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NORMAL CONES OF MONOMIAL PRIMES

REINHOLD HÜBL AND IRENA SWANSON

ABSTRACT. We explicitly calculate the normal cones of all monomial primes which define the curves of the form $(t^L, t^{L+1}, \ldots, t^{L+n})$, where $n \leq 4$. All of these normal cones are reduced and Cohen-Macaulay, and their reduction numbers are independent of the reduction. These monomial primes are new examples of integrally closed ideals for which the product with the maximal homogeneous ideal is also integrally closed.

Substantial use was made of the computer algebra packages Maple and Macaulay2.

Let $(R, \mathfrak{m})$ be a regular local or graded local ring and let $I \subseteq R$ be an ideal. In the case of a graded ring, I is assumed to be homogeneous. By $N_I = \bigoplus_{t \in \mathbb{N}} I^t/\mathfrak{m} \cdot I^t$ we denote the special fibre of the blow-up of I, i.e., the normal cone of I. When $I \subseteq R$ is an $\mathfrak{m}$–primary ideal (in which case $\operatorname{Spec}(N_I)$ is homeomorphic to the exceptional fibre of the blow–up of I), normal cones have been studied quite intensely (see for example the comprehensive reference [HIO]), and some of the results for $\mathfrak{m}$-primary ideals have been extended to equimultiple ideals (cf. [Sh1], [Sh2], [HSa], [CZ]). For more general ideals very little is known about the structure of their normal cones. If I is generated by a d-sequence, then N_I is a polynomial ring, cf. [Hu]. In particular this is the case if I is generated by a regular sequence. Conversely, a celebrated result of Cowsik and Nori [CN] asserts that an equidimensional radical ideal I with $\dim(N_I) = \operatorname{ht}(I) = \dim(R) - 1$ is a complete intersection ideal. Other than that, the structure of N_I has been determined only in some special cases ([MS], [G], [CZ]).

Our interest in the normal cones got sparked by their relations to evolutions and evolutionary stability of algebras. In [H] it was shown that whenever I is a radical ideal and N_I is reduced (or, more generally, N_I does not contain any nilpotent elements of degree 1), then the ideal $\mathfrak{m} \cdot I$ is integrally closed. If in addition R is essentially of finite type over a field k of characteristic zero, this in turn implies that R/I is evolutionarily stable as a k-algebra, thus answering a question of Mazur from [EM] in this case. In [HH] further connections between the normal cone of an ideal and the integral closedness of $\mathfrak{m} \cdot I$ have been established. For two-dimensional regular rings the product of any two integrally closed ideals is integrally closed; however, in general this is not the case. With the exception of radical ideals generated by d-sequences, few examples of classes of ideals I for

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which $\mathfrak{m} \cdot I$ is integrally closed are known. We provide many new examples in this paper.

The standard way of constructing N_I is as a homomorphic image of the associated graded ring of I. Good properties of the associated graded ring often imply similar good properties for the normal cone; however, often N_I satisfies properties that the associated graded ring does not. For example, whereas the associated graded ring of an ideal is rarely reduced, concrete examples show that this is the case quite frequently with N_I.

Another reason to study the normal cone is its relation to reductions and reduction numbers. Recall that a reduction of I is an ideal $J \subseteq I$ such that $JI^r = I^{r+1}$ for some $r \in \mathbb{N}$, and J is called a minimal reduction if it does not contain a proper reduction of I. The reduction number $r_J(I)$ of I with respect to J is defined to be the smallest r with $JI^r = I^{r+1}$. In general, it depends on the choice of a minimal reducton J of I. However, we have:

Proposition 1 (see also [CZ, Remark 4.5])**.** *If $k = R/\mathfrak{m}$ is an infinite field and if N_I is Cohen–Macaulay, then the reduction number $r_J(I)$ is independent of the choice of a minimal reduction J of I.*

Proof. Let $J = (x_1, \ldots, x_l) \subseteq I$ be a minimal reduction of I, and let $x_i^* := x_i + \mathfrak{m} \cdot I$. Then $x_1^*, \ldots, x_l^*$ are algebraically independent elements of N_I of degree 1 and

$$P = k[x_1^*, \ldots, x_l^*] \to N_I$$

is a homogeneous noetherian normalization of N_I (cf. [Va, page 100]). As N_I is Cohen–Macaulay, it is free as a graded P–module. Let $\{a_1, \ldots, a_m\}$ be a homogeneous P-basis of N_I with $\deg(a_i) \leq \deg(a_{i+1})$. Set

$$r := \max\{\deg(a_i)\} = \deg(a_m),$$

and for $0 \leq j \leq r$ set

$$t_j := |\{i \in \{1, \ldots, m\} : \deg(a_i) = j\}|.$$

Then, for the Hilbert function of the normal cone we have

$$H(N_I, \rho) := \dim_k(N_{I,\rho}) = \sum_{j=0}^{r} t_j \binom{l+\rho-1-j}{\rho-j}.$$

Hence if $\widetilde{J} = (y_1, \ldots, y_l) \subseteq I$ is another minimal reduction of I, and if we define $\widetilde{r}$, $\widetilde{t_j}$ in the analogous way, we get again

$$H(N_I, \rho) = \sum_{j=0}^{\widetilde{r}} \widetilde{t_j} \binom{l+\rho-1-j}{\rho-j}.$$

Comparing these two expressions, we conclude in particular that $r = \widetilde{r}$, hence

$$r_J(I) = r = \widetilde{r} = r_{\widetilde{J}}(I)$$

(cf. [Va, page 100]).

Thus there is ample reason to study the special fibre N_I. In this note we concentrate on the normal cones of the ideals of monomial curves of type $(t^L, t^{L+1}, \ldots, t^{L+n})$, $n \leq 4$. We determine the normal cones completely in these cases, and we show that N_I is reduced and Cohen–Macaulay. Moreover, we show that N_I is essentially determined by the residue class of L modulo n. In particular, in each of these cases the reduction number is independent of the choice of a minimal reduction, and $\mathfrak{m} \cdot I$ is integrally closed.

Our results are explicit and computationally intensive. We arrived at most of them after observing the patterns on the examples calculated with Macaulay2 and Maple. Though neither of these programs can actually prove anything about the structure of N_I in general, their help has been crucial to us. We used Macaulay2 to calculate normal cones for specific fields k and integers L and n (based on results of Patil and Singh). With the help of Maple we then manipulated these patterns to obtain general relations for large sets of L and arbitrary fields k. Finding these relations involved not just Maple but also some guessing. However, the obtained relations were subsequently all verified by traditional methods. Finally, with somewhat less computational algebra we subsequently proved that the relations we obtained in this way indeed generated the whole ideal of presentation of the normal cone. A few of the results were only verified by Macaulay2 for various k, n and L, and not proved. All such results are marked with * in the summary table at the end.

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Preliminaries and techniques

As mentioned earlier, a motivation for this work came from [H], in which properties (AR) and (MR) were defined:

Definition 2. *A radical ideal I in a local or graded local ring $(R, \mathfrak{m})$ satisfies property (MR) if whenever $x \in I \setminus \mathfrak{m} \cdot I$, then x is contained in a minimal reduction of I. A radical ideal I satisfies property (AR) if whenever $x \in I \setminus \mathfrak{m} \cdot I$, then $x + \mathfrak{m} \cdot I$ is not nilpotent in the normal cone N_I of I.*

Clearly (MR) implies (AR). The first author proved in [H] that if I satisfies (AR), then $\mathfrak{m} \cdot I$ is integrally closed. We show that all the monomial primes in this paper satisfy (AR) but many do not satisfy (MR).

Here is the set-up for this paper. Fix positive integers L, n, let

$$\alpha : R := k[X_0, X_1, \dots, X_n] \to k[t^L, t^{L+1}, \dots, t^{L+n}]$$

be the canonical map (with $\alpha(X_i) = t^{L+i}$), and let $\mathfrak{P}$ be the kernel of α. Set

$$G_n = \left\lceil \frac{L}{n} \right\rceil, \qquad s = G_n n - L.$$

Then $\mathfrak{P}$ can be determined completely, and specializing and slightly modifying results of Patil [P] and Patil–Singh [PS] to our situation gives:

Theorem 3. *i) Assume $L \leq n$. Then $\mathfrak{P}$ is minimally generated by the elements*

$$\begin{aligned} \xi_{i,j} &= X_i X_j - X_{i-1} X_{j+1} && \text{for } 1 \leq i \leq j \leq L-1, \\ \varphi_j &= X_{L+j} - X_0^{\left\lfloor \frac{L+j}{L} \right\rfloor} X_{j-L\left\lfloor \frac{j}{L} \right\rfloor} && \text{for } j \in \{0, \dots, n-L\}. \end{aligned}$$

ii) Assume $L > n$. Then $\mathfrak{P}$ is minimally generated by the elements

$$\begin{aligned} \xi_{i,j} &= X_i X_j - X_{i-1} X_{j+1} && \text{for } 1 \leq i \leq j \leq n-1, \\ \psi_j &= X_{n-s+j} X_n^{G_n - 1} - X_0^{G_n} X_j && \text{for } j \in \{0, \dots, s\}. \end{aligned}$$

In particular, the minimal number $\mu(\mathfrak{P})$ of generators of $\mathfrak{P}$ equals

$$\mu(\mathfrak{P}) = \begin{cases} \binom{L-1}{2} + n, & \text{if } L \leq n, \\ \binom{n}{2} + \left\lceil \frac{L}{n} \right\rceil n - L + 1, & \text{if } L > n. \end{cases}$$

From this we are able to derive an easy and straightforward algorithm to determine the normal cones of these primes. For example, in case $L > n$:

1. Set $S = k[X_0, \dots, X_n, \{X_{i,j}\}_{1 \leq i \leq j \leq n-1}, Y_0, \dots, Y_s]$ and $T = k[X_0, \dots, X_n, t]$.
2. Define $\varphi : S \to T$ by

$$\begin{aligned} \varphi(X_i) &= X_i, \\ \varphi(X_{i,j}) &= (X_i X_j - X_{i-1} X_{j-1}) \cdot t, \\ \varphi(Y_l) &= (X_{n-s+j} X_n^{G_n - 1} - X_0^{G_n} X_j) \cdot t. \end{aligned}$$

3. Determine $\mathfrak{Q} = \ker(\varphi)$.
4. Set $\mathfrak{m} = (X_0, \dots, X_n) \subseteq S$ and $J = \mathfrak{m} + \mathfrak{Q}$.

Proposition 4. *S/J is the normal cone of $\mathfrak{P}$ and $S/\mathfrak{Q}$ is the Rees–algebra of $\mathfrak{P}$.*

Proof. In view of the above theorem this is evident from the definitions.

The above algorithm can be made into an effective Macaulay2 code. For example, in the case that $n = 3$, $L = 3l - 2$, for various values of l, we input the corresponding monomial curve prime ideal as follows:

```
L = 3*l - 2
n = 3
v = apply(n+1, j -> L+j)
R = ZZ/31[x_0..x_n, Degrees => {1} | vv ]
P = ideal (x_1^ 2-x_0*x_2, x_1*x_2-x_0*x_3, x_2^ 2-x_1*x_3,
          x_1*x_3^(l-1)-x_0^ (l+1), x_2*x_3^ (l-1)-x_0^ l*x_1,
          x_3^ l-x_0^ l*x_2)
```

With this ideal P in a ring R we can now pass to the normal cone computation of P via the following routine:

Input: ring R, ideal P (not necessarily prime)
Output: the ideal presenting the normal cone of P

```
ncp = () -> (
      k = coefficientRing R;
      rr = gens R;
      n = numgens R;
      v = apply(n, i-> (degree(rr_i))_0);
      m = numgens P;
      vv = apply(m, i -> (degree(P_i))_0+1);
      S = k[rr,y_1..y_m, Degrees => join(v,vv), MonomialSize => 16];
      T = k[t,rr, Degrees => prepend(1,v), MonomialSize => 16];
      f = map(T,S, apply(n, i -> substitute(rr_i,T)) |
                apply(m, i -> t*substitute(P_i,T)));
```

```
K = ker f;
g = map(S,S, apply(n, i -> 0) | toList(y_1..y_m));
ideal mingens g(K)
)

-- example: R = ZZ/101[a,b,c];
--          P = ideal(a^ 2-b*c,a^ 3,b*c*d);
--          ncp ()
```

This algorithm and the examples obtained with it allowed us to find results about the structure of some classes of monomial primes (presented below), though in some cases additional calculations and algorithms have been needed.

Some easy cases

First we identify all the pairs (L, n) for which $\mu(\mathfrak{P}) = n$.

Theorem 5. *The number of generators of $\mathfrak{P}$ is n if and only if the normal cone $N_{\mathfrak{P}}$ of $\mathfrak{P}$ is a polynomial ring in n variables over k, and that is true if and only if one of the following conditions is satisfied:*

1. $L = 1 \leq n$,
2. $L = 2 \leq n$,
3. $n = 1 < L$,
4. $n = 2$ *and L is even.*

Proof. The dimension of the normal cone is at least the height of the ideal, which in these cases is n, and at most the number of generators of the ideal, which here is also n (see [HIO]). This proves that the first two statements are equivalent. That they are equivalent to the four conditions listed is an easy combinatorial exercise left to the reader.

By the Cowsik-Nori result [CN], the cases in Theorem 5 are exactly those cases for which the dimension of the normal cone is exactly n. Thus from now on it suffices to consider the cases for which $\mathfrak{P}$ is generated by at least $n+1$ elements, and in all these cases the dimension of the normal cone is at least $n+1$. However, the dimension of the normal cone is bounded above by the dimension $n+1$ of the ring (see [HIO]), so that from now on the dimension of the normal cones is exactly $n+1$. Moreover, the cases when $\mathfrak{P}$ is generated by at most $n+1$ elements are special:

Theorem 6. $\mu(\mathfrak{P}) \leq n+1$ *if and only if $N_{\mathfrak{P}}$ is a polynomial ring over k in $\mu(\mathfrak{P})$ variables.*

Proof. It is well-known that $\mathfrak{P}$ is generated by analytically independent elements if $N_{\mathfrak{P}}$ is a polynomial ring. But the number of analytically independent elements is bounded above by the dimension $n+1$ of the ring (see [HIO]), so that $\mathfrak{P}$ is generated by at most $n+1$ elements.

Conversely, assume that $\mu(\mathfrak{P}) \leq n+1$. By Theorem 5 we only need to consider the cases when $\mu(\mathfrak{P}) = n+1$. Then the normal cone is an $(n+1)$-dimensional quotient of a polynomial ring in $n+1$ variables over k, so that necessarily the normal cone equals the polynomial ring. This proves the theorem.

The Patil-Singh formulae help identify all the pairs (L, n) for which $\mu(\mathfrak{P}) = n+1$. When $L \leq n$, necessarily $\binom{L-1}{2} = 1$, so that $L = 3 \leq n$. When $L > n$ instead, $\binom{n}{2} + \lceil \frac{L}{n} \rceil n - L + 1 = n + 1$ if and only if $\binom{n}{2} - n = L - \lceil \frac{L}{n} \rceil n$, where the last quantity is non-positive. Thus $n\left(\frac{n-3}{2}\right) \leq 0$, which forces the following two cases:

1. $n = 3$ and $L = 3l$ for some integer $l > 1$,
2. $n = 2$ and L odd, $L \geq 3$.

Thus all in all, $N_{\mathfrak{P}}$ is a polynomial ring if and only if one of the following holds:

1. $L = 1 \leq n$,
2. $L = 2 \leq n$,
3. $n = 1 < L$,
4. $n = 2$ and L is even,
5. $n = 3$ and $L = 3l$ for some positive integer l,
6. $n = 2$ and L odd.

For all these cases, $\mathfrak{P}$ satisfies (MR), thus (AR), and thus $\mathfrak{m} \cdot \mathfrak{P}$ is integrally closed.

From now on we analyze the cases for which the number of generators of $\mathfrak{P}$ is at least $n+2$. The normal cones in these cases are, by the last theorem, not polynomial rings. They are of course quotients of polynomial rings, and the presenting relations have a very varied structure. These structures tend to be similar for the $\mathfrak{P}$ with the same n. We analyze the cases up to n equal to 4. So far we have proved that all the cases $n \leq 2$ yield polynomial normal cones, so we start with $n = 3$. Moreover, we may assume that $L \geq 3$.

Also, note that in all the previous cases the reduction number was zero as the $\mathfrak{P}$ were all basic. In all the cases with $\mu(\mathfrak{P}) \geq n + 2$, however, $\mathfrak{P}$ is not basic, so that the reduction numbers from now on will be positive integers.

The cases $n = 3$

Note that all the cases $L \leq n = 3$ yield polynomial rings as normal cones. So we may assume that $L > n$. The case when L is a multiple of 3 also yields polynomial rings as normal cones, so it remains to analyze the cases $L = 3l - 1$ and $L = 3l - 2$.

The case $n = 3, L = 3l - 1 > 3$. Necessarily $l \geq 2$. By the Patil-Singh theorem,

$$\mathfrak{P} = (x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3, x_2x_3^{l-1} - x_0^{l+1}, x_3^l - x_1x_0^l).$$

Let the five generators above be named $y_1, \ldots, y_5$ in the order given. One can verify by hand or by Maple that

$$y_3y_4^2 - y_2y_4y_5 + y_1y_5^2 - x_3^{l-2}y_5y_3^2 = x_0^{l-2}x_3^{l-2}(x_0y_2^3 + x_2y_1^2y_2 - x_1y_1y_2^2) - x_0^{l-1}y_1^2y_4.$$

When $l = 2$, set $F = y_3y_4^2 - y_2y_4y_5 + y_1y_5^2 - y_5y_3^2$, and when $l > 2$, set $F = y_3y_4^2 - y_2y_4y_5 + y_1y_5^2$. By the given relation, $N_{\mathfrak{P}}$ is a quotient ring of $k[y_1, y_2, y_3, y_4, y_5]/(F)$. It is easily verified that F is an irreducible polynomial in $k[y_1, y_2, y_3, y_4, y_5]$, so that $k[y_1, y_2, y_3, y_4, y_5]/(F)$ is a four-dimensional integral domain. If $N_{\mathfrak{P}}$ were a proper quotient ring of it, then $N_{\mathfrak{P}}$ would be three-dimensional, contradicting the earlier observation that the dimension of the normal cone has to be $n + 1 = 4$. Thus necessarily $N_{\mathfrak{P}} = k[y_1, y_2, y_3, y_4, y_5]/(F)$, so it is an integral domain, and in particular none of its elements of degree 1 is a zero-divisor.

One can calculate that $(y_1 - y_5, y_2, y_3, y_4)$ is a minimal reduction of $\mathfrak{P}$ with reduction number 2, so that by Proposition 1, the reduction number of $\mathfrak{P}$ equals 2.

The case $n = 3, L = 3l - 2 > 3$. Necessarily $l \geq 2$. By the Patil-Singh theorem,

$$\begin{aligned}\mathfrak{P} = (x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 \\ - x_1x_3, x_1x_3^{l-1} - x_0^{l+1}, x_2x_3^{l-1} - x_0^l x_1, x_3^l - x_0^l x_2).\end{aligned}$$

A combination of Macaulay2 and Maple produces two relations in S:

$$y_3y_4 - y_2y_5 + y_1y_6 = 0$$

and

$$\begin{aligned}y_5^2 - y_4y_6 - x_3^{l-2}y_3y_6 - x_0^{l-1}x_3^{l-2}y_2^2 - x_0^{l-2}x_2x_3^{l-2}y_1^2 \\ + x_0^{l-2}x_1x_3^{l-2}y_1y_2 + x_0^{l-1}y_1y_4 = 0\end{aligned}$$

for any value of l, where y_1, y_2, y_3, y_4, y_5 and y_6 are the given generators of $\mathfrak{P}$, in the given order. It is easy to check that these two relations are indeed relations on the given Patil-Singh generators of $\mathfrak{P}$, for an arbitrary integer l and the underlying field k. Set $F = y_3y_4 - y_2y_5 + y_1y_6$. Set $G = y_5^2 - y_4y_6 - y_3y_6$ if $l = 2$ and $G = y_5^2 - y_4y_6$ otherwise. F and G are clearly irreducible polynomials in $k[y_1, y_2, y_3, y_4, y_5, y_6]$ forming an ideal of height 2. By the two displayed relations, the normal cone of $\mathfrak{P}$ is a quotient ring of $A = k[y_1, y_2, y_3, y_4, y_5, y_6]/(F, G)$.

We first verify that A is an integral domain. Under the reverse lexicographic ordering, F and G are presented in their expansion in descending order. As their leading terms have no variables in common, their S-polynomial is zero, so that the Gröbner basis of (F, G) is $\{F, G\}$. Thus by the standard Gröbner bases result, $(F, G) \cap k[y_5, y_6] = (0)$. It is easy to see that

$$\begin{aligned}&(F, G)k(y_5, y_6)[y_1, \ldots, y_4] \\ &\quad = \left(y_1 + \frac{y_3y_4 - y_2y_5}{y_6}, y_4 + \delta_{l2}y_3 - \frac{y_5^2}{y_6}\right) k(y_5, y_6)[y_1, \ldots, y_4]\end{aligned}$$

is a prime ideal. Now in order to finish the proof that (F, G) is prime, it suffices to prove that $(F, G)k(y_5, y_6)[y_1, \ldots, y_4] \cap k[y_1, \ldots, y_6]$ equals (F, G) (see for example [GTZ, Lemma 4.2]). As the only elements of $k(y_5, y_6)$ which divide any of the leading terms of (F, G) are powers of y_5, by [GTZ, Proposition 3.7],

$$(F, G)k(y_5, y_6)[y_1, \ldots, y_4] \cap k[y_1, \ldots, y_6] = (F, G)k[y_1, \ldots, y_6]_{y_5} \cap k[y_1, \ldots, y_6].$$

But F, G, y_5 form a regular sequence in $k[y_1, \ldots, y_6]$, so that $(F, G)_{y_5} \cap k[y_1, \ldots, y_6]$ equals (F, G). Thus by [GTZ, Lemma 4.2] the ideal (F, G) is prime.

It follows that A is a four-dimensional complete intersection integral domain, and thus Gorenstein. As $n = 3$, the normal cone of $\mathfrak{P}$ has to have dimension 4, so necessarily the normal cone is A.

One can calculate that $(y_1 - y_6, y_2, y_3, y_4)$ is a minimal reduction of $\mathfrak{P}$ with reduction number 2, so that again by Proposition 1, the reduction number of $\mathfrak{P}$ equals 2.

We conclude in particular that $\mathfrak{P}$ satisfies the condition (MR) whenever $n = 3$, and consequenly $\mathfrak{m} \cdot \mathfrak{P}$ is integrally closed.

The cases $n = 4$

We have seen that the case $L = 3$ yields a polynomial normal cone of dimension 5. If instead $L = 4 = n$, by the Patil-Singh theorem,

$$\begin{aligned}\mathfrak{P} = (x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 \\ - x_0x_4, x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4, x_4 - x_0^2),\end{aligned}$$

and by Macaulay2 its normal cone is

$$k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]/(y_5^2 - y_3y_6 - y_4y_6, y_3y_4 - y_2y_5 + y_1y_6),$$

at least when $k = \mathbb{Z}/31\mathbb{Z}$. However, it is easy to see that the relations $y_5^2 - y_3y_6 - y_4y_6 = 0$ and $y_3y_4 - y_2y_5 + y_1y_6 = 0$ hold for arbitrary fields, and by analysis similar to the case $n = 3$, $L = 3l - 2 > 3$, we see that the two polynomials generate a prime ideal of height two. Thus

$$k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]/(y_5^2 - y_3y_6 - y_4y_6, y_3y_4 - y_2y_5 + y_1y_6)$$

is a 5-dimensional complete intersection domain, and by the dimension restraint there are no relations on the y_i outside of the ideal

$$(y_5^2 - y_3y_6 - y_4y_6, y_3y_4 - y_2y_5 + y_1y_6).$$

Thus

$$k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]/(y_5^2 - y_3y_6 - y_4y_6, y_3y_4 - y_2y_5 + y_1y_6)$$

is a normal cone of $\mathfrak{P}$. Hence every minimal generator of $\mathfrak{P}$ is a part of a minimal reduction of $\mathfrak{P}$, $\mathfrak{P}$ satisfies (MR), and $\mathfrak{m} \cdot \mathfrak{P}$ is integrally closed.

One of the minimal reductions is $(y_1 - y_6, y_2, y_3, y_4, y_7)$, and its reduction number is 2. Thus the reduction number of $\mathfrak{P}$ is 2, by Proposition 1.

Note that by the structure of the minimal generators of $\mathfrak{P}$ as given by Patil-Singh and by Proposition 4.2 in [HS], the normal cone for all cases $L = 4$, $n \geq 4$, is a polynomial ring in $n-4$ variables over the normal cone for the case $n = 4$. Thus any good property for the special case $n = 4$ also holds for the general case $n \geq 4$. In particular, for all the cases $L = 4$, $n \geq 4$, the normal cone is an $(n+1)$-dimensional complete intersection domain, with every minimal generator of $\mathfrak{P}$ being part of a minimal reduction of $\mathfrak{P}$, i.e., $\mathfrak{P}$ satisfies (MR).

We next analyze the cases when $n = 4$ and $L > n$.

As for the $n = 3$, $L > n$ cases, the normal cones depend on the congruence classes of L modulo n, but now in addition there are also a few special cases for low values of L.

An important ingredient for all these cases is the normal cone of the ideal $\mathfrak{Q}$ generated by the six quadric generators of all these ideals $\mathfrak{P}$: namely, let $\mathfrak{Q} = (x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_0x_4, x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4)$. Note that $\mathfrak{Q}$ is independent of L, so it is straightforward to calculate its associated graded ring and its normal cone:

Theorem 7. *The associated graded ring $gr_{\mathfrak{Q}}(R)$ of $\mathfrak{Q}$ has exactly two minimal primes. Note that $gr_{\mathfrak{Q}}(R)$ is a homomorphic image of*

$$S = k[x_0, x_1, x_2, x_3, x_4, y_1, \ldots, y_6],$$

where y_i maps to the i^{th} quadric generator of $\mathfrak{Q}$. With this notation, the two minimal primes in $gr_{\mathfrak{Q}}(R)$ are

$$\mathfrak{p}_1 = (x_0, x_1, x_2, x_3, x_4)gr_{\mathfrak{Q}}(R) \text{ and } \mathfrak{p}_2 = (G, x_2y_4 - x_1y_5 + x_0y_6)gr_{\mathfrak{Q}}(R),$$

where

$$G = y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6.$$

Moreover, in $gr_{\mathfrak{Q}}(R)$,

$$0 : x_4^\infty = 0 : x_0^\infty = \mathfrak{p}_2.$$

Also, the normal cone $N_{\mathfrak{Q}}$ *of* $\mathfrak{Q}$ *is isomorphic to* $\frac{k[y_1,\dots,y_6]}{(F)}$*, where* $F = y_3y_4 - y_2y_5 + y_1y_6$.

Proof. First of all, Macaulay2 calculates – for $k = \mathbb{Z}/31\mathbb{Z}$ – and one can verify manually that the Gröbner basis for the presenting ideal of the Rees algebra of $\mathfrak{Q}$ over an arbitrary field is indeed

$$\{x_4y_4 - x_3y_5 + x_2y_6, x_3y_4 - x_2y_5 + x_1y_6, x_4y_2 - x_3y_3 + x_1y_6, x_3y_2 - x_2y_3 + x_0y_6, \\ x_2y_2 - x_1y_3 - x_1y_4 + x_0y_5, x_4y_1 - x_2y_3 + x_1y_5, \\ x_3y_1 - x_1y_3 + x_0y_5, x_2y_1 - x_1y_2 + x_0y_4, y_3y_4 - y_2y_5 + y_1y_6\}$$

in $S = k[x_0, x_1, x_2, x_3, x_4, y_1, \dots, y_6]$. Thus it follows that the ideal presenting the normal cone of $\mathfrak{Q}$ is simply $(y_3y_4 - y_2y_5 + y_1y_6)$, and that the ideal K presenting the associated graded ring $gr_{\mathfrak{Q}}(R)$ is generated by all of these and by $x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_0x_4, x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4$.

Any minimal prime $\mathfrak{P}$ over K either contains x_0 or it does not. If x_0 is not in $\mathfrak{P}$, then after inverting x_0, $\mathfrak{P}S_{x_0}$ is minimal over

$$KS_{x_0} = (y_6 + \frac{x_1^3}{x_0^3}y_2 - \frac{x_1^2}{x_0^2}y_3, y_5 + \frac{x_1^3}{x_0^3}y_1 \\ - \frac{x_1}{x_0}y_3, y_4 + \frac{x_1^2}{x_0^2}y_1 - \frac{x_1}{x_0}y_2, x_2 - \frac{x_1^2}{x_0}, x_3 - \frac{x_1^3}{x_0^2}, x_4 - \frac{x_1^4}{x_0^3})S_{x_0},$$

which is a prime ideal. Let $\mathfrak{P}_2$ be the contraction of this prime ideal to S and $\mathfrak{p}_2$ the image of $\mathfrak{P}_2$ in $gr_{\mathfrak{Q}}(R)$. Both $\mathfrak{P}_2$ and $\mathfrak{p}_2$ are prime ideals. We have shown so far that $0 : x_0^\infty = \mathfrak{p}_2$. Note that x_4 is also not a zero divisor modulo $\mathfrak{p}_2$. It is then straightforward to show that also $0 : x_4^\infty = \mathfrak{p}_2$. Macaulay2 verifies, and one can do it manually as well, that $\mathfrak{P}_2$ and $\mathfrak{p}_2$ are as stated in the theorem.

If instead $x_0 \in \mathfrak{P}$, then $\mathfrak{P}$ is minimal over

$$(x_0, x_4y_4 - x_3y_5 + x_2y_6, x_3y_4 - x_2y_5 + x_1y_6, x_4y_2 - x_3y_3 + x_1y_6, x_3y_2 - x_2y_3, \\ x_2y_2 - x_1y_3 - x_1y_4, x_4y_1 - x_2y_3 + x_1y_5, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2, \\ y_3y_4 - y_2y_5 + y_1y_6, x_1^2, x_1x_2, x_1x_3, x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4)$$

and thus over

$$(x_0, x_1, x_2, x_3, x_4y_4, x_4y_2, x_4y_1, y_3y_4 - y_2y_5 + y_1y_6),$$

so that either

$$\mathfrak{P} = (x_0, x_1, x_2, x_3, x_4, y_3y_4 - y_2y_5 + y_1y_6),$$

or

$$\mathfrak{P} = (x_0, x_1, x_2, x_3, y_4, y_2, y_1).$$

However, the second prime ideal properly contains $\mathfrak{P}_2$. This proves that $gr_{\mathfrak{Q}}(R)$ indeed has only the two minimal primes.

Furthermore, by the structure of F and by Lemma 19.8 in Matsumura [M], the normal cone $N_{\mathfrak{Q}}$ of $\mathfrak{Q}$ is even a unique factorization domain. This ideal $\mathfrak{Q}$ is thus an example of an ideal for which the normal cone has much better properties than the associated graded ring: here $gr_{\mathfrak{Q}}(R)$ is not even reduced, whereas $N_{\mathfrak{Q}}$ is a unique factorization domain.

The case $n = 4, L = 4l > 4$. Necessarily $l \geq 2$. By the Patil-Singh theorem,

$$\begin{aligned}\mathfrak{P} = (&x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 \\ &- x_0x_4, x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4, x_4^l - x_0^{l+1}),\end{aligned}$$

and the combination of Macaulay2 and Maple establishes the following two relations:

$$\begin{aligned}&y_3y_4 - y_2y_5 + y_1y_6 = 0,\\ &(y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6)y_7\\ &\quad = x_4^{l-2}y_3^2y_6^2 - 2x_4^{l-2}y_3y_5^2y_6 + x_4^{l-2}y_5^4 + 2x_4^{l-2}y_2y_5y_6^2 - 2x_4^{l-2}y_4y_5^2y_6\\ &\qquad - 2x_4^{l-2}y_1y_6^3 + x_4^{l-2}y_4^2y_6^2 - x_0^{l-1}y_1^2y_3^2 + 2x_0^{l-1}y_1y_2^2y_3 - 2x_0^{l-2}x_1y_1^2y_2y_3\\ &\qquad + 2x_0^{l-2}x_2y_1^3y_3 - x_0^{l-1}y_2^4 + 2x_0^{l-2}x_1y_1y_2^3 - 2x_0^{l-2}x_2y_1^2y_2^2\\ &\qquad - x_0^{l-2}x_1y_1^2y_2y_4 + x_0^{l-2}x_2y_1^3y_4,\end{aligned}$$

where $y_1, y_2, y_3, y_4, y_5, y_6$ and y_7 are the generators of $\mathfrak{P}$ in the given order. When $l = 2$, this says that the normal cone is the quotient of $k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]$ by the two relations $y_3y_4 - y_2y_5 + y_1y_6$ and $-(y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6)y_7 + y_3^2y_6^2 - 2y_3y_5^2y_6 + y_5^4 + 2y_2y_5y_6^2 - 2y_4y_5^2y_6 - 2y_1y_6^3 + y_4^2y_6^2$. Indeed, by using Gianni-Trager-Zacharias [GTZ] techniques as for the case $n = 3$, $L = 3l - 2 > 3$, this quotient is a five-dimensional complete intersection domain, and thus by the dimension restriction it equals the normal cone. Thus again every minimal generator is part of a minimal reduction, and $\mathfrak{P}$ satisfies (MR). Furthermore, $\mathfrak{P}$ has reduction number 4.

When instead $l > 2$, the displayed relations imply that the normal cone of $\mathfrak{P}$ is a quotient ring of

$$\mathfrak{A} = \frac{k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]}{(y_3y_4 - y_2y_5 + y_1y_6, (y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6)y_7)}.$$

Note that the second relation of degree 4 is y_7 times the polynomial G from Theorem 7, and the first quadric relation is F from Theorem 7. Both F and G are in $k[y_1, y_2, y_3, y_4, y_5, y_6]$. It is not hard to see that $(F, y_7G) = (F, y_7) \cap (F, G)$ is a decomposition in $\mathfrak{A}$. Furthermore, as in our proof of the case $n = 3$, $L = 3l - 2 > 3$ using [GTZ], the two ideals (F, y_7) and (F, G) are primes. Thus if $N_{\mathfrak{P}}$ equals $\mathfrak{A}$, then $N_{\mathfrak{P}}$ is a complete intersection reduced ring, and y_7 is a minimal generator of $\mathfrak{P}$ which is not part of any minimal reduction of $\mathfrak{P}$. Therefore $\mathfrak{P}$ satisfies (AR) but not (MR).

Now we use Theorem 7 to prove that $N_{\mathfrak{P}} = \mathfrak{A}$. By the x-degree count, there is a canonical inclusion $N_{\mathfrak{Q}} \subseteq N_{\mathfrak{P}}$. As $N_{\mathfrak{Q}}$ is a unique factorization domain and the relations on $N_{\mathfrak{P}}$ contain F and y_7G, where $G \in N_{\mathfrak{Q}}$, it follows that

$$N_{\mathfrak{Q}} \subseteq N_{\mathfrak{P}} = \frac{N_{\mathfrak{Q}}[y_7]}{(y_7) \cap I}$$

for some ideal I in $N_{\mathfrak{Q}}[y_7]$ containing the canonical image of G. Let $\alpha \in N_{\mathfrak{P}}$, of degree j, and let i be a positive integer such that $\alpha y_7^i = 0$. Suppose we can prove that all such α are elements of $GN_{\mathfrak{P}}$. Then $I \subseteq GN_{\mathfrak{P}}$, so that

$$y_7GN_{\mathfrak{P}} \subseteq (y_7) \cap I \subseteq (y_7) \cap GN_{\mathfrak{P}} = y_7GN_{\mathfrak{P}}.$$

Hence equality holds throughout, proving that $N_{\mathfrak{P}} = \mathfrak{A}$. Thus it remains to prove that for all α as above, $\alpha \in GN_{\mathfrak{P}}$.

Of course, α can be written as $\alpha_0 + y_7\alpha_1$, where α_0 is the image of an element $\widetilde{\alpha}_0$ in $\mathfrak{Q}^j$ and α_1 is the image of an element $\widetilde{\alpha}_1$ in $\mathfrak{P}^{j-1}$. Note that either α_0 is 0 or $\widetilde{\alpha}_0$ may be taken to be a homogeneous element in the x's of degree exactly $2j$. By the assumption, if $\widetilde{y}_7$ denotes the preimage $x_4^l - x_0^{l+1}$ of y_7 in R, then

$$\begin{aligned}\widetilde{y}_7^i\widetilde{\alpha}_0 + \widetilde{y}_7^{i+1}\widetilde{\alpha}_1 &\in (x_0, x_1, x_2, x_3, x_4)\mathfrak{P}^{i+j} \\ &\subseteq \sum_{k=0}^{i-1}(x_0, x_1, x_2, x_3, x_4)\widetilde{y}_7^k\mathfrak{Q}^{i+j-k} + (x_0, x_1, x_2, x_3, x_4)\widetilde{y}_7^i\mathfrak{P}^j \\ &\subseteq \mathfrak{Q}^{j+1} + (x_0, x_1, x_2, x_3, x_4)(x_4^l, x_0^{l+1})^i(\mathfrak{Q} + (x_0^{l+1}, x_4^l))^j.\end{aligned}$$

Any x-monomial appearing in an element in the last ideal has degree at least $1 + li + 2j$, whereas on the left side there appear monomials of degree exactly $li + 2j$. Thus by taking the homogeneous part of degree $li + 2j$ we get that

$$x_4^{li}\widetilde{\alpha}_0 \in \mathfrak{Q}^{j+1},$$

so that by Theorem 7, $\alpha_0 \in GN_{\mathfrak{P}}$. This finishes the proof of the case $j = 0$, and when $j > 0$, it reduces the proof to showing that $y_7\alpha_1 \in GN_{\mathfrak{P}}$. Thus, as $y_7G = 0$ in $N_{\mathfrak{P}}$ and $\alpha_0 \in GN_{\mathfrak{P}}$, we have that $y_7^{i+1}\alpha_1 = 0$. But by induction on j, as α_1 lies in degree $j - 1$, we are done. This proves that $\mathfrak{A} = N_{\mathfrak{P}}$.

Then it is easy to verify that $(y_1 - y_6, y_2, y_3, y_4 - y_7, y_5)$ is a minimal reduction of $\mathfrak{P}$ with reduction number 4. Hence by Proposition 1, the reduction number of $\mathfrak{P}$ equals 4.

The case $n = 4, L = 4l - 1 > 4$. Note that necessarily $l \geq 2$. By the Patil-Singh theorem,

$$\begin{aligned}\mathfrak{P} = (&x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_0x_4, x_2^2 \\ &- x_1x_3, x_2x_3 - x_1x_4, x_3^2 - x_2x_4, x_3x_4^{l-1} - x_0^{l+1}, x_4^l - x_0^l x_1).\end{aligned}$$

The combination of Macaulay2, Maple, and some guessing produces the following relations on the generators $y_1, \ldots, y_8$ of $N_{\mathfrak{P}}$:

$$\begin{aligned}
&y_3y_4 - y_2y_5 + y_1y_6, \\
&y_6y_7^2 - y_5y_7y_8 + y_4y_8^2 - x_4^{l-2}y_6^2y_8 - x_0^{l-1}y_1y_3y_7 \\
&\qquad + x_0^{l-1}y_2^2y_7 - x_0^{l-1}x_4^{l-2}y_3(y_3y_5 - y_2y_6), \\
&y_5y_7^2 - y_3y_7y_8 - y_4y_7y_8 + y_2y_8^2 - x_4^{l-2}y_5y_6y_8 \\
&\qquad + x_0^{l-1}y_1y_2y_7 - x_0^{l-1}x_4^{l-2}y_3(y_3^2 - y_1y_6), \\
&y_4y_7^2 - y_2y_7y_8 + y_1y_8^2 + x_4^{l-2}(y_3y_6 - y_5^2)y_8 + x_0^{l-1}y_1^2y_7 \\
&\qquad + x_0^{l-2}x_4^{l-2}((x_1y_1 - x_0y_2)y_3^2 - x_0y_1y_2y_6 + x_0y_1y_4y_5 - x_3y_1y_2^2 + x_3y_1^2y_4), \\
&y_5^2y_7 - (y_3 + y_4)y_6y_7 - y_4y_5y_8 + y_2y_6y_8 + x_0^{l-1}(y_1y_3^2 - y_2^2y_3 + y_1y_2y_5 - y_1^2y_6), \\
&y_4y_5y_7 - y_2y_6y_7 - y_4^2y_8 + y_1y_6y_8 + x_4^{l-2}(y_3y_6 - y_5^2 + y_4y_6)y_6 \\
&\qquad + x_0^{l-1}y_2(y_1(y_3 + y_4) - y_2^2), \\
&y_4^2y_7 - y_1y_6y_7 - y_2y_4y_8 + y_1y_5y_8 + x_4^{l-2}y_5((y_3 + y_4)y_6 - y_5^2) \\
&\qquad + x_0^{l-1}y_1(y_1(y_3 + y_4) - y_2^2), \\
&y_2y_4y_7 - y_1y_5y_7 - y_2^2y_8 + y_1(y_3 + y_4)y_8 + x_4^{l-2}y_3((y_3 + y_4)y_6 - y_5^2).
\end{aligned}$$

Thus when $l = 2$, the normal cone $N_{\mathfrak{P}}$ is the quotient of $k[y_1, \dots, y_8]$ modulo the following induced relations:

$$
\begin{aligned}
&y_3y_4 - y_2y_5 + y_1y_6,\\
&y_6y_7^2 - y_5y_7y_8 + y_4y_8^2 - y_6^2y_8,\\
&y_5y_7^2 - y_3y_7y_8 - y_4y_7y_8 + y_2y_8^2 - y_5y_6y_8,\\
&y_4y_7^2 - y_2y_7y_8 + y_1y_8^2 + (y_3y_6 - y_5^2)y_8,\\
&y_5^2y_7 - (y_3 + y_4)y_6y_7 - y_4y_5y_8 + y_2y_6y_8,\\
&y_4y_5y_7 - y_2y_6y_7 - y_4^2y_8 + y_1y_6y_8 + (y_3y_6 - y_5^2 + y_4y_6)y_6,\\
&y_4^2y_7 - y_1y_6y_7 - y_2y_4y_8 + y_1y_5y_8 + y_5((y_3 + y_4)y_6 - y_5^2),\\
&y_2y_4y_7 - y_1y_5y_7 - y_2^2y_8 + y_1(y_3 + y_4)y_8 + y_3((y_3 + y_4)y_6 - y_5^2).
\end{aligned}
$$

Calculation by Macaulay2 for various finite fields k shows that the ideal generated by these elements is indeed the defining ideal of the normal cone $N_{\mathfrak{P}}$. It is unlikely that the relations would be any different for some field k; however, we provide no proof here. Similarly, it is also a result due to Macaulay2 that for various finite fields k, this ideal is prime, Cohen-Macaulay, and non-Gorenstein, so that each of the minimal generators of $\mathfrak{P}$ is part of a minimal reduction, i.e., $\mathfrak{P}$ satisfies (MR).

It is easy to verify that $(y_1 - y_6, y_3, y_5, y_2 - y_8, y_4 - y_7)$ is a minimal reduction of $\mathfrak{P}$ and that it has reduction number 2. Then by Proposition 1, the reduction number of $\mathfrak{P}$ equals 2. The same ideal is also a reduction with reduction number 2 in the case when $l > 2$.

Now let $l > 2$. Let J be the ideal in $k[y_1, \dots, y_8]$ generated by the images of the relations on the y_i calculated above, i.e.,

$$
\begin{aligned}
J = (&y_3y_4 - y_2y_5 + y_1y_6, y_6y_7^2 - y_5y_7y_8 + y_4y_8^2, y_5y_7^2 - y_3y_7y_8 - y_4y_7y_8 + y_2y_8^2,\\
&y_4y_7^2 - y_2y_7y_8 + y_1y_8^2, y_5^2y_7 - (y_3 + y_4)y_6y_7 - y_4y_5y_8 + y_2y_6y_8,\\
&y_4y_5y_7 - y_2y_6y_7 - y_4^2y_8 + y_1y_6y_8, y_4^2y_7 - y_1y_6y_7 - y_2y_4y_8 + y_1y_5y_8,\\
&y_2y_4y_7 - y_1y_5y_7 - y_2^2y_8 + y_1(y_3 + y_4)y_8).
\end{aligned}
$$

Thus $N_{\mathfrak{P}}$ is a homomorphic image of $\mathfrak{A} = k[y_1, \dots, y_8]/J$. We prove next that actually $N_{\mathfrak{P}}$ equals $\mathfrak{A}$. Certainly $N_{\mathfrak{P}} = k[y_1, \dots, y_8]/L$ for some ideal L containing J, and it suffices to prove that $L \subseteq J$.

As in the calculation of the normal cone of $\mathfrak{Q}$, also here one can show manually in a straightforward way that J has exactly two minimal primes:

$$
\begin{aligned}
\mathfrak{p}_1 &= (F, y_7, y_8)k[\underline{y}],\\
\mathfrak{p}_2 &= J : x_0^\infty = J : x_4^\infty,
\end{aligned}
$$

where F and G are the same as in the previous case: $F = y_3y_4 - y_2y_5 + y_1y_6$ and $G = y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6$. Macaulay2 calculated the generators of $\mathfrak{p}_2$ for $k = \mathbb{Z}/31\mathbb{Z}$, and it is actually a straightforward (even manually doable) Gröbner basis calculation that

$$\mathfrak{p}_2 = J + (G)k[\underline{y}],$$

whence it is easy to see that $J = \mathfrak{p}_1 \cap \mathfrak{p}_2$, so that J is reduced and equidimensional with two minimal primes.

By Theorem 7, as in the previous $n = 4$ case, as J contains Gy_7, Gy_8, we have

$$N_{\mathfrak{Q}} \subseteq N_{\mathfrak{P}} = \frac{N_{\mathfrak{Q}}[y_7, y_8]}{(y_7, y_8) \cap I}$$

for some ideal I containing G. This shows in particular that $L \subseteq \mathfrak{p}_1$. Thus to prove that $L = J$, it suffices to prove that $L \subseteq \mathfrak{p}_2$. We will use a similar method as in the previous case. The main difference is that the prime ideal $\mathfrak{p}_1$ here is generated by two variables, whereas before it was generated by only one. So we first reduce to one of the variables only, after which the proof is essentially the same as before.

Thus let α be a homogeneous element of $\mathfrak{A}$ of degree j which is contained in the image of L. To prove that $\alpha = 0$ it suffices to prove that $\alpha \in \mathfrak{p}_2$. Assume that $\alpha \neq 0$ and let i be the largest integer such that $\alpha \in (y_7, y_8)^i$. Note that $i > 0$, as $L \subseteq \mathfrak{p}_1$. Write $\alpha = \alpha_0 + \alpha_1$, with $\alpha_0 \in (y_1, \ldots, y_6)^{j-i}(y_7, y_8)^i R$ and $\alpha_1 \in (y_1, \ldots, y_8)^{j-i-1}(y_7, y_8)^{i+1} R$. As $y_2y_4 - y_1y_5 \notin \mathfrak{p}_2$, it suffices to prove that $(y_2y_4 - y_1y_5)^i \alpha \in \mathfrak{p}_2$, and even that $(y_2y_4 - y_1y_5)^i \alpha_k \in \mathfrak{p}_2$ for $k = 0, 1$. But modulo the element $(y_2y_4 - y_1y_5)y_7 - (y_2^2 - y_1(y_3 + y_4))y_8 \in J$, $(y_2y_4 - y_1y_5)^i \alpha_k = \beta_k y_8^i$, for some β_k of y-degree j, with $\beta_0 \in \mathfrak{Q}^j$. As $(y_2y_4 - y_1y_5)^i \alpha = (\beta_0 + \beta_1)y_8^i \in L$, the x-degree count gives (as in the previous case) that $\beta_0 \in (G)$. But then $(y_2y_4 - y_1y_5)^i \alpha_0 = \beta_0 y_8^i \in \mathfrak{p}_2$, so that $\alpha_0 \in \mathfrak{p}_2$. Hence it suffices to prove that $(y_2y_4 - y_1y_5)^i \alpha_1$ lies in $\mathfrak{p}_2$ or equivalently that α_1 lies in $\mathfrak{p}_2$. But $\alpha_1 \in (y_7, y_8)^{i+1}\mathfrak{P}^{j-i-1}$, so we have reduced the original α to one with strictly higher i. But necessarily i is bounded above by j, so this process eventually stops. This finishes the proof that $\mathfrak{A}$ is the normal cone of $\mathfrak{P}$.

Hence $\mathfrak{P}$ satisfies (AR) but not (MR). The calculation of the resolution by Macaulay2 shows that $N_{\mathfrak{P}}$ is Cohen-Macaulay but not Gorenstein.

The case $n = 4, L = 4l - 2 > 4$. Necessarily $l \geq 2$. By the Patil-Singh theorem,

$$\begin{aligned}\mathfrak{P} = (&x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_0x_4, x_2^2 - x_1x_3, x_2x_3 \\ &- x_1x_4, x_3^2 - x_2x_4, x_2x_4^{l-1} - x_0^{l+1}, x_3x_4^{l-1} - x_0^l x_1, x_4^l - x_0^l x_2).\end{aligned}$$

If these generators are $y_1, \ldots, y_9$, in the given order, it is easy to verify the following relations on them (we obtained them, as before, via Macaulay2 and Maple):

$$\begin{aligned}
F = y_3y_4 - y_2y_5 + y_1y_6 &= 0, \\
y_8^2 - y_7y_9 &= x_4^{l-2}y_6y_9 - x_0^{l-1}y_1y_7 + x_0^{l-1}x_4^{l-2}(y_3^2 - y_1y_6), \\
y_6y_7 - y_5y_8 + y_4y_9 &= -x_0^{l-1}(y_2^2 - y_1(y_3 + y_4)), \\
y_5y_7 - y_3y_8 - y_4y_8 + y_2y_9 &= 0, \\
y_4y_7 - y_2y_8 + y_1y_9 &= x_4^{l-2}(y_5^2 - y_3y_6 - y_4y_6).
\end{aligned}$$

When $l = 2$, Macaulay2 calculates that these relations indeed determine all the defining relations of the normal cone. Macaulay2 also determined that the normal cone is a Cohen-Macaulay integral domain which is not Gorenstein, and $\mathfrak{P}$ satisfies (MR). We do not provide a non-Macaulay2 proof of this case.

When $l > 2$, the left sides of these equations generate an ideal J, so that the normal cone $N_{\mathfrak{P}} = k[y_1, \ldots, y_9]/L$ is a homomorphic image of $k[y_1, \ldots, y_9]/J$. As in the previous $n = 4, l > 2$ cases, J is equidimensional and reduced with two minimal primes

$$\mathfrak{p}_1 = (F, y_7, y_8, y_9)k[\underline{y}],$$
$$\mathfrak{p}_2 = J + (G)k[\underline{y}].$$

Also as before, it suffices to prove that $L \subseteq \mathfrak{p}_2$, and as before, it suffices to be able to reduce elements $\alpha \in (y_7, y_8, y_9)^i \mathfrak{P}^{j-i}$ to ones in $y_9^i \mathfrak{P}^{j-i}$ modulo J, after possibly first multiplying by elements not in $\mathfrak{p}_2$. But this is straightforward:

1. As $y_6 \notin \mathfrak{p}_2$, by first multiplying by a power of y_6 and then by reducing modulo $y_6y_7 - y_5y_8 + y_4y_9 \in J$, we may assume that $\alpha \in (y_8, y_9)^i \mathfrak{P}^{j-i}$.
2. By first multiplying by a power of $y_4y_5 - y_2y_6 \notin \mathfrak{p}_2$ and then reducing modulo

$$\begin{aligned}(y_4y_5 - y_2y_6)y_8 &+ (y_1y_6 - y_4^2)y_9 \\ &= y_6(y_4y_7 - y_2y_8 + y_1y_9) - y_4(y_6y_7 - y_5y_8 + y_4y_9) \in J,\end{aligned}$$

without loss of generality $\alpha \in y_9^i \mathfrak{P}^{j-i}$, as was wanted.

With this the proof proceeds as in the previous $n = 4, l > 2$ cases.

This proves that indeed $N_{\mathfrak{P}}$ is defined by J. Thus $N_{\mathfrak{P}}$ is a reduced almost complete intersection with two minimal primes. Thus $\mathfrak{P}$ satisfies (AR) but not (MR). Macaulay2 also verifies that $N_{\mathfrak{P}}$ is Cohen-Macaulay.

Furthermore, one can verify that $(y_8 - y_4, y_1 - y_9, y_6 - y_7, y_2 - y_5, y_3)$ is a minimal reduction of $\mathfrak{P}$ with reduction number 2 for all values of $l \geq 2$. Thus by Proposition 1, the reduction number of $\mathfrak{P}$ equals 2.

The case $n = 4, L = 4l - 3 > 4$. Necessarily $l \geq 2$. By the Patil-Singh theorem,

$$\begin{aligned}\mathfrak{P} = (&x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_0x_4, x_2^2 - x_1x_3, x_2x_3 - x_1x_4, x_3^2 \\ &- x_2x_4, x_1x_4^{l-1} - x_0^{l+1}, x_2x_4^{l-1} - x_0^l x_1, x_3x_4^{l-1} - x_0^l x_2, x_4^l - x_0^l x_3).\end{aligned}$$

Let these generators be $y_1, \ldots, y_{10}$, in the given order. We find the following relations:

$$\begin{aligned}
y_3y_4 - y_2y_5 + y_1y_6 &= 0 \\
y_9^2 - y_8y_{10} &= x_4^{l-2}y_6y_{10} - x_0^{l-1}y_4y_7 \\
&\quad + x_0^{l-1}x_4^{l-2}((y_3 + y_4)y_5 - y_4y_5 - y_2y_6), \\
y_8y_9 - y_7y_{10} &= x_4^{l-2}y_5y_{10} - x_0^{l-1}y_2y_7 \\
&\quad + x_0^{l-1}x_4^{l-2}(y_3(y_3 + y_4) - y_2y_5), \\
y_8^2 - y_7y_9 &= x_4^{l-2}y_4y_{10} - x_0^{l-1}y_1y_7 \\
&\quad + x_0^{l-1}x_4^{l-2}(y_2(y_3 + y_4) - y_1y_5 - y_2y_4), \\
y_5y_8 - y_3y_9 - y_4y_9 + y_2y_{10} &= x_0^{l-1}(y_2^2 - y_1(y_3 + y_4)), \\
y_4y_8 - y_2y_9 + y_1y_{10} &= x_4^{l-2}(y_5^2 - (y_3 + y_4)y_6), \\
y_6y_8 - y_5y_9 + y_4y_{10} &= 0, \\
y_6y_7 - y_3y_9 + y_2y_{10} &= 0, \\
y_5y_7 - y_3y_8 + y_1y_{10} &= 0, \\
y_4y_7 - y_2y_8 + y_1y_9 &= 0.
\end{aligned}$$

When $l = 2$, these relations induce the following relations on the normal cone:

$$\begin{aligned} N_{\mathfrak{P}} = k[y_1, \dots, y_{10}] \text{ modulo } & (y_3y_4 - y_2y_5 + y_1y_6, \\ & y_9^2 - y_8y_{10} - y_6y_{10}, y_8y_9 - y_7y_{10} - y_5y_{10}, y_8^2 - y_7y_9 - y_4y_{10}, \\ & y_5y_8 - y_3y_9 - y_4y_9 + y_2y_{10}, y_4y_8 - y_2y_9 + y_1y_{10} - y_5^2 + (y_3 + y_4)y_6, \\ & y_6y_8 - y_5y_9 + y_4y_{10}, y_6y_7 - y_3y_9 + y_2y_{10}, \\ & y_5y_7 - y_3y_8 + y_1y_{10}, y_4y_7 - y_2y_8 + y_1y_9). \end{aligned}$$

Again we leave it as a Macaulay2 result (without human proof) that these relations define the normal cone and that the resulting normal cone is a Cohen-Macaulay, non-Gorenstein integral domain. Hence $\mathfrak{P}$ satisfies (MR).

When $l > 2$ instead, the relations above give that the normal cone is a quotient of $k[y_1, \dots, y_{10}]/J$, where of course

$$\begin{aligned} J = (& y_3y_4 - y_2y_5 + y_1y_6, y_9^2 - y_8y_{10}, y_8y_9 - y_7y_{10}, \\ & y_8^2 - y_7y_9, y_5y_8 - y_3y_9 - y_4y_9 + y_2y_{10}, \\ & y_4y_8 - y_2y_9 + y_1y_{10}, y_6y_8 - y_5y_9 + y_4y_{10}, y_6y_7 - y_3y_9 + y_2y_{10}, \\ & y_5y_7 - y_3y_8 + y_1y_{10}, y_4y_7 - y_2y_8 + y_1y_9). \end{aligned}$$

As in the previous $n = 4, l > 2$ cases, this J actually presents the normal cone. The main ingredient is again that J is reduced with two minimal primes, $\mathfrak{p}_1$ generated by y_7, y_8, y_9, y_{10} and F, and $\mathfrak{p}_2$ generated by G and J. As before, we need to be able to reduce the elements in $(y_7, y_8, y_9, y_{10})^i \mathfrak{P}^{j-i}$ to ones in $y_{10}^i \mathfrak{P}^{j-i}$ modulo J, but that is easy:

1. By multiplying by a power of $y_4y_6 \notin \mathfrak{p}_2$ and by reducing modulo $y_6y_7 - y_3y_9 + y_2y_{10}$ and $y_4y_8 - y_2y_9 + y_1y_{10}$ in J, we may assume that $\alpha \in (y_9, y_{10})^i \mathfrak{P}^{j-i}$.
2. By multiplying by a power of $y_4y_5 - y_2y_6 \notin \mathfrak{p}_2$ and by reducing modulo

$$\begin{aligned} & (y_4y_5 - y_2y_6)y_9 + (y_1y_6 - y_4^2)y_{10} \\ & \quad = y_6(y_4y_8 - y_2y_9 + y_1y_{10}) - y_4(y_6y_8 - y_5y_9 + y_4y_{10}) \end{aligned}$$

   in J, without loss of generality $\alpha \in y_{10}^i \mathfrak{P}^{j-i}$, as desired.

Then, as before, $N_{\mathfrak{P}}$ is presented by the ideal J, so that it is reduced and with two minimal primes. Furthermore, $\mathfrak{P}$ satisfies (AR) but not (MR).

Macaulay2 calculates that $N_{\mathfrak{P}}$ is also Cohen-Macaulay and non-Gorenstein.

Moreover, $(y_1 - y_{10}, y_2 - y_9, y_3 - y_4, y_5 - y_7, y_6 - y_8)$ is a minimal reduction of $\mathfrak{P}$ with reduction number 2 for all $l \geq 2$, so that by Proposition 1, the reduction number of $\mathfrak{P}$ equals 2.

Summary table of results on normal cones

Note that in all cases $\mathfrak{P}$ satisfies (AR), so that $\mathfrak{m} \cdot \mathfrak{P}$ is integrally closed.

The results "proved" by Macaulay2 rather than with a traditional proof are marked in the table with a star *.

n	$N_{\mathfrak{P}}$	L	red. no.	(MR)
1	$N_{\mathfrak{P}}$ is a polynomial ring in n variables	L arbitrary	0	yes
2	$N_{\mathfrak{P}}$ is a polynomial ring in n variables	L even or 1	0	yes
2	$N_{\mathfrak{P}}$ is a polynomial ring in $n+1$ variables	L odd, $L \geq 3$	0	yes
3	$N_{\mathfrak{P}}$ is a polynomial ring in n variables	$L = 1, 2$	0	yes
3	$N_{\mathfrak{P}}$ is a polynomial ring in $n+1$ variables	$L = 3l \geq 3$	0	yes
3	$N_{\mathfrak{P}} = k[y_1, y_2, y_3, y_4, y_5]$ modulo $(y_3y_4^2 - y_2y_4y_5 + y_1y_5^2 - \delta_{l2}y_5y_3^2)$ is a c.i. integral domain, dimension 4	$L = 3l - 1 > 3$	2	yes
3	$N_{\mathfrak{P}} = k[y_1, y_2, y_3, y_4, y_5, y_6]$ modulo $(y_5^2 - y_4y_6 - \delta_{l2}y_3y_6, y_3y_4 - y_2y_5 + y_1y_6)$ $N_{\mathfrak{P}}$ is a c.i. integral domain, dimension 4	$L = 3l - 2 > 3$	2	yes
4	$N_{\mathfrak{P}}$ is a polynomial ring in n variables	$L = 1, 2$	0	yes
4	$N_{\mathfrak{P}}$ is a polynomial ring in $n+1$ variables	$L = 3$	0	yes
4	$N_{\mathfrak{P}} = k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]$ modulo $(y_5^2 - y_4y_6 - y_3y_6, y_3y_4 - y_2y_5 + y_1y_6)$ $N_{\mathfrak{P}}$ is a c.i. integral domain of dimension 5	$L = 4$	2	yes
4*	$N_{\mathfrak{P}} = k[y_1, \ldots, y_{10}]$ modulo 10 relations $N_{\mathfrak{P}}$ is a Cohen-Macaulay, non-Gorenstein domain	$L = 5$	2	yes
4*	$N_{\mathfrak{P}} = k[y_1, \ldots, y_9]$ modulo 5 relations so almost complete intersection $N_{\mathfrak{P}}$ is a Cohen-Macaulay, non-Gorenstein domain	$L = 6$	2	yes
4*	$N_{\mathfrak{P}} = k[y_1, \ldots, y_8]$ modulo 8 relations $N_{\mathfrak{P}}$ is a Cohen-Macaulay, non-Gorenstein domain	$L = 7$	2	yes
4	$N_{\mathfrak{P}} = k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]$ with two relations $y_3y_4 - y_2y_5 + y_1y_6$, $(y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6)y_7$ $+y_3^2y_6^2 - 2y_3y_5^2y_6 + y_5^4$ $+2y_2y_5y_6^2 - 2y_4y_5^2y_6 - 2y_1y_6^3 + y_4^2y_6^2$ $N_{\mathfrak{P}}$ is a c.i. integral domain, dimension 5	$L = 8$	4	yes
4	$N_{\mathfrak{P}} = k[y_1, y_2, y_3, y_4, y_5, y_6, y_7]$ with two relations $y_3y_4 - y_2y_5 + y_1y_6$, $(y_4^3 + y_1y_5^2 + y_2^2y_6 - y_2y_4y_5 - y_1y_3y_6 - 2y_1y_4y_6)y_7$ $N_{\mathfrak{P}}$ is c.i. reduced, dimension 5 exactly two minimal primes y_7 is not part of any minimal reduction	$L = 4l \geq 12$	4	no
4	$N_{\mathfrak{P}} = k[y_1, \ldots, y_8]$ modulo 8 relations $N_{\mathfrak{P}}$ is Cohen-Macaulay, non-Gorenstein, reduced exactly two minimal primes y_7, y_8 are not part of any minimal reduction	$L = 4l - 1 > 7$	2	no
4	$N_{\mathfrak{P}} = k[y_1, \ldots, y_9]$ modulo 5 relations so almost complete intersection $N_{\mathfrak{P}}$ is Cohen-Macaulay*, non-Gorenstein*, reduced exactly two minimal primes y_7, y_8, y_9 are not part of any minimal reduction	$L = 4l - 2 > 6$	2	no
4	$N_{\mathfrak{P}} = k[y_1, \ldots, y_{10}]$ modulo 10 relations $N_{\mathfrak{P}}$ is Cohen-Macaulay*, non-Gorenstein*, reduced exactly two minimal primes y_7, y_8, y_9, y_{10} are not part of any minimal reduction	$L = 4l - 3 > 5$	2	no

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NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: `Reinhold.Huebl@sap.com`

Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003-8001
E-mail address: `iswanson@nmsu.edu`