N is that it serves as an efficient and reliable error indicator provided the exact solution is smooth enough.
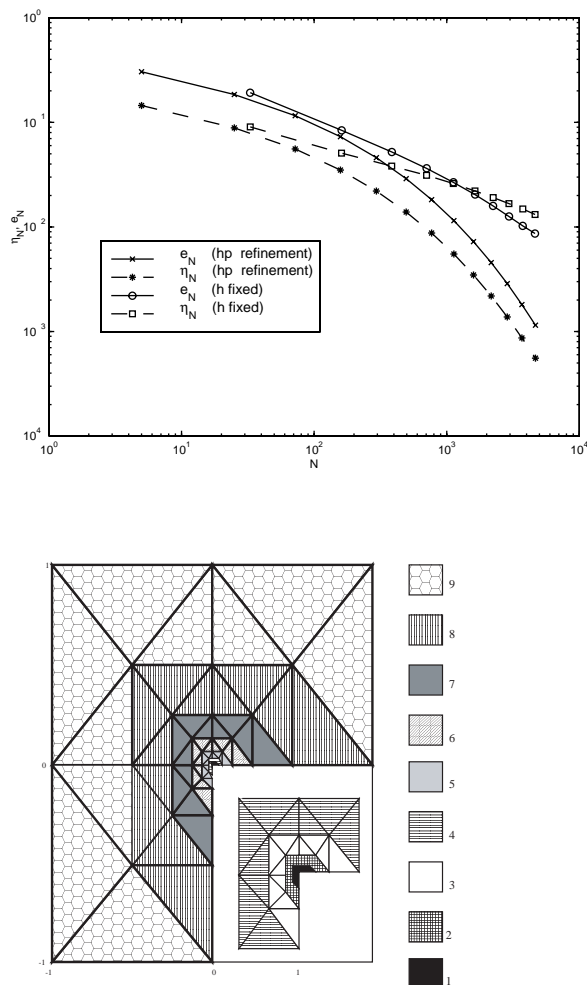
FIGURE 6. Error and error estimator for hp-adaptive mesh-refinement strategy in Example 5.3 and for uniform p-refinement on the twice uniformly refined mesh $\mathcal{T}_0$ (top). Distribution of the polynomial degrees on $\mathcal{T}_9$ and magnified re-entrant corner (bottom).

For nonsmooth exact solutions there is no equivalence for high polynomial degrees in general, but η_N is still reliable.

(ii) For smooth exact solutions, the constant in the efficiency estimate tends to one as the polynomial degree is increased. This behaviour is in agreement with Formula (4.27) of Theorem 4.3.

(iii) The numerical experiments show that adaptive mesh-refinement strategies of Algorithm $(A_\Theta^{\vartheta,\delta})$ yield considerable convergence improvements.

(iv) As is argued in [CB], the numerical examples support the theoretical prediction that the higher order terms in the reliability estimate depend on the smoothness of given right-hand sides, while those in the efficiency estimate depend on the smoothness of the exact solution.
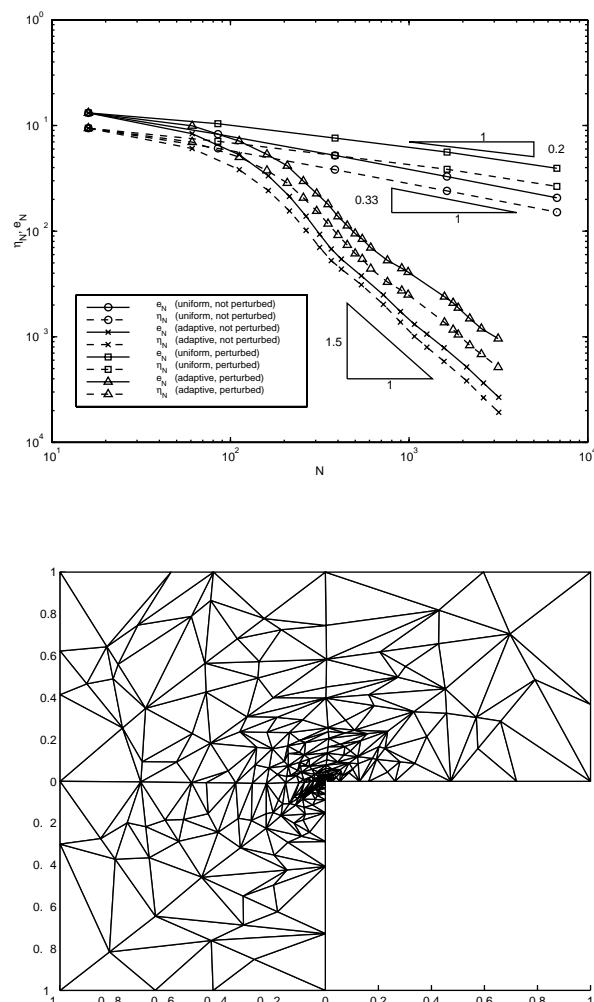
FIGURE 7. Error and error estimator for uniform and adaptive
mesh-refinement on perturbed and unperturbed meshes in Exam-
ple 5.3 (top) for $p = 3$. Triangulation $\mathcal{T}_{22}$ with 382 nodes obtained
from Algorithm $(A_{1/2}^{1,(0,0)})$ (bottom).

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