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L^2-ESTIMATE FOR THE DISCRETE PLATEAU PROBLEM

PAOLA POZZI

ABSTRACT. In this paper we prove the L^2 convergence rates for a fully discrete finite element procedure for approximating minimal, possibly unstable, surfaces.

Originally this problem was studied by G. Dziuk and J. Hutchinson. First they provided convergence rates in the H^1 and L^2 norms for the boundary integral method. Subsequently they obtained the H^1 convergence estimates using a fully discrete finite element method. We use the latter framework for our investigation.

1. Introduction

A disk-like *minimal surface* or *solution of the Plateau Problem* is a surface in $\mathbb{R}^n$ which has the topology of the unit disc, spans a given boundary curve $\Gamma \in \mathbb{R}^n$, and either minimizes, or more generally is stationary for, the area functional. By studying the problem in detail, it turns out that an equivalent and more convenient formulation is the following characterisation.

Let D be the unit disc in $\mathbb{R}^2$ and Γ be a smooth Jordan curve in $\mathbb{R}^n$. Let $\mathcal{F}$ be the class of *harmonic* maps $u : \overline{D} \to \mathbb{R}^n$ such that $u|_{\partial D} : \partial D \to \Gamma$ is monotone and satisfies a certain integral "three-point condition"; cf. (1). The function $u \in \mathcal{F}$ is said to be a *minimal surface* if u is stationary in $\mathcal{F}$ for the Dirichlet energy $\mathcal{D}(u) = \frac{1}{2}\int_D |\nabla u|^2$. Such a map u provides an harmonic conformal parametrisation of the corresponding minimal surface.

The formulation of the corresponding discrete problem is as follows. Let D_h be a quasi-uniform triangulation of D with grid size controlled by h. Let $\mathcal{F}_h$ be the class of *discrete harmonic* maps $u_h : \overline{D_h} \to \mathbb{R}^n$ for which $u_h(\phi_j) \in \Gamma$ whenever ϕ_j is a boundary node of D_h, and which satisfy an analogue of the previous integral "three-point condition". The function $u_h \in \mathcal{F}_h$ is said to be a *discrete minimal surface* if u_h is stationary within $\mathcal{F}_h$ for the Dirichlet energy $\mathcal{D}(u_h) = \frac{1}{2}\int_{D_h} |\nabla u_h|^2$ (see below for a precise formulation).

The main result proved in [4] is that if u is a nondegenerate minimal surface spanning Γ, then there exists a discrete minimal surface u_h, unique in a ball of "almost" constant radius $\epsilon_0|\log h|^{-1}$, such that $\|u - u_h\|_{H^1(D_h)} \leq ch$, where c depends on Γ and the nondegeneracy constant for u, but is independent of h (see Theorem 2.2 of this paper).

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In this paper, which can be considered a continuation of [3] and [4], we prove the additional estimate

$$\|u-u_h\|_{L^2(D_h)} \leq ch^2|\log h|^{3/2}.$$

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2. Preliminary estimates and theorems

In this section we will concisely recall some definitions, estimates, and theorems from the papers cited above.

2.1. **The smooth energy functional.** Let D be the open unit disc in $\mathbb{R}^2$, with boundary ∂D. Denote by S^1 another distinct copy of the unit circle. Let Γ be a Jordan curve in $\mathbb{R}^n$ with regular C^r-parametrisation $\gamma : S^1 \to \Gamma$ where $r \geq 3$. (Note that more regularity will be required when stating the main theorems.)

The reason for introducing S^1 and fixing a parametrisation γ is that each map $f : \partial D \to \Gamma$ can be uniquely written in the form $f = \gamma \circ s$, where $s : \partial D \to S^1$. It turns out that it is more convenient to make use of such a factorisation and work in the space of $\{s \mid s : \partial D \to S^1\}$. Recall also that we are interested in working in the class of harmonic functions and that information on the boundary alone is sufficient to completely determine such a function.

For $f : \partial D \to \mathbb{R}^n$, we denote by $\Phi(f) : \overline{D} \to \mathbb{R}^n$ its unique harmonic extension to D, specified by

$$\triangle\Phi(f) = 0 \text{ in } D, \qquad \Phi(f) = f \text{ on } \partial D.$$

Then $\Phi : H^{1/2}(\partial D, \mathbb{R}^n) \to H^1(D, \mathbb{R}^n)$ is a bounded linear map with bounded inverse.

We will use the Hilbert space H of functions defined by

$$H = \{\xi : \partial D \to \mathbb{R} \mid |\xi|_{H^{1/2}} \leq \infty \textit{ and } (1) \textit{ is satisfied}\},$$

where

$$(1) \qquad \int_0^{2\pi} \xi(\phi)\, d\phi = 0, \qquad \int_0^{2\pi} \xi(\phi)\cos\phi\, d\phi = 0, \qquad \int_0^{2\pi} \xi(\phi)\sin\phi\, d\phi = 0.$$

The norm on H is the usual norm $\|\cdot\|_{H^{1/2}}$, which by the first condition in (1) and Poincaré's inequality is equivalent to $|\cdot|_{H^{1/2}}$. The corresponding affine space of maps $s : \partial D \to S^1$ such that $s(\phi) = \phi + \sigma(\phi)$ for some $\sigma \in H$ is denoted by $\mathcal{H}$. We also need the Banach space T defined by $T = H \cap C^0(\partial D, \mathbb{R})$ with norm $\|\xi\|_T = \|\xi\|_{H^{1/2}} + \|\xi\|_{C^0}$. The corresponding affine space $\mathcal{T}$ is defined by $\mathcal{T} = \mathcal{H} \cap C^0(\partial D, S^1)$. With some abuse of notation we write $\|s\| = 1 + \|\sigma\|$ for various norms $\|\cdot\|$ on σ.

The energy functional E is defined on $\mathcal{H}$ by

$$(2) \qquad E(s) = \frac{1}{2}\int_D |\nabla\Phi(\gamma\circ s)|^2 = \mathcal{D}(\Phi(\gamma\circ s)).$$

Finiteness of E follows from (8).

We say that the harmonic function $u = \Phi(\gamma \circ s)$ is a *minimal surface spanning* Γ if s is monotone and stationary for E; i.e.,

$$(3) \qquad \langle E'(s), \xi\rangle = 0 \quad \forall \xi \in T.$$

We have the following regularity result (see [4, Proposition 2.1]).

Proposition 2.1. *If $\gamma \in C^{k,\alpha}$, where $k \geq 1$ and $0 < \alpha < 1$, and $s \in \mathcal{T}$ is monotone and stationary for E, then*

$$\|s\|_{C^{k,\alpha}} \leq c = c(\|\gamma\|_{C^{k,\alpha}}, \||\gamma'|^{-1}\|_{L^\infty}).$$

We next recall some properties of the energy functional from [3, Section 3.3]. Using the notation

$$u = \Phi(\gamma \circ s), \qquad v = \Phi(\gamma' \circ s\, \xi), \qquad w = \Phi(\gamma'' \circ s\, \xi^2), \tag{4}$$

we get by formal computation that

$$E(s) = \frac{1}{2}\int_D |\nabla u|^2 \,, \tag{5}$$

$$\langle E'(s), \xi\rangle = \frac{d}{dt}\Big|_{t=0} E(s + t\xi) = \frac{1}{2}\int_D \nabla u \nabla v \,, \tag{6}$$

$$E''(s)(\xi,\xi) = \frac{d^2}{dt^2}\Big|_{t=0} E(s + t\xi) = \int_D \nabla u \nabla w + \int_D |\nabla v|^2 \,, \tag{7}$$

with an analogous expression for $E''(s)(\xi,\eta)$ obtained by bilinearity in the case of distinct variations.

Proposition 2.2. *If γ is C^r the energy functional $E : \mathcal{T} \to \mathbb{R}$ is C^{r-1}. Let $s = id + \sigma$. Then*

$$E(s) \leq c(\|\gamma\|_{C^1})(1 + |\sigma|^2_{H^{1/2}}),$$

$$|d^j E(s)(\xi_1, \dots, \xi_j)| \leq c(\|\gamma\|_{C^{j+1}})(1 + |\sigma|^2_{H^{1/2}})\|\xi_1\|_T \cdot \dots \cdot \|\xi_j\|_T$$

for $1 \leq j \leq r - 1$.

Proof. See [2, Proposition 2.1]. □

The functional E is *not* differentiable on $\mathcal{H}$, but if γ and s are as smooth as is necessary for the following estimates, then we have

$$E(s) \leq c\|\gamma\|^2_{C^1}\|s\|^2_{H^{1/2}}, \tag{8}$$

$$|\langle E'(s), \xi\rangle| \leq c\|\gamma\|^2_{C^2}\|s\|^2_{C^1}\|\xi\|_{H^{1/2}}, \tag{9}$$

$$|E''(s)(\xi,\eta)| \leq c\|\gamma\|^2_{C^2}\|s\|^2_{C^1}\|\xi\|_{H^{1/2}}\|\eta\|_{H^{1/2}}. \tag{10}$$

In particular this will be used in case s is stationary for E.

It is important to consider the behaviour of the second derivatives of E near a stationary point $s \in \mathcal{T}$. The second derivative $E''(s)$ can be interpreted as a self-adjoint bounded map $\nabla^2 E(s) : H \to H$. Let

$$H = H^- \oplus H^0 \oplus H^+, \qquad \xi = \xi^- + \xi^0 + \xi^+ \quad \text{if } \xi \in H, \tag{11}$$

be the orthogonal decomposition generated by the eigenfunctions of $\nabla^2 E(s)$ having negative, zero, and positive eigenvalues, respectively.

For s monotone and stationary for E, we say s is *nondegenerate* if $H^0 = \{0\}$. The corresponding minimal surface $u = \Phi(\gamma \circ s)$ is also said to be *nondegenerate*. If s is nondegenerate, it follows that there exists a $\lambda > 0$ such that for $\xi \in H$

$$E''(s)(\xi, \xi^+ - \xi^-) = E''(s)(\xi^+,\xi^+) - E''(s)(\xi^-,\xi^-) \geq \lambda\|\xi\|^2_{H^{1/2}}. \tag{12}$$

We call λ a *nondegeneracy constant* for s.

2.2. **The discrete energy functional.** Let $\mathcal{G}_h$ be a quasi-uniform triangulation of D with grid size comparable to h. Let

$$\begin{aligned} D_h &= \bigcup\{G \mid G \in \mathcal{G}_h\}, \\ \partial D_h &= \bigcup\{E_j \mid 1 \le j \le M\} \quad \text{where the } E_j \text{ are the boundary edges,} \\ \mathcal{B}_h &= \{\phi_1, \dots, \phi_M\} \quad \text{be the set of boundary nodes.} \end{aligned}$$

The projection $\pi : \partial D \to \partial D_h$ is defined by

$$\pi\Big(e^{i((1-t)\phi_j + t\phi_{j+1})}\Big) = (1-t)e^{i\phi_j} + te^{i\phi_{j+1}}\,, \tag{13}$$

for $0 \le t \le 1$, $1 \le j \le M$.

In order to have a discrete analogue E_h of the functional E, we define the following discrete analogues of $H^1(D, \mathbb{R}^n)$, $H^{1/2}(\partial D, \mathbb{R}^n)$, H, T, $\mathcal{H}$ and $\mathcal{T}$:

$$X_h^n = \{u_h \in C^0(D_h, \mathbb{R}^n) \mid u_h \in P_1(G) \text{ for } G \in \mathcal{G}_h\}, \tag{14}$$

$$x_h^n = \{f_h \in C^0(\partial D_h, \mathbb{R}^n) \mid f_h \in P_1(E_j) \text{ for } 1 \le j \le M\}, \tag{15}$$

$$H_h = \{\xi_h \in C^0(\partial D, \mathbb{R}) \mid \xi_h \in P_1(\pi^{-1}(E_j)) \text{ if } 1 \le j \le M,\ \xi_h \text{ satisfies (1)}\}, \tag{16}$$

$$\mathcal{H}_h = \{s_h \in C^0(\partial D, S^1) \mid s_h(\phi) = \phi + \sigma_h(\phi) \text{ for some } \sigma_h \in H_h\}. \tag{17}$$

Thus $H_h \subset T \subset H$, $\mathcal{H}_h \subset \mathcal{T} \subset \mathcal{H}$, and the space of variations at $s_h \in \mathcal{H}_h$ is naturally identified with H_h. We write $X_h = X_h^1$ and $x_h = x_h^1$.

We have the following inverse-type estimates.

Proposition 2.3. *If $\xi_h \in H_h$, then*

$$\|\xi_h\|_{H^1} \le ch^{-1/2}\|\xi_h\|_{H^{1/2}}, \tag{18}$$

$$\|\xi_h\|_{H^{1/2}} \le \|\xi_h\|_T \le c|\ln h|^{1/2}\|\xi_h\|_{H^{1/2}} \tag{19}$$

for h small.

Proof. The first estimate is standard. The second is in [1, Proposition 5.3]. □

Suppose $f \in C^0(\partial D, \mathbb{R}^n)$. We define the "linear interpolants"

$$\begin{aligned} I_h f \in x_h^n, &\quad I_h f\big((1-t)e^{i\phi_j} + te^{\phi_{j+1}}\big) = (1-t)f(e^{\phi_j}) + tf(e^{\phi_{j+1}}), \\ I_h^{\partial D} f \in C^0(\partial D, \mathbb{R}^n), &\quad I_h^{\partial D} f\big(e^{i((1-t)\phi_j + t\phi_{j+1})}\big) = (1-t)f(e^{\phi_j}) + tf(e^{\phi_{j+1}}), \end{aligned}$$

where $0 \le t \le 1$, $1 \le j \le M$. Here and elsewhere, $\phi_{M+1} = \phi_1$. Note the different domains ∂D_h and ∂D of $I_h f$ and $I_h^{\partial D} f$, respectively. Note also that the image of $I_h(\gamma \circ s)$ is a polygonal approximation to Γ and that $I_h(\gamma \circ s)(\phi_j) = \gamma \circ s(\phi_j) \in \Gamma$ for $\phi_j \in \mathcal{B}_h$. Finally,

$$I_h^{\partial D} f = I_h f \circ \pi. \tag{20}$$

Another type of approximation operator we require is a map $p_h : T\ (\mathcal{T}) \to H_h\ (\mathcal{H}_h)$, which acts like an interpolation operator and preserves the normalisation condition (1). The proof of the following is essentially given in [1, Proposition 5.2].

Proposition 2.4. *There is a bounded linear operator $p_h : T \to H_h$, such that (in particular)*

$$\|\xi - p_h\xi\|_{H^s} \le ch^{k-s}\|\xi\|_{H^k} \tag{21}$$

for $s = 0, \frac{1}{2}, 1$ *and* $k = 1, \frac{3}{2}, 2$. *Moreover,*

$$\|\xi - p_h\xi\|_{C^{0,1}} \le ch\|\xi\|_{C^2}, \qquad \|p_h\xi\|_{C^{0,1}} \le c\|\xi\|_{C^{0,1}}, \tag{22}$$

$$\|\xi - p_h\xi\|_{C^0} \le ch^2\|\xi\|_{C^2}, \qquad \|\xi - p_h\xi\|_{C^0} \le ch\|\xi\|_{C^1}. \tag{23}$$

If $s \in \mathcal{T}$ *and* $s(\phi) = \phi + \sigma(\phi)$, *then* $p_h s$ *is defined by* $p_h s(\phi) = \phi + p_h\sigma(\phi)$ *and* $s - p_h s = \sigma - p_h\sigma$. *Hence* $p_h s$ *satisfies estimates similar to those for* $p_h\xi$.

For $f_h \in x_h$ the *discrete harmonic extension* $\Phi_h f_h \in X_h$ is defined by

$$\triangle_h \Phi_h f_h = 0 \text{ in } D_h, \qquad \Phi_h f_h = f_h \text{ on } \partial D_h. \tag{24}$$

Here $\triangle_h$ is the discrete Laplacian and so the first equation in (24) is interpreted as $\int_{D_h} \nabla(\Phi_h f_h)\nabla\psi_h = 0$ for all ψ_h in X_h such that $\psi_h = 0$ on ∂D_h. If $f_h \in x_h^n$ the discrete harmonic extension $\Phi_h f_h$ is defined componentwise.

For $s_h \in \mathcal{H}_h$ the *discrete energy functional* E_h is defined by

$$E_h(s_h) = \frac{1}{2}\int_{D_h} |\nabla\Phi_h I_h(\gamma \circ s_h)|^2 = \mathcal{D}_h(\Phi_h I_h(\gamma \circ s_h)). \tag{25}$$

Note that E_h is of course *not* the restriction of E to $\mathcal{H}_h$. The discrete harmonic function $u_h = \Phi_h I_h(\gamma \circ s_h)$ is said to be *a discrete minimal surface spanning* Γ, or *a solution of the discrete Plateau Problem for* Γ, if

$$\langle E_h'(s_h), \xi_h\rangle = 0 \quad \forall \xi_h \in H_h. \tag{26}$$

Note that we do not require monotonicity of s_h, as in the case for s in (3). The derivatives of E_h are given by

$$\begin{aligned} E_h(s_h) &= \frac{1}{2}\int_{D_h} |\nabla u_h|^2, \\ \langle E_h'(s_h), \xi_h\rangle &= \frac{1}{2}\int_{D_h} \nabla u_h \nabla v_h, \\ E_h''(s_h)(\xi_h, \xi_h) &= \int_{D_h} \nabla u_h \nabla w_h + \int_{D_h} |\nabla v_h|^2, \end{aligned}$$

where

$$u_h = \Phi_h I_h(\gamma \circ s_h), \qquad v_h = \Phi_h I_h(\gamma' \circ s_h\, \xi_h), \qquad w_h = \Phi_h I_h(\gamma'' \circ s_h\, \xi_h^2).$$

2.3. **The negative space.** Let us define $H^{-1/2}(\partial D)$ to be the dual space of $H^{1/2}(\partial D)$ with the usual operator norm. There is a natural imbedding $H^{1/2}(\partial D) \hookrightarrow H^{-1/2}(\partial D)$ given by

$$\langle \zeta, \eta\rangle = \int_{\partial D} \zeta\eta \quad \forall \eta \in H^{1/2}(\partial D),$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing of $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$. Thus

$$\|\zeta\|_{H^{-1/2}(\partial D)} = \sup_{\|\eta\|_{H^{1/2}(\partial D)}=1} \int_{\partial D} \zeta\eta.$$

We will need the interpolation result

$$\|\zeta\|_{L^2(\partial D)} \le c\|\zeta\|^{1/2}_{H^{-1/2}(\partial D)} \|\zeta\|^{1/2}_{H^{1/2}(\partial D)}, \tag{27}$$

which follows from the relevant definitions.

2.4. **Preliminary estimates.** We will make use of the following estimates.

Proposition 2.5. *Suppose* $f, g : \partial D \to \mathbb{R}$. *Then*

$$|fg|_{H^{1/2}} \leq \|f\|_{C^0}\, |g|_{H^{1/2}} + |f|_{H^{1/2}}\, \|g\|_{C^0}, \tag{28}$$

$$\|fg\|_{H^{1/2}} \leq c\|f\|_{C^{0,1}}\, \|g\|_{H^{1/2}}, \tag{29}$$

$$|fg|_{H^1} \leq \|f\|_{C^0}\, |g|_{H^1} + |f|_{H^1}\, \|g\|_{C^0}, \tag{30}$$

$$\|fg\|_{H^1} \leq c\|f\|_{C^{0,1}}\, \|g\|_{H^1}, \tag{31}$$

$$\|fg\|_{H^{3/2}} \leq c\|f\|_{C^2}\, \|g\|_{H^{3/2}}. \tag{32}$$

Proof. These follow by direct computation. See [4, Proposition 3.1]. □

The following proposition will typically be applied in case g is γ, γ' or γ'' (and in particular is C^1), and where either $s_1 = s_0$ and $s_2 = p_h s_0$, or $s_1 = p_h s_0$ and $s_2 = p_h s_0 + \eta_h$ for some $\eta_h \in H_h$.

Proposition 2.6. *Suppose* $s_i = id + \sigma_i : \partial D \to S^1$ *for* $i = 1, 2$, *and* $g : S^1 \to \mathbb{R}$. *Then*

$$\begin{aligned}
&|g \circ s_1 - g \circ s_2|_{H^{1/2}} \leq c\|g\|_{C^2}\, (\|s_1\|_{C^{0,1}} + \|s_1 - s_2\|_{C^0})\, \|s_1 - s_2\|_{H^{1/2}}, \\
&|g \circ s_1 - g \circ s_2|_{H^1} \leq c\|g\|_{C^2}\, \|s_1\|_{C^{0,1}}\, \|s_1 - s_2\|_{H^1}.
\end{aligned}$$

Proof. This follows by direct computation. See [4, Proposition 3.3]. □

Proposition 2.7. *If* $f \in H^s(\partial D, \mathbb{R}^n)$, *where* $s = 1, 3/2$, *then*

$$\begin{aligned}
|\Phi(f) - \Phi_h I_h(f)|_{H^1(D_h)} &\leq ch^{s-1/2}|f|_{H^s(\partial D)}, \\
|\Phi_h I_h(f)|_{H^1(D_h)} &\leq |f|_{H^{1/2}(\partial D)} + ch^{s-1/2}|f|_{H^s(\partial D)}.
\end{aligned}$$

Proof. See [4, Proposition 3.4]. Standard methods are used. □

Proposition 2.8. *If* $f \in H^s(\partial D, \mathbb{R}^n)$, *where* $s = 1, 3/2$, *then*

$$\begin{aligned}
\|\Phi(f) - \Phi_h I_h(f)\|_{L^2(D_h)} &\leq ch^{s+1/2}|f|_{H^s(\partial D)} + \|f - I_h^{\partial D}(f)\|_{L^2(\partial D)}, \\
\|\Phi_h I_h(f)\|_{L^2(D_h)} &\leq \|f\|_{L^2(\partial D)} + ch^s|f|_{H^s(\partial D)}.
\end{aligned}$$

Proof. See [5, Theorem 1]. An Aubin-Nitsche type of argument is used. □

Proposition 2.9. *Suppose* u *is harmonic in* D, *with trace* $u|_{\partial D}$ *in* $L^2(\partial D)$ *or in* $H^1(\partial D)$, *as appropriate. Then*

$$\|u\|_{L^2(D \setminus D_h)} \leq ch\|u\|_{L^2(\partial D)}, \tag{33}$$

$$\|\nabla u\|_{L^2(D \setminus D_h)} \leq ch|u|_{H^1(\partial D)}, \tag{34}$$

$$\|u - u \circ \pi\|_{L^2(\partial D)} \leq ch^2|u|_{H^1(\partial D)}, \tag{35}$$

$$\left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial D_h)} \leq c|u|_{H^1(\partial D)}. \tag{36}$$

Proof. See [4, Proposition 3.7]. □

2.5. **Main theorems.** The following theorems and lemma are the starting points for the proof of the L^2-estimate. Recall $\gamma \in C^r$. Define

$$H^{3/2}(\partial D) = \{\xi \in H^{1/2}(\partial D) \; : \; \xi' \in H^{1/2}(\partial D)\},$$

where ξ' is the distributional derivative of ξ. Define the seminorm

$$|\xi|_{H^{3/2}(\partial D)} = |\xi'|_{H^{1/2}(\partial D)}$$

and the norm

$$\|\xi\|_{H^{3/2}(\partial D)} = |\xi|_{H^{3/2}(\partial D)} + \|\xi\|_{L^2(\partial D)}.$$

Lemma 2.1. *Assume $r \geq 5$ and s is a nondegenerate stationary point for E. Suppose $\xi \in H$. Then the "adjoint" problem*

$$d^2E(s)(\phi_\xi, \eta) = \int_{\partial D} \xi\eta \qquad \forall\, \eta \in H \tag{37}$$

has a unique solution $\phi_\xi \in H$. Moreover, $\phi_\xi \in H^{3/2}(\partial D)$ and

$$|\phi_\xi|_{H^{3/2}(\partial D)} \leq c|\xi|_{H^{1/2}(\partial D)}. \tag{38}$$

The constant c depends on s.

Proof. See [2, Lemma 4.2]. □

Theorem 2.1. *Assume $\gamma \in C^4$. Let s be a monotone nondegenerate stationary point for E, with nondegeneracy constant λ. Then there exist positive constants h_0 and c_0 depending on $\|\gamma\|_{C^4}$ and $\||\gamma'|^{-1}\|_{L^\infty}$, and on λ in the case of h_0, such that if $0 < h \leq h_0$, then there exists $s_h \in \mathcal{H}_h$ which is stationary for E_h and satisfies*

$$\|s - s_h\|_{H^{1/2}} \leq c_0\lambda^{-1}h\,. \tag{39}$$

Moreover, there exists $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \||\gamma'|^{-1}\|_{L^\infty}, \lambda) > 0$ such that s_h is the unique stationary point for E_h, satisfying

$$\|s - s_h\|_{H^{1/2}} \leq \epsilon_0|\log h|^{-1}. \tag{40}$$

Proof. See [4, Theorem 5.4]. □

Corollary 2.1. *Under the same hypotheses and using the same notation of Theorem 2.1, we have*

$$\|s - s_h\|_T \leq ch|\ln h|^{1/2}, \tag{41}$$

where c is independent of h.

Proof. Recall that in the proof of Theorem 2.1 ([4, (118)]) the estimate

$$\|p_h s - s_h\|_{H^{1/2}} \leq ch \tag{42}$$

was established, and therefore by Proposition 2.3,

$$\|p_h s - s_h\|_T \leq c|\ln h|^{1/2}\|p_h s - s_h\|_{H^{1/2}} \leq ch|\ln h|^{1/2}. \tag{43}$$

Hence

$$\begin{aligned} \|s - s_h\|_T &\leq \|s - p_h s\|_T + \|p_h s - s_h\|_T \\ &\leq \|s - p_h s\|_{H^{1/2}} + \|s - p_h s\|_{C^0} + \|p_h s - s_h\|_T \\ &\leq ch^{3/2} + ch^2 + ch|\ln h|^{1/2} \leq ch|\ln h|^{1/2}, \end{aligned}$$

by Proposition 2.4 and the observation above. □

Theorem 2.2. *Assume $\gamma \in C^4$. Let u be a nondegenerate minimal surface spanning Γ with nondegeneracy constant λ. Then there exist positive constants h_0 and c_0 depending on $\|\gamma\|_{C^4}$ and $\||\gamma'|^{-1}\|_{L^\infty}$, and on λ in the case of h_0, such that if $0 < h \leq h_0$, then there is a discrete minimal surface u_h satisfying*

$$\|u - u_h\|_{H^1(D_h)} \leq c_0 \lambda^{-1} h \,. \tag{44}$$

Moreover, there exists $\epsilon_0 = \epsilon_0(\|\gamma\|_{C^4}, \||\gamma'|^{-1}\|_{L^\infty}, \lambda) > 0$ such that if $u = \Phi(\gamma \circ s)$ and $u_h = \Phi_h I_h(\gamma \circ s_h)$, then u_h is the unique discrete minimal surface satisfying

$$\|s - s_h\|_{H^{1/2}} \leq \epsilon_0 |\log h|^{-1}. \tag{45}$$

Proof. See [4, Theorem 5.5]. □

3. The L^2-estimates

Finally we are able to start discussing the L^2-estimate. We want to prove the following theorems.

Theorem 3.1. *With the same hypotheses and notation as in Theorem 2.1 and the additional assumption that $\gamma \in C^5$, we have that*

$$\|s - s_h\|_{L^2(\partial D)} \leq c h^{3/2} |\ln h|^{3/4},$$

where the constant c does not *depend on h.*

Theorem 3.2. *With the same hypotheses and notation as in Theorem 2.2 and the additional assumption that $\gamma \in C^5$, we have that*

$$\|u - u_h\|_{L^2(D_h)} \leq c h^2 |\ln h|^{3/2},$$

where the constant c does not *depend on h.*

The approach will initially be that of [2]; i.e., we will use Lemma 2.1 to estimate $\|s - s_h\|_{H^{-1/2}(\partial D)}$. Then by means of the inequality (27), an estimate for $\|s - s_h\|_{L^2(\partial D)}$ will follow. Finally, using trace theory results and Proposition 2.8, we will obtain Theorem 3.2.

Before beginning the proofs, we consider some estimates which will often be used.

Proposition 3.1. *Using the notation and the hypotheses of Theorem 3.1 and Lemma 2.1, we have*

$$\|s_h\|_{C^{0,1}(\partial D)} \leq c |\ln h|^{1/2}, \tag{46}$$

$$|\gamma \circ s_h - \gamma \circ s|_{H^{1/2}(\partial D)} \leq ch, \tag{47}$$

$$|\gamma \circ p_h s - \gamma \circ s|_{H^{1/2}(\partial D)} \leq ch^{3/2}, \tag{48}$$

$$|\gamma \circ p_h s - \gamma \circ s_h|_{H^{1/2}(\partial D)} \leq ch, \tag{49}$$

$$|\gamma \circ s_h - \gamma \circ p_h s|_{H^1(\partial D)} \leq ch^{1/2}, \tag{50}$$

$$|\gamma \circ p_h s - \gamma \circ s|_{H^1(\partial D)} \leq ch, \tag{51}$$

$$|\gamma \circ s - \gamma \circ s_h|_{H^1(\partial D)} \leq ch^{1/2}, \tag{52}$$

$$|\gamma' \circ s_h - \gamma' \circ s|_{H^{1/2}(\partial D)} \leq ch, \tag{53}$$

$$|\gamma' \circ s_h - \gamma' \circ s|_{H^1(\partial D)} \leq ch^{1/2}, \tag{54}$$

$$|(\gamma' \circ s_h - \gamma' \circ s)p_h\phi_\xi|_{H^{1/2}(\partial D)} \leq ch|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)}, \tag{55}$$

$$|(\gamma' \circ s_h - \gamma' \circ s)p_h\phi_\xi|_{H^1(\partial D)} \leq ch^{1/2}|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)}, \tag{56}$$

$$\Big(\sum_j |\gamma \circ s_h|^2_{H^2(\pi^{-1}(E_j))}\Big)^{1/2} \leq c|\ln h|, \tag{57}$$

$$\Big(\sum_j |(\gamma' \circ s_h)p_h\phi_\xi|^2_{H^2(\pi^{-1}(E_j))}\Big)^{1/2} \leq c|\ln h||\xi|_{H^{1/2}(\partial D)}. \tag{58}$$

Proof. First note that if we consider the space $V = \{v | v$ is piecewise (arcwise) linear on $S^1\}$, where S^1 has a fixed grid controlled by h, by using a rescaling argument and the fact that on a finite dimensional space all norms are comparable, we get $[v]_{C^{0,1}} \leq h^{-1}\|v\|_{C^0}\ \forall v \in V$ (where $[\cdot]_{C^{0,1}}$ is the $C^{0,1}$- seminorm). Therefore

$$\begin{aligned}
\|s_h\|_{C^{0,1}} &\leq \|s_h - p_h s\|_{C^0} + [s_h - p_h s]_{C^{0,1}} + c\|s\|_{C^{0,1}} && \text{by Prop. 2.4}\\
&\leq c|\ln h|^{1/2}h + ch^{-1}h|\ln h|^{1/2} + c\|s\|_{C^{0,1}} && \text{by (43)}\\
&\leq c|\ln h|^{1/2} && \text{for } h \text{ small.}
\end{aligned}$$

For (47), using Proposition 2.6, (41), and Theorem 2.1, we compute

$$|\gamma \circ s_h - \gamma \circ s|_{H^{1/2}} \leq c\|\gamma\|_{C^2}\left(\|s\|_{C^{0,1}} + \|s_h - s\|_{C^0}\right)\|s - s_h\|_{H^{1/2}} \leq ch.$$

In the same way we obtain (53).

For (48) we compute

$$\begin{aligned}
&|\gamma \circ p_h s - \gamma \circ s|_{H^{1/2}}\\
&\quad\leq c\|\gamma\|_{C^2}\left(\|s\|_{C^{0,1}} + \|s - p_h s\|_{C^0}\right)\|s - p_h s\|_{H^{1/2}} && \text{by Prop. 2.6}\\
&\quad\leq c\|\gamma\|_{C^2}\left(\|s\|_{C^{0,1}} + ch^2\|s\|_{C^2}\right)h^{3/2}\,\|s\|_{C^2} \leq ch^{3/2} && \text{by Prop. 2.4.}
\end{aligned}$$

Now (49) follows from the triangle inequality, (47), and (48).

For (50) we compute

$$\begin{aligned}
&|\gamma \circ s_h - \gamma \circ p_h s|_{H^1}\\
&\quad\leq c\|\gamma\|_{C^2}\,\|p_h s\|_{C^{0,1}}\,\|s_h - p_h s\|_{H^1} && \text{by (33)}\\
&\quad\leq c\|\gamma\|_{C^2}\,\|s\|_{C^{0,1}}h^{-1/2}\,\|s_h - p_h s\|_{H^{1/2}} && \text{by Prop. 2.4 and 2.3}\\
&\quad\leq c\|\gamma\|_{C^2}\,\|s\|_{C^{0,1}}\,h^{-1/2}h \leq ch^{1/2} && \text{by (42).}
\end{aligned}$$

For (51) we use (33) and Proposition 2.4 to compute

$$\begin{aligned}
|\gamma \circ p_h s - \gamma \circ s|_{H^1} &\leq c\|\gamma\|_{C^2}\,\|s\|_{C^{0,1}}\,\|s - p_h s\|_{H^1}\\
&\leq c\|\gamma\|_{C^2}\,\|s\|_{C^{0,1}}\,h\,\|s\|_{H^2} \leq ch.
\end{aligned}$$

Estimate (52) follows from the triangle inequality, (50), and (51). Estimate (54) is established in a similar way.

For (55) we compute

$$
\begin{aligned}
&|(\gamma' \circ s_h - \gamma' \circ s) p_h \phi_\xi|_{H^{1/2}} \\
&\quad \le \|\gamma' \circ s_h - \gamma' \circ s\|_{C^0} \, |p_h \phi_\xi|_{H^{1/2}} \\
&\qquad + |\gamma' \circ s_h - \gamma' \circ s|_{H^{1/2}} \, \|p_h \phi_\xi\|_{C^0} && \text{by Prop. 2.5} \\
&\quad \le \|\gamma\|_{C^2} \|s - s_h\|_{C^0} \, (\|p_h \phi_\xi - \phi_\xi\|_{H^{1/2}} + \|\phi_\xi\|_{H^{1/2}}) \\
&\qquad + |\gamma' \circ s_h - \gamma' \circ s|_{H^{1/2}} \, \|p_h \phi_\xi\|_{H^{1/2}} \, |\ln h|^{1/2} && \text{by Prop. 2.3} \\
&\quad \le c\|\gamma\|_{C^2} h \, |\ln h|^{1/2} \, (h \|\phi_\xi\|_{H^{3/2}} \|\phi_\xi\|_{H^{3/2}}) \\
&\qquad + c\|\gamma\|_{C^3} h \, |\ln h|^{1/2} \|p_h \phi_\xi\|_{H^{1/2}} && \text{by (41), Prop. 2.4, and (53)} \\
&\quad \le c\|\gamma\|_{C^3} h \, |\ln h|^{1/2} \, (h |\xi|_{H^{1/2}} \\
&\qquad + |\xi|_{H^{1/2}}) \le ch |\ln h|^{1/2} |\xi|_{H^{1/2}} && \text{by Lemma 2.1.}
\end{aligned}
$$

Note that we have also used the fact that $\|\cdot\|_{H^{3/2}}$ is equivalent to $|\cdot|_{H^{3/2}}$ on $H \cap H^{3/2}(\partial D)$.

For (56) we compute

$$
\begin{aligned}
&|(\gamma' \circ s_h - \gamma' \circ s) p_h \phi_\xi|_{H^1} \\
&\quad \le \|\gamma' \circ s_h - \gamma' \circ s\|_{C^0} \, |p_h \phi_\xi|_{H^1} \\
&\qquad + |\gamma' \circ s_h - \gamma' \circ s|_{H^1} \, \|p_h \phi_\xi\|_{C^0} && \text{by Prop. 2.5} \\
&\quad \le \|\gamma\|_{C^2} \|s - s_h\|_{C^0} \, (\|p_h \phi_\xi - \phi_\xi\|_{H^1} + |\phi_\xi|_{H^{3/2}}) \\
&\qquad + c|\gamma' \circ s_h - \gamma' \circ s|_{H^1} \, \|p_h \phi_\xi\|_{H^{1/2}} \, |\ln h|^{1/2} && \text{by Prop. 2.3} \\
&\quad \le c\|\gamma\|_{C^2} h \, |\ln h|^{1/2} |\xi|_{H^{1/2}} + c\|\gamma\|_{C^3} h^{1/2} |\ln h|^{1/2} |\xi|_{H^{1/2}} && \text{by (54) and (41)} \\
&\quad \le ch^{1/2} |\ln h|^{1/2} |\xi|_{H^{1/2}}.
\end{aligned}
$$

To prove the last two inequalities, we exploit the fact that the second derivatives of s_h and $p_h \phi_\xi$ vanish on each arc segment $\pi^{-1}(E_j)$ (recall that the E_j are the boundary edges). More precisely, on each arc segment we have that $(\gamma \circ s_h)'' = \gamma'' \circ s_h \, (s_h')^2$ and $((\gamma' \circ s_h) p_h \phi_\xi)'' = \gamma''' \circ s_h \, (s_h')^2 p_h \phi_\xi + 2\gamma'' \circ s_h \, s_h' \, p_h \phi_\xi'$. Therefore it follows from (46) that

$$
\begin{aligned}
&\Big(\sum_j |\gamma \circ s_h|^2_{H^2(\pi^{-1}(E_j))} \Big)^{\frac{1}{2}} \\
&\qquad \le \Big(\sum_j \|\gamma'' \circ s_h \, (s_h')^2\|^2_{L^2(\pi^{-1}(E_j))} \Big)^{\frac{1}{2}} \le c\|s_h\|^2_{C^{0,1}} \le c|\ln h|.
\end{aligned}
$$

Using (46), Proposition 2.4, and Lemma 2.1, we finally obtain

$$
\begin{aligned}
&\Big(\sum_j |(\gamma' \circ s_h) p_h \phi_\xi|^2_{H^2(\pi^{-1}(E_j))} \Big)^{\frac{1}{2}} \\
&\qquad \le c\|s_h\|^2_{C^{0,1}} \, \|p_h \phi_\xi\|_{L^2} + c\|s_h\|_{C^{0,1}} \, |p_h \phi_\xi|_{H^1} \\
&\qquad \le c|\ln h| \, \|p_h \phi_\xi\|_{H^1} \le c|\ln h| \, (\|p_h \phi_\xi - \phi_\xi\|_{H^1} + \|\phi_\xi\|_{H^1}) \\
&\qquad \le c|\ln h| |\phi_\xi|_{H^{3/2}} \le c|\ln h| |\xi|_{H^{1/2}}.
\end{aligned}
$$

$\square$

Proof of Theorem 3.1. As remarked above, the first step consists in finding an estimate for $\|s - s_h\|_{H^{-1/2}(\partial D)}$. By Lemma 2.1 we have

$$\begin{aligned}\int_{\partial D} \xi(s_h - s) &= d^2E(s)(\phi_\xi, s_h - s)\\ &= d^2E(s)(\phi_\xi - p_h\phi_\xi, s_h - s) + d^2E(s)(p_h\phi_\xi, s_h - s)\\ &\equiv A + B.\end{aligned}$$

First we estimate

$$\begin{aligned}|A| &= |d^2E(s)(\phi_\xi - p_h\phi_\xi, s_h - s)|\\ &\le c\|s - s_h\|_{H^{1/2}(\partial D)}\, \|\phi_\xi - p_h\phi_\xi\|_{H^{1/2}(\partial D)} \qquad \text{by (10)}\\ &\le ch^2|\phi_\xi|_{H^{3/2}(\partial D)} \le ch^2|\xi|_{H^{1/2}(\partial D)}\end{aligned}$$

by Theorem 2.1, Proposition 2.4, and Lemma 2.1. Then we calculate

$$\begin{aligned}|B| &= |d^2E(s)(p_h\phi_\xi, s_h - s)|\\ &\le |d^2E(s)(p_h\phi_\xi, s_h - s) + dE(s)(p_h\phi_\xi)\\ &\qquad - dE(s_h)(p_h\phi_\xi)| + |dE(s_h)(p_h\phi_\xi)|\\ &\le c\|s - s_h\|_T^2\, \|p_h\phi_\xi\|_T + |dE(s_h)(p_h\phi_\xi)| \quad \text{by Taylor's theorem and Prop. 2.2}\\ &\le ch^2|\ln h|^{3/2}\|p_h\phi_\xi\|_{H^{1/2}} + |dE(s_h)(p_h\phi_\xi)| \qquad \text{by (41) and Prop. 2.3}\\ &\le ch^2|\ln h|^{3/2}|\xi|_{H^{1/2}} + |dE(s_h)(p_h\phi_\xi)| \qquad \text{by Prop. 2.4 and Lemma 2.1.}\end{aligned}$$

Now we want to give an estimate for $|dE(s_h)(p_h\phi_\xi)|$. Since s_h is stationary for E_h, we know that $dE_h(s_h)(\xi_h) = 0$ for all $\xi_h \in H_h$. Hence

$$\begin{aligned}dE(s_h)(p_h\phi_\xi) &= dE(s_h)(p_h\phi_\xi) - dE_h(s_h)(p_h\phi_\xi)\\ &= \int_D \nabla u \nabla v\; - \int_{D_h} \nabla u_h \nabla v_h,\end{aligned}$$

where

$$\begin{aligned}u &= \Phi(\gamma \circ s_h), & u_h &= \Phi_h I_h(\gamma \circ s_h),\\ v &= \Phi((\gamma' \circ s_h)p_h\phi_\xi), & v_h &= \Phi_h I_h((\gamma' \circ s_h)p_h\phi_\xi).\end{aligned}$$

Next write

$$\begin{aligned}dE(s_h)(p_h\phi_\xi) &= \int_{D_h} \nabla u \nabla v - \int_{D_h} \nabla u_h \nabla v_h + \int_{D\setminus D_h} \nabla u \nabla v\\ &= \int_{D_h} \nabla(u - u_h)\nabla(v_h - v) + \int_{D_h} \nabla(u - u_h)\nabla v\\ &\quad + \int_{D_h} \nabla u \nabla(v - v_h) + \int_{D\setminus D_h} \nabla u \nabla v \equiv I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Estimate of I_1. We have

$$|I_1| = \Big|\int_{D_h} \nabla(u - u_h)\nabla(v - v_h)\Big| \le |u - u_h|_{H^1(D_h)}\, |v - v_h|_{H^1(D_h)}.$$

For the first term we calculate

$$\begin{aligned}
|u-u_h|_{H^1(D_h)} &= |\Phi(\gamma\circ s_h)-\Phi_h I_h(\gamma\circ s_h)|_{H^1(D_h)}\\
&\le |\Phi(\gamma\circ s_h-\gamma\circ s)-\Phi_h I_h(\gamma\circ s_h-\gamma\circ s)|_{H^1(D_h)}\\
&\quad + |\Phi(\gamma\circ s)-\Phi_h I_h(\gamma\circ s)|_{H^1(D_h)}\\
&\le ch^{1/2}|\gamma\circ s_h-\gamma\circ s|_{H^1(\partial D)} + ch|\gamma\circ s|_{H^{3/2}(\partial D)} && \text{by Prop. 2.7}\\
&\le ch + ch\|\gamma\|_{C^2}\|s\|_{H^{3/2}} \le ch && \text{by (52).}
\end{aligned}$$

For the second term we compute

$$\begin{aligned}
&|v-v_h|_{H^1(D_h)} = \big|\Phi\big((\gamma'\circ s_h)p_h\phi_\xi\big)-\Phi_h I_h\big((\gamma'\circ s_h)p_h\phi_\xi\big)\big|_{H^1(D_h)}\\
&\quad\le \big|\Phi\big((\gamma'\circ s_h)p_h\phi_\xi-(\gamma'\circ s)p_h\phi_\xi\big)-\Phi_h I_h\big((\gamma'\circ s_h)p_h\phi_\xi-(\gamma'\circ s)p_h\phi_\xi\big)\big|_{H^1(D_h)}\\
&\qquad + \big|\Phi\big((\gamma'\circ s)p_h\phi_\xi\big)-\Phi_h I_h\big((\gamma'\circ s)p_h\phi_\xi\big)\big|_{H^1(D_h)}\\
&\quad\le ch^{1/2}|(\gamma'\circ s_h)p_h\phi_\xi-(\gamma'\circ s)p_h\phi_\xi|_{H^1(\partial D)}\\
&\qquad + \big|\Phi\big((\gamma'\circ s)(p_h\phi_\xi-\phi_\xi)\big)-\Phi_h I_h\big((\gamma'\circ s)(p_h\phi_\xi-\phi_\xi)\big)\big|_{H^1(D_h)}\\
&\qquad + \big|\Phi\big((\gamma'\circ s)\phi_\xi\big)-\Phi_h I_h\big((\gamma'\circ s)\phi_\xi\big)\big|_{H^1(D_h)} \qquad \text{by Prop. 2.7}\\
&\quad\le ch|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)} + ch^{1/2}|(\gamma'\circ s)(p_h\phi_\xi-\phi_\xi)|_{H^1(\partial D)}\\
&\qquad + ch|(\gamma'\circ s)\phi_\xi|_{H^{3/2}(\partial D)} \qquad \text{by (56) and by Prop. 2.7}\\
&\quad\le ch|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)} + ch^{1/2}\|\gamma'\circ s\|_{C^{0,1}}\,\|p_h\phi_\xi-\phi_\xi\|_{H^1(\partial D)}\\
&\qquad + ch\|\gamma'\circ s\|_{C^2}\,\|\phi_\xi\|_{H^{3/2}(\partial D)} \qquad \text{by Prop. 2.5}\\
&\quad\le ch|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)} + ch|\phi_\xi|_{H^{3/2}(\partial D)} \qquad \text{by Prop. 2.4}\\
&\quad\le ch|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)} \qquad \text{by Lemma 2.1.}
\end{aligned}$$

Therefore we get

$$|I_1| \le ch^2|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)}.$$

Estimate of I_2. From integration by parts we obtain

$$I_2 = \int_{D_h}\nabla(u-u_h)\nabla v = \int_{\partial D_h}(u-u_h)\frac{\partial v}{\partial\nu}.$$

Therefore, by (36),

$$|I_2| \le \left\|\frac{\partial v}{\partial\nu}\right\|_{L^2(\partial D_h)}\|u-u_h\|_{L^2(\partial D_h)} \le c|v|_{H^1(\partial D)}\,\|u-u_h\|_{L^2(\partial D_h)}.$$

The first term is estimated by

$$\begin{aligned}
&|v|_{H^1(\partial D)} = |(\gamma'\circ s_h)p_h\phi_\xi|_{H^1(\partial D)}\\
&\quad\le |(\gamma'\circ s_h-\gamma'\circ s)p_h\phi_\xi|_{H^1(\partial D)} + |(\gamma'\circ s)p_h\phi_\xi|_{H^1(\partial D)}\\
&\quad\le ch^{1/2}|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)} + c\|\gamma'\circ s\|_{C^{0,1}}\|p_h\phi_\xi\|_{H^1(\partial D)} \text{ by (56) and Prop. 2.5}\\
&\quad\le ch^{1/2}|\ln h|^{1/2}|\xi|_{H^{1/2}(\partial D)} + c\big(ch^{1/2}|\xi|_{H^{1/2}(\partial D)} + |\xi|_{H^{1/2}(\partial D)}\big) \le c|\xi|_{H^{1/2}(\partial D)}.
\end{aligned}$$

For the second term we have

$$\begin{aligned}
&\|u-u_h\|_{L^2(\partial D_h)} = \|\Phi(\gamma\circ s_h) - \Phi_h I_h(\gamma\circ s_h)\|_{L^2(\partial D_h)} \\
&\quad\le \|\Phi(\gamma\circ s_h)\circ\pi - I_h(\gamma\circ s_h)\circ\pi\|_{L^2(\partial D)} \\
&\quad\le \|\Phi(\gamma\circ s_h)\circ\pi - (\gamma\circ s_h)\|_{L^2(\partial D)} + \|(\gamma\circ s_h) - I_h^{\partial D}(\gamma\circ s_h)\|_{L^2(\partial D)} \\
&\quad\le ch^2|\gamma\circ s_h|_{H^1(\partial D)} + ch^2\Big(\sum_j |\gamma\circ s_h|^2_{H^2(\pi^{-1}(E_j))}\Big)^{1/2}.
\end{aligned}$$

For the last inequality we have used (35) and standard interpolation results. We have $|u|_{H^1(\partial D)} = |\gamma\circ s_h|_{H^1(\partial D)} \le |\gamma\circ s_h - \gamma\circ s|_{H^1(\partial D)} + |\gamma\circ s|_{H^1(\partial D)} \le c$ by (52). Together with (57) we obtain

$$\|\Phi(\gamma\circ s_h) - \Phi_h I_h(\gamma\circ s_h)\|_{L^2(\partial D_h)} \le ch^2|\ln h|.$$

Hence

$$|I_2| \le ch^2|\ln h||\xi|_{H^{1/2}(\partial D)}.$$

Estimate of I_3. Again by integration by parts we get

$$\begin{aligned}
|I_3| &= \Big|\int_{D_h}\nabla u\nabla(v-v_h)\Big| \le \Big\|\frac{\partial u}{\partial\nu}\Big\|_{L^2(\partial D_h)}\|v-v_h\|_{L^2(\partial D_h)} \\
&\le c|u|_{H^1(\partial D)}\,\|v - I_h((\gamma'\circ s_h)p_h\phi_\xi)\|_{L^2(\partial D_h)} && \text{by (36)} \\
&\le c|\gamma\circ s_h|_{H^1(\partial D)}\,\|v\circ\pi - I_h((\gamma'\circ s_h)p_h\phi_\xi)\circ\pi\|_{L^2(\partial D)} \\
&\le c\left(\|v\circ\pi - v\|_{L^2(\partial D)} + \|v - I_h^{\partial D}((\gamma'\circ s_h)p_h\phi_\xi)\|_{L^2(\partial D)}\right) && \text{by (52)} \\
&\le ch^2|v|_{H^1(\partial D)} + c\|(\gamma'\circ s_h)p_h\phi_\xi - I_h^{\partial D}((\gamma'\circ s_h)p_h\phi_\xi)\|_{L^2(\partial D)} && \text{by (35)} \\
&\le ch^2|v|_{H^1(\partial D)} + ch^2\Big(\sum_j |(\gamma'\circ s_h)p_h\phi_\xi|^2_{H^2(\pi^{-1}(E_j))}\Big)^{1/2}
\end{aligned}$$

by standard interpolation results. By the calculation above we have that $|v|_{H^1(\partial D)} = |(\gamma'\circ s_h)p_h\phi_\xi|_{H^1(\partial D)} \le c|\xi|_{H^{1/2}(\partial D)}$. Together with (58) we obtain

$$|I_3| \le ch^2|\ln h||\xi|_{H^{1/2}(\partial D)}.$$

Estimate of I_4.

$$\begin{aligned}
|I_4| &= \Big|\int_{D\setminus D_h}\nabla u\nabla v\Big| \le |u|_{H^1(D\setminus D_h)}\,|v|_{H^1(D\setminus D_h)} \\
&\le ch^2|u|_{H^1(\partial D)}\,|v|_{H^1(\partial D)} \text{ by (34)} \\
&= ch^2|\gamma\circ s_h|_{H^1(\partial D)}\,|(\gamma'\circ s_h)p_h\phi_\xi|_{H^1(\partial D)} \le ch^2|\xi|_{H^{1/2}(\partial D)},
\end{aligned}$$

by what we remarked above.

From the estimates for I_1, I_2, I_3 and I_4, we finally obtain

$$|dE(s_h)(p_h\phi_\xi)| \le |I_1| + |I_2| + |I_3| + |I_4| \le ch^2|\ln h||\xi|_{H^{1/2}(\partial D)}.$$

This leads to

$$|B| \le ch^2|\ln h|^{3/2}|\xi|_{H^{1/2}(\partial D)} + |dE(s_h)(p_h\phi_\xi)| \le ch^2|\ln h|^{3/2}|\xi|_{H^{1/2}(\partial D)},$$

and therefore we can write

$$\int_{\partial D}\xi(s_h - s) \le |A| + |B| \le ch^2|\ln h|^{3/2}|\xi|_{H^{1/2}(\partial D)}.$$

It follows that

$$\|s - s_h\|_{H^{-1/2}} = \sup_{\|\xi\|_{H^{1/2}(\partial D)}=1} \int_{\partial D} \xi(s_h - s) \leq ch^2 |\ln h|^{3/2}. \tag{59}$$

The claim of Theorem 3.1 now follows from Theorem 2.1 and (27). ☐

Proof of Theorem 3.2. Following the notation of Theorem 2.2, let

$$u = \Phi(\gamma \circ s), \qquad u_h = \Phi_h I_h(\gamma \circ s_h).$$

We want to give an estimate for $\|u - u_h\|_{L^2(D_h)}$. Write

$$\begin{aligned}
&\|u - u_h\|_{L^2(D_h)} \\
&\quad \leq \|\Phi(\gamma \circ s) - \Phi(\gamma \circ s_h)\|_{L^2(D_h)} + \|\Phi(\gamma \circ s_h) - \Phi_h I_h(\gamma \circ s_h)\|_{L^2(D_h)} \\
&\quad \equiv C + D.
\end{aligned}$$

We have that

$$\begin{aligned}
C &= \|\Phi(\gamma \circ s) - \Phi(\gamma \circ s_h)\|_{L^2(D_h)} \leq \|\Phi(\gamma \circ s) - \Phi(\gamma \circ s_h)\|_{L^2(D)} \\
&\leq c\|\gamma \circ s - \gamma \circ s_h\|_{H^{-1/2}(\partial D)} \quad \text{by trace theory results} \\
&\leq c\|\gamma'(s)(s_h - s)\|_{H^{-1/2}(\partial D)} \\
&\quad + c\left\|(s - s_h)^2 \int_0^1 (1-q)\gamma''(s + q(s_h - s))dq\right\|_{H^{-1/2}(\partial D)} \text{ by Taylor} \\
&\leq c\|s_h - s\|_{H^{-1/2}(\partial D)} + c\|s_h - s\|_{C^0}\, \|s_h - s\|_{H^{-1/2}(\partial D)} \\
&\leq ch^2|\ln h|^{3/2} + h^3|\ln h|^2 \leq ch^2|\ln h|^{3/2} \quad \text{ by (59) and (41).}
\end{aligned}$$

Finally,

$$\begin{aligned}
D &= \|\Phi(\gamma \circ s_h) - \Phi_h I_h(\gamma \circ s_h)\|_{L^2(D_h)} \\
&\leq \|\Phi(\gamma \circ s_h - \gamma \circ s) - \Phi_h I_h(\gamma \circ s_h - \gamma \circ s)\|_{L^2(D_h)} \\
&\quad + \|\Phi(\gamma \circ s) - \Phi_h I_h(\gamma \circ s)\|_{L^2(D_h)} \\
&\leq ch^{3/2}|\gamma \circ s_h - \gamma \circ s|_{H^1(\partial D)} \\
&\quad + c\|(\gamma \circ s_h - \gamma \circ s) - I_h^{\partial D}(\gamma \circ s_h - \gamma \circ s)\|_{L^2(\partial D)} \\
&\quad + ch^2|\gamma \circ s|_{H^{3/2}(\partial D)} + c\|\gamma \circ s - I_h^{\partial D}(\gamma \circ s)\|_{L^2(\partial D)} \qquad \text{by Prop. 2.8} \\
&\leq ch^2 + c\|(\gamma \circ s_h - \gamma \circ s) - I_h^{\partial D}(\gamma \circ s_h - \gamma \circ s)\|_{L^2(\partial D)} \\
&\qquad\qquad\qquad\qquad \text{by (52) and standard interpolation results} \\
&\leq ch^2 + ch^2\Big(\sum_j |\gamma \circ s_h - \gamma \circ s|^2_{H^2(\pi^{-1}(E_j))}\Big)^{1/2} \leq ch^2|\ln h| \quad \text{ by (57).}
\end{aligned}$$

Theorem 3.2 now follows immediately from the estimates obtained for the terms C and D. ☐

Final remarks. In [3, Section 6] J. Hutchinson and G. Dziuk analyse the problem of the classical Enneper surface with parameter R and calculate the order of convergence between the smooth and the discrete solution. They study three different cases corresponding to different choices of R, and in each case a different grid is used in order to make the comparison more realistic. These experiments confirm the L^2 convergence rate established in Theorem 3.2.

References

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Centre for Mathematics and its Applications, MSI, Australian National University, Canberra, Australian Capital Territory 0200, Australia

E-mail address: Paola.Pozzi@maths.anu.edu.au