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THE TAME KERNEL OF IMAGINARY QUADRATIC FIELDS WITH CLASS NUMBER 2 OR 3

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ABSTRACT. This paper presents improved bounds for the norms of exceptional finite places of the group K_2O_F, where F is an imaginary quadratic field of class number 2 or 3. As an application we show that $K_2Z[\sqrt{-10}] = 1$.

1. Introduction

Tate [9] has determined the tame kernel of all imaginary quadratic Euclidean fields F and of $F = Q(\sqrt{-15})$. He and Bass [2] proved that when the norm of the finite place v of the field F is sufficiently large, then a certain associated map ∂_v (see section 2 below) is an isomorphism. It follows that in order to compute the tame kernel we need only investigate the remaining v's (those whose norms are smaller than the bound of the exceptional v's), and perform the appropriate computations with Steinberg symbols. Skalba [8] used a generalization of Thue's theorem to get a reasonable bound for norms of exceptional v's, and proved that $K_2O_F = 1$ when $F = Q(\sqrt{-19})$ and $F = Q(\sqrt{-20})$. Modifying the method of Tate, Qin [6, 7] did the same for $F = Q(\sqrt{-24})$ and $F = Q(\sqrt{-35})$. Recently Browkin [4] improved the estimates of Skalba and Qin for the bounds of norms of exceptional v's and applied his result to the case $F = Q(\sqrt{-23})$. In the present paper we present a certain adaptation of Tate, Skalba and Browkin's method for computing the tame kernel of imaginary quadratic fields with class number 2 or 3, and we get much smaller bounds for norms of exceptional v's. We apply this result in the case $F = Q(\sqrt{-40})$.

2. Notation

Let F be a number field and let $v_1, v_2, v_3, \dots$ be all finite places of F, ordered in a certain way. For $m \geq 0$ let $S_m = \{v_1, \dots, v_m\}$, thus $S_0 = \phi$. Denote the ring of S_m-integers of F by O_{S_m}, the group of S_m-units by U_{S_m}. Let P_m be the maximal ideal of $O_{S_{m-1}}$ corresponding to the place v_m. The residue field of the place v_m is denoted by k_{v_m} ($k_{v_m} = O_{S_{m-1}}/P_m$) and the norm of v_m is definded to be $N_{v_m} = \mathrm{card} k_{v_m}$. Thus O_{S_0} is the ring O_F of integers of F; we denote it by O below.

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Let $K_2^{S_m}(F)$ be the subgroup of K_2F generated by symbols $\{a,b\}$, where $a,b \in U_{s_m}$. Then $K_2F = \bigcup_{m=1}^{\infty} K_2^{S_m}(F)$.

Let $\partial_{v_m} : K_2F \to k_{v_m}^*$ be the tame symbol corresponding to v_m. Since $K_2^{S_{m-1}}(F) \subset \ker(\partial_{v_m})$, there is an induced map (also denoted by ∂_{v_m})

$$\partial_{v_m} : K_2^{S_m}(F)/K_2^{S_{m-1}}(F) \to k_{v_m}^*.$$

By Quillen's exact sequence

$$1 \to K_2O \to K_2F = \bigcup_{m=1}^{\infty} K_2^{S_m}(F) \xrightarrow{\partial} \coprod_{m=1}^{\infty} k_{v_m}^* \to 1, \tag{2.1}$$

where $\partial = \coprod_{m=1}^{\infty} \partial_{v_m}$, we know that if ∂_{v_j} are isomorphisms for all $j > m$, then $K_2O \subseteq K_2^{S_m}(F)$.

Since in the sequel we shall assume that m is fixed, we simplify the notation as follows:

$$S = S_{m-1}, \quad P = P_m \bigcap O, \quad v = v_m, \quad \partial_v = \partial_{v_m}, \quad U = U_{S_{m-1}}, \quad k^* = k_{v_m}^*.$$

Now let $F = Q(\sqrt{d})$ be the imaginary quadratic field with discriminant d and class number 2 or 3. Under the embedding $\sigma : F \to \mathcal{C} : a + b\sqrt{d} \to a + b\sqrt{|d|}i$ of the field F into $\mathcal{C}$, we can consider F as a subfield of $\mathcal{C}$. For all $x \in O$, $Nx = |x|^2 = x\overline{x}$.

Suppose that $\lambda = p$ is the least prime number such that p is not inertial in O and all factors in the prime decomposition of pO are nonprincipal. Let $pO = Q_1Q_2$, where Q_1,Q_2 are nonprincipal primes in O (Q_2 may equal Q_1 when the class number of F is 2). Set $A = \{Q_1, Q_2\}$. For any nonprincipal prime P in O, there is an element $Q \in A$ such that QP is principal and the norm of QP is $\lambda N(P)$.

Definition 2.1. Let P be a prime ideal of O. Define the principal norm $M(P)$ of P as follows:

$$M(P) = \begin{cases} N(P), & \text{if } P \text{ is principal in } O, \\ \lambda N(P), & \text{if } P \text{ is nonprincipal in } O. \end{cases}$$

For the finite place v corresponding to a prime ideal P of O, we write $M_v = M(P)$ and call M_v the principal norm of v. If P is principal in O, we say the place v is principal and we have $M_v = N_v$. Otherwise, v is nonprincipal and $M_v = \lambda N_v$.

For finite places v_i and v_j, if $i < j$, we say that v_i precedes v_j.

Let M be a real number satisfying $\lambda \le M$. We assume that the ordering on the set of all finite places of F satisfies the property: for all finite places v and v', if $M_v < M_{v'}$ and $M < M_{v'}$, then v precedes v'.

When the ordering on the set of finite places satisfies the preceding condition, we say that the ordering is normal for finite places with principal norm greater than M.

In the two sections below, we always assume that $M < M_v$. Thus the prime ideal PO_S corresponding to v in O_S is principal. We write PO_S as πO_S, where $\pi \in O$ and $N\pi = M_v$. Let β be the homomorphism from U to k^* given by $\beta(u) = u(\text{mod } \pi)$ (sometimes we write $u(\text{mod } \pi)$ as $u(\text{mod } v)$). Denote by U_1 the subgroup of U generated by $(1 + \pi U) \cap U$. It is easy to see that $U_1 \subseteq \ker\beta$. Let $a, b \in U$; then $a \sim b$ means $a \in bU_1$.

3. Preliminary information on the finite place v

Lemma 3.1. 1. *If $x \in O$ and $0 < Nx < M_v$, then $x \in U$.*
2. *$W = \{x \in O \cap U \mid Nx \leq M_v\}$ generates U.*

Proof. (1) Suppose that $(\ x\ ) = (\ y\ )(\ z\ )$ where $(\ y\ ) = P_1P_2\cdots P_r$, $(\ z\ ) = Q_1Q_2\cdots Q_s$, P_i $(i = 1,\dots,r)$ is principal prime in O and Q_j $(j = 1,\dots,s)$ is nonprincipal prime in O respectively. For P_i, we have $N(P_i) \leq Nx < M_v$ and for Q_j, we have $\lambda \leq N(Q_j) \leq Nz/\lambda \leq Nx/\lambda < M_v/\lambda$, i.e., $M(Q_j) < M_v$. That means the finite places corresponding to P_i or Q_j precede v, so $x \in U$.

(2) In fact U is generated by the set $\{a \in O \cap U \mid Na = 1 \text{ or } Na = M_{v'}, v' \text{ precedes } v\}$. □

Lemma 3.2. *Suppose v is principal. Let $a, b \in O \cap U$ satisfy $|a| + |b| < N_v$ and $a \equiv b (\operatorname{mod} v)$. Then $a \sim b$.*

Proof. Assume $a \neq b$. Since $a \equiv b(\operatorname{mod} v)$, $\pi | a - b$ (note that $PO_S = \pi O_S$). By $|a| + |b| < N_v$, we have $N(\frac{a-b}{\pi}) < N_v$. Then $\frac{a-b}{\pi} \in U$ by Lemma 3.1, so $a - b \in \pi U$, $a \sim b$. □

Now we assume v is nonprincipal and let Q be a nonprincipal prime in A such that QP is principal in O. Then $N(QP) = \lambda N_v = M_v$.

Lemma 3.3. *Suppose v is nonprincipal and let $a, b \in O \cap U$.*
1. *If $|a| + |b| < N_v$ and $a \equiv b(\operatorname{mod} v)$, then $a \sim b$.*
2. *Further, if $a, b \in Q$ satisfy $|a| + |b| < M_v$ and $a \equiv b(\operatorname{mod} v)$, then $a \sim b$.*

Proof. Assume $a \neq b$. As an ideal of O, $(a-b) = (\ y\ )(\ z\ )$, where $(\ y\ ) = P_1P_2\cdots P_r$, $(\ z\ ) = PQ_1\cdots Q_s$, P_i $(i = 1,\dots,r)$ is principal prime in O and Q_j $(j = 1,\dots,s, s \geq 1)$ is nonprincipal prime in O respectively and PQ_1 or PQ_1Q_2 is a principal ideal denoted by $(\ c\ )$.

(1) First we show that $PO_S = cO_S$, where $(\ c\ ) = PQ_1$ or PQ_1Q_2. This follows from the fact that the finite place corresponding to Q_j $(j = 1, 2, \dots, s)$ precedes v. In fact, $N(Q_j) \leq N(a-b)/N(P) < N_v^2/N_v = N_v$. Then $\frac{a-b}{c} \in U, a - b \in cU, a \sim b$.

(2) Similarly to the proof of (1), suppose $PQ = (\ c\ )$; then $Nc = M_v$, $c|a - b$. We have $N(\frac{a-b}{c}) < M_v$. By Lemma 3.1, $a - b \in cU, a \sim b$. □

4. Conditions for ∂_v being bijective

Tate [9] has proved the following useful result

Theorem 4.1 ([9, Prop. 1])**.** *Suppose that W, C and G are subsets of U such that*
1. *$W \subset CU_1$ and W generates U,*
2. *$CG \subset CU_1$ and $\beta(G)$ generates k^*,*
3. *$1 \in C \cap \ker\beta \subset U_1$.*

Then ∂_v is bijective.

In this section we will give some other conditions for ∂_v being bijective.

Proposition 4.2. *Suppose that W, D, E and E' are subsets of U such that*
1. *$W \subset DD^{-1}U_1$, where $D^{-1} = \{d^{-1} \mid d \in D\}$ and W generates U,*
2. *$1 \in E, E' \subset E$, and the map: $E' \times E' \times E \to k^* \times k^*$ given by $(e_1, e_2, e) \to (\beta(e_2/e_1), \beta(e/e_1))$ is surjective,*
3. *for any $x \in \{1\} \cup D, e_1, e_2, e' \in E'$ and $e \in E$,*

(a) *if* $xe_1 \equiv ee_2 (\operatorname{mod} v)$*, then* $xe_1 \sim ee_2$*,*
(b) *if* $e' \equiv e(\operatorname{mod} v)$*, then* $e' \sim e$*.*

Then ∂_v *is bijective.*

Proof. Let $u \in U$. Write $u \sim \prod_{i=1}^{r} d_i^{s_i}$, where $d_i \in D$ and $s_i = 1$ or $s_i = -1$ by condition (1). We want to show, by induction on r, that

(4.1) for any $u \in U$, there are $e' \in E'$ and $e \in E$ such that $u \sim e'/e$.

When $r = 1$ we have $u \sim d^s$, where $s = 1$ or $s = -1$. If $s = 1$, there are $e_1, e_2 \in E'$ such that $d \equiv e_2/e_1 (\operatorname{mod} v)$ by condition (2) (since $(\beta(d), 1)$ has preimage in $E' \times E' \times E$). We have $d \sim e_2/e_1$ from condition (3). So (4.1) holds (note that $E' \subset E$). When $s = -1$, replacing u by u^{-1}, we have the same proof.

Assume (4.1) is true for $r - 1$. Writing $u \sim d^s \prod_{i=2}^{r} d_i^{s_i}$, we have that there are $e' \in E'$ and $e \in E$ such that $u \sim d^s e'/e$, where $s = 1$ or $s = -1$, by the inductive assumption.

When $s = 1$, by condition (2) there are $e_1, e_2 \in E'$ and $e_3 \in E$ such that $d/e \equiv e_2/e_1 (\operatorname{mod} v)$, $1/e' \equiv e_3/e_1 (\operatorname{mod} v)$. Then we have $d/e \sim e_2/e_1$, $1/e' \sim e_3/e_1$ by condition (3). Hence $u \sim (e_2/e_1)/(e_3/e_1) = e_2/e_3$, and (4.1) holds. When $s = -1$, replacing $d/e \equiv e_2/e_1 (\operatorname{mod} v)$ and $1/e' \equiv e_3/e_1 (\operatorname{mod} v)$ by $1/e \equiv e_2/e_1 (\operatorname{mod} v)$ and $d/e' \equiv e_3/e_1 (\operatorname{mod} v)$, where $e_1, e_2 \in E'$ and $e_3 \in E$, respectively, we can prove it by the same line of argument as the case $s = 1$.

Further, we have $\ker\beta \subseteq U_1$ and hence $\ker\beta = U_1$, since if $u \sim e'/e \in \ker\beta$, i.e., $\beta(e'/e) = 1$, then $e'/e \sim 1$.

Let α be the homomorphism from U to $K_2^{S_m} F / K_2^{S_{m-1}} F$ given by $\alpha(u) = \{u, \pi\} (\operatorname{mod} K_2^{S_{m-1}})$, where $\pi \in O$ satisfies $PO_S = \pi O_S$. Then, as in the proof of Lemma 3.2 in [2], we have $\beta = \partial_v \circ \alpha$. To prove ∂_v is bijective, it suffices to show that

(i) α is surjective, and
(ii) β is surjective and $\ker\beta \subset \ker\alpha$.

From [2, pages 405-406], (i) is true. By condition (2), β is surjective. If $1 - u\pi \in U_1$, where $u \in U$, then $\{1 - u\pi, \pi\} = \{1 - u\pi, u\}^{-1} \in K_2^{S_{m-1}} F$ and so $\ker\beta = U_1 \subset \ker\alpha$. We are done. $\square$

Let $h > 0$ be a real number. Define $D(h) = \{x \in O \bigm| 0 < |x| \leq h\}$. For simplicity, let $K = \frac{2}{\pi}\sqrt{|d|}$.

Lemma 4.3. *Suppose that* v *is nonprincipal and* $N_v > K^2$*. Let* $h^2 = \lambda K N_v^{\frac{1}{2}}$ *and* $D = D(h) \cap Q \cap U$*, where* Q *is in* A *and such that* QP *is a principal ideal in* O*. For any* $a \in O \setminus P$*, there are* $x, y \in D$ *such that* $a \equiv y/x (\operatorname{mod} v)$*.*

Proof. We only sketch the line of the proof, since it is similar to that of Lemmas 1–3 in [8].

For any $a \in O$, define $F_a : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by the formula $(x, y) \to y - ax$. Let $K_a = F_a^{-1}(P) \cap (Q \times Q)$. Then K_a is a lattice in $R^4 \simeq \mathcal{C} \times \mathcal{C}$, and we have

$$\operatorname{vol} K_a = \operatorname{vol} K_0 = \operatorname{vol}(Q \times (Q \cap P)) = (\frac{\sqrt{|d|}}{2})^2 N(Q)^2 N_v.$$

Let $S_h = \{(x, y) \in \mathcal{C} \times \mathcal{C} \bigm| |x| \leq h, |y| \leq h\}$. We have $\operatorname{vol} S_h = (\pi h^2)^2$ and $2^4 \operatorname{vol} K_a = \operatorname{vol} S_h$. Applying Minkowski's theorem, we can get $x, y \in S_h \bigcap K_a$, where $(x, y) \neq (0, 0)$. Note that $Nx, Ny \leq h^2 < M_v$. If $a \in O \setminus P$, then $x, y \neq 0$.

By Lemma 3.1, $x, y \in U$. Then $a \equiv y/x (\bmod v)$ (see Skalba's GTT in [8, page 306]). $\square$

Lemma 4.4. *Suppose that v is nonprincipal and $M_v > K^3$. Let $\tilde{h} > 0$ and $\tilde{h}^2 = KM_v^{\frac{2}{3}}$ $(= K\lambda^{\frac{2}{3}}N_v^{\frac{2}{3}})$, and let $E = D(\tilde{h}) \cap U, E' = E \cap Q$. For any $a, b \in O \setminus P$, there exist $e_1, e_2 \in E'$ and $e \in E$ such that $a \equiv e_2/e_1 (\bmod v)$ and $b \equiv e/e_1 (\bmod v)$.*

Proof. For any $a, b \in O$, define $F_{a,b} : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ by the formula $(x, y, z) \to (y - ax, z - bx)$. Let $K_{a,b} = F_{a,b}^{-1}(P \times P) \cap (Q \times Q \times O)$. The remaining procedure is as in Lemma 4.3 (you also can refer to [5]). $\square$

Theorem 4.5. *Suppose that v is nonprincipal ($M_v = \lambda N_v$). If the inequalities*

1. $N_v > \max\{K^2, K^3/\lambda\}$,
2. $N_v > (1 + M_v^{-\frac{1}{2}})^4 \cdot K^2$,
3. $M_v > [(1 + \lambda^{\frac{1}{4}} M_v^{-\frac{1}{12}})K]^3$,
4. $N_v > \lambda^{\frac{1}{2}}(4K)^{\frac{3}{4}}$

hold, then ∂_v is bijective.

Proof. Let us check that the subsets W as in Lemma 3.1, D as in Lemma 4.3, and E, E' as in Lemma 4.4, of U satisfy the conditions of Proposition 4.2.

(i) Let $w \in W$. Since $N_v > \max\{K^2, K^3/\lambda\}$, by Lemma 4.3 there exist $d_1, d_2 \in D$ such that $w \equiv d_1/d_2 (\bmod v)$. The condition (2) above provides $|wd_2| + |d_1| < M_v$, so $w \in DD^{-1}U_1$ by Lemma 3.3.

(ii) That $E' \subset E, 1 \in E$ is clear. The condition (1) above allows us to apply Lemma 4.4 to get that the map from $E' \times E' \times E$ to $k^* \times k^*$ is surjective.

(iii) Let $x \in \{1\} \cup D$, $e_1, e_2, e' \in E'$ and $e \in E$. By conditions (3) and (4) above, the following inequalities hold:

$$|xe_1| + |ee_2| < M_v, \quad |e'| + |e| < N_v.$$

Applying Lemma 3.3, we know that condition (3) of Proposition 4.2 holds. $\square$

When v is principal, the results of Lemma 3.1 and Lemma 3.2 allow us to apply Theorem 1 in [5] to obtain the following result.

Theorem 4.6. *Suppose that v is principal and $|d| \geq 3$. If the inequalities*

1. $N_v > K^3$,
2. $N_v > (1 + N_v^{-\frac{1}{2}})^4 \cdot K^2$,
3. $N_v > [(1 + N_v^{-\frac{1}{12}})K]^3$

hold, then ∂_v is bijective.

Note. We take $a = 1$ and $\lambda = 1$ in Theorem 1 of [5].

In Table 1, for every discriminant d such that the class number of $F = Q(\sqrt{d})$ is 2 or 3, we give the estimates of N_v from Theorem 4.5 and Theorem 4.6

Note. 1) From [1], we know that all d such that the class number of $F = Q(\sqrt{d})$ is 3 are listed in the above table.

2) When computing the bound of N_v, note that the right sides of the inequalities (2) and (3) in Theorems 4.5 and 4.6 are decreasing functions on M_v and N_v respectively.

Remark 4.7. Comparing the estimates of N_v above with Browkin's in [4], we can see that the bound for norms of exceptional v's is much smaller here.

Table 1.

$-d$	23	24	31	35	40	51	52	59	83	88	91
λ	2	2	2	3	2	3	2	3	3	2	5
N_v (v principal)	132	140	198	234	281	391	402	478	763	827	866
N_v (v nonprincipal)	81	85	120	107	170	177	243	216	342	496	271
$-d$	107	115	123	139	148	187	211	232	235	267	283
λ	3	5	3	5	2	7	5	2	5	3	7
N_v (v principal)	1082	1195	1311	1551	1691	2336	2760	3147	3203	3822	4142
N_v (v nonprincipal)	483	371	583	479	1008	565	842	1865	975	1677	988
$-d$	307	331	379	403	427	499	547	643	883	907	
λ	7	5	5	11	7	5	11	7	13	13	
N_v (v principal)	4637	5146	6209	6761	7326	9096	10334	12940	20126	20893	
N_v (v nonprincipal)	1103	1551	1865	1160	1724	2712	1752	3004	2974	3084	

5. Computation of $K_2Z[\sqrt{-10}]$

In this section we apply the general method of previous sections to the special case $F = Q(\sqrt{-40})$. Note that the class number of F is 2.

Let $\omega = \sqrt{-10}$. Then $O = Z[\omega]$. Take $Q = (2, \omega)$; we have $Q^2 = (\ 2\ ), \lambda = N(Q) = 2$. O and Q can be considered as a lattice in $\mathcal{C}$.

Let δ and δ' denote the maximal distance from $\mathcal{C}$ to O and Q respectively. Then $\delta^2 = \frac{11}{4}, \delta'^2 = \frac{7}{2}$.

Lemma 5.1. *Let A be a principal ideal of O. Then there is a representative element c in every residue class modulo A such that $Nc \leq \frac{11}{4}N(A)$.*

Lemma 5.2. *Let A be a nonprincipal ideal of O. Then there is a representative element c in every residue class modulo A such that $Nc \leq \frac{7}{4}N(A)$.*

The proofs of the two lemmas are elementary.

Suppose that all finite places of F are ordered in the following way:

(1) v_1, v_2 and v_3 correspond to the prime ideals $(\ 3\ ), (2, \omega)$ and $(\ 5\ )$, respectively (these ideals are in O, O_{S_1} and O_{S_2}, respectively). The remaining finite places with principal norm not greater than 41 correspond to the ideal (a_m), $m = 4, 5, \ldots, 13$, in $O_{S_{m-1}}$ where $a_4 = 2 - \omega, a_6 = 1 + \omega, a_8 = 3 + \omega, a_{10} = 4 + \omega, a_{12} = 1 + 2\omega$, and $a_{2k+1} = \overline{a_{2k}}$ $(k = 2, 3, 4, 5, 6)$.

(2) The ordering is normal for finite places with principal norm greater than 41.

In fact this ordering is even normal on the set of finite places with principal norm greater than 14.

In the following propositions we set $G = \{x \in O \cap U \bigm| |x| \leq |g|\}$, where $|g|$ is the least number such that $\beta(G)$ generates k^* and W as in Lemma 3.1, and choose a suitable subset C of U satisfying the conditions of Theorem 4.1 so as to apply Theorem 4.1 to show ∂_v is bijective for some v with small norm.

Proposition 5.3. *If v is nonprincipal and $N_v > 13$ (i.e., $M_v > 26$), then ∂_v is bijective.*

Proof. According to the table for the bound of N_v in Section 4, we only need to consider the case $N_v \leq 170$.

Let $C = \{x \in O \bigm| 0 < Nx \leq \frac{7}{4}N_v\}$. By Lemma 3.1 we know that $C \subset U$. From Lemma 3.3 and Lemma 5.2 we see that if N_v satisfies the inequalities

(1) $\sqrt{2N_v} + \sqrt{\frac{7}{4}N_v} < N_v$, i.e., $7.5 \approx (\sqrt{2} + \sqrt{\frac{7}{4}})^2 < N_v$,

(2) $\sqrt{\frac{7}{4}N_v}(|g|+1) < N_v$, i.e., $\frac{7}{4}(|g|+1)^2 < N_v$,
then we can apply Theorem 4.1 to obtain that ∂_v is bijective. Thus for the following cases the consequence holds:

$$\begin{aligned}
&N_v = 23, 37, 47, 53, 103, 167, \text{ here } |g| = 2;\\
&N_v = 127, \text{ here } |g| = 3;\\
&N_v = 157, \text{ here } |g| = \sqrt{11}. \quad \square
\end{aligned}$$

Let $h > 0$. Denote $h' = \max\{|x| \bigm| x \in D(h)\}$ (note that $D(h) = \{x \in O \bigm| 0 < |x| \leq h\}$).

Lemma 5.4 (Lemma 3.2 in [5])**.** *Suppose $N_v > K^2$, $D_1 = D(h_1)$ and $D_2 = D(h_2)$, where $h_1, h_2 > 0$ satisfy the following conditions:*
(1) $h_1 h_2 = K N_v^{\frac{1}{2}}$,
(2) $\max(h_1'^2, h_2'^2) < N_v$.
Then for any $a \in O \setminus P$, there exist $x \in D_1$ and $y \in D_2$ such that $a \equiv x/y (\operatorname{mod} v)$.

Proposition 5.5. *If v is principal and $N_v > 41$, then ∂_v is bijective.*

Proof. We only need to consider the case $N_v < 281$ by the table in Section 4. Take h_1, h_2 and D_1, D_2 as in Lemma 5.4 and let $C = D_1 D_2^{-1}$. By Lemma 3.1 we know that $D_1, D_2 \subset U$, so $C \subset U$. From Lemma 3.2 and Lemma 5.4 we see that if the two inequalities
(1) $N_v^{\frac{1}{2}} h_2' + h_1' < N_v$,
(2) $(1 + |g|)h_1' h_2' < N_v$
hold, then ∂_v is bijective by Theorem 4.1.

It is easy to show the consequence holds for the following cases. Here we list the values of $N_v, |g|, h_2^2, h_1'^2$ and $h_2'^2$, where h_1 is determined by condition (1) in Lemma 5.4 (in fact the method was used by Skalba [8]).

N_v	$\lvert g\rvert$	h_2^2	$h_1'^2$	$h_2'^2$
251	3	18.99	211	16
211	2	58.49	56	56
179	2	53.87	49	49
139	2	18.99	115	16
131	2	18.99	110	16

For the remaining cases we apply Qin's method [6, 7] to construct C.

Let $C' = \{x \in O \bigm| 0 < Nx \leq \frac{11}{4} N_v\}$, $T = C' \setminus U = \{t_1, t_2, \ldots, t_r\}$. Choose $S = \{s_1, s_2, \ldots, s_r\} \subset O \cap U$, where $s_i \equiv t_i (\operatorname{mod} v)$, $i = 1, \ldots, r$. Let $C = (C' \setminus T) \cup S$ (the choice of elements in S should make $m(C) = \max\{|x| \bigm| x \in C\}$ as small as possible).

By Lemma 3.1 and Lemma 5.1, if the inequalities
(1) $m(C) + \sqrt{N_v} < N_v$,
(2) $m(C)(1 + |g|) < N_v$
hold, then ∂_v is bijective. So the result is true for the remaining cases:

N_v	241	89	59
$\lvert g\rvert$	$\sqrt{14}$	3	2
$m(C) \leq$	$\sqrt{2124}$	22	$\sqrt{304}$

$\square$

We have stated that the finite places with principal norm not greater than 41 correspond to the ideals $(a_m),\ m = 4, 5, \ldots, 13$, in $O_{S_{m-1}}$ (except for v_1, v_2, v_3). For convenience, we write ∂_{v_m} as ∂_m, where $m = 4, 5, \ldots, 13$.

Proposition 5.6. *∂_m is bijective for $m = 5, 6, \ldots, 13$.*

Proof. We will construct the subsets W, C and G of U directly. Let $a_0 = -1, a_1 = 2, a_2 = 3, a_3 = \omega$ (note that the meaning of $a_m,\ m = 4, 5, \ldots, 13$, has been given before). Set $W_m = \{a_j \mid j = 0, 1, \ldots, m-1\}$, where $m = 5, 6, \ldots, 13$. We will list the sets C and G just for ∂_m, where $m = 2r+1,\ r = 2, 3, 4, 5, 6$. For $m = 2r,\ r = 3, 4, 5, 6$, as the sets C and G for ∂_{2r} we take the conjugates of C and G for ∂_{2r+1} respectively. Let X be a number set. Denote by $\pm X$ the set $-X \cup X$, where $-X = \{-x \mid x \in X\}$.

$$\begin{aligned}
m = 13,\quad & C = \pm\{1, 2, 4, 8, 18, \omega, \omega+1, 2(\omega+1), 4(\omega+1), \omega+2, 2(\omega+2),\\
& \qquad 3(\omega+2), \omega+3, 2(\omega+3), 3(\omega+3), \omega+4, 2(\omega+4), \omega+5,\\
& \qquad 2(\omega+5), 9\omega(\omega+1)/2, 9\omega(\omega+1)\},\ G = \{2, 3\};\\
m = 11,\quad & C = \pm\{1, 2, \omega, 2\omega, -\omega/5, -10\},\ G = \{2\};\\
m = 9,\quad & C = \pm\{1, 2, 4, 5, 4/5, \omega/2, \omega/5, \omega/10, 2\omega, 4\omega, 2(\omega+2)\},\ G = \{2\};\\
m = 7,\quad & C = \pm\{1, 2, 5/2, \omega/2, \omega/3, \omega+2\},\ G = \{2\};\\
m = 5,\quad & C = \pm\{1, 2, 4\},\ G = \{-2\}.
\end{aligned}$$

We can use the method used in [5] and [8] to check that the subsets W, C and G of U satisfy the conditions of Theorem 4.1. □

Theorem 5.7. $K_2Z[\sqrt{-10}] = 1$.

Before proving the assertion, we list some identities on symbols and some results on the elements of K_2F which are useful for our proof.

1) Let $x \in F^*$ and assume that the polynomials of x make sense of the following symbols. We have

$$\{x, x\}^2 = \{x, -1\}^2 = 1, \{x, x+1\}^2 = 1, \{x, x^2 \pm 1\}^4 = 1, \{x, x^2+x+1\}^3 = 1.$$

2) If $x, y, z \in F^*$ and $z = x \pm y$, then $\{\frac{x}{z}, \frac{y}{z}\}^2 = 1$, and hence $\{x, y\}^2 \cdot \{y, z\}^2 = \{x, z\}^2$.

3) If $\{\frac{a}{b}, \frac{cc'}{d}\}^n = 1$, where $a, b, c, d \in U_{S_m},\ c' \in F^*$, then $\{a, c'\}^n \equiv \{b, c'\}^n \pmod{K_2^{S_m}(F)}$.

4) From the proof of Proposition 3 in [2], we know that there is no element of order 2 in K_2O. Note that the class number of $Q(\sqrt{3 \times 40})$ is prime to 3, so there is no element of order 3 in K_2O by the results on p-rank of K_2O in [3]. Thus if $x \in K_2O$ and $x^{2^r 3^k} \in A$, where A is a subgroup of K_2F and $r, k \in N$, we have $x \in A$.

Now let us complete the proof of Theorem 5.7.

By Quillen's exact sequence (see (2.1) in Section 2) and the preceding results, we have

$$K_2O \subseteq K_2^{S_4}(F). \tag{5.1}$$

a) By (5.1), K_2O consists of products of symbols of the form $\{a, b\}$, where $a, b \in \{-1, 2, \omega, 3, 2-\omega\}$. Since $\omega - (\omega - 2) = 2$, we have

$$\{\frac{\omega}{2}, \frac{\omega-2}{2}\}^2 = 1. \tag{5.2}$$

Since $(\omega-2)^2=\omega^2-4\omega+4=-(4\omega+6)$, we have

$$\{-\frac{4\omega}{6},-\frac{(\omega-2)^2}{6}\}=1. \tag{5.3}$$

From (5.2), (5.3), we obtain that $K_2O\subseteq\langle\{2,\omega-2\}\rangle K_2^{S_3}(F)$. Let $x=\frac{\omega-2}{4}$. Then $x^2+x+1=\frac{1}{8}$ and $\{x,\frac{1}{2}\}^9=\{x,\frac{1}{8}\}^3=1$. So

$$\{x,2\}^9=1,\ \{2,x\}^9=1,\ \text{and}\ \{2,\omega-2\}^9=1. \tag{5.4}$$

(Note that $\{2,4\}=\{2,2\}^2=1$.) Therefore $K_2O\subseteq K_2^{S_3}(F)$.

b) Since

$$\{2,\omega\}^8=\{2^4,\omega^2\}=\{16,-10\}=\{4,-10\}^2=\{4,5\}^2\{4,-2\}^2=1 \tag{5.5}$$

and

$$\{3,\omega\}^8=\{3^2,\omega^2\}^2=\{9,-10\}^2=1, \tag{5.6}$$

we have $K_2O\subseteq K_2^{S_2}(F)$.

Since $\{2,3\}^2=1$, we obtain $K_2O=1$. □

Remark 5.8. The result agrees with that conjectured in [3], and also that in [4].

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