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EXISTENCE AND ASYMPTOTIC STABILITY OF RELAXATION DISCRETE SHOCK PROFILES

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ABSTRACT. In this paper we study the asymptotic nonlinear stability of discrete shocks of the relaxing scheme for approximating the general system of nonlinear hyperbolic conservation laws. The existence of discrete shocks is established by suitable manifold construction, and it is shown that weak single discrete shocks for such a scheme are nonlinearly stable in L^2, provided that the sums of the initial perturbations equal zero. These results should shed light on the convergence of the numerical solution constructed by the relaxing scheme for the single shock solution of the system of hyperbolic conservation laws. These results are proved by using both a weighted norm estimate and a characteristic energy method based on the internal structures of the discrete shocks.

1. INTRODUCTION

We investigate the asymptotic stability of the numerical approximation of the following Riemann problem for the general system of nonlinear conservation laws

$$u_t + F(u)_x = 0, u(x,0) = u_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^m$, F is a smooth nonlinear mapping from $\mathbb{R}^m$ to $\mathbb{R}^m$, and $u_\pm$ are two constant vectors in $\mathbb{R}^m$. We assume that the system (1.1) is strictly hyperbolic in the sense that at each state $u \in \mathbb{R}^m$ the Jacobian $\nabla F(u)$ has m real and distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_m(u)$$

with corresponding left and right eigenvectors l_μ and r_μ, respectively, and the characteristic field is either genuinely nonlinear or linearly degenerate in the sense of Lax [12]; i.e., for $\mu = 1, \ldots, m$, the eigenvector r_μ satisfies $\bigtriangledown\lambda_\mu \cdot r_\mu \equiv 1$ or $\bigtriangledown\lambda_\mu \cdot r_\mu \equiv 0$. We normalize the eigenvectors so that $l_\mu r_\kappa = \delta_{\mu\kappa}$ and we denote the $m \times m$ matrices $L(u), R(u)$ and $\Lambda(u)$ by

$$L(u) = (l_1(u)^T, \ldots, l_m(u)^T)^T, R(u) = (r_1(u), \ldots, r_m(u)),$$
$$\Lambda(u) = \operatorname{diag}(\lambda_1(u), \ldots, \lambda_m(u)).$$

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Lax [12] showed that the Riemann problem (1.1) has a solution consisting of at most $(m+1)$ constant states separated by shock waves, centered rarefaction waves, and contact discontinuities, provided that the shock strength $|u_+ - u_-|$ is suitably small.

Assume that the Riemann problem (1.1) has a k-shock wave solution

$$u(x,t) = \begin{cases} u_-, & x < st, \\ u_+, & x > st \end{cases}$$

corresponding to the k-th genuinely nonlinear field. Here the constant states $u_\pm$ and the shock speed s satisfy the Rankine-Hugoniot condition

$$F(u_-) - F(u_+) = s(u_- - u_+)$$

and the Lax entropy condition

$$\lambda_k(u_+) < s < \lambda_k(u_-).$$

We approximate (1.1) with the relaxing scheme proposed by Jin and Xin [9],

$$\begin{aligned} u_j^{n+1} - u_j^n + \tfrac{k}{2h}(v_{j+1}^n - v_{j-1}^n) - \tfrac{k}{2h}A^{\frac{1}{2}}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0, \\ v_j^{n+1} - v_j^n + \tfrac{k}{2h}A(u_{j+1}^n - u_{j-1}^n) - \tfrac{k}{2h}A^{\frac{1}{2}}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) \\ = \kappa(F(u_j^{n+1}) - v_j^{n+1}). \end{aligned} \tag{1.2}$$

κ is sufficiently large, u_j^n is the approximation of $u(x_j, t^n)$, $x_j = j\Delta x$ and $t^n = n\Delta t$, Δx and Δt are the spatial and temporal grid sizes. $\lambda = \frac{k}{h}$ is sufficiently small, which satisfies the Courant-Friedrichs-Lwey (CFL) condition

$$\lambda \sup_\mu |\lambda_\mu(u)| \le 1.$$

$A = aI$ and $A^{\frac{1}{2}} = a^{\frac{1}{2}}I$. The relaxing scheme (1.2) satisfies the strict subcharacteristic condition which implies stability in some sense.

A relaxation discrete shock profile connecting $(u_-, F(u_-))$ and $(u_+, F(u_+))$ is a special solution of the difference equation (1.2) in the form

$$\begin{aligned} u_{x-\eta} - u_x + \tfrac{k}{2h}(v_{x+1} - v_{x-1}) - \tfrac{k}{2h}A^{\frac{1}{2}}(u_{x+1} - 2u_x + u_{x-1}) = 0, \\ v_{x-\eta} - v_x + \tfrac{k}{2h}A(u_{x+1} - u_{x-1}) - \tfrac{k}{2h}A^{\frac{1}{2}}(v_{x+1} - 2v_x + v_{x-1}) \\ = \kappa(F(u_{x-\eta}) - v_{x-\eta}) \end{aligned} \tag{1.3}$$

with

$$\begin{aligned} \lim_{x\to+\infty}(u_x, v_x) = (u_+, F(u_+)), \\ \lim_{x\to-\infty}(u_x, v_x) = (u_-, F(u_-)). \end{aligned}$$

When η is a rational number, $x \in \{n\eta + m | n, m \in \mathbb{Z}\}$; if η is an irrational number, $x \in \mathbb{R}$. Let u_- and u_+ satisfy the Rankine-Hugoniot condition

$$F(u_-) - F(u_+) = s(u_- - u_+).$$

The discrete shock speed η is related to s on the condition that $\lambda s = \eta$. In this paper we require that this discrete shock can be observed on the original grid; therefore, we assume that $\eta = p/q$ is rational. Thus, the discrete shock profiles

which we have studied are the solutions of the difference equations satisfying

$$\begin{gathered}\phi_j^{n+1}-\phi_j^n+\tfrac{k}{2h}(\psi_{j+1}^n-\psi_{j-1}^n)-\tfrac{k}{2h}A^{\frac{1}{2}}(\phi_{j+1}^n-2\phi_j^n+\phi_{j-1}^n)=0,\\ \psi_j^{n+1}-\psi_j^n+\tfrac{k}{2h}A(\phi_{j+1}^n-\phi_{j-1}^n)-\tfrac{k}{2h}A^{\frac{1}{2}}(\psi_{j+1}^n-2\psi_j^n+\psi_{j-1}^n)\\ =\kappa(F(\phi_j^{n+1})-\psi_j^{n+1}),\\ (\phi_j^{nq},\psi_j^{nq})=(\phi_{j-np}^0,\psi_{j-np}^0),\\ (\phi_j,\psi_j)\to(u_\pm,f(u_\pm)),\ j\to\pm\infty.\end{gathered}\tag{1.4}$$

The study of the existence and stability of discrete shock profiles is important for understanding the convergence behavior of numerical shock computations. In the study of discrete conservation laws, Jennings [10] showed the stability of discrete shock profiles of general scalar first-order monotone schemes. The existence of discrete shock profiles of high order scalar difference scheme was established in [4, 8] recently. For the study of the system of conservation laws, the existence of discrete shock profiles of finite-difference methods which are accurate to first order for systems of conservation laws was established by Majda and Ralson [21] using the center manifold construction and by Michelson in [22]. Liu and Xin [16, 17] studied the nonlinear stability of discrete shock profiles of system of conservation laws. Liu and Yu [19, 20] studied the case of more general shock speeds. There are several authors who have studied the stability of shock profiles; see [2, 3, 5, 6, 25, 26]. In the study of discrete relaxation conservation laws, most works have concentrated on the scalar case, such as Liu, Wang and Yang [13, 14, 15]. In this paper, we will concentrate on the case of systems. Stability theorems of discrete shock profiles follow. Their existence will be proved in section 2. Note that we assume that λ is suitably small in all the following theorems.

Theorem 1.1. *Suppose that* (1.1) *is a strictly hyperbolic system and that the k-characteristic field is genuinely nonlinear. Let* (ϕ_j,ψ_j) *be the stationary discrete shock profile of* (1.3) *connecting* $(u_-,f(u_-))$ *and* $(u_+,f(u_+))$. *Assume further that*

$$\textstyle\sum_{j=-\infty}^{\infty}(u_j^0-\phi_j,v_j^0-\psi_j)=0,\tag{1.5}$$

$$\epsilon=|u_+-u_-|\le c_1\tag{1.6}$$

and

$$\sum_{j=-\infty}^{\infty}(1+j^2)(|u_j^0-\phi_j|^2,|v_j^0-\psi_j|^2)\le c_2\tag{1.7}$$

for some (suitably small) positive constants c_1 *and* c_2. *Then, there exists a unique global solution* (u_j^n,v_j^n) *to the relaxing scheme with initial data* (u_j^0,v_j^0), *and it satisfies*

$$\lim_{n\to+\infty}\sum_{j=-\infty}^{\infty}(|u_j^n-\phi_j|^2,|v_j^n-\psi_j|^2)=0.\tag{1.8}$$

Theorem 1.2. *Suppose that* (1.1) *is a strictly hyperbolic system and that the k-characteristic field is genuinely nonlinear. Let* (ϕ_j^n,ψ_j^n) *be the nonstationary discrete shock profile of* (1.3) *connecting* $(u_-,f(u_-))$ *and* $(u_+,f(u_+))$. *Assume that* κ *is suitably large,*

$$\textstyle\sum_{j=-\infty}^{\infty}(u_j^0-\phi_j^0,v_j^0-\psi_j^0)=0,\tag{1.9}$$

$$\epsilon=|u_+-u_-|\le c_1\tag{1.10}$$

and

$$\sum_{j=-\infty}^{\infty}(1+j^2)(|u_j^0-\phi_j^0|^2,|v_j^0-\psi_j^0|^2)\le c_2 \tag{1.11}$$

for some (suitably small) positive constants c_1 and c_2. Then there exists a unique global solution (u_j^n,v_j^n) to the relaxing scheme with initial data (u_j^0,v_j^0), and it satisfies

$$\lim_{n\to+\infty}\sum_{j=-\infty}^{\infty}(|u_j^n-\phi_j^n|^2,|v_j^n-\psi_j^n|^2)=0. \tag{1.12}$$

Remark 1.1. For the stationary case, we do not need to assume that κ is suitably large. This condition is only imposed in the nonstationary case due to technical reasons when we estimate the nonstationary discrete shock profiles in (4.8). However, this condition also implies that the relaxing scheme (1.2) approximates to the conservation laws (1.1) more closely.

Remark 1.2. (1.5),(1.7),(1.9) and (1.11) imply that the L^2 norm of initial perturbation is sufficiently small; see [7, 16].

Remark 1.3. Although there are some essential monotone properties in the principal direction, the multistep phenomena result in some difficulties in studying the L^1 stability, which is kept for the future.

Remark 1.4. Only partial results are obtained, i.e., when η is a rational number, and there is no complete understanding if η is an arbitrary number. For a diophantine number [19], we need to use the Green function of the relaxing scheme, which is not very clear up to this point.

The outline of this paper is as follows. In Section 2, we study the existence of relaxation discrete shock profiles using the Majda-Ralston [21] theorem. Then, we obtain some good properties of the shock profiles using the theorems of Liu-Xin [16]. In Section 3, we study the L^2 stability of stationary discrete shock profiles using the suitable weighted functions proposed by Liu-Xin in [17]. In Section 4, we study the L^2 stability of nonstationary discrete shock profiles using the idea of vertical estimate.

2. Existence of discrete shock profiles

To establish the existence of discrete shock profiles for the relaxation scheme, we will use the center manifold construction proposed by Majda-Ralston [21]. Let $T:\mathbb{R}^N\to\mathbb{R}^N$ be a smooth map and let $x_0\in\mathbb{R}^N$ be a fixed point of T such that the following conditions are satisfied.

(1) There are neighborhoods μ and μ' of x_0 in $\mathbb{R}^N$ such that T maps μ diffeomorphically onto μ'.
(2) T fixes a smooth m-dimensional manifold Ω containing x_0.
(3) The algebraic eigenspace of $dT(x_0)$ associated with the eigenvalue 1 is the span of the tangent space of Ω, $\Gamma_{x_0}(\Omega)$ and a vector χ satisfying

$$dT(x_0)\chi=\chi+r, \tag{2.1}$$

where $r\in\Gamma_{x_0}(\Omega)$ and $r\neq 0$.
(4) $dT(x_0)$ has no eigenvalues in the set $\{z:|z|=1,z\neq 1\}$.

(5) Hypotheses (2) and (3) imply that given $\delta' > 0$, there is a $\delta > 0$ such that when $x \in \Omega$ and $|x - x_0| < \delta$, the eigenvalues of $dT(x)$ lying in $|z - 1| < \delta'$ are 1 with multiplicity m and real $\beta(x)$ with multiplicity 1 where $\beta(x) \in C^\infty(\Omega)$, and $\nabla\beta \cdot r|_{x_0} > 0$.

We summarize the center manifold construction for such an operator in the following proposition.

Proposition 2.1 ([21]). *Assume that the map T satisfies condition (1)–(5). Then there is an $\epsilon > 0$ such that if $x_- \in \Omega, \beta(x_-) > 1$ and $|x_- - x_0| < \epsilon$, then*

(1) *there is a C^1 segment γ_{x_-} beginning at x_- such that $T^{-1}\gamma_{x_-} \subset \gamma_{x_-}$ and $\bigcap_j T^{-j}\gamma_{x_-} = x_-$,*
(2) *for any $x \in \gamma_{x_-}, x \neq x_-$, the sequence $T^j x, j = 0, 1, \ldots$ converges to $x_+ \in \Omega$ with $\beta(x_+) < 1$,*
(3) *if the limit of $T^j x$ in (2) does not depend on x, then $\bigcup_{j=0} T^j\gamma_{x_-} = \Gamma_{x_-}$ is a C^1 curve segment beginning at $x_- \in \Omega$, $\beta(x_-) > 1$, and ending at $x_+ \in \Omega$, $\beta(x_+) < 1$.*

Remark 2.1. Item (3) is guaranteed here by the Rankine-Hugoniot condition and the Lax entropy condition $\lambda_k(u_-) > s > \lambda_k(u_+)$, and u_+ is uniquely determined by u_-. This will be clear in the remainder of this section.

Our strategy is using this proposition to prove the existence of discrete shock profiles of the u component first and then the v component. It follows from equation (1.3) that

$$\frac{\lambda}{2}(v_{x+1} - v_{x-1}) = \frac{\lambda}{2}A^{\frac{1}{2}}(u_{x+1} - 2u_x + u_{x-1}) - (u_{x-\eta} - u_x), \tag{2.2}$$

so the second equation of (1.3) yields

$$\begin{aligned} &\tfrac{\lambda}{2}(v_{x+1-\eta} - v_{x-1-\eta}) - \tfrac{\lambda}{2}(v_{x+1} - v_{x-1}) + \tfrac{\lambda^2}{4}A(u_{x+2} - 2u_x + u_{x-2}) \\ &\quad -\tfrac{\lambda}{2}A^{\frac{1}{2}}[\tfrac{\lambda}{2}(v_{x+2} - v_x) - \lambda(v_{x+1} - v_{x-1}) + \tfrac{\lambda}{2}(v_x - v_{x-2})] \\ &\quad = \tfrac{\kappa\lambda}{2}(F(u_{x+1-\eta}) - F(u_{x-1-\eta})) - \tfrac{\kappa\lambda}{2}(v_{x+1-\eta} - v_{x-1-\eta}). \end{aligned} \tag{2.3}$$

For simplicity, we only consider $\eta = \frac{1}{q}$ and $q > 0$, and the other cases are similar. Some remarks about the case of $s > 0$ will also be given later. Substituting (2.2) into (2.3), scaling the step size as $\Delta x/q$, and writing $\tilde{u}_j = u(j\Delta x/q)$, we have the following equation,

$$\begin{aligned} &[\frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}A^{\frac{1}{2}}]\tilde{u}_{j+2q+1} + [(1 - A^{\frac{1}{2}})(1+\kappa) + 1 - A^{\frac{1}{2}}\lambda]\tilde{u}_{j+q+1} \\ &+ [\frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}A^{\frac{1}{2}}]\tilde{u}_{j+1} - [1+\kappa]\tilde{u}_{j+q} + [2\lambda A^{\frac{1}{2}} - 2\lambda^2 A - 1]\tilde{u}_{j+q+2} \\ &+ [\lambda^2 A - \lambda A^{\frac{1}{2}}]\tilde{u}_{j+2} - \frac{\lambda}{2}\kappa[F(\tilde{u}_{j+2q+1}) - F(\tilde{u}_{j+1})] = [\lambda A^{\frac{1}{2}} - \lambda^2 A]\tilde{u}_{j+2q+2}, \end{aligned} \tag{2.4}$$

$j \in \mathbb{Z}$. Let

$$\begin{aligned} &H = \lambda A^{\frac{1}{2}} - \lambda^2 A, \bar{A} = \frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}A^{\frac{1}{2}} + \frac{\lambda}{2}\kappa F'(u), B = \lambda^2 A - \lambda A^{\frac{1}{2}}, \\ &D = (1 - A^{\frac{1}{2}}\lambda)(1+\kappa) + 1 - A^{\frac{1}{2}}\lambda, E = 2\lambda A^{\frac{1}{2}} - 2\lambda^2 A - 1, \\ &F = \frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}A^{\frac{1}{2}} - \frac{\lambda}{2}\kappa F'(u), C = -(1+\kappa). \end{aligned}$$

So we can rewrite (2.4) as

$$\begin{aligned}\tilde{u}_{2q+2} &= H^{-1}[\frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa)+\frac{\lambda}{2}A^{\frac{1}{2}}]\tilde{u}_{2q+1}+H^{-1}D\tilde{u}_{q+1}-H^{-1}[1+\kappa]\tilde{u}_q\\ &\quad+H^{-1}[\frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa)+\frac{\lambda}{2}A^{\frac{1}{2}}]\tilde{u}_1+H^{-1}B\tilde{u}_2\\ &\quad-\frac{\lambda}{2}\kappa H^{-1}[F(\tilde{u}_{2q+1})-F(\tilde{u}_1)]+H^{-1}E\tilde{u}_{q+2}\\ &\equiv G(\tilde{u}_1,\ldots,\tilde{u}_{2q+1}).\end{aligned}$$

Define the matching map associated with (2.4), $T:\mathbb{R}^{m(2q+1)}\to\mathbb{R}^{m(2q+1)}$, as

$$T\begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\\ .\\ .\\ .\\ \tilde{u}_{2q+1}\end{pmatrix}=\begin{pmatrix}\tilde{u}_2\\ \tilde{u}_3\\ .\\ .\\ .\\ G(\tilde{u}_1,\ldots,\tilde{u}_{2q+1})\end{pmatrix}.$$

The notation $\mathbf{u}$ in $\mathbb{R}^{m(2q+1)}$ is denoted by

$$\mathbf{u}=\begin{pmatrix}u\\ .\\ .\\ .\\ u\end{pmatrix}$$

for $u\in\mathbb{R}^m$ and let $\Delta=\{\mathbf{u}|u\in\mathbb{R}^m\}$. Using the nonlinear marching map T, it is obvious that proving the existence of the u component discrete shock profiles is equivalent to finding the initial data

$$\begin{pmatrix}\widetilde{u}_1\\ .\\ .\\ .\\ \widetilde{u}_{2q+1}\end{pmatrix}\in\mathbb{R}^{m(2q+1)}$$

such that

$$\lim_{j\to\infty}T^j\begin{pmatrix}\widetilde{u}_1\\ .\\ .\\ .\\ \widetilde{u}_{2q+1}\end{pmatrix}=\mathbf{u}_+,$$

$$\lim_{j\to-\infty}T^j\begin{pmatrix}\widetilde{u}_1\\ .\\ .\\ .\\ \widetilde{u}_{2q+1}\end{pmatrix}=\mathbf{u}_-.$$

We only need to check whether T satisfies the five conditions at the beginning of this section.

(1) Obviously, $T\mathbf{u}=\mathbf{u}$. Let $\mu_s\equiv\{\mathbf{u}|\lambda_k(u)=s\}$, $\mathbf{u_0}\in\mu_s$, ν denote a sufficiently small neighborhood of $\mathbf{u_0}$. Under the subcharacteristic condition, it is clear that T satisfies the first and second conditions.

(2) Suppose that $\lambda_k(\mathbf{u_0}) = s$ and λ_k is genuinely nonlinear. Then the algebraic eigenspace of $dT(\mathbf{u_0})$ belonging to the eigenvalue 1 is $m+1$-dimensional, which is spanned by the vectors

$$\{\mathbf{r_j}(u_0)\}_{j=1}^{m} \cup \{e_1(u_0)\},$$

where

$$e_1^t(u_0) = (r_k(u_0), 2r_k(u_0), \ldots, (2q+1)r_k(u_0)),$$

$(dT(\mathbf{u_0}))\mathbf{r_j} = \mathbf{r_j}$ and $dT(\mathbf{u_0})e_1 = e_1 + \mathbf{r_k}$.

Proof. Since

$$dT(\mathbf{u}) = \begin{pmatrix} 0 & I & & & & & & \\ & 0 & I & & & & & \\ & & \cdot & \cdot & \cdot & & & \\ H^{-1}\bar{A} & H^{-1}B & \cdots & H^{-1}C & H^{-1}D & H^{-1}E & \cdots & H^{-1}F \end{pmatrix},$$

the eigenvalues Z satisfy the equality

$$\det[Z^{2q}F - Z^{2q+1}H + Z^{q+1}E + Z^qD + Z^{q-1}C + ZB + \bar{A}] = 0.$$

When $Z = 1$,

$$F - H + E + D + C + B + \bar{A} = 0,$$

so

$$\det[(Z-1)(2qF - (2q+1)H + (q+1)E + qD + (q-1)C + B) + O(Z-1)^2] = 0;$$

i.e.,

$$\det[(Z-1)\kappa(1 - \lambda q F'(u)) + O(Z-1)^2] = 0.$$

Write

$$\mho F'(u)\mho^{-1} = \begin{pmatrix} \lambda_k & & & & & & & \\ & \lambda_1 & & & & & & \\ & & \ddots & & & & & \\ & & & \lambda_{k-1} & & & & \\ & & & & \lambda_{k+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \lambda_m & \end{pmatrix},$$

so

$$(Z-1)^m \det[\mho\kappa(1 - \lambda q F'(u))\mho^{-1} + O(Z-1)] = 0,$$

and when $\lambda_k(u_0) = s$, we have $1 - \lambda q \lambda_k(u_0) = 0$, and the first part follows. Since

$$H^{-1}\bar{A} + H^{-1}B + H^{-1}C + H^{-1}D + H^{-1}E + H^{-1}F = I,$$

it is easy to check that $dT(\mathbf{u})\mathbf{r}_j = \mathbf{r}_j$ and

$$dT(\mathbf{u}_0)e_1 = \begin{pmatrix} 2r_k \\ \vdots \\ (2q+1)r_k \\ (2q+2)r_k \end{pmatrix} = e_1 + \mathbf{r}_k.$$

□

Remark 2.2. For $\lambda s = \frac{p}{q}$ and $s > 0$, whatever the relations between q and p, the eigenvalue equation will be

$$\det[\bar{A} + Z^{q-p}C + Z^p B + Z^q D + Z^{q+p}E + Z^{2q}F - Z^{2q+p}H] = 0. \tag{2.5}$$

Then we can check the condition (2) in exactly the same way. When $s < 0$, we can also show this with a little modification.

(3) (Nonresonance condition) If λ_0 is sufficiently small, for $\lambda < \lambda_0$, there is a $\delta_1 > 0$ such that, for $|\mathbf{u} - \mathbf{u}_0| < \delta_1$, there is no eigenvalue of $dT(\mathbf{u})$ in the set $\{Z : |Z| = 1, Z \neq 1\}$.

Proof. The eigenvalues must satisfy the equation

$$\begin{aligned}
&Z^{2q}(\frac{\lambda}{2}a^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}a^{\frac{1}{2}} - \frac{\lambda}{2}\kappa\lambda_j) - Z^{2q+1}(\lambda a^{\frac{1}{2}} - \lambda^2 a)\\
&+ Z^{q+1}(2\lambda a^{\frac{1}{2}} - 2\lambda^2 a - 1) + Z^q((1 - a^{\frac{1}{2}}\lambda)(2+\kappa)) - Z^{q-1}(1+\kappa)\\
&+ Z(\lambda^2 a - \lambda a^{\frac{1}{2}}) + (\frac{\lambda}{2}a^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}a^{\frac{1}{2}} + \frac{\lambda}{2}\kappa\lambda_j) = 0,
\end{aligned}$$

for $j = 1, \ldots, m$. If there exist a $Z = e^{i\theta}$ for suitably small λ satisfying the equation, we can divide the equation by Z^{q-1} to get

$$\begin{aligned}
&Z^{q+1}(\frac{\lambda}{2}a^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}a^{\frac{1}{2}} - \frac{\lambda}{2}\kappa\lambda_j) - Z^{q+2}(\lambda a^{\frac{1}{2}} - \lambda^2 a)\\
&+ Z^2(2\lambda a^{\frac{1}{2}} - 2\lambda^2 a - 1) + Z((1 - a^{\frac{1}{2}}\lambda)(2+\kappa)) - (1+\kappa)\\
&+ Z^{2-q}(\lambda^2 a - \lambda a^{\frac{1}{2}}) + Z^{1-q}(\frac{\lambda}{2}a^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}a^{\frac{1}{2}} + \frac{\lambda}{2}\kappa\lambda_j) = 0,
\end{aligned}$$

for $j = 1, \ldots, m$. Let $\lambda \to 0$. The equation will be

$$Z^2 - (2+\kappa)Z + 1 + \kappa = 0.$$

It is impossible that $Z = e^{i\theta}$, so by perturbation analysis, $\exists\lambda_0$, for $\lambda < \lambda_0$, there is a $\delta_1 > 0$ such that $|u - u_0| < \delta_1$; eigenvalues do not belong to the set $\{Z : |Z| = 1, Z \neq 1\}$. □

Remark 2.3. If κ is sufficiently large, divide the equation by κ, and let $\lambda \to 0$. We have the equation $Z = 1$, which is a contradiction too.

Remark 2.4. For $\lambda s = \frac{p}{q}, s > 0$, the eigenvalue equations are

$$\begin{aligned}
&\frac{\lambda}{2}a^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}a^{\frac{1}{2}} + \frac{\lambda}{2}\kappa\lambda_j - (1+\kappa)Z^{q-p} + (\lambda^2 a - \lambda a^{\frac{1}{2}})Z^p\\
&\quad + (1 - a^{\frac{1}{2}}\lambda)(2+\kappa)Z^q + (2\lambda a^{\frac{1}{2}} - 2\lambda^2 a - 1)Z^{q+p}\\
&\quad + (\frac{\lambda}{2}a^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}a^{\frac{1}{2}} - \frac{\lambda}{2}\kappa\lambda_j)Z^{2q} - (\lambda a^{\frac{1}{2}} - \lambda^2 a)Z^{2q+p} = 0,
\end{aligned}$$

for $j = 1, \ldots, m$. Setting $\lambda \to 0$, we obtain the equation

$$Z^{2p} - (2+\kappa)Z^p + (1+\kappa) = 0.$$

Thus $e^{i\frac{2\pi n}{p}}$ are solutions if $|Z| = 1$, $n \in Z$. Substituting these solutions into the original equation, we get the equation

$$(-2\lambda^2 a - \kappa a^{\frac{1}{2}}\lambda)e^{i\frac{2q\pi n}{p}} + (\frac{\lambda}{2}\kappa a^{\frac{1}{2}} + \lambda^2 a - \frac{\lambda}{2}\kappa\lambda_j)e^{i\frac{4q\pi n}{p}} + (\lambda^2 a + \frac{\lambda}{2}a^{\frac{1}{2}}\kappa + \frac{\lambda}{2}\kappa\lambda_j) = 0.$$

This is impossible if $n \neq kp$, so we get a contradiction.

(4) There are $\delta_2 > 0$ and $r_2 > 0$ such that, for $|\mathbf{u} - \mathbf{u}_0| < \delta_2$, there are only $m + 1$ eigenvalues of $dT(\mathbf{u})$ lying in $|Z - 1| < r_2$. These eigenvalues are given by $1, 1, \ldots, 1, \beta(u)$, and $\beta(u)$ is real and satisfies

(a) $\beta(u) = 1$ whenever $\lambda_k(u) = s$,

(b) $\nabla_u\beta \cdot r_k(u)|_{u=u_0} > 0$ under the subcharacteristic and CFL conditions.

Proof. The first statement is guaranteed by (2) and (3), and (a) comes from (2) immediately. Assume that $q > 1$(the case $q = 1$ is similar). Then

$$\det[\mho\kappa(1 - \lambda qF'(u))\mho^{-1} + R(\beta(u) - 1) + O(\beta(u) - 1)^2] = 0,$$

where

$$\begin{aligned} R = {} & 2q(2q-1)(\frac{\lambda}{2}A^{\frac{1}{2}}(1+\kappa) + \frac{\lambda}{2}A^{\frac{1}{2}} - \frac{\lambda}{2}\kappa\Lambda) - (2q+1)2q(\lambda A^{\frac{1}{2}} - \lambda^2 A) \\ & + (q+1)q(2\lambda A^{\frac{1}{2}} - 2\lambda^2 A - 1) + q(q-1)(1 - A^{\frac{1}{2}})(2+\kappa) \\ & - (q-1)(q-2)(1+\kappa). \end{aligned}$$

Now expanding along the first row, one gets that

$$\begin{aligned} (2.6) \qquad & \kappa(1 - \lambda q\lambda_k(u)) + [q^2\lambda a^{\frac{1}{2}}\kappa - 2q^2\lambda\kappa\lambda_k(u) + 2q^2\lambda^2 a + 2q\kappa - 2 - 2\kappa \\ & + q\lambda\kappa\lambda_k(u)](\beta(u) - 1) + O(\beta(u) - 1)^2 = 0. \end{aligned}$$

Set

$$\tilde{\tau_1} = q^2\lambda a^{\frac{1}{2}}\kappa + 2q^2\lambda^2 a - 2 - k.$$

Then $\tilde{\tau_1} > 0$ under the subcharacteristic and CFL conditions. Thus the coefficient of $(\beta(u) - 1)$ is positive when $u = u_0$ in equation (2.6). By the implicit function theorem, $\exists\delta_2$, when $|\mathbf{u} - \mathbf{u}_0| < \delta_2$, there is a unique real smooth function $\beta(u)$ with $\beta(u_0) = 1$ provided that $\lambda_k(u_0) = s$. From equation (2.6), we have

$$\nabla_u\beta(u) \cdot r_k|_{u=u_0} = \frac{\lambda q\kappa\nabla_u\lambda_k \cdot r_k}{\tilde{\tau_1}}|_{u=u_0}. \qquad \square$$

Remark 2.5. For $\lambda s = \frac{p}{q}, s > 0$, equation (2.6) becomes

$$\begin{aligned} & \kappa(p - \lambda q\lambda_k(u)) + [(2qp - p^2 - p)\kappa - 2p^2 - q(2q-1)\lambda\kappa\lambda_k(u) + 2q^2\lambda^2 a \\ & + q^2\kappa\lambda a^{\frac{1}{2}}](\beta(u) - 1) + O(\beta(u) - 1)^2 = 0. \end{aligned}$$

When $u = u_0$, the coefficient of $\beta(u)$ is also positive under the subcharacteristic and CFL conditions. Thus we can prove the existence of the real function $\beta(u)$.

Remark 2.6. $\tilde{\tau_1} > 0$ and equation (2.6) are crucial for showing the strict monotonicity of the principal characteristic speed at the discrete shock profile and the asymptotic behavior of the discrete shock profiles. This monotonicity is crucial to our stability analysis (see [16]).

Remark 2.7. We need to assume that λ is suitably small, and it also implies that q is suitably large.

So, we have established the existence of entropy satisfying discrete shock profiles of the u-component. Furthermore, by Proposition 2.1 of Liu-Xin in [16], we have the following proposition.

Proposition 2.2. *There exists a family of discrete shock profiles ϕ_j^n of (2.4) connecting u_- and u_+, provided that $u_\pm$ satisfy the entropy condition, $\epsilon = |u_- - u_+|$ is suitably small and the subcharacteristic condition is satisfied. Furthermore, the discrete shock profiles ϕ_j^n have the following properties for $j = 0, \pm 1, \dots$:*

$$\lambda_k(\phi_j) > \lambda_k(\phi_{j+1}),$$

$$|\lambda_k(\phi_j) - s| \le c_1\epsilon,$$

$$c_2|\phi_j - \phi_{j+1}| \le \lambda_k(\phi_j) - \lambda_k(\phi_{j+1}) \le c_3|\phi_j - \phi_{j+1}|,$$

$$\nabla F(\phi_j)(\phi_{j+1} - \phi_j) = s(\phi_{j+1} - \phi_j) + O(\epsilon)|\phi_{j+1} - \phi_j|,$$

$$|\phi_{j+1} - 2\phi_j + \phi_{j-1}| \le c_4\epsilon|\phi_{j+1} - \phi_j|,$$

$$c_5\epsilon^2 e^{-c_6\epsilon|j|} \le \lambda_k(\phi_j) - \lambda_k(\phi_{j+1}) \le c_7\epsilon^2 e^{-c_8\epsilon|j|},$$

$$c_9\epsilon e^{-c_{10}\epsilon|j|} \le |\phi_j - u^\pm| \le c_{11}\epsilon e^{-c_{12}\epsilon|j|},$$

$$c_{12}|\phi_j - \phi_{j+q}| \le \lambda_k(\phi_j) - \lambda_k(\phi_{j+q}) \le c_{13}|\phi_j - \phi_{j+q}|,$$

$$\nabla F(\phi_j)(\phi_{j+q} - \phi_j) = s(\phi_{j+q} - \phi_j) + O(\epsilon)|\phi_{j+q} - \phi_j|,$$
$$|\phi_{j+q} - 2\phi_j + \phi_{j-q}| \le c_{14}\epsilon|\phi_{j+q} - \phi_j|$$

where $c_1, \dots, c_{14}$ are positive constants independent of ϵ and j.

Combining the above proposition and following Liu, Wang and Yang [13], we can also find the discrete shock profiles of the v component. To this end, summing the first equation of (1.3) over x from $y - 2N + 1$ to $y - 1$ with step size (2), we have

$$v_{y-2N} = v_y + \frac{2}{\lambda}\sum_{x=y-2N+1}^{y-1}(u_{x-\eta} - u_x) + A^{\frac{1}{2}}\sum_{x=y-2N+1}^{y-1}[(u_x - u_{x-1}) - (u_{x+1} - u_x)].$$

Since $|u_{x-x_0} - u_x| < c(\lambda_k(u_{x-x_0}) - \lambda_k(u_x))$ and $\sum_{x=-\infty}^{y_0}(\lambda_k(u_{x-x_0}) - \lambda_k(u_x)) < O(1)\epsilon$, then $\sum_{x=-\infty}^{y_0}(u_{x-x_0} - u_x)$ converges. Hence, v_{y-2N} converges as $N \to \infty$

for any y, which implies that $\lim_{x\to\pm\infty} v_x = F(u_\pm)$. Thus v_x can be expressed as

$$v_x = v_- + \frac{2}{\lambda}\sum_{m=-\infty}^{0}(u_{x+2m-1} - u_{x+2m-1-\eta}) + A^{\frac{1}{2}}\sum_{m=-\infty}^{0}(u_{x+2m} - 2u_{x+2m-1} + u_{x+2m-2}). \tag{2.7}$$

Similarly, as for Corollary 2.1 of Liu and Xin [16], we can also have the following theorem.

Theorem 2.3. *There exists a family of discrete shock profiles (ϕ_j^n, ψ_j^n) of (1.4), connecting $(u_-, f(u_-))$ and $(u_+, f(u_+))$, provided that $u_\pm$ satisfy the entropy condition, $\epsilon = |u_- - u_+|$ is suitably small and the subcharacteristic condition is satisfied. Furthermore, the discrete shock profiles ϕ_j^n have the following properties for $j = 0, \pm 1, \dots$ and $n = 0, 1, \dots$:*

$$\begin{gathered}
\lambda_k(\phi_j^n) > \lambda_k(\phi_{j+1}^n),\\
|\lambda_k(\phi_j^n) - s| \le C_1\epsilon,\\
C_2|\phi_j^n - \phi_{j+1}^n| \le \lambda_k(\phi_j^n) - \lambda_k(\phi_{j+1}^n) \le C_3|\phi_j^n - \phi_{j+1}^n|,
\end{gathered}$$

$$\begin{gathered}
\nabla F(\phi_j^n)(\phi_{j+1}^n - \phi_j^n) = s(\phi_{j+1}^n - \phi_j^n) + O(\epsilon)|\phi_{j+1}^n - \phi_j^n|,\\
|\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n| \le C_4\epsilon|\phi_{j+1}^n - \phi_j^n|,\\
C_5\epsilon^2 e^{-c_6\epsilon|j-\lambda sn|} \le \lambda_k(\phi_j^n) - \lambda_k(\phi_{j+1}^n) \le C_7\epsilon^2 e^{-c_8\epsilon|j-\lambda sn|},\\
C_9\epsilon e^{-c_{10}\epsilon|j-\lambda sn|} \le |\phi_j^n - u \pm| \le C_{11}\epsilon e^{-c_{12}\epsilon|j-\lambda sn|},\\
|\phi_j^{n\pm1} - \phi_j^n| \le C_{13}(\lambda_k(\phi_j^n) - \lambda_k(\phi_{j+1}^n)),\ n > 1,
\end{gathered}$$

where $C_1, \dots, C_{13}$ are positive constants independent of ϵ, j and n.

Remark 2.8. It is not difficult to show that there exists a family of stationary discrete shock profiles and these profiles satisfy the above properties too.

3. L^2 stability of stationary discrete shock profiles

In this section, we will prove Theorem 1.1. Let (u_j^n, v_j^n) be a solution of the relaxing scheme with initial data satisfying (1.5)–(1.7), which is assumed to exist up to $n \le n_1 < +\infty$. Denote the stationary discrete shock profile (ϕ, ψ) as

$$\begin{gathered}
\tfrac{\lambda}{2}(\psi_{j+1} - \psi_j) - \tfrac{\lambda}{2}A^{\frac{1}{2}}(\phi_{j+1} - 2\phi_j + \phi_{j-1}) = 0,\\
\tfrac{\lambda}{2}A(\phi_{j+1} - \phi_{j-1}) - \tfrac{\lambda}{2}A^{\frac{1}{2}}(\psi_{j+1} - 2\psi_j + \psi_{j-1}) = \kappa(F(\phi_j) - \psi_j),\\
\lim_{j\to\pm\infty}(\phi_j, \psi_j) = (u_\pm, f(u_\pm)).
\end{gathered}$$

Let

$$\bar{u}_j = u_j - \phi_j, \bar{v}_j = v_j - \psi_j.$$

Then

$$\begin{aligned}
&\bar{u}_j^{n+1} - \bar{u}_j^n + \frac{\lambda}{2}(\bar{v}_{j+1}^n - \bar{v}_{j-1}^n) - \frac{\lambda}{2}A^{\frac{1}{2}}(\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n) = 0,\\
&\bar{v}_j^{n+1} - \bar{v}_j^n + \frac{\lambda}{2}A(\bar{u}_{j+1}^n - \bar{u}_{j-1}^n) - \frac{\lambda}{2}A^{\frac{1}{2}}(\bar{v}_{j+1}^n - 2\bar{v}_j^n + \bar{v}_{j-1}^n)\\
&\qquad\qquad = \kappa(\nabla F(\phi_j)\bar{u}_j^{n+1} - \bar{v}_j^{n+1}) + \kappa Q(u_j^{n+1}, \phi_j).
\end{aligned}$$

Let $Q(u_j^{n+1}, \phi_j) = F(u_j^{n+1}) - F(\phi_j) - \nabla F(\phi_j)\bar{u}_j^{n+1}$. Then $|Q(u_j^{n+1}, \phi_j)| \le O(1)|\bar{u}_j^{n+1}|^2$ when u is in any bounded set. Formally let $\hat{u}_j^n = \sum_{i=-\infty}^{j} \bar{u}_i^n$, so $\bar{u}_j^n = \hat{u}_j^n - \hat{u}_{j-1}^n$ and

$$P_j^n \equiv -\frac{\lambda}{2}(\bar{v}_{j+1}^n + \bar{v}_j^n) = \hat{u}_j^{n+1} - \hat{u}_j^n - \frac{\lambda}{2}A^{\frac{1}{2}}(\hat{u}_{j+1}^n - 2\hat{u}_j^n + \hat{u}_{j-1}^n). \tag{3.1}$$

Immediately, we have

$$\begin{aligned}
&P_j^{n+1} - P_j^n + \frac{\kappa}{1+\kappa}P_j^n - \frac{\lambda^2 A}{4(1+\kappa)}(\hat{u}_{j+2}^n - 2\hat{u}_j^n + \hat{u}_{j-2}^n)\\
&\quad - \frac{\lambda A^{\frac{1}{2}}}{2(1+\kappa)}(P_{j+1}^n - 2P_j^n + P_{j-1}^n)\\
&\quad + \frac{\lambda\kappa}{2(1+\kappa)}\nabla F(\phi_j)(\hat{u}_j^{n+1} - \hat{u}_{j-1}^{n+1}) + \frac{\lambda\kappa}{2(1+\kappa)}\nabla F(\phi_{j+1})(\hat{u}_{j+1}^{n+1} - \hat{u}_j^{n+1})\\
&\quad = -\frac{\lambda}{2(1+\kappa)}[\kappa Q(u_j^{n+1}, \phi_j) + \kappa Q(u_{j+1}^{n+1}, \phi_{j+1})].
\end{aligned} \tag{3.2}$$

Use the notation

$$L_j = L_(\phi_j), \Lambda_j = \Lambda(\phi_j), R_j = R(\phi_j), \theta_j^n = Q(u_j^n, \phi_j).$$

We rewrite the system (3.2) in terms of the characteristic variables

$$P_j^n = R_j\tilde{P}_j^n, \hat{u}_j^n = R_j\tilde{u}_j^n, \tag{3.3}$$

and letting $\mu = \lambda a^{\frac{1}{2}}$, we have the equation

$$\begin{aligned}
&\tilde{P}_j^{n+1} - \tilde{P}_j^n + \frac{\kappa}{1+\kappa}\tilde{P}_j^n - \frac{\mu^2}{4(1+\kappa)}(\tilde{u}_{j+2}^n - 2\tilde{u}_j^n + \tilde{u}_{j-2}^n)\\
&\quad - \frac{\mu}{2(1+\kappa)}(\tilde{P}_{j+1}^n - 2\tilde{P}_j^n + \tilde{P}_{j-1}^n)\\
&\quad + \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_j(\tilde{u}_j^{n+1} - \tilde{u}_{j-1}^{n+1}) + \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_{j+1}(\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1})\\
&\quad = \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_j(L_{j+1} - L_{j-1})R_j\tilde{u}_j^{n+1} + e_j^n,
\end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
e_j^n = {} & \frac{\lambda\kappa}{2(1+\kappa)}\Delta_+\Lambda_j\Delta_0 L_j R_j \tilde{u}_j^{n+1} + \frac{\mu^2}{4(1+\kappa)} L_j(R_{j+2} - 2R_j + R_{j-2})\tilde{u}_j^n \\
& - \frac{\lambda\kappa}{2(1+\kappa)} L_j\Delta_+ R_j \Lambda_{j+1}\Delta_+ \tilde{u}_j^{n+1} + \frac{\mu}{2(1+\kappa)} L_j \Delta R_j \tilde{P}_j^n \\
& - \frac{\lambda\kappa}{2(1+\kappa)} \Lambda_j\Delta_- L_j R_{j-1}\Delta_- \tilde{u}_j^{n+1} + \frac{\mu^2}{4(1+\kappa)} L_j(R_j - R_{j-2})(\tilde{u}_j^n - \tilde{u}_{j-2}^n) \\
& + \frac{\mu^2}{4(1+\kappa)} L_j(R_{j+2} - R_j)(\tilde{u}_{j+2}^n - \tilde{u}_j^n) - \frac{\lambda\kappa}{2(1+\kappa)} \Lambda_j\Delta_- L_j\Delta_- R_j \tilde{u}_j^{n+1} \\
& + \frac{\mu}{2(1+\kappa)} L_j\Delta_+ R_j\Delta_+ \tilde{P}_j^n + \frac{\mu}{2(1+\kappa)} L_j\Delta_- R_j\Delta_- \tilde{P}_j^n \\
& + \frac{\lambda\kappa}{2(1+\kappa)} L_j\Delta_+ R_j \Lambda_{j+1}\Delta_+ L_j R_j \tilde{u}_j^{n+1} - \frac{\kappa\lambda}{2(1+\kappa)} L_j(\theta_j^{n+1} + \theta_{j+1}^{n+1}).
\end{aligned}
$$

Here, we have used the notation

$$
\begin{aligned}
\Delta_+ f_j &= f_{j+1} - f_j, \\
\Delta_- f_j &= f_j - f_{j-1}, \\
\Delta_0 f_j &= f_{j+1} - f_{j-1}, \\
\Delta f_j &= f_{j+1} - 2f_j + f_{j-1}, \\
\Delta^+ f^n &= f^{n+1} - f^n.
\end{aligned}
$$

Thus, to prove Theorem 1.1, we need to derive the basic prior L^2 estimate of the solution of (3.4). Denote the weight as W_j which will be defined later. It is used so the terms of the principal direction control the transverse terms. Now we need some basic energy estimates. Assume $2\tilde{P}_j^n W_j \times$ (3.4), and denote $|u_j^n|_w = (u_j^n . W_j u_j^n)^{\frac{1}{2}}$, because

$$
\begin{aligned}
& \sum_j 2\tilde{P}_j^n W_j(\tilde{P}_j^{n+1} - \tilde{P}_j^n) \\
& \qquad = \sum_j |\tilde{P}_j^{n+1}|_w^2 - \sum_j |\tilde{P}_j^n|_w^2 - \sum_j |\tilde{P}_j^{n+1} - \tilde{P}_j^n|_w^2,
\end{aligned}
$$

$$
\begin{aligned}
& \sum_j -\frac{\mu^2}{2(1+\kappa)} \tilde{P}_j^n W_j(\tilde{u}_{j+2}^n - 2\tilde{u}_j^n + \tilde{u}_{j-2}^n) \\
& \qquad = \frac{\mu^2}{2(1+\kappa)} \sum_j \tilde{P}_{j+1}^n (W_{j+1} - W_{j-1})(\tilde{u}_{j+1}^n - \tilde{u}_{j-1}^n) \\
& \qquad\quad + \frac{\mu^2}{2(1+\kappa)} \sum_j (\tilde{P}_{j+1}^n - \tilde{P}_{j-1}^n) W_{j-1}(\tilde{u}_{j+1}^n - \tilde{u}_{j-1}^n),
\end{aligned}
$$

$$\begin{aligned}
&\sum_j -\frac{\mu}{1+\kappa}\tilde{P}_j^n W_j(\tilde{P}_{j+1}-2\tilde{P}_j+\tilde{P}_{j-1})\\
&\quad=\frac{\mu}{2(1+\kappa)}\sum_j(\tilde{P}_{j+1}^n-\tilde{P}_j^n)(W_j+W_{j+1})(\tilde{P}_{j+1}^n-\tilde{P}_j^n)\\
&\qquad-\frac{\mu}{2(1+\kappa)}\sum_j\tilde{P}_j^n(W_{j+1}-2W_j+W_{j-1})\tilde{P}_j^n.
\end{aligned}$$

Let

$$M_j^n=\tilde{u}_{j+1}^n-\tilde{u}_j^n, N_j^n=\tilde{u}_{j+1}^n-\tilde{u}_{j-1}^n.$$

After summation by parts,

$$\begin{aligned}
&\sum_j|\tilde{P}_j^{n+1}|_w^2-\sum_j|\tilde{P}_j^n|_w^2\\
&\qquad+\frac{2\kappa}{1+\kappa}\sum_j|\tilde{P}_j^n|_w^2\\
&\qquad+\frac{\mu^2}{2(1+\kappa)}\sum_j\tilde{P}_{j+1}^n\Delta_0 W_j N_j^n\\
&\qquad+\frac{\mu}{2(1+\kappa)}\sum_j(\tilde{P}_{j+1}^n-\tilde{P}_j^n)(W_j+W_{j+1})(\tilde{P}_{j+1}^n-\tilde{P}_j^n)\\
&\quad=\sum_j|\tilde{P}_j^{n+1}-\tilde{P}_j^n|_w^2\\
&\qquad-\frac{\mu^2}{2(1+\kappa)}\sum_j(\tilde{P}_{j+1}^n-\tilde{P}_{j-1}^n)W_{j-1}N_j^n\\
&\qquad+\frac{\mu}{2(1+\kappa)}\sum_j\tilde{P}_j^n\Delta W_j\tilde{P}_j^n\\
&\qquad-\sum_j\frac{\lambda\kappa}{1+\kappa}(\tilde{P}_{j+1}^n W_{j+1}+\tilde{P}_j^n W_j)\Lambda_{j+1}(\tilde{u}_{j+1}^{n+1}-\tilde{u}_j^{n+1})\\
&\qquad+\sum_j\frac{\lambda\kappa}{1+\kappa}\tilde{P}_j^n W_j\Lambda_j(L_{j+1}-L_{j-1})R_j\tilde{u}_j^{n+1}\\
&\qquad+\sum_j 2\tilde{P}_j^n W_j e_j^n.
\end{aligned}\tag{3.5}$$

Because of (3.1), we have the identities

$$\begin{aligned}
\tilde{u}_j^{n+1}&=(1-\mu)\tilde{u}_j^n+\tilde{P}_j^n+\frac{\mu}{2}L_jR_{j+1}\tilde{u}_{j+1}^n+\frac{\mu}{2}L_jR_{j-1}\tilde{u}_{j-1}^n,\\
\tilde{u}_{j+1}^{n+1}-\tilde{u}_j^{n+1}&=(1-\mu)M_j^n+\tilde{P}_{j+1}^n-\tilde{P}_j^n\\
&\quad+\frac{\mu}{2}(L_{j+1}R_{j+2}\tilde{u}_{j+2}^n-L_jR_{j+1}\tilde{u}_{j+1}^n)\\
&\quad+\frac{\mu}{2}(L_{j+1}R_j\tilde{u}_j^n-L_jR_{j-1}\tilde{u}_{j-1}^n).
\end{aligned}$$

Next, assume $2\tilde{u}_j^{n+1}W_j \times (3.4)$, because of the identities

$$
\begin{aligned}
&\sum_j \frac{2\kappa}{1+\kappa}\tilde{u}_j^{n+1}W_j\tilde{P}_j^n \\
&\quad= \sum_j \frac{2\kappa}{1+\kappa}\tilde{u}_j^{n+1}W_j(\tilde{u}_j^{n+1}-\tilde{u}_j^n-\frac{\mu}{2}L_j(R_{j+1}\tilde{u}_{j+1}^n-2R_j\tilde{u}_j^n+R_{j-1}\tilde{u}_{j-1}^n)) \\
&\quad= \sum_j \frac{\kappa}{1+\kappa}\tilde{u}_j^{n+1}W_j\tilde{u}_j^{n+1}-\sum_j \frac{\kappa}{1+\kappa}\tilde{u}_j^nW_j\tilde{u}_j^n \\
&\qquad+\sum_j \frac{\kappa}{1+\kappa}\triangle^+\tilde{u}_j^nW_j\triangle^+\tilde{u}_j^n \\
&\qquad-\frac{\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^nW_jL_j(R_{j+1}\tilde{u}_{j+1}^n-2R_j\tilde{u}_j^n+R_{j-1}\tilde{u}_{j-1}^n) \\
&\qquad+\frac{\kappa\mu}{2(1+\kappa)}\sum_j \Delta_+\tilde{u}_j^n(W_j+W_{j+1})\Delta_+\tilde{u}_j^n \\
&\qquad-\frac{\kappa\mu}{2(1+\kappa)}\sum_j \tilde{u}_j^n\Delta W_j\tilde{u}_j^n \\
&\qquad-\frac{\kappa\mu}{1+\kappa}\sum_j \tilde{u}_j^nW_jL_j\Delta_+R_j\Delta_+\tilde{u}_j^n \\
&\qquad-\frac{\kappa\mu}{1+\kappa}\sum_j \tilde{u}_j^nW_jL_j\Delta_-R_j\Delta_-\tilde{u}_j^n \\
&\qquad-\frac{\kappa\mu}{1+\kappa}\sum_j \tilde{u}_j^nW_jL_j\Delta R_j\tilde{u}_j^n
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_j -\frac{\mu^2}{2(1+\kappa)}\tilde{u}_j^{n+1}W_j(N_{j+1}^n-N_{j-1}^n) \\
&\quad=-\frac{\mu^2}{2(1+\kappa)}\sum_j \Delta^+\tilde{u}_j^nW_j\Delta_0N_j^n+\frac{\mu^2}{2(1+\kappa)}\sum_j N_j^nW_jN_j^n \\
&\qquad+\frac{\mu^2}{2(1+\kappa)}\sum_j \tilde{u}_{j+1}^n\Delta_+W_jN_j^n+\frac{\mu^2}{2(1+\kappa)}\sum_j \tilde{u}_{j-1}^n\Delta_-W_jN_j^n, \\
&\sum_j -\frac{\mu}{1+\kappa}\tilde{u}_j^{n+1}W_j\Delta\tilde{P}_j^n \\
&\quad=\sum_j -\frac{\mu}{1+\kappa}\Delta^+\tilde{u}_j^nW_j\Delta\tilde{P}_j^n \\
&\qquad+\sum_j \frac{\mu}{1+\kappa}\Delta_+\tilde{u}_j^nW_j\Delta_+\tilde{P}_j^n+\sum_j \frac{\mu}{1+\kappa}\tilde{u}_{j+1}^n\Delta_+W_j\Delta_+\tilde{P}_j^n.
\end{aligned}
$$

Combining these terms and performing summation by parts, we get

$$
\begin{aligned}
(3.6)\qquad & 2\sum_j \tilde{u}_j^{n+1} W_j \tilde{P}_j^{n+1} - 2\sum_j \tilde{u}_j^n W_j \tilde{P}_j^n \\
& - 2\sum_j \Delta^+ \tilde{u}_j^n W_j \tilde{P}_j^n \\
& + \sum_j \frac{\kappa}{1+\kappa} \tilde{u}_j^{n+1} W_j \tilde{u}_j^{n+1} - \sum_j \frac{\kappa}{1+\kappa} \tilde{u}_j^n W_j \tilde{u}_j^n \\
& + \sum_j \frac{\kappa}{1+\kappa} \Delta^+ \tilde{u}_j^n W_j \Delta^+ \tilde{u}_j^n \\
& - \frac{\kappa\mu}{1+\kappa} \sum_j \Delta^+ \tilde{u}_j^n W_j L_j (R_{j+1} \tilde{u}_{j+1}^n - 2R_j \tilde{u}_j^n + R_{j-1} \tilde{u}_{j-1}^n) \\
& - \frac{\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n W_j L_j \Delta_+ R_j \Delta_+ \tilde{u}_j^n \\
& - \frac{\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n W_j L_j \Delta_- R_j \Delta_- \tilde{u}_j^n \\
& + \frac{\kappa\mu}{2(1+\kappa)} \sum_j \Delta_+ \tilde{u}_j^n (W_j + W_{j+1}) \Delta_+ \tilde{u}_j^n \\
& - \frac{\kappa\mu}{2(1+\kappa)} \sum_j \tilde{u}_j^n \Delta W_j \tilde{u}_j^n \\
& - \frac{\mu^2}{2(1+\kappa)} \sum_j \Delta^+ \tilde{u}_j^n W_j \Delta_0 N_j^n + \frac{\mu^2}{2(1+\kappa)} \sum_j N_j^n W_j N_j^n \\
& + \frac{\mu^2}{2(1+\kappa)} \sum_j \tilde{u}_{j+1}^n \Delta_+ W_j N_j^n + \frac{\mu^2}{2(1+\kappa)} \sum_j \tilde{u}_{j-1}^n \Delta_- W_j N_j^n \\
& + \sum_j \frac{\mu}{1+\kappa} \Delta_+ \tilde{u}_j^n W_j \Delta_+ \tilde{P}_j^n + \sum_j \frac{\mu}{1+\kappa} \tilde{u}_{j+1}^n \Delta_+ W_j \Delta_+ \tilde{P}_j^n \\
& + \sum_j \frac{\lambda\kappa}{1+\kappa} (\tilde{u}_{j+1}^{n+1} W_{j+1} + \tilde{u}_j^{n+1} W_j) \Lambda_{j+1} \Delta_+ \tilde{u}_j^{n+1} \\
& - \frac{\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n W_j L_j \Delta R_j \tilde{u}_j^n \\
& - \sum_j \frac{\mu}{1+\kappa} \Delta^+ \tilde{u}_j^n W_j \Delta \tilde{P}_j^n \\
& = \sum_j \frac{\lambda\kappa}{1+\kappa} \tilde{u}_j^{n+1} W_j \Lambda_j \Delta_0 L_j R_j \tilde{u}_j^{n+1} + 2\sum_j \tilde{u}_j^{n+1} e_j^n.
\end{aligned}
$$

Let τ be an $O(1)$ number to be decided. Combining (3.5) and (3.6), we get

$$E(n+1) - E(n) + \bar{L} = \bar{R} + \bar{X},$$

where

$$
\begin{aligned}
E(n+1) &= \sum_j [|\tilde{P}_j^{n+1}|_w^2 + 2\tau \tilde{u}_j^{n+1} W_j \tilde{P}_j^{n+1} + \tau \frac{\kappa}{1+\kappa} |\tilde{u}_j^{n+1}|_w^2],\\
E(n) &= \sum_j [|\tilde{P}_j^{n}|_w^2 + 2\tau \tilde{u}_j^{n} W_j \tilde{P}_j^{n} + \tau \frac{\kappa}{1+\kappa} |\tilde{u}_j^{n}|_w^2],\\
\bar{L} &= \frac{2\kappa}{1+\kappa} \sum_j |\tilde{P}_j^n|_w^2 + \frac{\mu}{2(1+\kappa)} \sum_j \Delta_+ \tilde{P}_j^n (W_j + W_{j+1}) \Delta_+ \tilde{P}_j^n\\
&\quad + \frac{\kappa\tau}{1+\kappa} \sum_j \Delta^+ \tilde{u}_j^n W_j \Delta^+ \tilde{u}_j^n + \frac{\tau\mu^2}{2(1+\kappa)} \sum_j N_j^n W_j N_j^n\\
&\quad + \frac{\tau\kappa\mu}{2(1+\kappa)} \sum_j M_j^n (W_j + W_{j+1}) M_j^n,\\
\bar{X} &= \sum_j |\tilde{P}_j^{n+1} - \tilde{P}_j^n|_w^2 - \frac{\mu^2}{2(1+\kappa)} \sum_j \tilde{P}_{j+1}^n \Delta_0 W_j N_j^n\\
&\quad - \frac{\mu^2}{2(1+\kappa)} \sum_j \Delta_0 \tilde{P}_j^n W_{j-1} N_j^n + \frac{\mu}{2(1+\kappa)} \sum_j \tilde{P}_j^n \Delta W_j \tilde{P}_j^n\\
&\quad + \frac{\lambda\kappa}{1+\kappa} \sum_j \tilde{P}_j^n W_j \Lambda_j \Delta_0 L_j R_j \tilde{u}_j^{n+1} + 2\tau \sum_j \Delta^+ \tilde{u}_j^n W_j \tilde{P}_j^n\\
&\quad + \frac{\tau\kappa\mu}{1+\kappa} \sum_j \Delta^+ \tilde{u}_j^n W_j L_j (R_{j+1} \tilde{u}_{j+1}^n - 2R_j \tilde{u}_j^n + R_{j-1} \tilde{u}_{j-1}^n)\\
&\quad + \frac{\tau\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n W_j L_j \Delta_+ R_j \Delta_+ \tilde{u}_j^n + \frac{\tau\mu^2}{2(1+\kappa)} \sum_j \Delta^+ \tilde{u}_j^n W_j \Delta_0 N_j^n\\
&\quad - \frac{\tau\mu^2}{2(1+\kappa)} \sum_j \tilde{u}_{j+1}^n \Delta_+ W_j N_j^n - \frac{\tau\mu^2}{2(1+\kappa)} \sum_j \tilde{u}_{j-1}^n \Delta_- W_j N_j^n\\
&\quad - \frac{\tau\mu}{1+\kappa} \sum_j \Delta_+ \tilde{u}_j^n W_j \Delta_+ \tilde{P}_j^n - \frac{\tau\mu}{1+\kappa} \sum_j \tilde{u}_{j+1}^n \Delta_+ W_j \Delta_+ \tilde{P}_j^n\\
&\quad - \frac{\tau\mu}{1+\kappa} \sum_j \Delta^+ \tilde{u}_j^n W_j \Delta \tilde{P}_j^n + \sum_j (2\tilde{P}_j^n + 2\tau \tilde{u}_j^{n+1}) W_j e_j^n,\\
\bar{R} &= - \frac{\lambda\kappa}{1+\kappa} \sum_j (\tilde{P}_{j+1}^n W_{j+1} + \tilde{P}_j^n W_j) \Lambda_{j+1} \Delta_+ \tilde{u}_j^{n+1}\\
&\quad + \frac{\tau\kappa\mu}{2(1+\kappa)} \sum_j \tilde{u}_j^n \Delta W_j \tilde{u}_j^n + \frac{\tau\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n W_j L_j \Delta R_j \tilde{u}_j^n\\
&\quad - \frac{\tau\kappa\lambda}{1+\kappa} \sum_j (\tilde{u}_{j+1}^{n+1} W_{j+1} + \tilde{u}_j^{n+1} W_j) \Lambda_{j+1} \Delta_+ \tilde{u}_j^{n+1}\\
&\quad + \frac{\tau\lambda\kappa}{1+\kappa} \sum_j \tilde{u}_j^{n+1} W_j \Lambda_j \Delta_0 L_j R_j \tilde{u}_j^{n+1}.
\end{aligned}
$$

The terms in $\bar{R}$ are labelled as $\bar{R}_i$, and the terms in $\bar{X}$ are labelled as $\bar{X}_i$. $\bar{L}$ is positive, so it makes the small perturbation to the discrete shock profile decay to zero. We want to show that $\bar{R}$ produces a negative term of $|\tilde{u}_j^{n+1}|_w$ if we choose

the weight W carefully. We can do that, because the discrete shock profiles have the strong compressibility property. The other terms of $\bar{R}$ and $\bar{X}$ can be estimated by $\bar{L}$ and the positive term of $|\tilde{u}_j^{n+1}|_w$ depending on the properties of W_j and ϕ_j. For this purpose, some terms in $\bar{R}$ should be changed.

$$\begin{aligned}
\bar{R}_1 =& \sum_j \frac{\lambda\kappa}{1+\kappa}\tilde{P}_j^n(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j)\tilde{P}_j^n + \sum_j \frac{\lambda\kappa}{1+\kappa}\tilde{P}_j^n\Delta_+ W_j\Lambda_{j+1}M_j^{n+1} \\
&- \sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde{P}_{j+1}^n + \tilde{P}_j^n)W_{j+1}\Lambda_{j+1}[(1-\mu)M_j^n + \frac{\mu}{2}(L_{j+1}R_{j+2}\tilde{u}_{j+2}^n \\
&- L_jR_{j+1}\tilde{u}_{j+1}) + \frac{\mu}{2}(L_{j+1}R_j\tilde{u}_j^n - L_jR_{j-1}\tilde{u}_{j-1}^n)], \\
\bar{R}_4 =& \frac{\tau\kappa\lambda}{1+\kappa}\sum_j \tilde{u}_j^{n+1}(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j)\tilde{u}_j^{n+1} \\
&+ \frac{\tau\lambda\kappa}{1+\kappa}\sum_j \tilde{u}_j^{n+1}\Delta_+ W_j\Lambda_{j+1}\Delta_+\tilde{u}_j^{n+1}.
\end{aligned}$$

Because the scheme is implicit, $\tilde{P}_j^n$ is controlled by $\bar{L}_1$ very well, and $\tilde{u}_j^{n+1}$ will appear in the transverse terms. We need to change the terms which have $\tilde{u}_j^n$:

$$\begin{aligned}
\bar{R}_2 =& \frac{\tau\kappa\mu}{2(1+\kappa)}\sum_j \Delta^+\tilde{u}_j^n\Delta W_j\Delta^+\tilde{u}_j^n - \frac{\tau\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n\Delta W_j\tilde{u}_j^{n+1} \\
&+ \frac{\tau\kappa\mu}{2(1+\kappa)}\sum_j \tilde{u}_j^{n+1}\Delta W_j\tilde{u}_j^{n+1}, \\
\bar{R}_3 =& \frac{\tau\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n W_jL_j\Delta R_j\Delta^+\tilde{u}_j^n - \frac{2\tau\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n W_jL_j\Delta R_j\tilde{u}_j^{n+1} \\
&+ \frac{\tau\kappa\mu}{2(1+\kappa)}\sum_j \tilde{u}_j^{n+1}W_jL_j\Delta R_j\tilde{u}_j^{n+1}.
\end{aligned}$$

Now, combining the above terms, we have

$$\begin{aligned}
\bar{R} =& \sum_j \frac{\tau\kappa}{1+\kappa}\tilde{u}_j^{n+1}[\lambda(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j) + \lambda W_j\Lambda_j\Delta_0 L_jR_j + \frac{\mu}{2}\Delta W_j \\
&+ \frac{\mu}{2}W_jL_j\Delta R_j]\tilde{u}_j^{n+1} + \frac{\tau\lambda\kappa}{1+\kappa}\sum_j \tilde{u}_j^{n+1}\Delta_+ W_j\Lambda_{j+1}\Delta_+\tilde{u}_j^{n+1} \\
&+ \frac{\tau\kappa\mu}{2(1+\kappa)}\sum_j \Delta^+\tilde{u}_j^n\Delta W_j\Delta^+\tilde{u}_j^n - \frac{\tau\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n\Delta W_j\tilde{u}_j^{n+1} \\
&+ \frac{\tau\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n W_jL_j\Delta R_j\Delta^+\tilde{u}_j^n - \frac{2\tau\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n W_jL_j\Delta R_j\tilde{u}_j^{n+1} \\
&+ \frac{\lambda\kappa}{1+\kappa}\sum_j \tilde{P}_j^n(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j)\tilde{P}_j^n + \sum_j \frac{\lambda\kappa}{1+\kappa}\tilde{P}_j^n\Delta_+ W_j\Lambda_{j+1}M_j^{n+1} \\
&- \frac{\lambda\kappa}{1+\kappa}\sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n)W_{j+1}\Lambda_{j+1}[(1-\mu)M_j^n + \frac{\mu}{2}(L_{j+1}R_{j+2}\tilde{u}_{j+2}^n \\
&- L_jR_{j+1}\tilde{u}_{j+1}^n) + \frac{\mu}{2}(L_{j+1}R_j\tilde{u}_j^n - L_jR_{j-1}\tilde{u}_{j-1}^n)].
\end{aligned}$$

Label the above terms as $\bar{R}_i$, where i is from 1 to 9. We want to show that $\bar{R}_1$ is negative, so choosing the weights [16]

$$W_j = \mathrm{diag}\{w_{1,j}, w_{2,j}, \dots, w_{m,j}\},$$

as

$$w_{\mu,j} = \frac{c_{1,\mu}}{|\lambda_{\mu,j}|} \prod_{i=-\infty}^{j-1} (1 - c_{2,\mu}\frac{\lambda_{k,i} - \lambda_{k,i+1}}{\lambda_{\mu,i}}), \mu \neq \kappa$$

and

$$w_{\kappa,j} \equiv 1,$$

we have the following lemma.

Lemma 3.1 ([16]). *Let W_j be the weights defined above. Then we can choose $c_{1,\mu}$ and $c_{2,\mu}$ appropriately so that*

$$W_{j+1} - W_j = O(1)(\lambda_{k,j} - \lambda_{k,j+1})W_j,$$
$$W_{j+1} - 2W_j + W_{j-1} = O(\epsilon)(\lambda_{k,j} - \lambda_{k,j+1})W_j,$$

and

$$\lambda(W_{j+1}\Lambda_{j+1} - W_j\Lambda_j) + \lambda W_j\Lambda_j(L_{j+1} - L_{j-1})R_j + \frac{\mu}{2}(W_{j+1} - 2W_j + W_{j-1}) \leq -\frac{\lambda}{2}(\lambda_{k,j} - \lambda_{k,j+1})W_j,$$

provided that ϵ is suitably small.

The position to show is that the transverse terms are controlled by the terms of the principal direction. Using Lemma 3.1, we have these estimates of $\bar{R}_1$ and $\bar{R}_7$

$$\bar{R}_1 \leq -\frac{\lambda}{2}(\lambda_{k,j} - \lambda_{k,j+1})\frac{\tau\kappa}{1+\kappa}\sum_j |\tilde{u}_j^{n+1}|_w^2,$$

$$\bar{R}_7 \leq -\frac{\lambda}{2}(\lambda_{k,j} - \lambda_{k,j+1})\frac{\kappa}{1+\kappa}\sum_j |\tilde{P}_j^{n}|_w^2.$$

We only need to estimate $\bar{X}_1, \bar{X}_6, \bar{X}_{12}, \bar{X}_{15}$, and the other terms can be handled easily. It follows from (3.4) that

$$\begin{aligned}
&\sum_j |\tilde{P}_j^{n+1} - \tilde{P}_j^n|_w^2 \\
&\leq \frac{5\kappa}{4(1+\kappa)}\sum_j |\tilde{P}_j^n|_w^2 + O(1)\frac{\mu^2}{(1+\kappa)}\sum_j |\tilde{P}_{j+1}^n - \tilde{P}_j^n|_w^2 + O(1)\frac{\mu^4}{1+\kappa}\sum_j |N_j^n|_w^2 \\
&\quad + O(1)\frac{\kappa\mu^2}{1+\kappa}\sum_j |M_j^n|_w^2 + O(1)\frac{\kappa\mu^2}{1+\kappa}\sum_j |\tilde{u}_j^{n+1} - \tilde{u}_j^n|_w^2 \\
&\quad + O(\epsilon)\sum_j (\lambda_{k,j} - \lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2 + O(1)\sum_j |e_j^n|_w^2.
\end{aligned}$$

Set

$$\bar{M}(n_1) = \sup_{n \leq n_1} (\sum_j |\tilde{u}_j^n|^2 + \sum_j |\tilde{P}_j^n|^2)^{\frac{1}{2}},$$

and assume that $\bar{M}(n_1)$ is small enough. Clearly, we have

$$\sup_{n\le n_1,j}(|\tilde{u}_j^n|+|\tilde{P}_j^n|)\le \bar{M}(n_1),$$

and because

$$|\theta_j^{n+1}|\le O(1)(|\tilde{u}_j^{n+1}-\tilde{u}_{j+1}^{n+1}|^2+(\lambda_{k,j}-\lambda_{k,j+1})^2|\tilde{u}_j^{n+1}|^2), \tag{3.7}$$

immediately, by Theorem 2.3, we have

$$\begin{aligned}\sum_j|e_j^n|_w^2 \le{}& O(1)\sum_j\frac{\kappa\mu^2}{1+\kappa}|(\lambda_{k,j}-\lambda_{k,j+1})\tilde{u}_j^{n+1}|_w^2+O(\epsilon)\sum_j\frac{\kappa\mu^4}{1+\kappa}|M_j^n|_w^2\\&+O(1)\sum_j\frac{\kappa\mu^2}{1+\kappa}|(\lambda_{k,j}-\lambda_{k,j+1})M_j^{n+1}|_w^2\\&+O(1)\sum_j\frac{\mu^2}{1+\kappa}|(\lambda_{k,j}-\lambda_{k,j+1})(\tilde{P}_{j+1}^n-\tilde{P}_j^n)|_w^2\\&+O(1)\sum_j\frac{\mu^2}{1+\kappa}|(\lambda_{k,j}-\lambda_{k,j+1})\tilde{P}_j^n|_w^2\\&+O(\epsilon)\sum_j\frac{\mu^4}{1+\kappa}|\Delta^+\tilde{u}_j^n|_w^2+\sum_j O(1)\bar{M}(n_1)^2\frac{\kappa\mu^2}{1+\kappa}|M_j^{n+1}|_w^2,\end{aligned} \tag{3.8}$$

so

$$\begin{aligned}&\sum_j|\tilde{P}_j^{n+1}-\tilde{P}_j^n|_w^2\\&\le\frac{5\kappa}{4(1+\kappa)}\sum_j|\tilde{P}_j^n|_w^2+O(1)\frac{\kappa\mu^2}{1+\kappa}\sum_j|\Delta^+\tilde{u}_j^n|_w^2\\&\quad+O(1)\sum_j\frac{\kappa\mu^2}{1+\kappa}|M_j^n|_w^2+\frac{\lambda}{8}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2\\&\quad+O(1)\frac{\mu^2}{1+\kappa}\sum_j|\tilde{P}_{j+1}^n-\tilde{P}_j^n|_w^2+O(\epsilon)\sum_j\frac{\mu^2}{1+\kappa}|\tilde{P}_j^n|_w^2\\&\quad+O(1)\sum_j\bar{M}(n_1)^2\frac{\kappa\mu^2}{1+\kappa}|M_j^{n+1}|_w^2.\end{aligned}$$

$\bar{X}_6$ and $\bar{X}_{12}$ should be dealt with more carefully. We can estimate these terms as

$$\begin{aligned}\bar{X}_6\le{}&(2\tau+O(1)\tau\mu+O(1)\tau\mu^{\frac{1}{2}})\sum_j|\tilde{P}_j^n|_w^2+\frac{\tau\kappa\mu}{4(1+\kappa)}\sum_j|M_j^n|_w^2\\&+\frac{\mu^{\frac{3}{2}}}{2}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2+\frac{\mu}{2}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\Delta^+\tilde{u}_j^n|_w^2,\\\bar{X}_{12}\le{}&\frac{\mu}{1+\kappa}[\frac{1}{2}\sum_j\tilde{P}_j^nW_j\tilde{P}_j^n+\frac{\tau^2}{2}\sum_j|M_j^n|_w^2].\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}\bar{X}_{15} \le{}& \frac{\lambda}{8}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2+\frac{\lambda}{8}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{P}_j^n|_w^2\\&+\frac{\lambda\kappa}{1+\kappa}(O(\epsilon)+\bar{M}(n_1))\sum_j(|M_j^n|_w^2+|\tilde{u}_j^{n+1}-\tilde{u}_j^n|_w^2)\\&+O(1)\frac{\mu^2}{1+\kappa}\sum_j|\tilde{u}_{j+1}^n-\tilde{u}_j^n|_w^2+O(\epsilon)\frac{\mu}{1+\kappa}\sum_j|\tilde{P}_{j+1}^n-\tilde{P}_j^n|_w^2.\end{aligned}$$

Two typical terms can be estimated as follows:

$$\begin{aligned}&\sum_j|\tilde{u}_j^{n+1}W_jL_j(R_{j+1}-R_j)\Lambda_{j+1}(\tilde{u}_{j+1}^{n+1}-\tilde{u}_j^{n+1})|\\&\quad\le O(1)\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}.W_j(\tilde{u}_{j+1}^{n+1}-\tilde{u}_j^{n+1})|\\&\quad\le\sum_j\frac{1}{48}(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2+\sum_jO(\epsilon)|\tilde{u}_{j+1}^{n+1}-\tilde{u}_j^{n+1}|_w^2,\\&\sum_j|\tilde{u}_j^{n+1}\cdot W_jL_j\theta_j^{n+1}|\\&\quad\le O(1)\sum_j|W_j\tilde{u}_j^{n+1}||\tilde{u}_j^{n+1}-\tilde{u}_{j+1}^{n+1}|^2+\sum_jO(\epsilon)(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2\\&\quad\le O(1)\bar{M}(n_1)\sum_j|\tilde{u}_j^{n+1}-\tilde{u}_{j+1}^{n+1}|_w^2+O(\epsilon)\sum_j(\lambda_{k,j}-\lambda_{k,j+1})|\tilde{u}_j^{n+1}|_w^2.\end{aligned}$$

Set $\tau=\frac{\kappa}{4(1+\kappa)}$. If ϵ and λ are suitably small, we will obtain

$$\begin{aligned}&E(n+1)-E(n)+\frac{\lambda}{4}\sum_j(\lambda_{k,j}-\lambda_{k,j+1})[|\tilde{u}_j^{n+1}|_w^2+|\tilde{P}_j^n|_w^2]+O(1)\sum_j|\tilde{P}_j^n|_w^2\\&\quad+O(1)[\sum_j|\tilde{u}_j^n-\tilde{u}_{j+1}^n|_w^2+\sum_j|\tilde{P}_j^n-\tilde{P}_{j+1}^n|_w^2+\sum_j|\tilde{u}_j^{n+1}-\tilde{u}_j^n|_w^2]\le 0.\end{aligned}$$

Since the weights are bounded by some positive constants, if ϵ and λ are suitably small, we have the basic a priori estimate.

Proposition 3.2 (A priori estimate)**.** *Let $(\tilde{u}^n,\tilde{P}^n)$ be a solution of* (3.4) *for $n\le n_1$. Then there exists a positive constant C independent of n_1 and ϵ such that for $n_2\le n_1$*

$$\begin{aligned}(3.9)\quad&\sum_j(|\tilde{u}_j^{n_2}|^2+|\tilde{P}_j^{n_2}|^2)+\sum_{n\le n_2}\sum_j|\lambda_{k,j}-\lambda_{k,j+1}|[|\tilde{u}_j^n|^2+|\tilde{P}_j^n|^2]\\&\quad+\sum_{n\le n_2}\sum_j[|\tilde{P}_j^n|^2+|\tilde{P}_j^n-\tilde{P}_{j+1}^n|^2+|\tilde{u}_j^n-\tilde{u}_{j+1}^n|^2]\le C\sum_j[|\tilde{u}_j^0|^2+|\tilde{P}_j^0|^2],\end{aligned}$$

provided that ϵ,λ and $\bar{M}(n_1)$ are suitably small.

In fact, if the initial data is small enough, i.e., $\bar{M}(0)$ is small enough, then (3.9) is true a little longer and hence forever by the standard continuity argument. Then the following proposition holds.

Proposition 3.3. *Assume that ϵ and $\bar{M}(0)$ are suitably small. Then the problem (3.4) has a unique global solution $(\tilde{u}_j^n, \tilde{P}_j^n)$. For any $n \geq 0$,*

$$\begin{aligned}\sup_n \sum_j (|\tilde{u}_j^n|^2 + |\tilde{P}_j^n|^2) + \sum_{j,n} |\lambda_{k,j} - \lambda_{k,j+1}|[|\tilde{u}_j^n|^2 + |\tilde{P}_j^n|^2] \\ + \sum_{j,n} [|\tilde{P}_j^n|^2 + |\tilde{P}_j^n - \tilde{P}_{j+1}^n|^2 + |\tilde{u}_j^n - \tilde{u}_{j+1}^n|^2] \leq C\bar{M}(0)^2,\end{aligned} \tag{3.10}$$

where C is a positive constant independent of n and j.

Now it remains to estimate the $\bar{v}_j^n$, because

$$\begin{aligned}(1+\kappa)\bar{v}_j^{n+1} = \bar{v}_j^n + A^{\frac{1}{2}}\lambda(\bar{v}_{j+1}^n - 2\bar{v}_j^n + \bar{v}_{j-1}^n) - \frac{\lambda A}{2}(\bar{u}_{j+1}^n - \bar{u}_{j-1}^n) \\ + \kappa(\nabla F(\phi_j)\bar{u}_j^{n+1}) + \kappa O(1)|\bar{u}_j^{n+1}|^2.\end{aligned} \tag{3.11}$$

Multiply the above equation by $\bar{v}_j^{n+1}$, and perform summation by parts. Immediately, we have the estimate

$$\sum_j |\bar{v}_j^{n+1}|^2 - \nu \sum_j |\bar{v}_j^n|^2 + O(1) \sum_j |\bar{v}_j^n - \bar{v}_{j+1}^n|^2 \leq C \sum_j |\hat{u}_j^n - \hat{u}_{j+1}^n|^2$$

provided that λ and $\bar{M}(0)$ are suitably small, where $\nu < 1$. Note that $O(1)$ and C are independent of n. Combining this with (3.10), we have the global estimate

$$\sup_n \sum_j |\bar{v}_j^n|^2 + O(1) \sum_{j,n} |\bar{v}_j^n - \bar{v}_{j+1}^n|^2 \leq C\bar{M}^2(0),$$

from above. Under the assumption of initial conditions, similarly to [16], we have

$$\lim_{n \to +\infty} [\sum_j |u_j^n - \phi_j|^2 + |v_j^n - \psi_j|^2] = 0.$$

This is our Theorem 1.1.

4. L^2 stability of nonstationary discrete shock profiles

In this section we will prove the L^2 stability of the single discrete shock profile. Here we assume the discrete shock speed $s > 0$ (the case of $s < 0$ can be handled by a little modification). Because of the complex structure of the nonstationary discrete shock profile, the method discussed in the last section cannot be used, so we use the idea of vertical estimate. As before, let (ϕ_j^n, ψ_j^n) be the single discrete shock profile in the k-field of the equilibrium system as follows:

$$\begin{aligned}\phi_j^{n+1} - \phi_j^n + \frac{k}{2h}(\psi_{j+1}^n - \psi_{j-1}^n) - \frac{k}{2h}A^{\frac{1}{2}}(\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n) = 0, \\ \psi_j^{n+1} - \psi_j^n + \frac{k}{2h}A(\phi_{j+1}^n - \phi_{j-1}^n) - \frac{k}{2h}A^{\frac{1}{2}}(\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n) \\ = \kappa(F(\phi_j^{n+1}) - \psi_j^{n+1}).\end{aligned} \tag{4.1}$$

The definition of $\bar{u}_j^n, P_j^n, \hat{P}_j^n$ is the same as in the stationary case. Also let

$$P_j^n = R_j^n \tilde{P}_j^n, \hat{u}_j^n = R_j^n \tilde{u}_j^n.$$

Immediately, we have

$$
\begin{aligned}
(4.2)\qquad \tilde{P}_j^{n+1} - \tilde{P}_j^n + \frac{\kappa}{1+\kappa}\tilde{P}_j^n - \frac{\mu^2}{4(1+\kappa)}(\tilde{u}_{j+2}^n - 2\tilde{u}_j^n + \tilde{u}_{j-2}^n) \\
- \frac{\mu}{2(1+\kappa)}(\tilde{P}_{j+1}^n - 2\tilde{P}_j^n + \tilde{P}_{j-1}^n) \\
+ \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_j^{n+1}(\tilde{u}_j^{n+1} - \tilde{u}_{j-1}^{n+1}) \\
+ \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_{j+1}^{n+1}(\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1}) \\
= \frac{1}{1+\kappa}(L_j^{n+1} - L_j^n)R_j^n\tilde{P}_j^n \\
+ \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_j^{n+1}(L_{j+1}^{n+1} - L_{j-1}^{n+1})R_j^{n+1}\tilde{u}_j^{n+1} + e_j^n,
\end{aligned}
$$

where

$$
\begin{aligned}
(4.3)\qquad e_j^n =& \frac{\lambda\kappa}{2(1+\kappa)}\Delta_+\Lambda_j^{n+1}\Delta_+L_j^{n+1}R_j^{n+1}\tilde{u}_j^{n+1} \\
&+ \frac{\mu}{2(1+\kappa)}L_j^n\Delta_+R_j^n\tilde{P}_{j+1}^n \\
&- \frac{\lambda\kappa}{2(1+\kappa)}L_j^{n+1}\Delta_+R_j^{n+1}\Lambda_{j+1}^{n+1}\Delta_+\tilde{u}_j^{n+1} \\
&+ \frac{\lambda\kappa}{2(1+\kappa)}L_j^{n+1}\Delta_+R_j^{n+1}\Lambda_{j+1}^{n+1}\Delta_+L_j^{n+1}R_j^{n+1}\tilde{u}_j^{n+1} \\
&- \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_j^{n+1}\Delta_-L_j^{n+1}R_{j-1}^{n+1}\Delta_-\tilde{u}_j^{n+1} \\
&- \frac{\lambda\kappa}{2(1+\kappa)}\Lambda_j^{n+1}\Delta_-L_j^{n+1}\Delta_-R_j^{n+1}\tilde{u}_j^{n+1} \\
&+ \frac{\mu^2}{4(1+\kappa)}\Delta^+L_j^n(R_{j+2}^n\tilde{u}_{j+2}^n - 2R_j^n\tilde{u}_j^n + R_{j-2}^n\tilde{u}_{j-2}^n) \\
&+ \frac{\mu^2}{4(1+\kappa)}L_j^n(R_{j+2}^n - R_j^n)\tilde{u}_{j+2}^n \\
&+ \frac{\mu^2}{4(1+\kappa)}L_j^n(R_{j-2}^n - R_j^n)\tilde{u}_{j-2}^n \\
&+ \frac{\mu}{2(1+\kappa)}\Delta^+L_j^n(R_{j+1}^n\tilde{P}_{j+1}^n - 2R_j^n\tilde{P}_j^n + R_{j-1}^n\tilde{P}_{j-1}^n) \\
&- \frac{\mu}{2(1+\kappa)}L_j^n\Delta_-R_j^n\tilde{P}_{j-1}^n \\
&- \frac{\kappa\lambda}{2(1+\kappa)}L_j^{n+1}[\theta_j^{n+1} + \theta_{j+1}^{n+1}].
\end{aligned}
$$

Note that

$$
(4.4)\qquad P_j^n = R_j^{n+1}\tilde{u}_j^{n+1} - R_j^n\tilde{u}_j^n - \frac{\lambda}{2}A^{\frac{1}{2}}(R_{j+1}^n\tilde{u}_{j+1}^n - 2R_j^n\tilde{u}_j^n + R_{j-1}^n\tilde{u}_{j-1}^n).
$$

Next, assume $2\tilde{P}_j^n \times$ (4.2). Then performing summation by parts leads to

$$
\begin{aligned}
(4.5)\qquad & \sum_j |\tilde{P}_j^{n+1}|^2 - \sum_j |\tilde{P}_j^n|^2 - \sum_j |\tilde{P}_j^{n+1} - \tilde{P}_j^n|^2 + \frac{2\kappa}{1+\kappa}\sum_j |\tilde{P}_j^n|^2 \\
& + \frac{\mu}{1+\kappa}\sum_j |\tilde{P}_{j+1}^n - \tilde{P}_j^n|^2 \\
& + \sum_j \frac{\mu^2}{2(1+\kappa)}(\tilde{P}_{j+1}^n - \tilde{P}_{j-1}^n)N_j^n \\
& + \sum_j \frac{\lambda\kappa}{1+\kappa}\tilde{P}_j^n(\Lambda_j^{n+1} - \Lambda_{j+1}^{n+1})\tilde{P}_j^n \\
& + \frac{\lambda\kappa}{1+\kappa}\sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n)\Lambda_{j+1}^{n+1}[\Delta_0 L_j^{n+1}\tilde{P}_{j+1}^n - \Delta^+ L_j^n \tilde{P}_j^n] \\
& + \frac{\lambda\kappa}{1+\kappa}(1-\mu)\sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n)\Lambda_{j+1}^{n+1}(L_j^{n+1}R_j^n\tilde{u}_j^n - L_{j+1}^{n+1}R_{j+1}^n\tilde{u}_{j+1}^n) \\
& + \frac{\lambda\kappa\mu}{2(1+\kappa)}\sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n)\Lambda_{j+1}^{n+1}(L_j^{n+1}R_{j+1}^n\tilde{u}_{j+1}^n - L_{j+1}^{n+1}R_{j+2}^n\tilde{u}_{j+2}^n) \\
& + \frac{\lambda\kappa\mu}{2(1+\kappa)}\sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n)\Lambda_{j+1}^{n+1}(L_j^{n+1}R_{j-1}^n\tilde{u}_{j-1}^n - L_{j+1}^{n+1}R_j^n\tilde{u}_j^n) \\
= & \sum_j \frac{2}{1+\kappa}\tilde{P}_j^n\Delta^+ L_j^n R_j^n \tilde{P}_j^n \\
& + \sum_j \frac{\lambda\kappa}{1+\kappa}\tilde{P}_j^n\Lambda_j^{n+1}\Delta_0 L_j^{n+1}R_j^{n+1}\tilde{u}_j^{n+1} \\
& + \sum_j 2\tilde{P}_j^n e_j^n.
\end{aligned}
$$

The key part to deriving the above equation is

$$
I = \sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde{P}_{j+1}^n\Lambda_{j+1}^{n+1} + \tilde{P}_j^n\Lambda_{j+1}^{n+1})(\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1}).
$$

Because

$$
\begin{aligned}
\tilde{u}_{j+1}^{n+1} - \tilde{u}_j^{n+1} = & \tilde{P}_{j+1}^n - \tilde{P}_j^n + \Delta^+ L_{j+1}^n\tilde{P}_{j+1}^n - \Delta^+ L_j^n\tilde{P}_j^n \\
& + \frac{\mu}{2}(L_j^{n+1}R_{j+1}^n\tilde{u}_{j+1}^n - L_{j+1}^{n+1}R_{j+2}^n\tilde{u}_{j+2}^n) \\
& + (L_j^{n+1}R_j^n\tilde{u}_j^n - L_{j+1}^{n+1}R_{j+1}^n\tilde{u}_{j+1}^n)(1-\mu) \\
& + \frac{\mu}{2}(L_j^{n+1}R_{j-1}^n\tilde{u}_{j-1}^n - L_{j+1}^{n+1}R_j^n\tilde{u}_j^n),
\end{aligned}
$$

Then

$$
\begin{aligned}
I = &\sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde P^n_{j+1}+\tilde P^n_j)\Lambda^{n+1}_{j+1}\Delta_+\tilde P^n_j\\
&+\sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde P^n_{j+1}+\tilde P^n_j)\Lambda^{n+1}_{j+1}[\Delta^+L^n_{j+1}\tilde P^n_{j+1}-\Delta^+L^n_j\tilde P^n_j]\\
&+\sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde P^n_{j+1}+\tilde P^n_j)\Lambda^{n+1}_{j+1}(L^{n+1}_jR^n_j\tilde u^n_j-L^{n+1}_{j+1}R^n_{j+1}\tilde u^n_{j+1})(1-\mu)\\
&+\sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde P^n_{j+1}+\tilde P^n_j)\Lambda^{n+1}_{j+1}(L^{n+1}_jR^n_{j+1}\tilde u^n_{j+1}-L^{n+1}_{j+1}R^n_{j+2}\tilde u^n_{j+2})\frac{\mu}{2}\\
&+\sum_j \frac{\lambda\kappa}{1+\kappa}(\tilde P^n_{j+1}+\tilde P^n_j)\Lambda^{n+1}_{j+1}(L^{n+1}_jR^n_{j-1}\tilde u^n_{j-1}-L^{n+1}_{j+1}R^n_j\tilde u^n_j)\frac{\mu}{2}.
\end{aligned}
$$

Next, $2\tilde u^{n+1}_j\times$ (4.2) can be rewritten using summation by parts:

$$
\begin{aligned}
&2\sum_j \tilde u^{n+1}_j\tilde P^{n+1}_j-2\sum_j \tilde u^n_j\tilde P^n_j+2\sum_j(\tilde u^n_j-\tilde u^{n+1}_j)\tilde P^n_j\\
&+\sum_j\frac{\kappa}{1+\kappa}|\tilde u^{n+1}_j|^2-\sum_j\frac{\kappa}{1+\kappa}|\tilde u^n_j|^2\\
&+\sum_j\frac{\kappa}{1+\kappa}|\Delta^+\tilde u^n_j|^2-\sum_j\frac{2\kappa}{1+\kappa}\tilde u^{n+1}_j\Delta^+L^n_jR^{n+1}_j\tilde u^{n+1}_j\\
&-\frac{\kappa\mu}{1+\kappa}\sum_j\Delta^+\tilde u^n_jL^n_j(R^n_{j+1}\tilde u^n_{j+1}-2R^n_j\tilde u^n_j+R^n_{j-1}\tilde u^n_{j-1})\\
&-\frac{\kappa\mu}{1+\kappa}\sum_j\tilde u^n_jL^n_j\Delta_+R^n_j\Delta_+\tilde u^n_j\\
&-\frac{\kappa\mu}{1+\kappa}\sum_j\tilde u^n_jL^n_j\Delta_-R^n_j\Delta_-\tilde u^n_j\\
&-\frac{\kappa\mu}{1+\kappa}\sum_j\tilde u^n_jL^n_j\Delta R^n_j\tilde u^n_j\\
&+\frac{\kappa\mu}{1+\kappa}\sum_j|\tilde u^n_{j+1}-\tilde u^n_j|^2\\
&-\sum_j\frac{\mu^2}{2(1+\kappa)}\Delta^+\tilde u^n_j(N^n_{j+1}-N^n_{j-1})+\frac{\mu^2}{2(1+\kappa)}\sum_j|N^n_j|^2\\
&-\sum_j\frac{\mu}{1+\kappa}\Delta^+\tilde u^n_j\Delta\tilde P^n_j+\sum_j\frac{\mu}{1+\kappa}\Delta_+\tilde u^n_j\Delta_+\tilde P^n_j\\
&-\sum_j\frac{\lambda\kappa}{1+\kappa}\tilde u^{n+1}_j\Delta_+\Lambda^{n+1}_j\tilde u^{n+1}_j-\sum_j\frac{2}{1+\kappa}\tilde u^{n+1}_j\Delta^+L^n_jR^n_j\tilde P^n_j\\
&=\frac{\lambda\kappa}{1+\kappa}\sum_j\tilde u^{n+1}_j\Lambda^{n+1}_j\Delta_0L^{n+1}_jR^{n+1}_j\tilde u^{n+1}_j+\sum_j2\tilde u^{n+1}_je^n_j.
\end{aligned}
\tag{4.6}
$$

The only term which we should be careful of is

$$\begin{aligned}
J \equiv & \sum_j \frac{2\kappa}{1+\kappa}\tilde{P}_j^n \tilde{u}_j^{n+1} \\
= & \sum_j \frac{2\kappa}{1+\kappa}\tilde{u}_j^{n+1}(L_j^n R_j^{n+1}\tilde{u}_j^{n+1} - \tilde{u}_j^n \\
& - \frac{\mu}{2}L_j^n(R_{j+1}^n \tilde{u}_{j+1}^n - 2R_j^n\tilde{u}_j^n + R_{j-1}^n \tilde{u}_{j-1}^n)) \\
= & \sum_j \frac{\kappa}{1+\kappa}|\tilde{u}_j^{n+1}|^2 - \sum_j \frac{\kappa}{1+\kappa}|\tilde{u}_j^n|^2 + \sum_j \frac{\kappa}{1+\kappa}|\Delta^+\tilde{u}_j^n|^2 \\
& - \sum_j \frac{2\kappa}{1+\kappa}\tilde{u}_j^{n+1}\Delta^+ L_j^n R_j^{n+1}\tilde{u}_j^{n+1} \\
& - \frac{\kappa\mu}{1+\kappa}\sum_j \tilde{u}_j^n L_j^n \Delta_+ R_j^n \Delta_+ \tilde{u}_j^n \\
& - \frac{\kappa\mu}{1+\kappa}\sum_j \Delta^+\tilde{u}_j^n L_j^n (R_{j+1}^n \tilde{u}_{j+1}^n - 2R_j^n\tilde{u}_j^n + R_{j-1}^n\tilde{u}_{j-1}^n) \\
& + \frac{\kappa\mu}{1+\kappa}\sum_j |\Delta_+\tilde{u}_j^n|^2 - \frac{\kappa\mu}{1+\kappa}\sum_j \tilde{u}_j^n L_j^n \Delta_- R_j^n \Delta_- \tilde{u}_j^n \\
& - \frac{\kappa\mu}{1+\kappa}\sum_j \tilde{u}_j^n L_j^n \Delta R_j^n \tilde{u}_j^n.
\end{aligned}$$

It follows from (4.5) + $\tau \times$ (4.6) that

$$E(n+1) - E(n) + L(n) = R(n), \tag{4.7}$$

where

$$\begin{aligned}
E(n+1) = & \sum_j [|\tilde{P}_j^{n+1}|^2 + 2\tau\tilde{u}_j^{n+1}\tilde{P}_j^{n+1} + \frac{\kappa\tau}{1+\kappa}|\tilde{u}_j^{n+1}|^2], \\
E(n) = & \sum_j [|\tilde{P}_j^n|^2 + 2\tau\tilde{u}_j^n\tilde{P}_j^n + \frac{\kappa\tau}{1+\kappa}|\tilde{u}_j^n|^2], \\
L(n) = & \frac{2\kappa}{1+\kappa}\sum_j |\tilde{P}_j^n|^2 + \frac{\mu}{1+\kappa}\sum_j |\tilde{P}_{j+1}^n - \tilde{P}_j^n|^2 \\
& + \frac{\lambda\kappa}{1+\kappa}\sum_j (\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})|\tilde{P}_{k,j}^n|^2 \\
& + \frac{\tau\kappa}{1+\kappa}\sum_j |\tilde{u}_j^{n+1} - \tilde{u}_j^n|^2 \\
& + \frac{\tau\kappa\mu}{1+\kappa}\sum_j |M_j^n|^2 + \frac{\tau\mu^2}{2(1+\kappa)}\sum_j |N_j^n|^2 \\
& + \frac{\tau\lambda\kappa}{1+\kappa}\sum_j (\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})|\tilde{u}_{k,j}^{n+1}|^2,
\end{aligned}$$

and

$$
\begin{aligned}
R(n) =& \sum_j |\tilde{P}_j^{n+1} - \tilde{P}_j^n|^2 - \sum_j \frac{\mu^2}{2(1+\kappa)} \Delta_0 \tilde{P}_j^n N_j^n \\
&- \frac{\lambda\kappa}{1+\kappa} \sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n) \Lambda_{j+1}^{n+1} [(L_{j+1}^{n+1} - L_{j-1}^{n+1}) \tilde{P}_{j+1}^n - \Delta^+ L_j^n \tilde{P}_j^n] \\
&- \frac{\lambda\kappa}{1+\kappa} (1-\mu) \sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n) \Lambda_{j+1}^{n+1} (L_j^{n+1} R_j^n \tilde{u}_j^n - L_{j+1}^{n+1} R_{j+1}^n \tilde{u}_{j+1}^n) \\
&- \frac{\lambda\kappa\mu}{2(1+\kappa)} \sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n) \Lambda_{j+1}^{n+1} (L_j^{n+1} R_{j+1}^n \tilde{u}_{j+1}^n - L_{j+1}^{n+1} R_{j+2}^n \tilde{u}_{j+2}^n) \\
&- \frac{\lambda\kappa\mu}{2(1+\kappa)} \sum_j (\tilde{P}_{j+1}^n + \tilde{P}_j^n) \Lambda_{j+1}^{n+1} (L_j^{n+1} R_{j-1}^n \tilde{u}_{j-1}^n - L_{j+1}^{n+1} R_j^n \tilde{u}_j^n) \\
&+ \sum_j \frac{2}{1+\kappa} \tilde{P}_j^n \Delta^+ L_j^n R_j^n \tilde{P}_j^n + \frac{\lambda\kappa}{1+\kappa} \sum_j \tilde{P}_j^n \Lambda_j^{n+1} \Delta_0 L_j^{n+1} R_j^{n+1} \tilde{u}_j^{n+1} \\
&+ 2\tau \sum_j \Delta^+ \tilde{u}_j^n \tilde{P}_j^n + 2 \sum_j \frac{\tau\kappa}{1+\kappa} \tilde{u}_j^{n+1} \Delta^+ L_j^n R_j^{n+1} \tilde{u}_j^{n+1} \\
&+ \frac{\tau\kappa\mu}{1+\kappa} \sum_j \Delta^+ \tilde{u}_j^n L_j^n (R_{j+1}^n \tilde{u}_{j+1}^n - 2R_j^n \tilde{u}_j^n + R_{j-1}^n \tilde{u}_{j-1}^n) \\
&+ \frac{\tau\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n L_j^n \Delta_+ R_j^n \Delta_+ \tilde{u}_j^n + \frac{\tau\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n L_j^n \Delta_- R_j^n \Delta_- \tilde{u}_j^n \\
&+ \frac{\tau\kappa\mu}{1+\kappa} \sum_j \tilde{u}_j^n L_j^n \Delta R_j^n \tilde{u}_j^n + \frac{\tau\mu^2}{2(1+\kappa)} \sum_j \Delta^+ \tilde{u}_j^n \Delta_0 N_j^n \\
&+ \frac{\tau\mu}{1+\kappa} \sum_j \Delta^+ \tilde{u}_j^n \Delta P_j^n - \frac{\tau\mu}{1+\kappa} \sum_j \Delta_+ \tilde{u}_j^n \Delta_+ \tilde{P}_j^n \\
&+ \frac{2\tau}{1+\kappa} \sum_j \tilde{u}_j^{n+1} \Delta^+ L_j^n R_j^n \tilde{P}_j^n + \frac{\tau\lambda\kappa}{1+\kappa} \sum_j \tilde{u}_j^{n+1} \Lambda_j^{n+1} \Delta_0 L_j^{n+1} R_j^{n+1} \tilde{u}_j^{n+1} \\
&- \frac{\lambda\kappa}{1+\kappa} \sum_{\mu \neq k} \tilde{P}_{\mu,j} (\Lambda_{\mu,j}^{n+1} - \Lambda_{\mu,j+1}^{n+1}) \tilde{P}_{\mu,j}^n \\
&- \frac{\tau\lambda\kappa}{1+\kappa} \sum_{\mu \neq k} \tilde{u}_{\mu,j}^{n+1} (\Lambda_{\mu,j}^{n+1} - \Lambda_{\mu,j+1}^{n+1}) \tilde{u}_{\mu,j}^{n+1} + \sum_j 2(\tau \tilde{u}_j^{n+1} + \tilde{P}_j^n) e_j^n.
\end{aligned}
$$

Label the terms in $R(n)$ as $R_1, \ldots, R_{23}$. Actually, we only need to handle the terms which have $\tilde{u}_j^{n+1}$, among which the most difficult terms are R_{10} and R_{19}. We have

$$
\begin{aligned}
L_j^{n+1} - L_j^n + \frac{\lambda s}{2} (L_{j+1}^n - L_{j-1}^n) \\
= (1 + O(\epsilon)) \nabla L(\phi_j^n) \times (\phi_j^{n+1} - \phi_j^n + \frac{\lambda s}{2} (\phi_{j+1}^n - \phi_{j-1}^n))
\end{aligned} \tag{4.8}
$$

and the identity

$$\begin{aligned} &L_j^{n+1} - L_j^n + \frac{\lambda}{2}\Lambda_j^n(L_{j+1}^n - L_{j-1}^n) \\ &\quad = L_j^{n+1} - L_j^n + \frac{\lambda s}{2}(L_{j+1}^n - L_{j-1}^n) + \frac{\lambda}{2}(\Lambda_j^n - s)(L_{j+1}^n - L_{j-1}^n). \end{aligned}$$

According to the discrete shock profile equation (4.1), we have

$$\frac{\lambda}{2}|\psi_{j+1}^n - \psi_{j-1}^n| \le c(\lambda_{k,j}^n - \lambda_{k,j+1}^n).$$

It is obvious that

$$\begin{aligned} R_{10} + R_{19} = &\frac{2\tau\kappa}{1+\kappa}\sum_j \tilde{u}_j^{n+1}[L_j^{n+2} - L_j^{n+1} + \frac{\lambda}{2}\Lambda_j^{n+1}(L_{j+1}^{n+1} - L_{j-1}^{n+1})]R_j^{n+1}\tilde{u}_j^{n+1} \\ &- \frac{\tau\lambda\kappa}{1+\kappa}\sum_j \tilde{u}_j^{n+1}(L_j^{n+2} - 2L_j^{n+1} + L_j^n)R_j^{n+1}\tilde{u}_j^{n+1}. \end{aligned}$$

Because

$$\begin{aligned} &\phi_j^{n+2} - \phi_j^{n+1} + \frac{\lambda}{2}(F(\phi_{j+1}^{n+1}) - F(\phi_{j-1}^{n+1})) - \frac{\lambda}{2}A^{\frac{1}{2}}(\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}) \\ =&\frac{\lambda}{2\kappa}[\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1} - (\psi_{j+1}^n - \psi_{j-1}^n) + \frac{\lambda}{2}A(\phi_{j+2}^n - 2\phi_j^n + \phi_{j-2}^n) \\ &- \frac{\lambda}{2}A^{\frac{1}{2}}(\psi_{j+2}^n - \psi_j^n - 2(\psi_{j+1}^n - \psi_{j-1}^n) + \psi_j^n - \psi_{j-2}^n)] \end{aligned}$$

and by Theorem 2.3, we have

$$\begin{aligned} &\phi_j^{n+2} - \phi_j^{n+1} + \frac{\lambda}{2}s(\phi_{j+1}^{n+1} - \phi_{j-1}^{n+1}) \\ &\qquad \le O(\epsilon)(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1}) + \frac{O(1)}{\kappa}(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1}). \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} R_{10} + R_{19} \le &\sum_j \frac{\lambda\tau\kappa}{8(1+\kappa)}(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})|\tilde{u}_{k,j}^{n+1}|^2 \\ &+ O(1)\frac{\tau\kappa}{1+\kappa}\sum_{j,\mu\neq k}(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})|\tilde{u}_{\mu,j}^{n+1}|^2 \end{aligned} \tag{4.9}$$

provided that κ is suitably large, which is a reasonable assumption. Set

$$\bar{M}(n_1) = \sup_{n\le n_1}(\sum_j |\tilde{u}_j^n|^2 + \sum_j |\tilde{P}_j^n|^2)^{\frac{1}{2}},$$

and assume a priori that $\bar{M}(n_1)$ is suitably small. Then clearly we have

$$\sup_{(j,n),n\le n_1} |\tilde{u}_j^n| \le \bar{M}(n_1).$$

Under this prior assumption, the other terms can be estimated as in the stationary case. The details are skipped. Now we reach the following inequality.

$$\begin{aligned} E(n+1) - E(n) & \\ &+ \frac{\lambda\tau\kappa}{4(1+\kappa)}\sum_j(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})|\tilde{u}_{k,j}^{n+1}|^2 \\ &+ \frac{\lambda\kappa}{4(1+\kappa)}\sum_j(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})|\tilde{P}_{k,j}^{n}|^2 \\ &+ O(1)[\sum_j |\tilde{P}_j^n|^2 + \sum_j |\tilde{P}_{j+1}^n - \tilde{P}_j^n|^2 \\ &\qquad + \sum_j |\tilde{u}_j^{n+1} - \tilde{u}_j^n|^2 + \sum_j |M_j^n|^2 + \sum_j |N_j^n|^2] \\ &\le O(1)\sum_j(\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1})\sum_{\mu\ne k}|\tilde{u}_{\mu,j}^{n+1}|^2. \end{aligned} \tag{4.10}$$

The waves in the transversal directions are bounded by the last term on the right-hand side of the above equation. We should estimate the transversal term. Here we will use the idea of vertical estimate, and the remaining part of this section is the estimate about this term.

It was mentioned in [16] that the estimates on transversal waves are technically quite involved due to the wave interactions and the fact that our shocks are non-stationary. The idea of vertical estimate is to relate a sum along time-like discrete segments to a sum over all the grid points in a strip of the space-time plane.

Suppose $\mu > k$. It is the same with another case. For any fixed index j_0, we define the k-th time-like grid line originated at $(j_0, 0)$ to be

$$J(j_0) = \{(n, j_n)|j_n = j_0 + (l-1)p \quad \text{for} \quad (l-1)|q| \le n < l|q|, l = 1, 2, \ldots\}.$$

For any integer n, $0 \le n \le Nq - 1$, because the terms which have $\tilde{P}$ can be estimated easily, we multiply the μ-th equation in (4.2) by $2\tilde{u}_{\mu,j}^{n+1}$ and sum by parts over j from $-\infty$ to j_n and n from 1 to $Nq - 1$. Under the assumption of the prior

estimate, we obtain the inequality

(4.11)
$$\begin{aligned}
&\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}\frac{\kappa}{1+\kappa}(\tilde{u}_{\mu,j}^{n+1})^2-\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}\frac{\kappa}{1+\kappa}(\tilde{u}_{\mu,j}^{n})^2+\sum_{n=0}^{Nq-1}\frac{\kappa}{1+\kappa}\frac{p}{q}(\tilde{u}_{\mu,j_n}^{n+1})^2\\
&\quad+\sum_{n=0}^{Nq-1}\frac{\kappa\lambda}{1+\kappa}(\lambda_{\mu,j_n}^{n+1}-s)(\tilde{u}_{\mu,j_n}^{n+1})^2+\frac{c_1\kappa\mu}{1+\kappa}\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}(\tilde{u}_{\mu,j}^{n}-\tilde{u}_{\mu,j-1}^{n})^2\\
&\quad+\frac{c_2\mu^2}{2(1+\kappa)}\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}|N_{\mu,j}^n|^2+\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}\frac{c_0\kappa}{1+\kappa}(\tilde{u}_{\mu,j}^{n+1}-\tilde{u}_{\mu,j}^{n})^2\\
&\le-\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}2\tilde{u}_{\mu,j}^{n+1}(\tilde{P}_{\mu,j}^{n+1}-\tilde{P}_{\mu,j}^{n})+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{u}_{j}^{n+1}|^2\\
&\quad+\sum_{n=0}^{Nq-1}\frac{\kappa}{1+\kappa}(\mu-\lambda_{\mu,j_n+1}^{n+1})\tilde{u}_{\mu,j_n}^{n+1}(\tilde{u}_{\mu,j_n+1}^{n+1}-\tilde{u}_{\mu,j_n}^{n+1})+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|M_j^n|^2\\
&\quad+\sum_{n=0}^{Nq-1}\frac{\mu^2}{2(1+\kappa)}\tilde{u}_{\mu,j_n+1}^{n}N_{\mu,j_n}^n-\sum_{n=0}^{Nq-1}\frac{\mu^2}{2(1+\kappa)}\tilde{u}_{\mu,j_n}^{n}N_{\mu,j_n+1}^n\\
&\quad+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}(\tilde{P}_{j+1}^{n}-\tilde{P}_{j}^{n})^2+\sum_{n=0}^{Nq-1}\frac{\mu}{1+\kappa}\tilde{u}_{\mu,j_n+1}^{n}(\tilde{P}_{\mu,j_n+1}^{n}-\tilde{P}_{\mu,j_n}^{n})\\
&\quad+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{+\infty}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{P}_{j}^{n}|^2+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{+\infty}|\Delta^+u_j^n|^2,
\end{aligned}$$

where c_0, c_1, c_2 are positive constants. Let

$$(E_{\mu,j}^n)^2=|\tilde{P}_{\mu,j}^n|^2+2\tau\tilde{u}_{\mu,j}^n\tilde{P}_{\mu,j}^n+\frac{\tau\kappa}{1+\kappa}|\tilde{u}_{\mu,j}^n|^2 \tag{4.12}$$

and $\tau=\frac{\kappa}{4(1+\kappa)}$. We have

$$\begin{aligned}
&\frac{1}{\tau}\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}[(E_{\mu,j}^{n+1})^2-(E_{\mu,j}^{n})^2]+\frac{3\kappa}{4(1+\kappa)}\sum_{n=0}^{Nq-1}\frac{p}{q}|\tilde{u}_{\mu,j_n}^{n+1}|^2\\
&\quad+C_0\sum_{n=0}^{Nq-1}|\tilde{u}_{\mu,j_n}^{n+1}|^2+(\text{the same remaining terms})\\
&\le O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}[|\tilde{P}_{\mu,j}^n|^2+|\tilde{P}_{\mu,j}^{n+1}-\tilde{P}_{\mu,j}^n|^2]+(\text{the same remaining terms}),
\end{aligned} \tag{4.13}$$

where C_0 is a positive constant. Hence, we should estimate

$$\frac{1}{\tau}\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{+\infty}[(E_{\mu,j}^{n+1})^2-(E_{\mu,j}^{n})^2]+\frac{3\kappa}{4(1+\kappa)}\frac{p}{q}\sum_{n=0}^{Nq-1}|\tilde{u}_{\mu,j_n}^{n+1}|^2. \tag{4.14}$$

Immediately, we have

$$\begin{aligned}(4.15)\quad &\frac{1}{\tau}\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{j_n}[(E_{\mu,j}^{n+1})^2-(E_{\mu,j}^{n})^2]+\frac{3\kappa}{4(1+\kappa)}\frac{p}{q}\sum_{n=0}^{Nq-1}|\tilde{u}_{\mu,j_n}^{n+1}|^2\\ &=\frac{1}{\tau}\sum_{j=-\infty}^{j_0+Np}(E_{\mu,j}^{Nq})^2-\frac{1}{\tau}\sum_{j=-\infty}^{j_0-p}(E_{\mu,j}^{0})^2+\frac{3\kappa}{4(1+\kappa)}\frac{1}{q}\sum_{n=0}^{N-1}(p\sum_{\alpha=1}^{q}(\tilde{u}_{\mu,j_{nq}}^{nq+\alpha})^2)\\ &\quad-\frac{1}{q}\sum_{n=0}^{N-1}q\sum_{\beta=0}^{p-1}(E_{\mu,j_{nq}-\beta}^{nq})^2\frac{1}{\tau}\end{aligned}$$

and

$$(4.16)\qquad \frac{1}{\tau}(E_{\mu,j_{nq}-\beta}^{nq})^2>\frac{3\kappa}{4(1+\kappa)}(\tilde{u}_{\mu,j_{nq}-\beta}^{nq})^2,$$

so

$$\begin{aligned}(4.17)\quad &\frac{3\kappa}{4(1+\kappa)}\sum_{n=0}^{Nq-1}\frac{p}{q}|\tilde{u}_{\mu,j_n}^{n+1}|^2-\frac{1}{q}\sum_{n=0}^{N-1}q\sum_{\beta=0}^{p-1}(E_{\mu,j_{nq}-\beta}^{nq})^2\frac{1}{\tau}\\ &\qquad\le\frac{3\kappa}{4(1+\kappa)}[\frac{1}{q}\sum_{n=0}^{N-1}(p\sum_{\alpha=1}^{q}(\tilde{u}_{\mu,j_{nq}}^{nq+\alpha})^2-q\sum_{\beta=0}^{p-1}(\tilde{u}_{\mu,j_{nq}-\beta}^{nq})^2)].\end{aligned}$$

The last term on the right can be estimated as

$$\begin{aligned}&\sum_{n=0}^{N-1}(p\sum_{\alpha=1}^{q}(\tilde{u}_{\mu,j_{nq}}^{nq+\alpha})^2-q\sum_{\beta=0}^{p-1}(\tilde{u}_{\mu,j_{nq}-\beta}^{nq})^2)\\ &\quad=p\sum_{n=0}^{N-1}\sum_{\alpha=1}^{q}(\tilde{u}_{\mu,j_{nq}}^{nq+\alpha}-\tilde{u}_{\mu,j_{nq}}^{nq})^2+2p\sum_{n=0}^{N-1}\sum_{\alpha=1}^{q}(\tilde{u}_{\mu,j_{nq}}^{nq+\alpha}-\tilde{u}_{\mu,j_{nq}}^{nq})\tilde{u}_{\mu,j_{nq}}^{nq}\\ &\quad\quad-q\sum_{n=0}^{N-1}\sum_{\beta=0}^{p-1}(\tilde{u}_{\mu,j_{nq}-\beta}^{nq}-\tilde{u}_{\mu,j_{nq}}^{nq})^2-2q\sum_{n=0}^{N-1}\sum_{\beta=0}^{p-1}(\tilde{u}_{\mu,j_{nq}-\beta}^{nq}-\tilde{u}_{\mu,j_{nq}}^{nq})\tilde{u}_{\mu,j_{nq}}^{nq}\\ &\quad\le\frac{O(1)}{\delta}\sum_{j,n}(\tilde{u}_{\mu,j}^{n}-\tilde{u}_{\mu,j+1}^{n})^2+\frac{O(1)}{\delta}\sum_{j,n}(\tilde{u}_{\mu,j}^{n+1}-\tilde{u}_{\mu,j}^{n})^2+\delta\sum_{n=0}^{N-1}(\tilde{u}_{\mu,j_{nq}}^{nq})^2\\ &\quad\le\frac{O(1)}{\delta}\sum_{j,n}(\tilde{u}_{\mu,j}^{n}-\tilde{u}_{\mu,j+1}^{n})^2+\frac{O(1)}{\delta}\sum_{j,n}(\tilde{u}_{\mu,j}^{n+1}-\tilde{u}_{\mu,j}^{n})^2\\ &\quad\quad+\delta\sum_{n=0}^{N-1}\sum_{\alpha=1}^{q}(\tilde{u}_{\mu,j_{nq}}^{nq+\alpha})^2+\delta|u_{\mu,j_0}^{0}|^2\\ &\quad=\frac{O(1)}{\delta}\sum_{j,n}(\tilde{u}_{\mu,j}^{n}-\tilde{u}_{\mu,j+1}^{n})^2+\frac{O(1)}{\delta}\sum_{j,n}(\tilde{u}_{\mu,j}^{n+1}-\tilde{u}_{\mu,j}^{n})^2\\ &\quad\quad+\delta\sum_{n=0}^{Nq-1}(\tilde{u}_{\mu,j_n}^{n+1})^2+\delta|u_{\mu,j_0}^{0}|^2,\end{aligned}$$

so choosing suitably small δ, we have

$$
\begin{aligned}
\sum_{n=0}^{Nq-1}(\tilde{u}_{\mu,j_n}^{n+1})^2 \le\ & O(1)E(0)\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{u}_j^{n+1}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{u}_{j+1}^{n}-\tilde{u}_j^{n}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{u}_j^{n+1}-\tilde{u}_j^{n}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{P}_{j+1}^{n}-\tilde{P}_j^{n}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\tilde{u}_{\mu,j_n+1}^{n}(\tilde{P}_{\mu,j_n+1}^{n}-\tilde{P}_{\mu,j_n}^{n})\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}[|\tilde{P}_j^{n+1}-\tilde{P}_j^{n}|^2+|\tilde{P}_j^{n}|^2]\\
&+O(1)\sum_{n=0}^{Nq-1}\tilde{u}_{\mu,j_n+1}^{n}N_{\mu,j_n}^{n}-O(1)\sum_{n=0}^{Nq-1}\tilde{u}_{\mu,j_n}^{n}N_{\mu,j_n+1}^{n}\\
&+O(1)\sum_{n=0}^{Nq-1}\tilde{u}_{\mu,j_n}^{n+1}(\tilde{u}_{\mu,j_n+1}^{n+1}-\tilde{u}_{\mu,j_n}^{n+1}).
\end{aligned}
\tag{4.18}
$$

Apply Cauchy's inequality to the above equation and use previous estimates

$$
\begin{aligned}
\sum_{n=0}^{Nq-1}(\tilde{u}_{\mu,j_n}^{n+1})^2 \le\ & O(1)E(0)\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{u}_j^{n+1}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{u}_{j+1}^{n}-\tilde{u}_j^{n}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{u}_j^{n+1}-\tilde{u}_j^{n}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{P}_{j+1}^{n}-\tilde{P}_j^{n}|^2\\
&+O(1)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{P}_j^{n}|^2.
\end{aligned}
$$

Multiply both sides by $\lambda^0_{k,j_0}-\lambda^0_{k,j_0+1}$ and sum over j_0 from $-\infty$ to ∞. Consequently one will obtain

$$\begin{aligned}
&\sum_{j_0}(\lambda^0_{k,j_0}-\lambda^0_{k,j_0+1})\sum_{n=0}^{Nq-1}(\tilde{u}^n_{\mu,j_n})^2\\
&\quad\le O(\epsilon)E(0)+O(\epsilon)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}(\lambda^{n+1}_{k,j}-\lambda^{n+1}_{k,j+1})|\tilde{u}^{n+1}_j|^2\\
&\qquad+O(\epsilon)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{u}^n_{j+1}-\tilde{u}^n_j|^2\\
&\qquad+O(\epsilon)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{u}^{n+1}_j-\tilde{u}^n_j|^2\\
&\qquad+O(\epsilon)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{P}^n_{j+1}-\tilde{P}^n_j|^2\\
&\qquad+O(\epsilon)\sum_{n=0}^{Nq-1}\sum_{j=-\infty}^{\infty}|\tilde{P}^n_j|^2.
\end{aligned}$$

According to the properties of the discrete shock profile,

$$\begin{aligned}
&\sum_{n=0}^{Nq-1}\sum_j(\lambda^{n+1}_{k,j}-\lambda^{n+1}_{k,j+1})(\tilde{u}^{n+1}_{\mu,j})^2\\
&\quad=\sum_{j_0}\sum_{n=0}^{Nq-1}(\lambda^{n+1}_{k,j_n}-\lambda^{n+1}_{k,j_n+1})(\tilde{u}^{n+1}_{\mu,j_n})^2\\
&\quad\le(1+O(\epsilon))\sum_{j_0}(\lambda^0_{k,j_0}-\lambda^0_{k,j_0+1})\sum_{n=0}^{Nq-1}(\tilde{u}^{n+1}_{\mu,j_n})^2.
\end{aligned}$$

Combining this estimate, we have

$$\begin{aligned}
&\sum_{j,n}(\lambda^{n+1}_{k,j}-\lambda^{n+1}_{k,j+1})(\tilde{u}^{n+1}_{\mu,j})^2\\
&\quad\le\ O(\epsilon)E(0)+O(\epsilon)\sum_{j,n}(\lambda^{n+1}_{k,j}-\lambda^{n+1}_{k,j+1})|\tilde{u}^{n+1}_j|^2\\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{u}^n_{j+1}-\tilde{u}^n_j|^2\\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{u}^{n+1}_j-\tilde{u}^n_j|^2\\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{P}^n_{j+1}-\tilde{P}^n_j|^2\\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{P}^n_j|^2.
\end{aligned}\tag{4.19}$$

Similarly for $\mu < k$; therefore, summing over μ yields

$$\begin{aligned}
&\sum_{j,n}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})\sum_{\mu\neq k}(\tilde{u}_{\mu,j}^{n+1})^2 \\
&\quad\le O(\epsilon)E(0)+O(\epsilon)\sum_{j,n}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{u}_j^{n+1}|^2 \\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{u}_{j+1}^n-\tilde{u}_j^n|^2+O(\epsilon)\sum_{j,n}|\tilde{u}_j^{n+1}-\tilde{u}_j^n|^2 \\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{P}_{j+1}^n-\tilde{P}_j^n|^2 \\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{P}_j^n|^2.
\end{aligned} \tag{4.20}$$

It follows that

$$\begin{aligned}
&\sum_{j,n}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})\sum_{\mu\neq k}(\tilde{u}_{\mu,j}^{n+1})^2 \\
&\quad\le O(\epsilon)E(0)+O(\epsilon)\sum_{j,n}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{u}_{k,j}^{n+1}|^2 \\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{u}_{j+1}^n-\tilde{u}_j^n|^2+O(\epsilon)\sum_{j,n}|\tilde{u}_j^{n+1}-\tilde{u}_j^n|^2 \\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{P}_{j+1}^n-\tilde{P}_j^n|^2 \\
&\qquad+O(\epsilon)\sum_{j,n}|\tilde{P}_j^n|^2.
\end{aligned}$$

After this estimate, combining (4.10), we prove the following basic estimate.

Proposition 4.1 (A priori estimate)**.** *Let $(\tilde{u},\tilde{P})$ be the solution of (4.2) for $n_2 \le n_1$. Then there exist positive constants C, c_0 independent of n_1 and ϵ such that*

$$\begin{aligned}
&E(n_2)+\sum_{j,n\le n_2}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{u}_{k,j}^{n+1}|^2 \\
&\qquad+\sum_{j,n\le n_2}(\lambda_{k,j}^{n+1}-\lambda_{k,j+1}^{n+1})|\tilde{P}_{k,j}^n|^2 \\
&\qquad+c_0\Big[\sum_{j,n\le n_2}|\tilde{P}_j^n|^2+\sum_{j,n\le n_2}|\tilde{P}_{j+1}^n-\tilde{P}_j^n|^2 \\
&\qquad\qquad+\sum_{j,n\le n_2}|\tilde{u}_j^{n+1}-\tilde{u}_j^n|^2 \\
&\qquad\qquad+\sum_{j,n\le n_2}|M_j^n|^2+\sum_{j,n\le n_2}|N_j^n|^2\Big] \\
&\quad\le CE(0)
\end{aligned} \tag{4.21}$$

for all $n_2 \le n_1$, provided that ϵ, λ and $\bar{M}(n_1)$ are suitably small and κ is suitably large.

By standard continuity argument, the following proposition exists.

Proposition 4.2. *Assume that* $\epsilon, \frac{1}{\kappa}$ *and* $\bar{M}(0)$ *are suitably small. Then problem* (4.2) *has a unique global solution* $(\tilde{u}_j^n, \tilde{P}_j^n)$ *satisfying*

$$\sup_n E(n) + \sum_{j,n} (\lambda_{k,j}^{n+1} - \lambda_{k,j+1}^{n+1}) |\tilde{u}_{k,j}^{n+1}|^2 + \sum_{j,n} |\tilde{P}_j^n|^2 + \sum_{j,n} |\tilde{P}_{j+1}^n - \tilde{P}_j^n|^2 + \sum_{j,n} |\tilde{u}_j^{n+1} - \tilde{u}_j^n|^2 + \sum_{j,n} |M_j^n|^2 + \sum_{j,n} |N_j^n|^2 \le C\bar{M}(0)^2$$

for any $n \ge 0$, *where* C *is a positive constant independent of* n *and* j.

With this estimate, as before, we can get the L^2 estimate of $\bar{v}_j^n$, which is Theorem 1.2.

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