MATHEMATICS OF COMPUTATION
Volume 73, Number 248, Pages 1623–1653
S 0025-5718(04)01650-3
Article electronically published on March 23, 2004

LOCALIZED POINTWISE ERROR ESTIMATES FOR MIXED FINITE ELEMENT METHODS

ALAN DEMLOW

ABSTRACT. In this paper we give weighted, or localized, pointwise error estimates which are valid for two different mixed finite element methods for a general second-order linear elliptic problem and for general choices of mixed elements for simplicial meshes. These estimates, similar in spirit to those recently proved by Schatz for the basic Galerkin finite element method for elliptic problems, show that the dependence of the pointwise errors in both the scalar and vector variables on the derivative of the solution is mostly local in character or conversely that the global dependence of the pointwise errors is weak. This localization is more pronounced for higher order elements. Our estimates indicate that localization occurs except when the lowest order Brezzi-Douglas-Marini elements are used, and we provide computational examples showing that the error is indeed not localized when these elements are employed.

1. INTRODUCTION

We consider mixed finite element methods for approximating solutions to a general second-order linear elliptic scalar problem for $u(x)$, where $x \in \Omega$, a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Written in "divergence" form, the problem is to find u satisfying

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega,$$

or, with the matrix $A = [a_{ij}]$ and the vector $\vec{b} = [b_i]$,

$$-\operatorname{div}(A\nabla u) + \vec{b}\nabla u + cu = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{1.1}$$

Under minimal smoothness assumptions on the coefficient $\vec{b}$, the problem may equivalently be formulated in the "conservation" form

$$-\operatorname{div}(A\nabla u - \vec{b}u) + c^* u = f \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega. \tag{1.2}$$

Here $c^* = c - \operatorname{div}\vec{b}$. We shall restrict our attention to the case where the coefficients A, $\vec{b}$ and c and the boundary $\partial\Omega$ are smooth.

In this paper we investigate the pointwise convergence of the "natural" mixed finite element methods corresponding to (1.1) and (1.2). In particular, we prove

Received by the editor May 22, 2002.
2000 *Mathematics Subject Classification.* Primary 65N30, 65N15.
Key words and phrases. Mixed finite element methods, pointwise error estimates.
This material is based on work supported under a National Science Foundation graduate fellowship and under NSF grant DMS-0071412.

weighted, or localized, maximum norm estimates in the spirit of [Sch98]. There, instead of proving a typical global almost-best-approximation result of the form

$$\|u - u_h\|_{L_\infty(\Omega)} \le C \min_{\chi \in S_h} \|u - \chi\|_{L_\infty(\Omega)}, \tag{1.3}$$

where $u_h \in S_h^r$ is a standard Galerkin approximation to u, Schatz considered the error at a single point x_0. Modulo a logarithmic factor, he proved results of the form

$$|(u - u_h)(x_0)| \le C \min_{\chi \in S_h} \|\sigma^s (u - \chi)\|_{L_\infty(\Omega)}. \tag{1.4}$$

Here r is the order of approximation given by S_h^r in L_p $(1 \le p \le \infty)$, $0 \le s \le r-2$, and h is taken to be the diameter of an element in a globally quasi-uniform mesh. The weight σ is defined by

$$\sigma(y) = \frac{h}{|y - x_0| + h}. \tag{1.5}$$

Note that σ is 1 at x_0 and it is $O(h)$ a unit distance away from x_0.

Heuristically these estimates indicate that standard Galerkin methods are "localized" (except in the piecewise linear case $r = 2$) and that higher order methods are more localized. By "localized", we mean roughly that the pointwise error $(u - u_h)(x_0)$ is dependent mostly on the best possible approximation error $u - \chi$ near the point x_0 and is only weakly dependent on $u - \chi$ a unit distance away from x_0. More precisely, we can see from the estimate (1.4) and the definition (1.5) that the dependence of the finite element error $(u - u_h)(x_0)$ on the approximation error $(u - \chi)(y)$ is weakened by a factor of h^{r-2} when y is a unit distance away from x_0. We finally note that the pointwise estimate (1.4) reduces to the global almost-best-approximation estimate (1.3) when $s = 0$, so that (1.4) is a generalization of (1.3).

Mixed finite element methods for approximating solutions to (1.1) and (1.2) when the coefficient $\vec{b}$ is nonzero were introduced in [DR82]. Let $\vec{Q}_h \times V_h \subset H(\mathrm{div};\Omega) \times L_2(\Omega)$ be a mixed approximating subspace, denote by $(\cdot,\cdot)$ the $L_2(\Omega)$ or $[L_2(\Omega)]^n$ inner product, and denote by $\langle \cdot,\cdot \rangle$ the $L_2(\partial\Omega)$ inner product. Then the mixed finite element method corresponding to the divergence form (1.1) is as follows: Find a pair $\{\vec{p}_h, u_h\} \in \vec{Q}_h \times V_h$ such that

$$\begin{aligned} (A^{-1}\vec{p}_h, \vec{q}_h) - (\operatorname{div} \vec{q}_h, u_h) &= -\langle g, \vec{q}_h \cdot \vec{n} \rangle, \\ (\operatorname{div} \vec{p}_h, v_h) - (\vec{b}A^{-1}\vec{p}_h, v_h) + (cu_h, v_h) &= (f, v_h) \end{aligned} \tag{1.6}$$

for all $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$. Here the vector variable $\vec{p}_h$ approximates $\vec{p} = -A\nabla u$. The corresponding mixed method for the conservation form problem (1.2) is as follows: Find $\{\tilde{p}_h, u_h^*\} \in \vec{Q}_h \times V_h$ such that

$$\begin{aligned} (A^{-1}\tilde{p}_h, \vec{q}_h) - (\operatorname{div} \vec{q}_h, u_h^*) - (A^{-1}\vec{b}u_h^*, \vec{q}_h) &= -\langle g, \vec{q}_h \cdot \vec{n} \rangle, \\ (\operatorname{div} \tilde{p}_h, v_h) + (c^* u_h^*, v_h) &= (f, v_h) \end{aligned} \tag{1.7}$$

for all $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$. The vector variable $\tilde{p}_h$ now approximates $\tilde{p} = -(A\nabla u - \vec{b}u)$. We also emphasize that the scalar variable u is fixed, but it is approximated by u_h in the divergence form mixed method and by u_h^* in the conservation form mixed method, and $u_h \neq u_h^*$ in general.

Existence, uniqueness, and optimal-order L_2 error estimates for the vector and scalar variables were proved in [DR82] for the methods (1.6) and (1.7) when $\vec{Q}_h \times V_h$ is taken to be one of the Raviart-Thomas family of spaces, and this analysis was extended to encompass all of the mixed finite element spaces usually used in this context in [Dem02]. In the latter paper, computational examples were presented showing that the convergence of the vector approximation $\tilde{p}_h$ to $\tilde{p}$ in the conservation form method (1.7) is of suboptimal order when one of the Brezzi-Douglas-Marini, or BDM, family of spaces is used and $\vec{b}$ is nonzero. Finally, optimal order global maximum norm estimates were proved for the conservation form method employing any member of the Raviart-Thomas family of elements in [GN88] and for methods using general choices of element spaces to approximate solutions to a restricted model problem with no lower-order terms in [GN89].

In the following analysis we accomplish three major goals. The first is to extend Schatz's sharply localized pointwise analysis of basic Galerkin methods to the mixed finite element methods (1.6) and (1.7). Secondly, the previous global maximum norm analyses cited above do not admit fully general choices of simplicial mixed elements and linear differential operators. We fill these gaps here. Finally, we continue the comparison between varying choices of mixed finite element methods and mixed elements for approximating solutions to elliptic problems that was begun in [Dem02], where it was shown that suboptimal convergence occurs if the BDM elements are used in the conservation form method (1.7). In this paper, we employ asymptotic error expansion inequalities derived from our localized pointwise results to make the slightly more subtle observation that the lowest-order BDM elements do not in general yield a localized approximation, while the lowest-order Raviart-Thomas elements do.

We first prove general estimates for errors in the vector variables. Our results are valid for the divergence and the conservation form methods (1.6) and (1.7) using any of the usual choices of simplicial mixed finite elements. We let Π_h be a local interpolant for $\vec{Q}_h$ which approximates to order k, and we let P_h be the L_2-projection onto V_h, with V_h approximating to order j, $j = k$ (e.g., in the case of the Raviart-Thomas elements) or $j = k-1$ (e.g., in the case of the BDM elements). We recall that $\vec{p}_h$ approximates $\vec{p} = -A\nabla u$ in the divergence form method (1.6) and $\tilde{p}_h$ approximates $\tilde{p} = -(A\nabla u - \vec{b}u)$ in the conservation form method (1.7).

Theorem 1.1. *Let the general assumptions of §2 concerning the differential operators, Π_h, P_h, and the mesh be satisfied. Then there exists a constant C independent of u, $\vec{p}$, $\tilde{p}$, and h such that for any $x_0 \in \Omega$, $0 \le s \le j$, and $0 \le t \le j-1$,*

$$\begin{aligned} |(\vec{p} - \vec{p}_h)(x_0)| \le C[&\ell_{h,1}\|\sigma^s(\vec{p} - \Pi_h\vec{p})\|_{[L_\infty(\Omega)]^n} \\ &+ h(\|\sigma^t(u - P_h u)\|_{L_\infty(\Omega)} + \ell_{h,2}\|\sigma^t(\operatorname{div}\vec{p} - P_h \operatorname{div}\vec{p})\|_{L_\infty(\Omega)})] \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} |(\tilde{p} - \tilde{p}_h)(x_0)| \le C[&\ell_{h,1}(\|\sigma^s(\tilde{p} - \Pi_h\tilde{p})\|_{L_\infty(\Omega)} \\ &+ \|\sigma^s(u - P_h u)\|_{L_\infty(\Omega)}) + h\ell_{h,2}\|\sigma^t(\operatorname{div}\tilde{p} - P_h \operatorname{div}\tilde{p})\|_{L_\infty(\Omega)}]. \end{aligned} \tag{1.9}$$

Here $\ell_{h,1} = 1$ if $s < j$ and $\ell_{h,1} = \log\frac{1}{h}$ if $s = j$, and $\ell_{h,2} = 1$ if $t < j-1$ and $\ell_{h,2} = \log\frac{1}{h}$ if $t = j-1$.

We next show that for the scalar variable u, the mixed approximations u_h and u_h^* are superclose to the interpolant $P_h u$. We recall the Kronecker delta δ_{ij}, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

Theorem 1.2. *Let the same assumptions hold as for Theorem 1.1. Also let* $0 \leq s \leq j-1$ *and* $0 \leq t \leq j-2$ *(*$t = 0$ *if* $j = 1$*). Then for* h *small enough,*

$$\begin{aligned}|(P_h u - u_h)(x_0)| &\leq C[h\ell_{h,3}\|\sigma^s(\vec{p} - \Pi_h \vec{p})\|_{L_\infty(\Omega)} \\ &\quad + h^{2-\delta_{1j}}(\|\sigma^t(u - P_h u)\|_{L_\infty(\Omega)} + \ell_{h,4}\|\sigma^t(\operatorname{div}\vec{p} - P_h \operatorname{div}\vec{p})\|_{L_\infty(\Omega)})]\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}|(P_h u - u_h^*)(x_0)| &\leq C[h\ell_{h,3}(\|\sigma^s(\tilde{p} - \Pi_h \tilde{p})\|_{L_\infty(\Omega)} \\ &\quad + \|\sigma^s(u - P_h u)\|_{L_\infty(\Omega)}) + h^{2-\delta_{1j}}\ell_{h,4}\|\sigma^t(\operatorname{div}\vec{p} - P_h \operatorname{div}\vec{p})\|_{L_\infty(\Omega)}].\end{aligned} \tag{1.11}$$

Here $\ell_{h,3} = 1$ *if* $s < j-1$ *and* $\ell_{h,3} = \log\frac{1}{h}$ *if* $s = j-1$*, and* $\ell_{h,4} = 1$ *if* $t < j-2$ *or* $j = 1$ *and* $\ell_{h,4} = \log\frac{1}{h}$ *if* $t = j-2$*.*

Finally we state asymptotic error expansion inequalities for the vector variables. These results follow from Theorems 1.1 and 1.2 via an elementary argument nearly identical to Schatz's proof, given in [Sch98], of the corresponding inequalities for the standard Galerkin method. We do not give the proof here.

Theorem 1.3. *Let* (1.8) *and* (1.9) *hold, along with the approximation assumptions given in §2.3. Let* s *be an integer satisfying* $1 \leq s \leq j$*. Then for the divergence form problem,*

$$\begin{aligned}|(\vec{p} - \vec{p}_h)(x_0)| &\leq C\ell_{h,1}[\sum_{k \leq |\alpha| \leq k+s-1} h^{|\alpha|}|D^\alpha \vec{p}(x_0)| \\ &\quad + \sum_{j \leq |\alpha| \leq j+s-2} h^{|\alpha|+1}(|D^\alpha u(x_0)| + |D^\alpha \operatorname{div}\vec{p}(x_0)|) \\ &\quad + h^{k+s}\|\vec{p}\|_{W_\infty^{k+s}(\Omega)} + h^{j+s}(\|u\|_{W_\infty^{j+s-1}(\Omega)} + \|\operatorname{div}\vec{p}\|_{W_\infty^{j+s-1}(\Omega)})],\end{aligned} \tag{1.12}$$

where the second term on the right-hand side is taken to be 0 *if* $s = 1$*. For the conservation form problem,*

$$\begin{aligned}|(\tilde{p} - \tilde{p}_h)(x_0)| &\leq C\ell_{h,1}[\textstyle\sum_{k \leq |\alpha| \leq k+s-1} h^{|\alpha|}|D^\alpha \tilde{p}(x_0)| \\ &\quad + \textstyle\sum_{j \leq |\alpha| \leq j+s-1} h^{|\alpha|}|D^\alpha u(x_0)| + \textstyle\sum_{j \leq |\alpha| \leq j+s-2} h^{|\alpha|+1}|D^\alpha \operatorname{div}\tilde{p}(x_0)| \\ &\quad + h^{k+s}\|\tilde{p}\|_{W_\infty^{k+s}(\Omega)} + h^{j+s}\|u\|_{W_\infty^{j+s}(\Omega)} + h^{j+s}\|\operatorname{div}\tilde{p}\|_{W_\infty^{j+s-1}(\Omega)}].\end{aligned} \tag{1.13}$$

Here $\ell_{h,1}$ *is defined as above.*

We now consider the estimate (1.12) for the divergence form method for two choices of element space. If we let $\vec{Q}_h \times V_h$ be the lowest-order Raviart-Thomas space RT_0, then $j = k = 1$, that is, the vector and scalar variables both are approximated to first order. Then with $s = j = 1$, (1.12) becomes

$$\begin{aligned}|(\vec{p} - \vec{p}_h)(x_0)| &\leq C\log\tfrac{1}{h}[\textstyle\sum_{|\alpha|=1} h|D^\alpha \vec{p}(x_0)| \\ &\quad + h^2\|\vec{p}\|_{W_\infty^2(\Omega)} + h^2(\|u\|_{W_\infty^1(\Omega)} + \|\operatorname{div}\vec{p}\|_{W_\infty^1(\Omega)})].\end{aligned}$$

Here all terms of lowest order (order one) are dependent only upon u and $\vec{p}$ and their derivatives at the point x_0, while all global terms are multiplied by a factor of h^2 and are thus of higher order. The divergence form method is therefore localized when the lowest order Raviart-Thomas spaces are used. A similar conclusion may be drawn for the conservation form method using the lowest order Raviart-Thomas spaces.

We next consider (1.12) for the divergence form method when the lowest order Brezzi-Douglas-Marini space BDM_1 is used. Here $k = 2$ and $j = 1$, that is, the vector finite element space is able to approximate to one order higher than the scalar space. With $j = s = 1$, (1.12) becomes

$$\begin{aligned} |(\vec{p} - \vec{p}_h)(x_0)| \le C \log \tfrac{1}{h} [\textstyle\sum_{|\alpha|=2} h^2 |D^\alpha \vec{p}(x_0)| \\ + h^3 \|\vec{p}\|_{W^3_\infty(\Omega)} + h^2 (\|u\|_{W^1_\infty(\Omega)} + \|\operatorname{div} \vec{p}\|_{W^1_\infty(\Omega)})]. \end{aligned} \tag{1.14}$$

Here there are global terms of lowest order, indicating that the method is not localized. In §5 we present numerical experiments confirming that this estimate is sharp in that $|(\vec{p} - \vec{p}_h)(x_0)| \ge ch^2$ even if $D^\alpha \vec{p}(x_0) = 0$ for all $|\alpha| \le 2$ and $D^\alpha u(x_0) = 0$ for all $|\alpha| \le 3$. Thus (1.14) is sharp with respect to localization, and (1.8) and (1.9) are sharp with respect to the orders of weights allowed. We note that loss of localization occurs only when using the lowest order BDM elements and not when using the higher order BDM elements.

We summarize here our findings regarding the use of the BDM family of elements for simplicial meshes in mixed methods for elliptic problems. We first note that the BDM family of elements was constructed with the goal of approximating ∇u (or $\vec{p}$ or $\tilde{p}$) with maximum efficiency as this is normally the variable of greatest interest in mixed methods. It was previously shown in [Dem02] that this goal is not met when $\vec{b} \neq 0$ and the conservation form method is used as $\tilde{p}_h$ converges to $\tilde{p}$ at a suboptimal rate for any choice of BDM space for simplicial elements. Here we have shown that if one uses the lowest-order BDM space in the divergence form method, one loses the property of localization (as defined above) while retaining the intended gain in efficiency over the Raviart-Thomas or other elements for which $\vec{Q}_h$ does not consist of $[P_k]^n$, i.e., of a full order of polynomials. We note that localized estimates have been used in [HSWW01] in the analysis of asymptotically exact pointwise a posteriori error estimators for basic Galerkin methods for elliptic problems, and we expect our results for mixed methods to be similarly applicable. Such an analysis is precluded, or at the least greatly complicated, when localization is lost as in the case of the lowest-order BDM elements. Thus use of the lowest order BDM elements in the divergence form method involves a tradeoff in which one gains efficiency while losing localization properties.

Finally, we note that the divergence and conservation form mixed finite element methods for approximating solutions of (1.1) and (1.2) behave in a fashion which is sometimes significantly different than their counterpart when lower-order terms are not present. Put in other words, Poisson's problem is not generally a suitable model problem for studying the behavior of mixed approximations to solutions of (1.1) and (1.2). This is in particular the case when the first-order term $\vec{b}$ is nonzero. In addition to adding complication to the structure of the error bounds obtained for these methods (and as already mentioned leading to suboptimal bounds when the BDM elements are used in the conservation form method), the proofs of these error bounds are somewhat more complicated when lower-order terms are present. In the present case, the first-order term $\vec{b}$ prevents a clean development of the technical details in our analysis below, particularly in the conservation form method.

2. Preliminaries and assumptions

In this section we describe our notation, outline the assumptions under which we shall prove our results, and state results concerning some of the technical tools essential for proving our main theorems.

2.1. **Notation.**

Here we describe notation and conventions that we shall use in the course of our proofs. We begin by defining a shorthand notation for weighted L_p norms; here we shall use the conventions $\|v\|_{L_p,x_0,s} = \|\sigma^s v\|_{L_p(\Omega)}$ and $\|\vec{q}\|_{L_p,x_0,s} = \|\sigma^s \vec{q}\|_{[L_p(\Omega)]^n}$. Also, for any domain $D \subseteq \Omega$, we define $\|v\|_D = \|v\|_{L_2(D)}$ and $\|\vec{q}\|_D = \|\vec{q}\|_{[L_2(D)]^n}$. By $a \lesssim b$, we shall mean that $a \leq Cb$ for some constant C which is independent of essential quantities. Also, we shall employ the Sobolev spaces W_p^m and $H_2^m = W_2^m$ and the associated norms and seminorms. We do not give definitions here as they are standard; cf. [GT98, Chap. 7]. Finally, we shall denote by $H_0^1(D)$ the functions in $H_2^1(D)$ with zero trace on ∂D.

It will be convenient in our technical development to prove certain facts for the conservation and divergence form problems at the same time, and for the sake of convenience we gather here the necessary notational conventions. We shall let $\hat{u}_h \in V_h$ denote either u_h or u_h^*, depending on whether we are considering the divergence or conservation form case, respectively, and similarly we shall let $\hat{c}$ denote either c or c^*. Also, we shall let p denote $\vec{p}$ or $\tilde{p}$ as appropriate, and similarly we let p_h denote either $\vec{p}_h$ or $\tilde{p}_h$. Finally, we let $\vec{b}_1 = \vec{b}$ and $\vec{b}_2 = 0$ in the divergence form case and $\vec{b}_1 = 0$ and $\vec{b}_2 = -\vec{b}$ in the conservation form case.

2.2. **Assumptions on the partial differential equation.**

We assume that the matrix of coefficients $A = [a_{ij}(x)]$ satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C_l|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad x \in \Omega.$$

We also assume that the coefficients a_{ij}, b_i and c are bounded and smooth and that the boundary $\partial\Omega$ is smooth. We do not assume coercivity, but rather we only require that solutions of (1.1) and (1.2) exist and that they be unique.

Of particular importance in the proofs of our results are the adjoint problems to (1.1) and (1.2). Here we employ the problem

$$L^*\phi = -\operatorname{div}(A^*\nabla\phi + \vec{b}_1\phi) + \vec{b}_2\nabla\phi + \hat{c}\phi = f \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega \tag{2.1}$$

with associated bilinear form

$$\begin{aligned}\mathcal{L}^*(u,v) = &\textstyle\int_\Omega A^*(x)\nabla u(x)\nabla v(x)\,\mathrm{dx} + \int_\Omega \vec{b}_1(x)u(x)\nabla v(x)\mathrm{d}x \\ &+ \textstyle\int_\Omega \vec{b}_2(x)\nabla u(x)v(x)\,\mathrm{dx} + \int_\Omega \hat{c}(x)u(x)v(x)\,\mathrm{dx}\,.\end{aligned}$$

In the divergence form case, we shall take $\vec{b}_1 = \vec{b}$, $\vec{b}_2 = 0$, and $\hat{c} = c$ so that (2.1) reduces to the adjoint of (1.1). In the conservation form case, we shall take $\vec{b}_1 = 0$, $\vec{b}_2 = -\vec{b}$, and $\hat{c} = c^*$ so that (2.1) reduces to the adjoint of (1.2). We shall also use the following regularity results for u satisfying (1.1) and (1.2) and ϕ satisfying (2.1). For $f \in L_2(\Omega)$, we require the estimate

$$\|u\|_{H_2^2(\Omega)} + \|\phi\|_{H_2^2(\Omega)} \lesssim \|f\|_{L_2(\Omega)}, \tag{2.2}$$

and in the case that $f = \operatorname{div} \vec{z}$ for some $\vec{z} \in H(\operatorname{div}; \Omega)$, we record the easily proven energy-type estimate

$$\|u\|_{H_2^1(\Omega)} + \|\phi\|_{H_2^1(\Omega)} \lesssim \|\vec{z}\|_{[L_2(\Omega)]^n}. \tag{2.3}$$

We finally state a lemma concerning the Green's functions for the problems (1.1) and (1.2) and their adjoint (2.1). With slight abuse of notation, we shall denote by $G(x, y)$ a function satisfying either $\mathcal{L}^*(u, G(x, \cdot)) = u(x)$ or $\mathcal{L}^*(G(x, \cdot), u) = u(x)$, and $G(x, y) = 0$ for $y \in \partial\Omega$.

Lemma 2.1. *There exists a constant C such that for x and y in Ω,*

$$|D_x^\alpha D_y^\beta G(x, y)| \lesssim |x - y|^{2-n-|\alpha+\beta|} \quad \text{for } |\alpha + \beta| > 0 \tag{2.4}$$

and

$$|G(x, y)| \lesssim |x - y|^{1-n}. \tag{2.5}$$

Here C depends only on Ω and the coefficients A, $\vec{b}$, and c.

Proof. The inequality (2.4) is proven in [Kra69]. The inequality (2.5) follows in the current context of smooth coefficients and boundary by an elementary argument using the fundamental theorem of calculus and (2.4). We note that (2.5), although very weak, is sufficient for our purposes here, and we do not state a sharp result in order to avoid then having to distinguish between $n = 2$ and $n \geq 3$. $\square$

2.3. **Error equations.** In this section we state error equations for the mixed methods (1.6) and (1.7). We recall our convention that $\vec{b}_1 = \vec{b}$, $\vec{b}_2 = 0$, and $\hat{c} = c$ in the divergence form method and associated problems and $\vec{b}_1 = 0$, $\vec{b}_2 = -\vec{b}$, and $\hat{c} = c^*$ in the conservation form method and associated problems. We also let p denote either $\vec{p}$ (in the divergence form case) or $\tilde{p}$ (in the conservation form case), and we similarly let $\hat{u}_h$ denote either u_h or u_h^*. Using this convention and combining (1.1) and (1.2) with (1.6) and (1.7) under the assumption that all integrals (boundary integrals in particular) are exact, we have

$$(A^{-1}(p - p_h), \vec{q}_h) - (\operatorname{div} \vec{q}_h, u - \hat{u}_h) + (A^{-1}\vec{b}_2(u - \hat{u}_h), \vec{q}_h) = 0, \tag{2.6a}$$

$$(\operatorname{div}(p - p_h), v_h) - (\vec{b}_1 A^{-1}(p - p_h), v_h) + (\hat{c}(u - \hat{u}_h), v_h) = 0 \tag{2.6b}$$

for all pairs $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$.

2.4. **Assumptions on the finite element spaces.** Let $\{\tau_h\}$ be a partition of Ω into triangles or simplices having maximum diameter h. Boundary elements are allowed to have one curved face as described in, for example, [DR85] and [BDM85]. We assume that the simplices τ_h each satisfy the quasi-uniformity condition $ch^n \leq \operatorname{Vol}(\tau_h) \leq Ch^n$.

Let the finite element space $\vec{Q}_h \times V_h \subset H(\operatorname{div}; \Omega) \times L_2(\Omega)$ be defined with respect to $\{\tau_h\}$. Here $H(\operatorname{div}; \Omega) = \{\vec{q} \in [L_2(\Omega)]^n \text{ s.t. } \operatorname{div} \vec{q} = \sum_{i=1}^n \frac{\partial \vec{q}_i}{\partial x_i} \in L_2(\Omega)\}$. For technical reasons we also state the requirement that V_h locally contain precisely a full order of polynomials, i.e., $V_h|_{\tau_h} = P^{j-1}$ on each element τ_h.

We first require that the commuting diagram property be satisfied. We let $W = H(\operatorname{div}; \Omega) \cap [L^p(\Omega)]^n$ for some fixed $p > 2$ and we let $P_h : L_2(\Omega) \to V_h$ be the local (elementwise) L_2 projection. We then require that there exist a local projection operator $\Pi_h : W \to \vec{Q}_h$ (also acting elementwise) such that the following diagram commutes.

$$\begin{tikzcd}
W \arrow[r, "\Pi_h"] \arrow[d, "\mathrm{div}"] & \vec{Q}_h \arrow[d, "\mathrm{div}"] \\
L_2(\Omega) \arrow[r, "P_h"] & V_h
\end{tikzcd}$$

The commuting diagram property can also be stated in the form

$$\operatorname{div} \Pi_h \vec{q} = P_h \operatorname{div} \vec{q} \tag{2.7}$$

for $\vec{q} \in W$.

We next state approximation properties. Let $D \subseteq \Omega$, and let $D' = \bigcup_{\tau_h : \tau_h \cap D \neq \emptyset} \tau_h$. For the vector finite element space $\vec{Q}_h$ with interpolant Π_h, we assume there exists an integer k such that for all $1 \leq s \leq k$ and $1 \leq p \leq \infty$,

$$\|\vec{q} - \Pi_h \vec{q}\|_{[L_p(D)]^n} \lesssim h^s |\vec{q}|_{[W_p^s(D')]^n}, \tag{2.8}$$

for all $\vec{q} \in [W_p^s(\Omega)]^n$. We then let $j = k$ (as in for example the Raviart-Thomas elements) or $j = k - 1$ (as in the Brezzi-Douglas-Marini elements) and we require that for the scalar L_2-projection P_h onto V_h,

$$\|v - P_h v\|_{L_p(D)} \lesssim h^s |v|_{W_p^s(D')} \tag{2.9}$$

for all $v \in W_p^s(\Omega)$, all $1 \leq s \leq j$, and all $1 \leq p \leq \infty$. We also recall here that P_h is stable in any W_p^m norm for any $1 \leq p \leq \infty$ and any $m \geq 0$, that is,

$$\|P_h v\|_{W_p^m(D)} \leq \|v\|_{W_p^m(D')},$$

where the norm on the left-hand side is computed piecewise over the elements.

Next we state inverse assumptions. With domains D and D' as above, we require that

$$\|v_h\|_{W_q^m(D)} \lesssim h^{n/q - n/p - m} \|v_h\|_{L_p(D')}, \tag{2.10a}$$

$$\|\vec{q}_h\|_{[W_q^m(D)]^n} \lesssim h^{n/q - n/p - m} \|\vec{q}_h\|_{[L_p(D')]^n} \tag{2.10b}$$

for $1 \leq p \leq q \leq \infty$ and $0 \leq m \leq k$. These estimates follow via the usual proofs under the assumption of quasi-uniformity of the mesh in the cases of the elements normally used in this context.

We finally require the following superapproximation properties. For any smooth function ω and any $v_h \in V_h$, we require that for $1 \leq p \leq \infty$,

$$\|\omega v_h - P_h(\omega v_h)\|_{L_p(\Omega)} \lesssim h \|\omega\|_{W_\infty^j(\Omega)} \|v_h\|_{L_p(\Omega)}. \tag{2.11}$$

If $\operatorname{supp}(\omega) \subset D$, $\|D^\alpha \omega\|_{L_\infty(\Omega)} \leq \tilde{C} d^{-|\alpha|}$ for $1 \leq |\alpha| \leq k+1$ and $\tilde{c} h \leq d$ for some $\tilde{c}$ and $\tilde{C}$, then we require that

$$\|\omega v_h - P_h(\omega v_h)\|_{L_p(D)} \lesssim \frac{h}{d} \|v_h\|_{L_p(D')}, \tag{2.12a}$$

$$\|\omega \vec{q}_h - \Pi_h(\omega \vec{q}_h)\|_{[L_p(D)]^n} \lesssim \frac{h}{d} \|\vec{q}_h\|_{[L_p(D')]^n} \tag{2.12b}$$

for $1 \leq p \leq \infty$. We remark that for standard element choices, these properties follow by standard proofs (using a sharp form of the Bramble-Hilbert Lemma) given the previously made assumptions concerning inverse properties; cf. [Sch83].

2.5. **Discrete δ-functions.** In this subsection we introduce discrete δ functions for both the scalar and vector subspaces. We refer the reader to [SW95] and [Wah91] for proofs of the existence of such functions and their stated properties, respectively, in representative finite element contexts.

In the scalar case, we assume that given $x_0 \in \tau_h$, there exists a function $\delta^0 \in V_h$ having support in τ_h such that

$$(\delta^0, v_h) = v_h(x_0) \quad \text{for all } v_h \in V_h. \tag{2.13}$$

We furthermore require that

$$\|\delta^0\|_{L_p(\Omega)} \lesssim h^{n(1/p-1)}, \quad 1 \le p \le \infty.$$

In the vector case we require the existence of $\vec{\delta}_i^0 \in \vec{Q}_h$ such that given $x_0 \in \tau$,

$$(A^{-1}\vec{q}_h, \vec{\delta}_i^0) = [\vec{q}_h(x_0)]_i \quad \text{for all } \vec{q}_h \in \vec{Q}_h, \tag{2.14}$$

where $[\vec{q}_h(x_0)]_i$ is the i-th component of the vector $\vec{q}_h(x_0)$. The existence of such a $\vec{\delta}_i^0$ is proven by constructing a continuous $\vec{\delta}_i$ such that $(\vec{\delta}_i, \vec{q}_h) = [\vec{q}_h(x_0)]_i$ in the usual manner and then taking $\vec{\delta}_i^0$ to be the weighted L_2 projection of $\vec{\delta}_i$ into $\vec{Q}_h$ with respect to the form $(A^{-1}\cdot, \cdot)$. With only slight modifications of the proof given in [Wah91], we can then show that

$$\|\vec{\delta}_i^0\|_{[L_p(\Omega)]^n} + h\|\operatorname{div}\vec{\delta}_i^0\|_{L_p(\Omega)} \lesssim h^{n(1/p-1)}, \tag{2.15}$$

and for any $y \in \Omega$,

$$|\vec{\delta}_i^0(y)| + h|\operatorname{div}\vec{\delta}_i^0(y)| \lesssim h^{-n}e^{-c|y-x_0|/h}. \tag{2.16}$$

3. Global and local L_2 estimates for the dual problem

In this section we introduce a mixed finite element method corresponding to the adjoint problem (2.1), state global L_2 estimates for the error in the method, and prove a local L_2 estimate essential to the proofs of Theorem 1.1 and Theorem 1.2.

3.1. **The dual mixed finite element method.** We first state the mixed finite element method for approximating solutions to (2.1), which reduces to (1.6) or (1.7) (with different coefficients and right-hand side) for the allowed choices of $\vec{b}_1$ and $\vec{b}_2$ and $\hat{c}$. We seek a pair $\{\vec{r}_h, \phi_h\} \in \vec{Q}_h \times V_h$ such that

$$(A^{-*}\vec{r}_h, \vec{q}_h) - (\operatorname{div}\vec{q}_h, \phi_h) + (A^{-*}\vec{b}_1\phi_h, \vec{q}_h) = 0, \tag{3.1a}$$

$$(\operatorname{div}\vec{r}_h, v_h) - (\vec{b}_2 A^{-*}\vec{r}_h, v_h) + (\hat{c}\phi_h, v_h) = (\gamma, v_h) \tag{3.1b}$$

for all pairs $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$.

We next state global L_2 results for the above method.

Lemma 3.1. *For h small enough,*

(3.2a) $\|\vec{r} - \vec{r}_h\|_\Omega \lesssim \|\vec{r} - \Pi_h\vec{r}\|_\Omega + \|\phi - P_h\phi\|_\Omega$,

(3.2b) $\|\phi - \phi_h\|_\Omega \lesssim \|\phi - P_h\phi\|_\Omega + h\|\vec{r} - \Pi_h\vec{r}\|_\Omega + h^{2-\delta_{1j}}\|\operatorname{div}\vec{r} - P_h\operatorname{div}\vec{r}\|_\Omega$,

(3.2c) $\|\phi - \phi_h\|_\Omega \lesssim \|\phi - P_h\phi\|_\Omega + \|\vec{r} - \Pi_h\vec{r}\|_\Omega$.

The above three results were proved in [Dem02], where they can be found in Theorems 1.2 and 1.4, Corollary 3.4, and Remark 3.5, respectively.

3.2. **Statement of local L_2 results.** In our proofs of Theorem 1.1 and Theorem 1.2, we shall employ the following local L_2 result.

Lemma 3.2. *Assume that ϕ satisfies*

$$-\operatorname{div}(A^*\nabla\phi + \vec{b}_1\phi) + \vec{b}_2\nabla\phi + \hat{c}\phi = \gamma \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega \tag{3.3}$$

where $\gamma \in V_h$. Let $\vec{r} = -(A^*\nabla\phi + \vec{b}_1\phi)$, and let the pair $\{\vec{r}_h, \phi_h\} \in \vec{Q}_h \times V_h$ satisfy the mixed finite element equations (3.1a) *and* (3.1b)*. Assume also that D is an annulus centered at some point $x \in \Omega$ (or the intersection of such an annulus with Ω) and that D has radius $C_1 d$, where $\tilde{c} \le C_1 \le \tilde{C}$ for some constants $\tilde{c}$ and $\tilde{C}$. Also, let $D_d = \{x \in \Omega$ s.t. $\operatorname{dist}(x, D) < d\}$. Then if $d \ge Kh$ (where $K > 1$ is some constant),*

$$\begin{aligned}
&\frac{1}{d}\|\phi - \phi_h\|_D + \|\vec{r} - \vec{r}_h\|_D \\
&\quad \lesssim \|\vec{r} - \Pi_h\vec{r}\|_{D_d} + \frac{1}{d}(\|\phi - P_h\phi\|_{D_d} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{D_d}) \\
&\quad\quad + \frac{h}{d}(\|\vec{r} - \vec{r}_h\|_{D_d} + \|\phi - \phi_h\|_{D_d} + \|\vec{b}_2A^{-*}\vec{r} - P_h(\vec{b}_2A^{-*}\vec{b})\|_{D_d}) \\
&\quad\quad + h^j d^{-n/2-j}(\|\phi - \phi_h\|_{L_1(\Omega)} + \|\vec{r} - \vec{r}_h\|_{[L_1(\Omega)]^n}) \\
&\quad\quad + h^j d^{1-n/2-j}(\|\phi - P_h\phi\|_{L_1(\Omega)} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(\Omega)} \\
&\quad\quad + \|\vec{b}_2A^{-*}\vec{r} - P_h(\vec{b}_2A^{-*}\vec{r})\|_{L_1(\Omega)}).
\end{aligned} \tag{3.4}$$

Remark 3.3. It is possible to give an alternate proof of Lemma 3.2 which is lengthier but in some sense more traditional. One may first prove general local L_2 bounds of the form

$$\begin{aligned}
\frac{1}{d}\|\phi - \phi_h\|_D + \|\vec{r} - \vec{r}_h\|_D &\lesssim \frac{1}{d}\|\phi - P_h\phi\|_{D_d} + \|\vec{r} - \Pi_h\vec{r}\|_{D_d} \\
&\quad + \frac{1}{d^2}\|\phi - \phi_h\|_{H^{-1}(D_d)} + \frac{1}{d}\|\vec{r} - \vec{r}_h\|_{H^{-1}(D_d)},
\end{aligned} \tag{3.5}$$

where $\|\cdot\|_{H^{-1}(D_d)}$ is a suitably defined negative norm, γ in (3.3) is a general right-hand side which is not required to be in V_h, and D is any domain contained in Ω. One may then suitably bound the negative norm terms in (3.5) in order to prove (3.4).

3.3. **Proof of Lemma 3.2.** We begin by bounding $\|\Pi_h\vec{r} - \vec{r}_h\|_D$. Let ω be a smooth cutoff function which satisfies $\omega \equiv 1$ on D, $\omega \equiv 0$ on $\Omega \backslash D_{.5d}$, $0 \le \omega \le 1$, and $\|D^\alpha\omega\|_{L_\infty(\Omega)} \le \frac{C}{d^{|\alpha|}}$ for $1 \le |\alpha| \le k+1$. Noting that A^{-1} is uniformly positive definite with uniformly bounded entries since A is, we find that

$$\begin{aligned}
\|\Pi_h\vec{r} - \vec{r}_h\|_D^2 &\le (\omega(\Pi_h\vec{r} - \vec{r}_h), \omega(\Pi_h\vec{r} - \vec{r}_h)) \\
&\lesssim (A^{-*}\omega(\Pi_h\vec{r} - \vec{r}_h), \omega(\Pi_h\vec{r} - \vec{r}_h)) \\
&\lesssim (A^{-*}\omega(\Pi_h\vec{r} - \vec{r}), \omega(\Pi_h\vec{r} - \vec{r}_h)) + (A^{-*}\omega(\vec{r} - \vec{r}_h), \omega(\Pi_h\vec{r} - \vec{r}_h)) \\
&\equiv I + II.
\end{aligned} \tag{3.6}$$

We next combine the mixed form of (3.3) with the mixed finite element equations (3.1a) and (3.1b) to obtain the error equations

$$(A^{-*}(\vec{r} - \vec{r}_h), \vec{q}_h) - (\operatorname{div}\vec{q}_h, \phi - \phi_h) + (A^{-*}\vec{b}_1(\phi - \phi_h), \vec{q}_h) = 0, \tag{3.7a}$$

$$(\operatorname{div}(\vec{r} - \vec{r}_h), v_h) - (\vec{b}_2A^{-*}(\vec{r} - \vec{r}_h), v_h) + (\hat{c}(\phi - \phi_h), v_h) = 0, \tag{3.7b}$$

for all pairs $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$. Now

$$II = (A^{-*}(\vec{r}-\vec{r}_h), \omega^2(\Pi_h\vec{r}-\vec{r}_h) - \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h))) + (A^{-*}(\vec{r}-\vec{r}_h), \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h))). \tag{3.8}$$

Using the error equation (3.7a) and the commuting diagram property (2.7), we find that

$$\begin{aligned}
&(A^{-*}(\vec{r}-\vec{r}_h), \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h)))\\
&\quad= (\operatorname{div}(\Pi_h\omega^2(\Pi_h\vec{r}-\vec{r}_h)), \phi-\phi_h) - (A^{-*}\vec{b}_1(\phi-\phi_h), \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h)))\\
&\quad= (P_h \operatorname{div}(\omega^2(\Pi_h\vec{r}-\vec{r}_h)), \phi-\phi_h)\\
&\qquad-(A^{-*}\vec{b}_1(\phi-\phi_h), \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h)) - \omega^2(\Pi_h\vec{r}-\vec{r}_h))\\
&\qquad-(A^{-*}\vec{b}_1(\phi-\phi_h), \omega^2(\Pi_h\vec{r}-\vec{r}_h))\\
&\quad= (2\nabla\omega\omega(\Pi_h\vec{r}-\vec{r}_h) + \omega^2 \operatorname{div}(\Pi_h\vec{r}-\vec{r}_h), P_h\phi-\phi_h)\\
&\qquad-(A^{-*}\vec{b}_1(\phi-\phi_h), \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h)) - \omega^2(\Pi_h\vec{r}-\vec{r}_h))\\
&\qquad-(A^{-*}\vec{b}_1(\phi-\phi_h), \omega^2(\Pi_h\vec{r}-\vec{r}_h)).
\end{aligned} \tag{3.9}$$

We next use the commuting diagram property (2.7) and the error equation (3.7b) to obtain

$$\begin{aligned}
&(\omega^2\operatorname{div}(\Pi_h\vec{r}-\vec{r}_h), P_h\phi-\phi_h) = (P_h(\operatorname{div}(\vec{r}-\vec{r}_h)), \omega^2(P_h\phi-\phi_h))\\
&\quad= (\operatorname{div}(\vec{r}-\vec{r}_h), P_h(\omega^2(P_h\phi-\phi_h)))\\
&\quad= (\vec{b}_2A^{-*}(\vec{r}-\vec{r}_h), P_h(\omega^2(P_h\phi-\phi_h))) - (\hat{c}(\phi-\phi_h), P_h(\omega^2(P_h\phi-\phi_h)))\\
&\quad= (\vec{b}_2A^{-*}(\vec{r}-\vec{r}_h), P_h(\omega^2(P_h\phi-\phi_h)) - \omega^2(P_h\phi-\phi_h))\\
&\qquad+(\vec{b}_2A^{-*}(\vec{r}-\vec{r}_h), \omega^2(P_h\phi-\phi_h)) - (\hat{c}(\phi-\phi_h), P_h(\omega^2(P_h\phi-\phi_h))).
\end{aligned} \tag{3.10}$$

Combining (3.6), (3.8), (3.9) and (3.10) yields

$$\begin{aligned}
I + II \lesssim{}& \|\vec{r}-\Pi_h\vec{r}\|_{D_{.5d}}\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}\\
&+ \|\vec{r}-\vec{r}_h\|_{D_{.5d}}\|\omega^2(\Pi_h\vec{r}-\vec{r}_h) - \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h))\|_{D_{.5d}}\\
&+ \|\nabla\omega\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}\|P_h\phi-\phi_h\|_{D_{.5d}}\\
&+ \|\vec{r}-\vec{r}_h\|_{D_{.5d}}\|P_h(\omega^2(P_h\phi-\phi_h)) - \omega^2(P_h\phi-\phi_h)\|_{D_{.5d}}\\
&+ \|\omega^2(\vec{r}-\vec{r}_h)\|_{D_{.5d}}\|P_h\phi-\phi_h\|_{D_{.5d}} + \|\phi-\phi_h\|_{D_{.5d}}\|P_h\phi-\phi_h\|_{D_{.5d}}\\
&+ \|\phi-\phi_h\|_{D_{.5d}}\|\omega^2(\Pi_h\vec{r}-\vec{r}_h) - \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h))\|_{D_{.5d}}\\
&+ \|\phi-\phi_h\|_{D_{.5d}}\|\omega^2(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}.
\end{aligned} \tag{3.11}$$

We next use the superapproximation assumptions (2.12b) and (2.12a) and the L_2 stability of P_h to obtain

$$\begin{aligned}
\|\omega^2(\Pi_h\vec{r}-\vec{r}_h) - \Pi_h(\omega^2(\Pi_h\vec{r}-\vec{r}_h))\|_{D_{.5d}} &\lesssim \tfrac{h}{d}\|\Pi_h\vec{r}-\vec{r}_h\|_{D_d}\\
&\lesssim \tfrac{h}{d}(\|\Pi_h\vec{r}-\vec{r}\|_{D_d} + \|\vec{r}-\vec{r}_h\|_{D_d})
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\|P_h(\omega^2(P_h\phi-\phi_h))-\omega^2(P_h\phi-\phi_h)\|_{D_{.5d}}\\ \lesssim \tfrac{h}{d}\|P_h\phi-\phi_h\|_{D_{.75d}} \lesssim \tfrac{h}{d}\|\phi-\phi_h\|_{D_d}.\end{aligned} \tag{3.13}$$

Recalling that $|\nabla\omega| \lesssim \frac{1}{d}$, we also find

$$\|\nabla\omega\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}} \lesssim \frac{1}{d}\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}. \tag{3.14}$$

We next insert (3.12), (3.13) and (3.14) into (3.11) while again noting that $\|P_h\phi-\phi_h\|_{D_{.5d}} = \|P_h(\phi-\phi_h)\|_{D_{.5d}} \le \|\phi-\phi_h\|_{D_d}$ and $\frac{h}{d} \le 1$ in order to obtain

$$\begin{aligned} I+II &\lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_{.5d}}\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}\\ &\quad+\|\vec{r}-\vec{r}_h\|_{D_{.5d}}\tfrac{h}{d}(\|\Pi_h\vec{r}-\vec{r}\|_{D_d}+\|\vec{r}-\vec{r}_h\|_{D_d})\\ &\quad+\tfrac{1}{d}\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}\|\phi-\phi_h\|_{D_d}+\|\vec{r}-\vec{r}_h\|_{D_{.5d}}\tfrac{h}{d}\|\phi-\phi_h\|_{D_d}\\ &\quad+(\|\vec{r}-\Pi_h\vec{r}\|_{D_{.5d}}+\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}})\|\phi-\phi_h\|_{D_d}+\|\phi-\phi_h\|_{D_d}^2\\ &\quad+\|\phi-\phi_h\|_{D_{.5d}}\tfrac{h}{d}(\|\Pi_h\vec{r}-\vec{r}\|_{D_d}+\|\vec{r}-\vec{r}_h\|_{D_d})\\ &\quad+\|\phi-\phi_h\|_{D_{.5d}}\|\omega^2(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}\\ &\lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}(\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_d}+\tfrac{h}{d}\|\vec{r}-\vec{r}_h\|_{D_d}+\|\phi-\phi_h\|_{D_d})+\tfrac{h}{d}\|\vec{r}-\vec{r}_h\|_{D_d}^2\\ &\quad+\|\phi-\phi_h\|_{D_d}(\tfrac{1}{d}\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_d}+\|\phi-\phi_h\|_{D_d}+\tfrac{h}{d}\|\vec{r}-\vec{r}_h\|_{D_d}).\end{aligned}$$

Then taking $\epsilon_i > 0$, $i=1,2$, recalling that $\frac{h}{d} \le 1$, and noting that $d \lesssim 1$ yields

$$\begin{aligned} I+II &\lesssim (1+\tfrac{1}{\epsilon_1})\|\vec{r}-\Pi_h\vec{r}\|_{D_d}^2+\tfrac{h}{d}\|\vec{r}-\vec{r}_h\|_{D_d}^2\\ &\quad+(\tfrac{1}{\epsilon_2}+1)\tfrac{1}{d^2}\|\phi-\phi_h\|_{D_d}^2+(\epsilon_1+\epsilon_2)\|\omega(\Pi_h\vec{r}-\vec{r}_h)\|_{D_{.5d}}^2.\end{aligned} \tag{3.15}$$

Inserting (3.15) into (3.6) and taking ϵ_1 and ϵ_2 small enough to kick back the final term above, we find that

$$\|\Pi_h\vec{r}-\vec{r}_h\|_D^2 \lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}^2+\frac{h}{d}\|\vec{r}-\vec{r}_h\|_{D_d}^2+\frac{1}{d^2}\|\phi-\phi_h\|_{D_d}^2. \tag{3.16}$$

Employing the triangle inequality along with (3.16), we obtain

$$\|\vec{r}-\vec{r}_h\|_D \lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}+\sqrt{\frac{h}{d}}\|\vec{r}-\vec{r}_h\|_{D_d}+\frac{1}{d}\|\phi-\phi_h\|_{D_d}. \tag{3.17}$$

Iterating (3.17) $n+2j$ times yields

$$\|\vec{r}-\vec{r}_h\|_D \lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_{(n+2j)d}}+\left(\frac{h}{d}\right)^{n/2+j}\|\vec{r}-\vec{r}_h\|_{D_{(n+2j)d}}+\frac{1}{d}\|\phi-\phi_h\|_{D_{(n+2j)d}},$$

and changing the domain $D_{(n+2j)d}$ to $D_{.5d}$ (which will change the constant in the above inequality by a constant factor not dependent upon d) gives

$$\|\vec{r}-\vec{r}_h\|_D \lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_{.5d}}+\left(\frac{h}{d}\right)^{n/2+j}\|\vec{r}-\vec{r}_h\|_{D_{.5d}}+\frac{1}{d}\|\phi-\phi_h\|_{D_{.5d}}.$$

We next use the triangle inequality, the inverse property (2.10b), and Hölder's inequality while recalling that $h \le d$ to find that

$$\begin{aligned}(3.18)\qquad \left(\tfrac{h}{d}\right)^{n/2+j}\|\vec{r}-\vec{r}_h\|_{D_{.5d}} &\lesssim \left(\tfrac{h}{d}\right)^{n/2+j}(\|\vec{r}-\Pi_h\vec{r}\|_{D_{.5d}}+\|\Pi_h\vec{r}-\vec{r}_h\|_{D_{.5d}})\\ &\lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}+h^{-n/2}\left(\tfrac{h}{d}\right)^{n/2+j}\|\Pi_h\vec{r}-\vec{r}_h\|_{[L_1(D_d)]^n}\\ &\lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}+h^jd^{-n/2-j}(\|\vec{r}-\vec{r}_h\|_{[L_1(D_d)]^n}+d^{n/2}\|\vec{r}-\Pi_h\vec{r}\|_{D_d})\\ &\lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}+h^jd^{-n/2-j}\|\vec{r}-\vec{r}_h\|_{[L_1(D_d)]^n}.\end{aligned}$$

Inserting (3.18) into (3.17) yields

$$(3.19)\qquad \|\vec{r}-\vec{r}_h\|_D \lesssim \|\vec{r}-\Pi_h\vec{r}\|_{D_d}+h^jd^{-n/2-j}\|\vec{r}-\vec{r}_h\|_{[L_1(D_d)]^n}+\frac{1}{d}\|\phi-\phi_h\|_{D_d}.$$

We shall next bound $\|P_h\phi-\phi_h\|_D$. We note that

$$(3.20)\qquad \|P_h\phi-\phi_h\|_D=\sup_{v\in L_2(D),\|v\|_D=1}(P_h\phi-\phi_h,v).$$

Thus we let v be supported in D with $\|v\|_D=1$, and we then seek to bound $\frac{1}{d}(P_h\phi-\phi_h,v)$ by the right-hand side of (3.4).

We first proceed with a duality argument. We let m solve

$$-\operatorname{div}(A\nabla m+\vec{b}_2m)+\vec{b}_1\nabla m+\hat{c}m=v \text{ in } \Omega,\quad m=0 \text{ on } \partial\Omega.$$

Then, with $\vec{z}=-(A\nabla m+\vec{b}_2m)$ and recalling that $\vec{b}_1=0$ or $\vec{b}_2=0$, we have

$$(3.21)\qquad (P_h\phi-\phi_h,v)=(P_h\phi-\phi_h,\operatorname{div}\vec{z}-\vec{b}_1A^{-1}\vec{z}+\hat{c}m)\equiv I+II+III.$$

Using the commuting diagram property (2.7), we note that

$$(3.22)\qquad I=(P_h\phi-\phi_h,P_h\operatorname{div}\vec{z})=(\phi-\phi_h,\operatorname{div}\Pi_h\vec{z}).$$

We next find that

$$\begin{aligned}(3.23)\qquad II&=-(P_h\phi-\phi_h,P_h(\vec{b}_1A^{-1}\vec{z}))\\ &=-(\phi-\phi_h,P_h(\vec{b}_1A^{-1}\vec{z})-\vec{b}_1A^{-1}\vec{z}+\vec{b}_1A^{-1}\vec{z}-\vec{b}_1A^{-1}\Pi_h\vec{z})\\ &\quad-(\phi-\phi_h,\vec{b}_1A^{-1}\Pi_h\vec{z}).\end{aligned}$$

Combining (3.22) with (3.23) and rearranging terms, we thus find that

$$\begin{aligned}(3.24)\qquad I+II&=(\operatorname{div}\Pi_h\vec{z},\phi-\phi_h)-(A^{-*}\vec{b}_1(\phi-\phi_h),\Pi_h\vec{z})\\ &\quad-(\phi-\phi_h,P_h(\vec{b}_1A^{-1}\vec{z})-\vec{b}_1A^{-1}\vec{z}+\vec{b}_1A^{-1}(\vec{z}-\Pi_h\vec{z})).\end{aligned}$$

Using (3.7a), we deduce that

$$\begin{aligned}(3.25)\qquad (\operatorname{div}\Pi_h\vec{z},\phi-\phi_h)-(A^{-*}\vec{b}_1(\phi-\phi_h),\Pi_h\vec{z})&=(A^{-*}(\vec{r}-\vec{r}_h),\Pi_h\vec{z})\\ &=(A^{-*}(\vec{r}-\vec{r}_h),\Pi_h\vec{z}-\vec{z})+(A^{-*}(\vec{r}-\vec{r}_h),\vec{z}).\end{aligned}$$

Combining (3.21), (3.24) and (3.25) yields

$$\begin{aligned}(3.26)\qquad I+II+III&=(A^{-*}(\vec{r}-\vec{r}_h),\Pi_h\vec{z}-\vec{z})+(A^{-*}(\vec{r}-\vec{r}_h),\vec{z})\\ &\quad-(\phi-\phi_h,P_h(\vec{b}_1A^{-1}\vec{z})-\vec{b}_1A^{-1}\vec{z}+\vec{b}_1A^{-1}(\vec{z}-\Pi_h\vec{z}))+(P_h\phi-\phi_h,\hat{c}m).\end{aligned}$$

We recall that $\vec{z} = -(A\nabla m + \vec{b}_2 m)$ and integrate by parts to find

$$(A^{-*}(\vec{r} - \vec{r}_h), \vec{z}\,) = (A^{-*}(\vec{r} - \vec{r}_h), -A\nabla m - \vec{b}_2 m) \\ = (\operatorname{div}(\vec{r} - \vec{r}_h), m) - (\vec{b}_2 A^{-*}(\vec{r} - \vec{r}_h), m). \tag{3.27}$$

Recalling that $\gamma \in V_h$, we next deduce from (3.1b) and (3.3) that

$$\operatorname{div} \vec{r} - \vec{b}_2 A^{-*}\vec{r} + \hat{c}\phi = \gamma = \operatorname{div} \vec{r}_h - P_h(\vec{b}_2 A^{-*}\vec{r}_h) + P_h(\hat{c}\phi_h)$$

and thus

$$\operatorname{div}(\vec{r} - \vec{r}_h) - \vec{b}_2 A^{-*}(\vec{r} - \vec{r}_h) = \vec{b}_2 A^{-*}\vec{r}_h - P_h(\vec{b}_2 A^{-*}\vec{r}_h) - \hat{c}\phi + P_h(\hat{c}\phi_h). \tag{3.28}$$

Combining (3.26), (3.27), and (3.28) and then rearranging terms yield

$$\begin{aligned} I + II + III &= (A^{-*}(\vec{r} - \vec{r}_h), \Pi_h \vec{z} - \vec{z}\,) \\ &\quad + (\vec{b}_2 A^{-*}\vec{r}_h - P_h(\vec{b}_2 A^{-*}\vec{r}_h), m) + (-\hat{c}\phi + P_h(\hat{c}\phi_h), m) \\ &\quad - (\phi - \phi_h, P_h(\vec{b}_1 A^{-1}\vec{z}\,) - \vec{b}_1 A^{-1}\vec{z} + \vec{b}_1 A^{-1}(\vec{z} - \Pi_h \vec{z}\,)) + (P_h\phi - \phi_h, \hat{c}m) \\ &= (-\hat{c}\phi + \hat{c}P_h\phi + P_h(\hat{c}\phi_h) - \hat{c}\phi_h, m) + (A^{-*}(\vec{r} - \vec{r}_h), \Pi_h \vec{z} - \vec{z}\,) \\ &\quad + (\phi - \phi_h, \vec{b}_1 A^{-1}\vec{z} - P_h(\vec{b}_1 A^{-1}\vec{z}\,)) - (\phi - \phi_h, \vec{b}_1 A^{-1}(\vec{z} - \Pi_h \vec{z}\,)). \end{aligned} \tag{3.29}$$

We finally note that

$$\begin{aligned} (-\hat{c}\phi + \hat{c}P_h\phi + P_h(\hat{c}\phi_h) - \hat{c}\phi_h, m) &= (P_h\phi - \phi, \hat{c}m) + (P_h(\hat{c}\phi_h) - \hat{c}\phi_h, m) \\ &= (P_h\phi - \phi, \hat{c}m - P_h(\hat{c}m)) + (-\hat{c}\phi_h, m - P_h m) \\ &= (P_h\phi - \phi, \hat{c}m - P_h(\hat{c}m)) + (-\hat{c}\phi_h + \hat{c}\phi, m - P_h m) \\ &\quad - (\hat{c}\phi - P_h(\hat{c}\phi), m - P_h m) \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} (\vec{b}_2 A^{-*}\vec{r}_h - P_h(\vec{b}_2 A^{-*}\vec{r}_h), m) &= (\vec{b}_2 A^{-*}\vec{r}_h, m - P_h m) \\ &= (\vec{b}_2 A^{-*}(\vec{r}_h - \vec{r}), m - P_h m) + (\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r}), m - P_h m), \end{aligned} \tag{3.31}$$

so that inserting (3.30) and (3.31) into (3.29) and then inserting (3.29) into (3.21), we obtain

$$\begin{aligned} (P_h\phi - \phi_h, v) = {} & [(P_h\phi - \phi, \hat{c}m - P_h(\hat{c}m)) \\ & + (-\hat{c}\phi_h + \hat{c}\phi, m - P_h m) - (\hat{c}\phi - P_h(\hat{c}\phi), m - P_h m) \\ & + (\vec{b}_2 A^{-*}(\vec{r}_h - \vec{r}), m - P_h m) + (\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r}), m - P_h m)] \\ & + [(A^{-*}(\vec{r} - \vec{r}_h), \Pi_h \vec{z} - \vec{z}\,) - (\phi - \phi_h, \vec{b}_1 A^{-1}(\vec{z} - \Pi_h \vec{z}\,)) \\ & + (\phi - \phi_h, \vec{b}_1 A^{-1}\vec{z} - P_h(\vec{b}_1 A^{-1}\vec{z}\,))] \equiv I + II. \end{aligned} \tag{3.32}$$

We next find that

$$
\begin{aligned}
I \lesssim\ & \|\phi - P_h\phi\|_{D_{2d}} \|\hat{c}m - P_h(\hat{c}m)\|_{D_{2d}} \\
& + \|\phi - P_h\phi\|_{L_1(\Omega)} \|\hat{c}m - P_h(\hat{c}m)\|_{L_\infty(\Omega\setminus D_{2d})} \\
& + (\|\phi - \phi_h\|_{D_{2d}} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{D_{2d}} + \|\vec{r} - \vec{r}_h\|_{D_{2d}} \\
& + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{D_{2d}}) \|m - P_h m\|_{D_{2d}} \\
& + (\|\phi - \phi_h\|_{L_1(\Omega)} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(\Omega)} + \|\vec{r} - \vec{r}_h\|_{[L_1(\Omega)]^n} \\
& + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{L_1(\Omega)}) \|m - P_h m\|_{L_\infty(\Omega\setminus D_{2d})}.
\end{aligned} \tag{3.33}
$$

Using approximation properties, global regularity, and recalling that $\|v\|_D = 1$, we obtain

$$
\|cm - P_h(cm)\|_{D_{2d}} + \|m - P_h m\|_{D_{2d}} \lesssim h\|m\|_{H_2^1(\Omega)} \lesssim h\|v\|_D \lesssim h. \tag{3.34}
$$

We next note that since the coefficients A, $\vec{b}$, and c are smooth and v is supported in D, m must be smooth in $\Omega \setminus D_d$. Then using the approximation property (2.9), we deduce that

$$
\begin{aligned}
& \|cm - P_h(cm)\|_{L_\infty(\Omega\setminus D_{2d})} + \|m - P_h m\|_{L_\infty(\Omega\setminus D_{2d})} \\
& \qquad \lesssim h^j \textstyle\sum_{|\alpha|\le j} \|D^\alpha m\|_{L_\infty(\Omega\setminus D_d)}.
\end{aligned} \tag{3.35}
$$

Now for any fixed $x \in \Omega \setminus D_d$,

$$
m(x) = \int_\Omega G(x,y)v(y)dy = \int_D G(x,y)v(y)dy,
$$

and for any multiindex α with $|\alpha| \le j$,

$$
D^\alpha m(x) = \int_D D_x^\alpha G(x,y)v(y)dy.
$$

We next observe that $|y - x| \ge d$ for $y \in D$, so we may apply Lemma 2.1 while recalling that $d \le C$ to obtain

$$
|D^\alpha m(x)| \lesssim \|D^\alpha G(x,\cdot)\|_{L_\infty(D)} \|v\|_{L_1(D)} \lesssim d^{2-n-j} d^{n/2} \|v\|_D \lesssim d^{2-n/2-j} \tag{3.36}
$$

for any multiindex $|\alpha| \le j$. Combining (3.33), (3.34), (3.35), and (3.36) yields

$$
\begin{aligned}
I \lesssim\ & h(\|\phi - P_h\phi\|_{D_{2d}} + \|\phi - \phi_h\|_{D_{2d}} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{D_{2d}} \\
& + \|\vec{r} - \vec{r}_h\|_{D_{2d}} + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{D_{2d}}) \\
& + h^j d^{2-n/2-j} (\|\phi - P_h\phi\|_{L_1(\Omega)} + \|\phi - \phi_h\|_{L_1(\Omega)} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(\Omega)} \\
& + \|\vec{r} - \vec{r}_h\|_{[L_1(\Omega)]^n} + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{L_1(\Omega)}).
\end{aligned} \tag{3.37}
$$

We next find that

$$
\begin{aligned}
II \lesssim\ & (\|\vec{r} - \vec{r}_h\|_{D_{2d}} + \|\phi - \phi_h\|_{D_{2d}})(\|\vec{z} - \Pi_h\vec{z}\|_{D_{2d}} \\
& + \|\vec{b}_1 A^{-1}\vec{z} - P_h(\vec{b}_1 A^{-1}\vec{z})\|_{D_{2d}}) \\
& + (\|\vec{r} - \vec{r}_h\|_{[L_1(\Omega)]^n} + \|\phi - \phi_h\|_{L_1(\Omega)})(\|\vec{z} - \Pi_h\vec{z}\|_{L_\infty(\Omega\setminus D_{2d})} \\
& + \|\vec{b}_1 A^{-1}\vec{z} - P_h(\vec{b}_1 A^{-1}\vec{z})\|_{\Omega\setminus D_{2d}}).
\end{aligned} \tag{3.38}
$$

Using the approximation properties (2.8) and (2.9) along with global regularity yields

$$\begin{aligned}\|\vec{z}-\Pi_h\vec{z}\|_{D_{2d}}+\|\vec{b}_1A^{-1}\vec{z}-P_h(\vec{b}_1A^{-1}\vec{z})\|_{D_{2d}} &\lesssim h\|\vec{z}\|_{[H_2^1(\Omega)]^n}\\ &\lesssim h\|m\|_{H_2^2(\Omega)}\lesssim h\|v\|_{D_d}\lesssim h.\end{aligned}\tag{3.39}$$

Proceeding as in (3.35) through (3.36) while recalling that $j \le k$, we obtain

$$\begin{aligned}\|\vec{z}-\Pi_h\vec{z}\|_{L_\infty(\Omega\setminus D_{2d})}&+\|\vec{b}_1A^{-1}\vec{z}-P_h(\vec{b}_1A^{-1}\vec{z})\|_{\Omega\setminus D_{2d}}\\ &\lesssim h^j\textstyle\sum_{|\alpha|\le j+1}\|D^\alpha m\|_{L_\infty(\Omega\setminus D_d)}\lesssim h^j d^{1-n/2-j}.\end{aligned}\tag{3.40}$$

Combining (3.38), (3.39), and (3.40), we find

$$\begin{aligned}II &\lesssim h(\|\vec{r}-\vec{r}_h\|_{D_{2d}}+\|\phi-\phi_h\|_{D_{2d}})\\ &\quad+h^jd^{1-n/2-j}(\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n}+\|\phi-\phi_h\|_{L_1(\Omega)}).\end{aligned}\tag{3.41}$$

Combining (3.20), (3.32), (3.37), and (3.41) while multiplying by $\frac{1}{d}$, changing D_{2d} to D_d (which is again an inconsequential change of notation), and applying the triangle inequality yield

$$\begin{aligned}\tfrac{1}{d}\|\phi-\phi_h\|_D &\lesssim \tfrac{1}{d}(\|\phi-P_h\phi\|_{D_d}+\|\hat{c}\phi-P_h(\hat{c}\phi)\|_{D_d})\\ &\quad+\tfrac{h}{d}(\|\vec{r}-\vec{r}_h\|_{D_d}+\|\phi-\phi_h\|_{D_d}+\|\vec{b}_2A^{-*}\vec{r}-P_h(\vec{b}_2A^{-*}\vec{b})\|_{D_d})\\ &\quad+h^jd^{-n/2-j}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\\ &\quad+h^jd^{1-n/2-j}(\|\phi-P_h\phi\|_{L_1(\Omega)}+\|\hat{c}\phi-P_h(\hat{c}\phi)\|_{L_1(\Omega)}\\ &\quad+\|\vec{b}_2A^{-*}\vec{r}-P_h(\vec{b}_2A^{-*}\vec{r})\|_{L_1(\Omega)}).\end{aligned}\tag{3.42}$$

In order to complete the proof of (3.4), we insert (3.42) (with D changed to D_d and D_d changed to D_{2d}) into (3.19), change D_{2d} in the result to D_d, and finally add the resulting equation to (3.42). $\square$

4. Proofs of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2. We first state and prove some lemmas and then carry out the proofs.

4.1. **Discrete Green's functions.** In proving Theorems 1.1 and 1.2, we shall use discrete Green's functions in order to represent the errors $[(\vec{p}-\vec{p}_h)(x_0)]_i$, $[(\tilde{p}-\tilde{p}_h)(x_0)]_i$, $(P_hu-u_h)(x_0)$, and $(P_hu-u_h^*)(x_0)$ and thereby reduce the problem to bounding certain weighted L_1 norms of the errors in mixed finite element approximations to these discrete Green's functions. We first write down the problems the two necessary discrete Green's functions will solve. In proving Theorem 1.1, we shall employ the problem

$$\begin{aligned}-\operatorname{div}(A^*\nabla\phi+\vec{b}_1\phi)+\vec{b}_2\nabla\phi+\hat{c}\phi=\operatorname{div}\vec{\delta}_i^0-P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)\text{ in }\Omega,\\ \phi=0\text{ on }\partial\Omega.\end{aligned}\tag{4.1}$$

With $\vec{r}=-(A^*\nabla\phi+\vec{b}\phi)$ and recalling our convention that either $\vec{b}_2=0$ (in the divergence form case) or $\vec{b}_1=0$ (in the conservation form case), (4.1) has mixed form

$$(A^{-*}\vec{r},\vec{q}) - (\operatorname{div}\vec{q},\phi) + (A^{-*}\vec{b}_1\phi,\vec{q}) = 0, \tag{4.2a}$$

$$(\operatorname{div}\vec{r},v) - (\vec{b}_2A^{-*}\vec{r},v) + (\hat{c}\phi,v) = (\operatorname{div}\vec{\delta}_i^0 - P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0),v) \tag{4.2b}$$

for all $\{\vec{q},v\} \in H(\operatorname{div};\Omega) \times L_2(\Omega)$, with corresponding mixed finite element equations

$$(A^{-*}\vec{r}_h,\vec{q}_h) - (\operatorname{div}\vec{q}_h,\phi_h) + (A^{-*}\vec{b}_1\phi_h,\vec{q}_h) = 0, \tag{4.3a}$$

$$(\operatorname{div}\vec{r}_h,v_h) - (\vec{b}_2A^{-*}\vec{r}_h,v_h) + (\hat{c}\phi_h,v_h) = (\operatorname{div}\vec{\delta}_i^0 - P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0),v_h) \tag{4.3b}$$

for all $\{\vec{q}_h,v_h\} \in \vec{Q}_h \times V_h$.

Similarly, in proving Theorem 1.2 we shall employ the problem

$$-\operatorname{div}(A^*\nabla\psi + \vec{b}_1\psi) + \vec{b}_2\nabla\psi + \hat{c}\psi = \delta^0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega. \tag{4.4}$$

With $\vec{z} = -(A^*\nabla\psi + \vec{b}_1\psi)$, (4.4) has mixed form

$$(A^{-*}\vec{z},\vec{q}) - (\operatorname{div}\vec{q},\psi) + (A^{-*}\vec{b}_1\psi,\vec{q}) = 0, \tag{4.5a}$$

$$(\operatorname{div}\vec{z},v) - (\vec{b}_2A^{-*}\vec{z},v) + (\hat{c}\psi,v) = (\delta^0,v) \tag{4.5b}$$

for all $\{\vec{q},v\} \in H(\operatorname{div};\Omega) \times L_2(\Omega)$, with corresponding mixed finite element equations

$$(A^{-*}\vec{z}_h,\vec{q}_h) - (\operatorname{div}\vec{q}_h,\psi_h) + (A^{-*}\vec{b}_1\psi_h,\vec{q}_h) = 0, \tag{4.6a}$$

$$(\operatorname{div}\vec{z}_h,v_h) - (\vec{b}_2A^{-*}\vec{z}_h,v_h) + (\hat{c}\psi_h,v_h) = (\delta^0,v_h) \tag{4.6b}$$

for all $\{\vec{q}_h,v_h\} \in \vec{Q}_h \times V_h$.

We now state a lemma representing the errors in the mixed finite element approximations in terms of the discrete Green's functions defined above.

Lemma 4.1. *Let p denote either $\vec{p}$ (in the divergence form case, so that $\vec{b}_1 = \vec{b}$, $\vec{b}_2 = 0$, and $\hat{c} = c$) or $\tilde{p}$ (in the conservation form case, so that $\vec{b}_1 = 0$, $\vec{b}_2 = -\vec{b}$, and $\hat{c} = c^*$). Then for any $x_0 \in \Omega$,*

$$\begin{aligned}
[(p-p_h)(x_0)]_i = {} & [(p-\Pi_hp)(x_0)]_i + (A^{-1}(\Pi_hp-p),\vec{\delta}_i^0) \\
& + (A^{-1}(p-\Pi_hp),\vec{r}_h-\vec{r}) + (\vec{b}_1A^{-1}(p-\Pi_hp),\phi_h-\phi) \\
& - (u-P_hu,\vec{b}_2A^{-*}\vec{\delta}_i^0) + (\vec{b}_2A^{-*}(\vec{r}_h-\vec{r}),u-P_hu) \\
& + (\vec{b}_2A^{-*}\vec{r} - P_h(\vec{b}_2A^{-*}\vec{r}),u-P_hu) \\
& - (u-P_hu,\hat{c}(\phi_h-P_h\phi) - P_h(\hat{c}(\phi_h-P_h\phi))) \\
& - (u-P_hu,\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)) + (\operatorname{div}p - P_h\operatorname{div}p,\phi-P_h\phi).
\end{aligned} \tag{4.7}$$

Also, let $\hat{u}_h$ denote u_h (in the divergence form case) or u_h^* (in the conservation form case). Then

$$\begin{aligned}
(P_hu-\hat{u}_h)(x_0) = {} & (A^{-1}(p-\Pi_hp),\vec{z}_h-\vec{z}) + (\vec{b}_1A^{-1}(p-\Pi_hp),\psi_h-\psi) \\
& -(u-P_hu,\hat{c}(\psi_h-P_h\psi) - P_h(\hat{c}(\psi_h-P_h\psi))) \\
& -(u-P_hu,\hat{c}P_h\psi - P_h(\hat{c}P_h\psi)) + (\vec{b}_2A^{-*}(\vec{z}_h-\vec{z}),u-P_hu) \\
& +(\vec{b}_2A^{-*}\vec{z} - P_h(\vec{b}_2A^{-*}\vec{z}),u-P_hu) + (\operatorname{div}p - P_h\operatorname{div}p,\psi-P_h\psi).
\end{aligned} \tag{4.8}$$

Proof. We begin the proof of (4.7) by using (2.14) to write

$$\begin{aligned}[(p-p_h)\,(x_0)]_i &= [(p-\Pi_h p)(x_0)]_i + (A^{-1}(\Pi_h p - p_h), \vec{\delta}_i^0) \\ &= [(p-\Pi_h p)(x_0)]_i + (A^{-1}(\Pi_h p - p), \vec{\delta}_i^0) + (A^{-1}(p-p_h), \vec{\delta}_i^0).\end{aligned} \tag{4.9}$$

Using (2.6a), noting that $\operatorname{div}\vec{\delta}_i^0 \in V_h$, and employing (4.3b), we next find that

$$\begin{aligned}(A^{-1}(p-p_h), \vec{\delta}_i^0) &= (\operatorname{div}\vec{\delta}_i^0, u-\hat{u}_h) - (A^{-1}\vec{b}_2(u-\hat{u}_h), \vec{\delta}_i^0) \\ &= (\operatorname{div}\vec{\delta}_i^0 - P_h(\vec{b}_2 A^{-*}\vec{\delta}_i^0), P_h u - \hat{u}_h) + (\vec{b}_2 A^{-*}\vec{\delta}_i^0, P_h u - \hat{u}_h) \\ &\quad -(u-\hat{u}_h, \vec{b}_2 A^{-*}\vec{\delta}_i^0) \\ &= (\operatorname{div}\vec{\delta}_i^0 - P_h(\vec{b}_2 A^{-*}\vec{\delta}_i^0), P_h u - \hat{u}_h) - (u - P_h u, \vec{b}_2 A^{-*}\vec{\delta}_i^0) \\ &= (\operatorname{div}\vec{r}_h, P_h u - \hat{u}_h) - (\vec{b}_2 A^{-*}\vec{r}_h, P_h u - \hat{u}_h) + (\hat{c}\phi_h, P_h u - \hat{u}_h) \\ &\quad -(u - P_h u, \vec{b}_2 A^{-*}\vec{\delta}_i^0) \\ &= [(\operatorname{div}\vec{r}_h, u-\hat{u}_h) - (A^{-1}\vec{b}_2(u-\hat{u}_h), \vec{r}_h)] + [(\hat{c}\phi_h, P_h u - \hat{u}_h)] \\ &\quad +[(\vec{b}_2 A^{-*}\vec{r}_h, u - P_h u) - (u - P_h u, \vec{b}_2 A^{-*}\vec{\delta}_i^0)] \equiv I + II + III.\end{aligned} \tag{4.10}$$

Again using the error equation (2.6a) yields

$$I = (\operatorname{div}\vec{r}_h, u-\hat{u}_h) - (A^{-1}\vec{b}_2(u-\hat{u}_h), \vec{r}_h) = (A^{-1}(p-p_h), \vec{r}_h). \tag{4.11}$$

Employing the dual mixed equation (4.3a), the error equation (2.6b), the commuting diagram property (2.7), and finally (4.3a) once more, we next deduce that

$$\begin{aligned}(A^{-1}p_h, \vec{r}_h) &= (A^{-*}\vec{r}_h, p_h) = (\operatorname{div} p_h, \phi_h) - (A^{-*}\vec{b}_1\phi_h, p_h) \\ &= (\operatorname{div} p_h, \phi_h) - (\vec{b}_1 A^{-1} p_h, \phi_h) \\ &= (\operatorname{div} p, \phi_h) - (\vec{b}_1 A^{-1} p, \phi_h) + (\hat{c}(u-\hat{u}_h), \phi_h) \\ &= (\operatorname{div}\Pi_h p, \phi_h) - (\vec{b}_1 A^{-1} p, \phi_h) + (\hat{c}(u-\hat{u}_h), \phi_h) \\ &= (A^{-*}\vec{r}_h, \Pi_h p) + (A^{-*}\vec{b}_1\phi_h, \Pi_h p) - (\vec{b}_1 A^{-1} p, \phi_h) + (\hat{c}(u-\hat{u}_h), \phi_h) \\ &= (A^{-1}\Pi_h p, \vec{r}_h) + (\vec{b}_1 A^{-1}(\Pi_h p - p), \phi_h) + (\hat{c}(u-\hat{u}_h), \phi_h).\end{aligned} \tag{4.12}$$

Inserting (4.12) into (4.11), we find that

$$I = (A^{-1}(p-\Pi_h p), \vec{r}_h) + (\vec{b}_1 A^{-1}(p-\Pi_h p), \phi_h) - (\hat{c}(u-\hat{u}_h), \phi_h). \tag{4.13}$$

We then combine (4.13) with term II from (4.10), rearrange terms, use the dual mixed equation (4.2a), and finally employ the commuting diagram property (2.7) to deduce that

$$
\begin{aligned}
I + II &= (A^{-1}(p - \Pi_h p), \vec{r}_h) + (\vec{b}_1 A^{-1}(p - \Pi_h p), \phi_h) - (\hat{c}(u - P_h u), \phi_h) \\
&= (A^{-1}(p - \Pi_h p), \vec{r}_h - \vec{r}\,) + (\vec{b}_1 A^{-1}(p - \Pi_h p), \phi_h - \phi) \\
&\quad + (A^{-1}(p - \Pi_h p), \vec{r}\,) + (\vec{b}_1 A^{-1}(p - \Pi_h p), \phi) - (u - P_h u, \hat{c}\phi_h) \\
&= (A^{-1}(p - \Pi_h p), \vec{r}_h - \vec{r}\,) + (\vec{b}_1 A^{-1}(p - \Pi_h p), \phi_h - \phi) \\
&\quad + (A^{-*}\vec{r}, p - \Pi_h p) + (A^{-*}\vec{b}_1\phi, p - \Pi_h p) - (u - P_h u, \hat{c}\phi_h) \\
&= (A^{-1}(p - \Pi_h p), \vec{r}_h - \vec{r}\,) + (\vec{b}_1 A^{-1}(p - \Pi_h p), \phi_h - \phi) \\
&\quad + (\operatorname{div}(p - \Pi_h p), \phi) - (u - P_h u, \hat{c}(\phi_h - P_h\phi)) - (u - P_h u, \hat{c}P_h\phi) \\
&= (A^{-1}(p - \Pi_h p), \vec{r}_h - \vec{r}\,) + (\vec{b}_1 A^{-1}(p - \Pi_h p), \phi_h - \phi) \\
&\quad + (\operatorname{div} p - P_h \operatorname{div} p, \phi - P_h\phi) \\
&\quad - (u - P_h u, \hat{c}(\phi_h - P_h\phi) - P_h(\hat{c}(\phi_h - P_h\phi))) \\
&\quad - (u - P_h u, \hat{c}P_h\phi - P_h(\hat{c}P_h\phi)).
\end{aligned} \tag{4.14}
$$

We finally note that

$$
\begin{aligned}
III &= (\vec{b}_2 A^{-*}(\vec{r}_h - \vec{r}\,), u - P_h u) \\
&\quad + (\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r}\,), u - P_h u) - (u - P_h u, \vec{b}_2 A^{-*}\vec{\delta}_i^{\,0}).
\end{aligned} \tag{4.15}
$$

Combining (4.9), (4.10), (4.14), and (4.15) completes the proof of (4.7).

In order to prove (4.8), we note from (2.13) and (4.6b) that

$$
\begin{aligned}
(P_h u - \hat{u}_h)(x_0) &= (P_h u - \hat{u}_h, \delta^0) \\
&= (\operatorname{div} \vec{z}_h, P_h u - \hat{u}_h) - (\vec{b}_2 A^{-*}\vec{z}_h, P_h u - \hat{u}_h) + (\hat{c}\psi_h, P_h u - \hat{u}_h)
\end{aligned}
$$

and we proceed as in (4.10) through (4.15) with appropriate slight modifications. □

4.2. **A partition of Ω.** We begin this section by partitioning Ω into special subdomains. Recall that we are seeking to estimate the errors in various finite element approximations to u, $\vec{p}$, and $\tilde{p}$ at some point $x_0 \in \Omega$. We let $M > 0$ be an arbitrary constant which will later be taken to be large enough and define $B_{Mh} = \{y \in \Omega : |y - x_0| < Mh\}$. We next let $d_i = 2^{-i}$ for $i = 0, 1, 2, \ldots$ and define

$$
\begin{aligned}
\Omega_i &= \{y \in \Omega : d_{i+1} < |y - x_0| < d_i\}, \\
\Omega_i^{'} &= \{y \in \Omega : d_{i+2} < |y - x_0| < d_{i-1}\}, \\
\Omega_i^{''} &= \{y \in \Omega : d_{i+3} < |y - x_0| < d_{i-2}\},
\end{aligned}
$$

etc. Thus the Ω_i's are annuli centered at x_0, with a larger subscript i indicating a smaller radius and a larger number of primes indicating a thicker annulus. For notational ease we shall assume that Ω has unit radius. We then let J be the smallest integer such that $\Omega = B_{Mh} \cup (\bigcup_{i=0}^{J} \Omega_i)$.

We also collect here a few simple arithmetic results. We first note that for $M > 0$, $J \lesssim \log \frac{1}{h}$ since $2^J Mh \approx 1$. Since $d_i = c2^{J-i}Mh$ for some $1 \le c \le 2$,

$$\begin{aligned} \sum_{i=0}^J \left(\frac{h}{d_i}\right)^\ell &\lesssim \sum_{i=0}^J \left(\frac{h}{2^{J-i}Mh}\right)^\ell \lesssim \frac{1}{M^\ell} \sum_{i=0}^J \frac{1}{2^{i\ell}} \lesssim \frac{1}{M^\ell} J^{\delta_{0\ell}} \\ &\lesssim \frac{1}{M^\ell} \left(\log \frac{1}{h}\right)^{\delta_{0\ell}} \end{aligned} \tag{4.16}$$

and

$$\sum_{i=0}^J d_i \left(\frac{h}{d_i}\right)^\ell \lesssim \sum_{i=0}^J d_i \left(\frac{h}{2^{J-i}Mh}\right)^\ell \lesssim \frac{1}{M^\ell} \sum_{i=0}^J 2^{-i} \lesssim \frac{1}{M^\ell} \tag{4.17}$$

for $\ell \ge 0$. We also note that

$$\sum_{i=0}^J \left(\frac{d_i}{h}\right)^\ell e^{-cd_i/h} \lesssim \int_0^\infty x^\ell e^{-cx} dx \lesssim 1. \tag{4.18}$$

Finally we recall that $d_J = cMh$ for some $1 \le c \le 2$ so that $\left(\frac{d_J}{h}\right)^s d_J^{n/2} \frac{h}{d_J} = C(M)h^{n/2}$ and $\left(\frac{d_J}{h}\right)^s d_J^{n/2} \frac{1}{d_J} = C(M)h^{n/2-1}$.

4.3. **Another lemma: Bounds for approximation errors.**

Lemma 4.2. *Let ϕ and $\vec{r}$ be given by* (4.2a) *and* (4.2b)*, let $0 \le m \le j-1$, and let $0 \le \ell \le j$. Then*

$$\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{\Omega_i} + \|\phi - P_h\phi\|_{\Omega_i} \lesssim h d_i^{-n/2} \left((\tfrac{d_i}{h})^n e^{-cd_i/h} + (\tfrac{h}{d_i})^{j-1} \right), \tag{4.19}$$

$$\begin{aligned} &\|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{\Omega_i} + \|\vec{r} - \Pi_h \vec{r}\|_{\Omega_i} \\ &\qquad \lesssim d_i^{-n/2} \left((\tfrac{d_i}{h})^n e^{-cd_i/h} + (\tfrac{h}{d_i})^j \right), \end{aligned} \tag{4.20}$$

$$\|\phi - P_h\phi\|_{L_1,x_0,-m} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1,x_0,-m} \lesssim h \left(\log \tfrac{1}{h}\right)^{\delta_{j-1,m}}, \tag{4.21}$$

$$\|\phi - P_h\phi\|_{L_1,x_0,-\ell} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1,x_0,-\ell} \lesssim 1, \tag{4.22}$$

$$\|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{L_1,x_0,-\ell} \lesssim \left(\log \tfrac{1}{h}\right)^{\delta_{j\ell}}, \tag{4.23}$$

$$\|\vec{\delta}_i^0\|_{L_1,x_0,-\ell} + \|\phi\|_{L_1(\Omega)} \lesssim 1, \tag{4.24}$$

$$\begin{aligned} &\|\hat{c}(\phi_h - P_h\phi) - P_h(\hat{c}(\phi_h - P_h\phi))\|_{L_1,x_0,-m} \\ &\qquad + \|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1,x_0,-m} \lesssim h(1 + \|\phi - \phi_h\|_{L_1,x_0,-m}). \end{aligned} \tag{4.25}$$

Proof. In order to prove (4.19) and (4.20), we shall need to introduce two new mixed problems. We let ω be a smooth cutoff function which is 1 on Ω_i'', 0 on $\Omega \setminus \Omega_i'''$, satisfies $0 \le \omega \le 1$, and has a bounded first derivative, i.e., $\|\nabla \omega\|_{L_\infty(\Omega)} \lesssim \frac{1}{d_i}$. We then let $\phi_{i,1}$ and $\phi_{i,2}$ satisfy

$$\begin{aligned} -\operatorname{div}(A^*\phi_{i,1} + \vec{b}_1\phi_{i,1}) + \vec{b}_2\nabla\phi_{i,1} + \hat{c}\phi_{i,1} &= \operatorname{div}(\omega\vec{\delta}_i^0) - \omega P_h(\vec{b}_2 A^{-*}\vec{\delta}_i^0) \text{ in } \Omega, \\ \phi_{i,1} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} &-\operatorname{div}(A^*\phi_{i,2} + \vec{b}_1\phi_{i,2}) + \vec{b}_2\nabla\phi_{i,2} + \hat{c}\phi_{i,2} \\ &\qquad = \operatorname{div}((1-\omega)\vec{\delta}_i^0) - (1-\omega)P_h(\vec{b}_2 A^{-*}\vec{\delta}_i^0) \text{ in } \Omega, \quad \phi_{i,2} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.27}$$

We also let $\vec{r}_{i,1} = -(A^*\phi_{i,1} + \vec{b}_1\phi_{i,1})$ and $\vec{r}_{i,2} = -(A^*\phi_{i,2} + \vec{b}_1\phi_{i,2})$. By linearity and uniqueness, we then have $\phi = \phi_{i,1} + \phi_{i,2}$ and $\vec{r} = \vec{r}_{i,1} + \vec{r}_{i,2}$. Also, by the linearity of P_h and Π_h, we have

$$\begin{aligned}\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{\Omega_i} + \|\phi - P_h\phi\|_{\Omega_i} &\le \|\hat{c}\phi_{i,1} - P_h(\hat{c}\phi_{i,1})\|_{\Omega_i} \\ &+\|\hat{c}\phi_{i,2} - P_h(\hat{c}\phi_{i,2})\|_{\Omega_i} + \|\phi_{i,1} - P_h\phi_{i,1}\|_{\Omega_i} + \|\phi_{i,2} - P_h\phi_{i,2}\|_{\Omega_i}\end{aligned} \tag{4.28}$$

and

$$\begin{aligned}&\|\vec{b}_2A^{-*}\vec{r} - P_h(\vec{b}_2A^{-*}\vec{r})\|_{\Omega_i} + \|\vec{r} - \Pi_h\vec{r}\|_{\Omega_i} \\ &\quad\le \|\vec{b}_2A^{-*}\vec{r}_{i,1} - P_h(\vec{b}_2A^{-*}\vec{r}_{i,1})\|_{\Omega_i} + \|\vec{b}_2A^{-*}\vec{r}_{i,2} - P_h(\vec{b}_2A^{-*}\vec{r}_{i,2})\|_{\Omega_i} \\ &\quad\quad+\|\vec{r}_{i,1} - \Pi_h\vec{r}_{i,1}\|_{\Omega_i} + \|\vec{r}_{i,2} - \Pi_h\vec{r}_{i,2}\|_{\Omega_i}.\end{aligned} \tag{4.29}$$

Thus we must bound the right-hand sides of (4.28) and (4.29) by the right-hand sides of (4.19) and (4.20), respectively.

We first use the approximation property (2.9), the regularity assumptions (2.2) and (2.3), Hölder's inequality, and finally (2.16) to find that

$$\begin{aligned}\|\hat{c}\phi_{i,1} - P_h(\hat{c}\phi_{i,1})\|_{\Omega_i} + \|\phi_{i,1} - P_h\phi_{i,1}\|_{\Omega_i} &\lesssim h\|\phi_{i,1}\|_{H_2^1(\Omega)} \\ &\lesssim h(\|\omega\vec{\delta}_i^0\|_\Omega + \|\omega P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)\|_\Omega) \lesssim d_i^{n/2}h\|\vec{\delta}_i^0\|_{L\infty(\Omega_i''')} \\ &\lesssim hd_i^{n/2}h^{-n}e^{-cd_i/h} \lesssim hd_i^{-n/2}\left(\tfrac{d_i}{h}\right)^n e^{-cd_i/h}.\end{aligned} \tag{4.30}$$

We next use Hölder's inequality and the approximation assumption (2.9) to deduce that

$$\|\hat{c}\phi_{i,2} - P_h(\hat{c}\phi_{i,2})\|_{\Omega_i} + \|\phi_{i,2} - P_h\phi_{i,2}\|_{\Omega_i} \lesssim d_i^{n/2}h^j\|\phi_{i,2}\|_{W_\infty^j(\Omega_i')}. \tag{4.31}$$

We note that $\operatorname{div}[(1-\omega)\vec{\delta}_i^0] - (1-\omega)P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0) \equiv 0$ on Ω_i'', so $\phi_{i,2}$ is smooth on Ω_i'. For $x \in \Omega_i'$ and $|\alpha| \le j$, we use integration by parts and Lemma 2.1 along with (2.15) and the stability of P_h in L_1 to obtain

$$\begin{aligned}D^\alpha\phi_{i,2}(x) &= D^\alpha \int_{\Omega\setminus\Omega_i''} G(x,y)(\operatorname{div}[(1-\omega)\vec{\delta}_i^0] - (1-\omega)P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0))dy \\ &= -D^\alpha \int_{\Omega\setminus\Omega_i''}[\nabla_yG(x,y)(1-\omega)\vec{\delta}_i^0 + G(x,y)(1-\omega)P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)]dy \\ &= -\int_{\Omega\setminus\Omega_i''} D_x^\alpha[\nabla_yG(x,y)(1-\omega)\vec{\delta}_i^0 + G(x,y)(1-\omega)P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)]dy \\ &\lesssim d_i^{2-n-j-1}\|\vec{\delta}_i^0\|_{L_1(\Omega)} \lesssim d_i^{1-n-j}.\end{aligned} \tag{4.32}$$

Combining (4.32) with (4.31) and (4.30) yields (4.19). The proof of (4.20) is very similar, and we omit it here.

In order to prove (4.21), we first break (4.1) into two problems with homogeneous Dirichlet boundary conditions, one with right-hand side $\operatorname{div}\vec{\delta}_i^0$ and solution ϕ_1 and the other with right-hand side $-P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)$ and solution ϕ_2. By linearity and uniqueness, $\phi = \phi_1 + \phi_2$. We then use the approximation assumption (2.9), the regularity assumptions (2.2) and (2.3), and finally (2.15) to find that

$$\begin{aligned}&\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{B_{Mh}} + \|\phi - P_h\phi\|_{B_{Mh}} \lesssim h\|\phi\|_{H_2^1(\Omega)} \\ &\quad\lesssim h(\|\phi_1\|_{H_2^1(\Omega)} + \|\phi_2\|_{H_2^1(\Omega)}) \lesssim h(\|\vec{\delta}_i^0\|_\Omega + \|P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)\|_\Omega) \lesssim h^{1-n/2}.\end{aligned} \tag{4.33}$$

We next use Hölder's inequality, (4.33) and (4.19), and finally (4.16) and (4.18) to find

$$
\begin{aligned}
&\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1,x_0,-m} + \|\phi - P_h\phi\|_{L_1(\Omega),x_0,-m} \\
&\quad\lesssim \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(B_{Mh})} + \|\phi - P_h\phi\|_{L_1(B_{Mh})} \\
&\qquad + \textstyle\sum_{i=0}^J \left(\frac{d_i}{h}\right)^m (\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(\Omega_i)} + \|\phi - P_h\phi\|_{L_1(\Omega_i)}) \\
&\quad\lesssim (Mh)^{n/2}(\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{B_{Mh}} + \|\phi - P_h\phi\|_{B_{Mh}}) \\
&\qquad + \textstyle\sum_{i=0}^J \left(\frac{d_i}{h}\right)^m d_i^{n/2}(\|\hat{c}\phi - P_h(\hat{c}\phi)\|_{\Omega_i} + \|\phi - P_h\phi\|_{\Omega_i}) \\
&\quad\lesssim h^{n/2}h^{1-n/2} + \textstyle\sum_{i=0}^J \left(\frac{d_i}{h}\right)^m d_i^{n/2} h d_i^{-n/2}((\frac{d_i}{h})^n e^{-cd_i/h} + (\frac{h}{d_i})^{j-1}) \\
&\quad\lesssim h + h\textstyle\sum_{i=0}^J (\frac{d_i}{h})^m((\frac{d_i}{h})^n e^{-cd_i/h} + (\frac{h}{d_i})^{j-1}) \\
&\quad\lesssim h + h\textstyle\sum_{i=0}^J \left(\frac{d_i}{h}\right)^{m+n} e^{-cd_i/h} + (\frac{h}{d_i})^{j-1-m} \lesssim h\left(\log\frac{1}{h}\right)^{\delta_{j-1,m}}.
\end{aligned}
\tag{4.34}
$$

The proofs of (4.22) and (4.23) are similar.

In order to prove (4.24), we first use Hölder's inequality and the triangle inequality to obtain

$$
\begin{aligned}
\|\vec{\delta}_i^0\|_{L_1,x_0,-\ell} + \|\phi\|_{L_1(\Omega)} &\lesssim \|\vec{\delta}_i^0\|_{L_1(B_{Mh})} + (Mh)^{n/2}\|\phi\|_{B_{Mh}} \\
&+ \textstyle\sum_{i=0}^J ((\frac{d_i}{h})^\ell d_i^n \|\vec{\delta}_i^0\|_{L_\infty(\Omega_i)} + d_i^{n/2}\|\phi_{i,1}\|_{\Omega_i} + d_i^n\|\phi_{i,2}\|_{L_\infty(\Omega_i)}).
\end{aligned}
\tag{4.35}
$$

We next find via (2.15) and computations as in (4.33) that

$$\|\vec{\delta}_i^0\|_{L_1(B_{Mh})} + (Mh)^{n/2}\|\phi\|_{B_{Mh}} \lesssim 1. \tag{4.36}$$

We may also deduce as in (4.30) that

$$\|\phi_{i,1}\|_{\Omega_i} \lesssim d_i^{-n/2}\left(\frac{d_i}{h}\right)^n e^{-cd_i/h} \tag{4.37}$$

and as in (4.32) that

$$\|\phi_{i,2}\|_{L_\infty(\Omega_i)} \lesssim d_i^{1-n}. \tag{4.38}$$

We finally note using (2.16) that

$$\|\vec{\delta}_i^0\|_{L_\infty(\Omega_i)} \lesssim h^{-n}e^{-cd_i/h}. \tag{4.39}$$

Inserting (4.36), (4.37), (4.38), and (4.39) into (4.35) and summing using (4.17) and (4.18) yield (4.24).

In order to prove (4.25), we first use the superapproximation assumption (2.11) and the L_1 stability of P_h to find that for any $0 \le m \le j-1$,

$$
\begin{aligned}
&\|\hat{c}(\phi_h - P_h\phi) - P_h(\hat{c}(\phi_h - P_h\phi))\|_{L_1,x_0,-m} \\
&\quad\lesssim \|\hat{c}(\phi_h - P_h\phi) - P_h(\hat{c}(\phi_h - P_h\phi))\|_{L_1(B_{Mh})} \\
&\qquad + \textstyle\sum_{i=0}^J (\frac{d_i}{h})^m \|\hat{c}(\phi_h - P_h\phi) - P_h(\hat{c}(\phi_h - P_h\phi))\|_{L_1(\Omega_i)} \\
&\quad\lesssim h\|\phi_h - P_h\phi\|_{L_1(B_{2Mh})} + \textstyle\sum_{i=0}^J (\frac{d_i}{h})^m h\|\phi_h - P_h\phi\|_{L_1(\Omega_i')} \\
&\quad\lesssim h\|\phi_h - \phi\|_{L_1(B_{2Mh})} + \textstyle\sum_{i=0}^J (\frac{d_i}{h})^m h\|\phi_h - \phi\|_{L_1(\Omega_i')} \\
&\quad\lesssim h\|\phi - \phi_h\|_{L_1,x_0,-m}.
\end{aligned}
\tag{4.40}
$$

We next obtain

$$
\begin{aligned}
&\|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1,x_0,-m} \\
&\quad\lesssim \|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1(B_{Mh})} + \textstyle\sum_{i=0}^{J} \left(\frac{d_i}{h}\right)^m \|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1(\Omega_i)}.
\end{aligned} \tag{4.41}
$$

Employing the superapproximation assumption (2.11), the L_1 stability of P_h, and (4.24), we find

$$
\|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1(B_{Mh})} \lesssim h\|P_h\phi\|_{L_1(B_{2Mh})} \lesssim h\|\phi\|_{L_1(\Omega)} \lesssim h.
$$

Recalling from (4.26) and (4.27) the definitions of $\phi_{i,1}$ and $\phi_{i,2}$, we note that

$$
\begin{aligned}
&\|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1(\Omega_i)} \\
&\qquad \le \|\hat{c}P_h\phi_{i,1} - P_h(\hat{c}P_h\phi_{i,1})\|_{L_1(\Omega_i)} + \|\hat{c}P_h\phi_{i,2} - P_h(\hat{c}P_h\phi_{i,2})\|_{L_1(\Omega_i)}.
\end{aligned}
$$

Using Hölder's inequality, the superapproximation assumption (2.11), the L_2 stability of P_h, and proceeding as in (4.30), we next find

$$
\begin{aligned}
\|\hat{c}P_h\phi_{i,1} - P_h(\hat{c}P_h\phi_{i,1})\|_{L_1(\Omega_i)} &\lesssim d_i^{n/2} h\|P_h\phi_{i,1}\|_{L_2((\Omega_i)_h)} \\
&\lesssim h d_i^{n/2}\|\phi_{i,1}\|_{\Omega_i'} \lesssim h\left(\frac{d_i}{h}\right)^n e^{-cd_i/h}.
\end{aligned}
$$

We next use Hölder's inequality and the approximation assumption (2.9) while noting that $D^\alpha P_h\phi \equiv 0$ on each element for $|\alpha| = j$ since $P_h\phi$ is a polynomial of order $j-1$ in order to deduce

$$
\begin{aligned}
&\|\hat{c}P_h\phi_{i,2} - P_h(\hat{c}P_h\phi_{i,2})\|_{L_1(\Omega_i)} \lesssim d_i^n h^j |\hat{c}P_h\phi_{i,2}|_{W_\infty^j((\Omega_i)_h)} \\
&\quad\lesssim d_i^n h^j \textstyle\sum_{|\alpha+\beta|=j, |\beta|<j} |D^\alpha \hat{c} D^\beta P_h\phi_{i,2}|_{L_\infty((\Omega_i)_h)} \lesssim d_i^n h^j \|P_h\phi_{i,2}\|_{W_\infty^{j-1}((\Omega_i)_h)}.
\end{aligned}
$$

Proceeding from above while using the stability of P_h in W_∞^{j-1} and calculating as in (4.32) except now with $|\alpha| \le j-1$ yields

$$
\|\hat{c}P_h\phi_{i,2} - P_h(\hat{c}P_h\phi_{i,2})\|_{L_1(\Omega_i)} \lesssim d_i^n h^j \|\phi\|_{W_\infty^{j-1}(\Omega_i')} \lesssim d_i^n h^j d_i^{2-n-j}. \tag{4.42}
$$

Combining (4.41) through (4.42) and employing (4.17) and (4.18) while recalling that $0 \le m \le j-1$, we obtain

$$
\begin{aligned}
\|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1,x_0,-m} &\lesssim h + \textstyle\sum_{i=0}^{J} \left(\frac{d_i}{h}\right)^m (h(\frac{d_i}{h})^n e^{-cd_i/h} + h^j d_i^{2-j}) \\
&\lesssim h(1 + \textstyle\sum_{i=0}^{J}((\frac{d_i}{h})^{m+n} e^{-cd_i/h} + d_i(\frac{h}{d_i})^{j-1-m})) \lesssim h.
\end{aligned} \tag{4.43}
$$

Combining (4.40) and (4.43) yields (4.25). $\square$

Lemma 4.3. *Let ψ and $\vec{z}$ be given by* (4.5a) *and* (4.5b)*, let $m = 0$ if $j = 1$ and $0 \le m \le j-2$ otherwise, and let $0 \le \ell \le j-1$. Then*

$$
\|\hat{c}\psi - P_h(\hat{c}\psi)\|_{\Omega_i} + \|\psi - P_h\psi\|_{\Omega_i} \lesssim h^2 d_i^{-n/2}(h/d_i)^{j-2}, \tag{4.44}
$$

$$
\|\vec{b}_2 A^{-*}\vec{z} - P_h(\vec{b}_2 A^{-*}\vec{z})\|_{\Omega_i} + \|\vec{z} - \Pi_h\vec{z}\|_{\Omega_i} \lesssim h d_i^{-n/2}(h/d_i)^{j-1}, \tag{4.45}
$$

$$
\|\hat{c}\psi - P_h(\hat{c}\psi)\|_{L_1,x_0,-m} + \|\psi - P_h\psi\|_{L_1,x_0,-m} \lesssim h^{2-\delta_{1j}}(\log\tfrac{1}{h})^{\delta_{j-2,m}}, \tag{4.46}
$$

$$
\|\vec{b}_2 A^{-*}\vec{z} - P_h(\vec{b}_2 A^{-*}\vec{z})\|_{L_1,x_0,-\ell} \lesssim h(\log\tfrac{1}{h})^{\delta_{j-1,\ell}}, \tag{4.47}
$$

$$
\|\psi\|_{L_1(\Omega)} \lesssim 1. \tag{4.48}
$$

$$
\begin{aligned}
&\|\hat{c}(\psi_h - P_h\psi) - P_h(\hat{c}(\psi_h - P_h\psi))\|_{L_1,x_0,-m} \\
&\qquad + \|\hat{c}P_h\psi - P_h(\hat{c}P_h\psi)\|_{L_1,x_0,-m} \lesssim h^{2-\delta_{1j}} + h\|\psi - \psi_h\|_{L_1,x_0,-m}.
\end{aligned} \tag{4.49}
$$

Proof. The proof is essentially the same as that of the previous lemma with the simplification that the support of δ^0 is localized to a single element. □

4.4. **Proof of Theorem 1.1.** In this subsection we complete the proof of Theorem 1.1. We first state our central lemma, which gives bounds for the errors

$$\|\vec{r} - \vec{r}_h\|_{L_1,x_0,-s}$$

and

$$\|\phi - \phi_h\|_{L_1,x_0,-m}.$$

Lemma 4.4. *Let $\vec{r}$ and ϕ and their mixed finite element approximations $\vec{r}_h$ and ϕ_h be as defined above. Then*

$$\|\vec{r} - \vec{r}_h\|_{L_1,x_0,-s} \lesssim (\log\frac{1}{h})^{\delta_{js}}, \quad 0 \le s \le j, \tag{4.50a}$$

$$\|\phi - \phi_h\|_{L_1,x_0,-m} \lesssim 1, \quad 0 \le m \le j, \tag{4.50b}$$

$$\|\phi - \phi_h\|_{L_1,x_0,-m} \lesssim h(\log\frac{1}{h})^{\delta_{m,j-1}}, \quad j \ge 2 \text{ and } 0 \le m \le j-1. \tag{4.50c}$$

Before proving Lemma 4.4, we complete the proof of Theorem 1.1 assuming that Lemma 4.4 holds. In order to prove the inequality (1.8) for the divergence form method, we first apply Hölder's inequality to (4.7) while recalling that $\vec{b}_1 = \vec{b}$, $\vec{b}_2 = 0$, and $\hat{c} = c$. We next use Lemma 4.4 along with (4.21), (4.24), and (4.25) of Lemma 4.2 to find

$$\begin{aligned}
|[(\vec{p} - &\vec{p}_h)(x_0)]_i| \le |(\vec{p} - \Pi_h\vec{p})(x_0)| \\
&+ \|\vec{p} - \Pi_h\vec{p}\|_{L_\infty,x_0,s}[\|\vec{\delta}_i^0\|_{L_1,x_0,-s} \\
&+ \|\vec{r} - \vec{r}_h\|_{L_1,x_0,-s} + \|\phi - \phi_h\|_{L_1,x_0,-s}] \\
&+ \|u - P_h u\|_{L_\infty,x_0,t}[\|c(\phi_h - P_h\phi) - P_h(c(\phi_h - P_h\phi))\|_{L_1,x_0,-t} \\
&+ \|cP_h\phi - P_h(cP_h\phi)\|_{L_1,x_0,-t}] \\
&+ \|\operatorname{div}(\vec{p} - \Pi_h\vec{p})\|_{L_\infty,x_0,t}\|\phi - P_h\phi\|_{L_1,x_0,-t} \\
\lesssim\ & (\log\tfrac{1}{h})^{\delta_{js}}\|\vec{p} - \Pi_h\vec{p}\|_{L_\infty,x_0,s} \\
&+ h[\|u - P_h u\|_{L_\infty,x_0,t} + (\log\tfrac{1}{h})^{\delta_{j-1,t}}\|\operatorname{div}(\vec{p} - \Pi_h\vec{p})\|_{L_\infty,x_0,t}]
\end{aligned} \tag{4.51}$$

for any $0 \le s \le j$ and $0 \le t \le j-1$, thus completing the proof of (1.8).

In order to prove the inequality (1.9) for the conservation form method, we first note that $\|v\|_{L_1,x_0,-t-1} \le \frac{1}{h}\|v\|_{L_1,x_0,-t}$, so that from (4.25) we deduce

$$\begin{aligned}
&\|\hat{c}(\phi_h - P_h\phi) - P_h(\hat{c}(\phi_h - P_h\phi))\|_{L_1,x_0,-\ell} \\
&\qquad + \|\hat{c}P_h\phi - P_h(\hat{c}P_h\phi)\|_{L_1,x_0,-\ell} \lesssim 1
\end{aligned}$$

for $0 \le \ell \le j$. We then apply Hölder's inequality to (4.7) while recalling that $\vec{b}_1 = 0$, $\vec{b}_2 = -\vec{b}$, and $\hat{c} = c^*$ and then we use Lemma 4.4 along with (4.21), (4.23), and (4.24) of Lemma 4.2 to find

$$
\begin{aligned}
[(\tilde{p}-\tilde{p}_h)(x_0)]_i \le{}& [(\tilde{p}-\Pi_h\tilde{p})(x_0)]_i \\
&+ \|\tilde{p}-\Pi_h\tilde{p}\|_{L_\infty,x_0,s}[\|\vec{\delta}_i^0\|_{L_1,x_0,-s} + \|\vec{r}-\vec{r}_h\|_{L_1,x_0,-s}] \\
&+ \|u-P_hu\|_{L_\infty,x_0,s}[\|\vec{\delta}_i^0\|_{L_1,x_0,-s} + \|\vec{r}-\vec{r}_h\|_{L_1,x_0,-s} \\
&+ \|\vec{b}A^{-*}\vec{r}-P_h(\vec{b}A^{-*}\vec{r})\|_{L_1,x_0,-s} \\
&+ \|\hat{c}(\phi_h-P_h\phi)-P_h(\hat{c}(\phi_h-P_h\phi))\|_{L_1,x_0,-s} + \|\hat{c}P_h\phi-P_h(\hat{c}P_h\phi)\|_{L_1,x_0,-s}] \\
&+ \|\operatorname{div}\tilde{p}-\operatorname{div}\Pi_h\tilde{p}\|_{L_\infty,x_0,t}\|\phi-P_h\phi\|_{L_1,x_0,-t} \\
\lesssim{}& (\log\tfrac{1}{h})^{\delta_{js}}[\|\tilde{p}-\Pi_h\tilde{p}\|_{L_\infty,x_0,s} + \|u-P_hu\|_{L_\infty,x_0,s}] \\
&+ h(\log\tfrac{1}{h})^{\delta_{j-1,t}}\|\operatorname{div}\tilde{p}-\operatorname{div}\Pi_h\tilde{p}\|_{L_\infty,x_0,t}.
\end{aligned}
$$

Thus we complete the proof of (1.9) and therefore of Theorem 1.1.

Proof of Lemma 4.4. We begin by noting that

$$
\|\vec{r}-\vec{r}_h\|_{L_1,x_0,-s} \lesssim \|\vec{r}-\vec{r}_h\|_{L_1(B_{Mh})} + \sum_{i=0}^{J}\left(\frac{d_i}{h}\right)^s \|\vec{r}-\vec{r}_h\|_{L_1(\Omega_i)}. \tag{4.52}
$$

We next use Hölder's inequality, the global L_2 bounds given in Lemma 3.1, the approximation assumptions (2.8) and (2.9), (2.2), and (2.15) to obtain

$$
\begin{aligned}
&\|\vec{r}-\vec{r}_h\|_{[L_1(B_{Mh})]^n} + \|\phi-\phi_h\|_{L_1(B_{Mh})} \\
&\quad\lesssim h^{n/2}(\|\vec{r}-\vec{r}_h\|_\Omega + \|\phi-\phi_h\|_\Omega) \\
&\quad\lesssim h^{n/2}(\|\vec{r}-\Pi_h\vec{r}\|_\Omega + \|\phi-P_h\phi\|_\Omega) \\
&\quad\lesssim h^{n/2+1}\|\phi\|_{H_2^2(\Omega)} \\
&\quad\lesssim h^{n/2+1}\|\operatorname{div}\vec{\delta}_i^0 - P_h(\vec{b}_2A^{-*}\vec{\delta}_i^0)\|_\Omega \\
&\quad\lesssim 1.
\end{aligned} \tag{4.53}
$$

We next insert a "dummy" term for later use in a kickback argument, and then we use the local L_2 bound (3.4) from Lemma 3.2 to deduce that

$$
\begin{aligned}
\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s\|\vec{r}-\vec{r}_h\|_{L_1(\Omega_i)} \le{}& \textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i} + \|\vec{r}-\vec{r}_h\|_{\Omega_i}) \\
\lesssim{}& \textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}[\|\vec{r}-\Pi_h\vec{r}\|_{\Omega_i'} + \frac{1}{d_i}(\|\phi-P_h\phi\|_{\Omega_i'} + \|\hat{c}\phi-P_h(\hat{c}\phi)\|_{\Omega_i'}) \\
&+\tfrac{h}{d_i}(\|\vec{r}-\vec{r}_h\|_{\Omega_i'} + \|\phi-\phi_h\|_{\Omega_i'} + \|\vec{b}_2A^{-*}\vec{r}-P_h(\vec{b}_2A^{-*}\vec{r})\|_{\Omega_i'}) \\
&+h^j d_i^{-n/2-j}(\|\phi-\phi_h\|_{L_1(\Omega)} + \|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n}) \\
&+h^j d_i^{1-n/2-j}(\|\phi-P_h\phi\|_{L_1(\Omega)} + \|\hat{c}\phi-P_h(\hat{c}\phi)\|_{L_1(\Omega)} \\
&+\|\vec{b}_2A^{-*}\vec{r}-P_h(\vec{b}_2A^{-*}\vec{r})\|_{L_1(\Omega)})]
\end{aligned} \tag{4.54}
$$

and then we rearrange terms in (4.54) while recalling that $d_J = cMh$ to find

$$\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\quad\lesssim (Mh)^{n/2}(\|\phi-\phi_h\|_{B_{Mh}}+\|\vec{r}-\vec{r}_h\|_{B_{Mh}})\\
&\qquad+\textstyle\sum_{i=0}^{J+1}(\frac{d_i}{h})^s d_i^{n/2}[\|\vec{b}_2A^{-*}\vec{r}-P_h(\vec{b}_2A^{-*}\vec{r})\|_{\Omega_i}+\|\vec{r}-\Pi_h\vec{r}\|_{\Omega_i}\\
&\qquad\quad+\tfrac{1}{d_i}(\|\phi-P_h\phi\|_{\Omega_i}+\|\hat{c}\phi-P_h(\hat{c}\phi)\|_{\Omega_i})]\\
&\qquad+\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}\frac{h}{d_i}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\qquad+(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\textstyle\sum_{i=0}^{J}(\frac{h}{d_i})^{j-s}\\
&\qquad+(\|\phi-P_h\phi\|_{L_1(\Omega)}+\|\hat{c}\phi-P_h(\hat{c}\phi)\|_{L_1(\Omega)}\\
&\qquad+\|\vec{b}_2A^{-*}\vec{r}-P_h(\vec{b}_2A^{-*}\vec{r})\|_{L_1(\Omega)})\textstyle\sum_{i=0}^{J}d_i(\frac{h}{d_i})^{j-s}.
\end{aligned}$$

Using in turn the relevant bounds from Lemma 4.2 along with (4.53) while recalling that $\frac{h}{d_i} \le M$ yields

$$\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\lesssim 1\\
&+\textstyle\sum_{i=0}^{J+1}(\frac{d_i}{h})^s d_i^{n/2}d_i^{-n/2}[((\frac{d_i}{h})^n e^{-cd_i/h}+(\frac{h}{d_i})^j)+\frac{1}{d_i}h((\frac{d_i}{h})^n e^{-cd_i/h}+(\frac{h}{d_i})^{j-1})]\\
&+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&+(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\textstyle\sum_{i=0}^{J}(\frac{h}{d_i})^{j-s}+\sum_{i=0}^{J}d_i(\frac{h}{d_i})^{j-s}.
\end{aligned}$$

We now collect and rearrange terms and then use (4.16), (4.17), and (4.18) while recalling that $j \le k$ to obtain

$$\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\quad\lesssim 1+\textstyle\sum_{i=0}^{J+1}[(\frac{d_i}{h})^{s+n}e^{-cd_i/h}+(\frac{h}{d_i})^{j-s}]\\
&\qquad+(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\textstyle\sum_{i=0}^{J}(\frac{h}{d_i})^{j-s}+\sum_{i=0}^{J}d_i(\frac{h}{d_i})^{j-s}\\
&\qquad+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\quad\lesssim (\log\tfrac{1}{h})^{\delta_{js}}[1+\tfrac{1}{M^{j-s}}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})]\\
&\qquad+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i}).
\end{aligned}\tag{4.55}$$

We next take M large enough to kick back the last term in (4.55) and thus deduce that

$$\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^s d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\quad\lesssim (\log\tfrac{1}{h})^{\delta_{js}}[1+\tfrac{1}{M^{j-s}}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{L_1(\Omega)})].
\end{aligned}\tag{4.56}$$

Then we insert (4.56) into (4.54) and in turn insert the resulting inequality and (4.53) into (4.52) to find that

$$\|\vec{r}-\vec{r}_h\|_{L_1,x_0,-s}\lesssim (\log\tfrac{1}{h})^{\delta_{js}}[1+\tfrac{1}{M^{j-s}}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})].\tag{4.57}$$

When $s = 0$ (so that $s < j$, i.e., $j - s > 0$ and $\delta_{js} = 0$), (4.57) reduces to

$$\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n}\lesssim 1+\frac{1}{M}\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n}+\frac{1}{M}\|\phi-\phi_h\|_{L_1(\Omega)}.$$

Taking M large enough to kick back the second term above and inserting the result into (4.57), we finally find that for $0 \le s \le j$,

$$\|\vec{r} - \vec{r}_h\|_{L_1,x_0,-s} \lesssim \left(\log \frac{1}{h}\right)^{\delta_{js}} [1 + \frac{1}{M^{j-s}} \|\phi - \phi_h\|_{L_1(\Omega)}]. \tag{4.58}$$

We next note that

$$\|\phi - \phi_h\|_{L_1,x_0,-m} \lesssim \|\phi - \phi_h\|_{L_1(B_{Mh})} + \sum_{i=0}^{J} \left(\frac{d_i}{h}\right)^m \|\phi - \phi_h\|_{L_1(\Omega_i)}. \tag{4.59}$$

We now use Hölder's inequality, the bound (3.2b) from Lemma 3.1, the approximation properties (2.8) and (2.9), global regularity, and (2.15) to obtain

$$\begin{aligned} \|\phi - \phi_h\|_{L_1(B_{Mh})} &\lesssim h^{n/2} \|\phi - \phi_h\|_\Omega \\ &\lesssim h^{n/2} [\|\phi - P_h\phi\|_\Omega + h\|\vec{r} - \Pi_h\vec{r}\|_\Omega + h^{2-\delta_{1j}} \|\operatorname{div} \vec{r} - P_h \operatorname{div} \vec{r}\|_\Omega] \\ &\lesssim h^{n/2} h^{2-\delta_{1j}} \|\phi\|_{H_2^2(\Omega)} \\ &\lesssim h^{n/2} h^{2-\delta_{1j}} \|\operatorname{div} \vec{\delta}_i^0 - P_h(\vec{b}_2 A^{-*} \vec{\delta}_i^0)\|_\Omega \lesssim h^{1-\delta_{1j}}. \end{aligned} \tag{4.60}$$

We next insert a "dummy" term for later use in a kickback argument and then use the local L_2 bound (3.4) from Lemma 3.2 to deduce that

$$\begin{aligned} &\textstyle\sum_{i=0}^J (\frac{d_i}{h})^m \|\phi - \phi_h\|_{L_1(\Omega_i)} \le \sum_{i=0}^J (\frac{d_i}{h})^m d_i^{n/2} (\|\phi - \phi_h\|_{\Omega_i} + d_i \|\vec{r} - \vec{r}_h\|_{\Omega_i}) \\ &\quad \lesssim \textstyle\sum_{i=0}^J (\frac{d_i}{h})^m d_i^{n/2} [d_i \|\vec{r} - \Pi_h \vec{r}\|_{\Omega_i'} + (\|\phi - P_h\phi\|_{\Omega_i'} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{\Omega_i'}) \\ &\qquad + h(\|\vec{r} - \vec{r}_h\|_{\Omega_i'} + \|\phi - \phi_h\|_{\Omega_i'} + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{\Omega_i'}) \\ &\qquad + h^j d_i^{1-n/2-j} (\|\phi - \phi_h\|_{L_1(\Omega)} + \|\vec{r} - \vec{r}_h\|_{[L_1(\Omega)]^n}) \\ &\qquad + h^j d_i^{2-n/2-j} (\|\phi - P_h\phi\|_{L_1(\Omega)} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(\Omega)} \\ &\qquad + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{L_1(\Omega)})] \end{aligned}$$

and then rearrange terms to find that

$$\begin{aligned} &\textstyle\sum_{i=0}^J (\frac{d_i}{h})^m d_i^{n/2} (\|\phi - \phi_h\|_{\Omega_i} + d_i \|\vec{r} - \vec{r}_h\|_{\Omega_i}) \\ &\quad \lesssim (Mh)^{n/2+1} (\|\phi - \phi_h\|_{B_{Mh}} + \|\vec{r} - \vec{r}_h\|_{B_{Mh}}) \\ &\qquad + \textstyle\sum_{i=0}^{J+1} (\frac{d_i}{h})^m d_i^{n/2} [d_i (\|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{\Omega_i} + \|\vec{r} - \Pi_h\vec{r}\|_{\Omega_i}) \\ &\qquad + (\|\phi - P_h\phi\|_{\Omega_i} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{\Omega_i})] \\ &\qquad + \textstyle\sum_{i=0}^J (\frac{d_i}{h})^m d_i^{n/2} \frac{h}{d_i} (\|\phi - \phi_h\|_{\Omega_i} + d_i \|\vec{r} - \vec{r}_h\|_{\Omega_i}) \\ &\qquad + (\|\phi - \phi_h\|_{L_1(\Omega)} + \|\vec{r} - \vec{r}_h\|_{[L_1(\Omega)]^n}) \textstyle\sum_{i=0}^J d_i (\frac{h}{d_i})^{j-m} \\ &\qquad + (\|\phi - P_h\phi\|_{L_1(\Omega)} + \|\hat{c}\phi - P_h(\hat{c}\phi)\|_{L_1(\Omega)} \\ &\qquad + \|\vec{b}_2 A^{-*}\vec{r} - P_h(\vec{b}_2 A^{-*}\vec{r})\|_{L_1(\Omega)}) \textstyle\sum_{i=0}^J d_i^2 (\frac{h}{d_i})^{j-m}. \end{aligned}$$

Using in turn each of the relevant bounds from Lemma 4.2 along with (4.53) while recalling that $\frac{h}{d_i} \le M$ yields

$$
\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i}) \\
&\quad\lesssim h+\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}d_i^{-n/2}d_i[((\frac{d_i}{h})^n e^{-cd_i/h}+(\frac{h}{d_i})^j) \\
&\qquad+\tfrac{1}{d_i}h((\frac{d_i}{h})^n e^{-cd_i/h}+(\frac{h}{d_i})^{j-1})] \\
&\qquad+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i}) \\
&\qquad+(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\textstyle\sum_{i=0}^{J}d_i(\frac{h}{d_i})^{j-m} \\
&\qquad+\textstyle\sum_{i=0}^{J}d_i^2(\frac{h}{d_i})^{j-m}.
\end{aligned}
\tag{4.61}
$$

In order to complete the proof of (4.50b), we collect and rearrange terms and then use (4.16), (4.17) and (4.18) while recalling that $j \le k$ to deduce that

$$
\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i}) \\
&\quad\lesssim 1+\textstyle\sum_{i=0}^{J}[d_i(\frac{d_i}{h})^{m+n}e^{-cd_i/h}+d_i(\frac{h}{d_i})^{j-m}] \\
&\qquad+(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\textstyle\sum_{i=0}^{J}d_i(\frac{h}{d_i})^{j-m} \\
&\qquad+\textstyle\sum_{i=0}^{J}d_i^2(\frac{h}{d_i})^{j-m}+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i}) \\
&\quad\lesssim 1+\tfrac{1}{M^{j-m}}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n}) \\
&\qquad+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i}).
\end{aligned}
\tag{4.62}
$$

Next we take M large enough to kick back the last term in (4.62), and then we insert the result and (4.60) into (4.59) to obtain

$$
\|\phi-\phi_h\|_{L_1,x_0,-m} \lesssim 1+\frac{1}{M^{j-m}}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n}). \tag{4.63}
$$

Inserting (4.58) (with $s=0$) into (4.63), we find that

$$
\|\phi-\phi_h\|_{L_1,x_0,-m} \lesssim 1+\frac{1}{M^{j-m}}\|\phi-\phi_h\|_{L_1(\Omega)}. \tag{4.64}
$$

Employing (4.64) with $m=0$ (so that $j-m>0$) and taking M large enough to perform a kickback argument yields

$$
\|\phi-\phi_h\|_{L_1(\Omega)} \lesssim 1. \tag{4.65}
$$

The proof of (4.50b) is completed by inserting (4.65) into (4.64), and the proof of (4.50a) is in turn completed by inserting (4.50b) (with $m=0$) into (4.58).

In order to complete the proof of (4.50c), we collect and rearrange terms in (4.61) and then use (4.16), (4.17), and (4.18) while recalling that $j \leq k$ to obtain

$$
\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\lesssim h+\textstyle\sum_{i=0}^{J}[h(\frac{d_i}{h})^{m+n+1}e^{-cd_i/h}+h(\frac{h}{d_i})^{j-m-1}]\\
&\quad+(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})\textstyle\sum_{i=0}^{J}h(\frac{h}{d_i})^{j-m-1}\\
&\quad+\textstyle\sum_{i=0}^{J}hd_i(\frac{h}{d_i})^{j-m-1}+\frac{1}{M}\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\lesssim h(\log\tfrac{1}{h})^{\delta_{j-1,m}}[1+\tfrac{1}{M^{j-m-1}}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})]\\
&\quad+\tfrac{1}{M}\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\frac{1}{d_i}\|\phi-\phi_h\|_{\Omega_i}+\|\vec{r}-\vec{r}_h\|_{\Omega_i}).
\end{aligned}
\tag{4.66}
$$

We next take M large enough to kick back the last term of (4.66), yielding

$$
\begin{aligned}
&\textstyle\sum_{i=0}^{J}(\frac{d_i}{h})^m d_i^{n/2}(\|\phi-\phi_h\|_{\Omega_i}+d_i\|\vec{r}-\vec{r}_h\|_{\Omega_i})\\
&\qquad\lesssim h(\log\tfrac{1}{h})^{\delta_{m,j-1}}[1+(\tfrac{1}{M})^{j-1-m}(\|\phi-\phi_h\|_{L_1(\Omega)}+\|\vec{r}-\vec{r}_h\|_{[L_1(\Omega)]^n})].
\end{aligned}
\tag{4.67}
$$

Inserting (4.50b) and (4.50a) with $m = s = 0$ into (4.67) and in turn inserting (4.67) and (4.60) (with $j \geq 2$, so $1-\delta_{1j} = 1$) into (4.59) yields (4.50c). ☐

4.5. **Proof of Theorem 1.2.** The proof of Theorem 1.2 is analogous to that of Theorem 1.1. We omit the details. ☐

5. Numerical results

In this section we present numerical results confirming that the lowest-order BDM elements do not in general yield a localized approximation when used in mixed methods for elliptice problems.

We first recall the relevant parameters and estimates. In the lowest order BDM space BDM_1 in $\mathbb{R}^2$, $\vec{Q}_h$ consists of the bi-piecewise linear functions with continuous normal traces, while V_h consists of the piecewise constants. Thus the vector variable is approximated to order $k = 2$ by BDM_1, while the scalar variable is approximated to order $j = 1$. In Theorem 1.1, the parameter s is thus allowed to be 1, while t must be 0. Applying approximation properties to (1.8) with these allowed choices of s and t yields

$$
\begin{aligned}
|(\vec{p}-\vec{p}_h)(x_0)| &\lesssim [\|\sigma(\vec{p}-\Pi_h\vec{p})\|_{[L_\infty(\Omega)]^n}\\
&\qquad+h(\|u-P_hu\|_{L_\infty(\Omega)}+\|\operatorname{div}\vec{p}-P_h\operatorname{div}\vec{p}\|_{L_\infty(\Omega)})]\\
&\lesssim \text{localized } O(h^2)+\text{global } O(h^2).
\end{aligned}
\tag{5.1}
$$

Note that we have ignored logarithmic factors as they have little effect on the observed rate of convergence. We wish to show that t cannot be larger, in particular that $t = 1$ is not an allowed choice in this case. If $t = 1$ *were* an allowed choice, (4.1) could be reduced to

$$|(\vec{p}-\vec{p}_h)(x_0)| \lesssim \text{localized } O(h^2)$$

and the error expansion inequality (1.14) would instead read

$$|(\vec{p}-\vec{p}_h)(x_0)| \lesssim \sum_{\alpha\leq 3} h^2|D^\alpha u(x_0)|+h^3\|u\|_{W_\infty^4(\Omega)} \lesssim \text{localized } O(h^2).$$

Thus if $t = 1$ were allowed and we chose u so that

$$\sum_{|\alpha| \le 3} |D^\alpha u(x_0)| = 0, \tag{5.2}$$

we would have

$$|(\vec{p} - \vec{p}_h)(x_0)| \lesssim \text{global } O(h^3),$$

that is, the method would be superconvergent at the point x_0. Our calculations confirm that no such superconvergence occurs.

For comparison we also performed computations using the lowest-order Raviart-Thomas elements RT_0. Here $j = k = 1$, and we recall from (1.14) that

$$|(\vec{p} - \vec{p}_h)(x_0)| \lesssim \sum_{|\alpha|=1} h|D^\alpha \vec{p}(x_0)| + h^2 \|u\|_{W^3_\infty(\Omega)}.$$

Thus if $\sum_{|\alpha|=1} h|D^\alpha \vec{p}(x_0)| = 0$, our theory predicts that

$$|(\vec{p} - \vec{p}_h)(x_0)| \lesssim \text{global } O(h^2),$$

and our experiments confirm that now this superconvergence indeed occurs.

In our numerical experiments, we chose u so that the derivatives of u up to third order vanish at a specific point x_0, that is, so that (5.2) holds. In particular, we took $u(x, y) = x^4(1 - x^2 - y^2)$, $x_0 = (0, 0)$, $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, $A = \text{diag}(2 + y, 3 + x)$, and $\vec{b} = c = 0$. We note here that $\partial\Omega$ is smooth and $u|_{\partial\Omega} = 0$.

For each value of the mesh size h, the domain Ω was meshed with a standard mesh, then perturbed randomly twenty times in order to place the origin at different points within elements and rule out superconvergence due to mesh symmetry about x_0. The mesh sizes were taken to be $h = \frac{1}{\sqrt{2}2^\ell}$, $\ell = 1, ..., 6$. Computations were performed on each perturbed mesh, and for each value of h, $\text{err}_{h,\max}$ was taken to be the largest value of the error $|(\vec{p} - \vec{p}_h)(0, 0)|$ obtained over the twenty mesh perturbations. Estimated rates of convergence were then calculated by

$$r_h = \log_2 \left(\frac{\text{err}_{h,\max}}{\text{err}_{h/2,\max}} \right).$$

Curved elements were used at the boundary. The triangular portions of all elements were integrated using a seven-point quadrature rule found in [SF73, p. 184] which is exact for polynomials of up to degree 5. The "skin layer" of the curved elements was integrated using 1-dimensional quadrature along the natural linear element edges and from the natural linear element edges to the actual curved element boundary. The quadrature rule, found in [HTB95, p. 522], is exact for polynomials of up to degree 8. A standard interelement Lagrange multiplier scheme for mixed methods (as described in [BDM85], for example) was employed in order to enable use of an iterative solver. The results of our computations are displayed in Table 1.

We recall that our theory predicts $O(h^2)$ convergence at $x_0 = (0, 0)$ when using either RT_0 or BDM_1. It is clear from our computations that superconvergence is indeed occurring at $(0, 0)$ when RT_0 is used; that is, $|(\vec{p} - \vec{p}_h)(x_0)| \lesssim h^2$, whereas RT_0 approximates to first order in general. It is also clear that a full order of superconvergence does not occur at $(0, 0)$ when BDM_1 is used; that is, $|(\vec{p} - \vec{p}_h)(x_0)| \lesssim h^3$ does not hold, and it appears that the rate of convergence is decreasing to 2 as the mesh is subdivided. Thus the sharpness of our theory is confirmed.

TABLE 1.

	RT_0		BDM_1	
ℓ	$\mathrm{err}_{h,\max}$	r_h	$\mathrm{err}_{h,\max}$	r_h
1	.1509	4.011	.05535	2.486
2	.009350	1.219	.009884	2.080
3	.003586	2.630	.002338	2.412
4	.0005791	2.317	.0004393	2.046
5	.0001162	2.365	.0001064	2.273
6	.00002256		.00002201	

Acknowledgments

This work constitutes part of the author's doctoral thesis. He thanks his advisor, Lars Wahlbin, for his many helpful suggestions during the completion of this work.

References

[BDM85] Franco Brezzi, Jim Douglas, Jr., and Donatella Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math. **47** (1985), 217–235. MR **87g:**65133

[Dem02] Alan Demlow, *Suboptimal and optimal convergence in mixed finite element methods*, SIAM J. Numer. Anal. **39** (2002), no. 6, 1938–1953. MR **2003e:**65216

[DR82] Jim Douglas, Jr. and Jean E. Roberts, *Mixed finite element methods for second order elliptic problems*, Mat. Apl. Comput. **1** (1982), no. 1, 91–103. MR **84b:**65111

[DR85] ———, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp. **44** (1985), no. 169, 39–52. MR **86b:**65122

[GN88] Lucia Gastaldi and Ricardo H. Nochetto, *On L^∞-accuracy of mixed finite element methods for second order elliptic problems*, Mat. Apl. Comput. **7** (1988), no. 1, 13–39. MR **90d:**65192a

[GN89] ———, *Sharp maximum norm error estimates for general mixed finite element approximations to second order elliptic problems*, RAIRO Modél. Math. Anal. Numér. **23** (1989), no. 1, 103–128. MR **91b:**65125

[GT98] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, Berlin, 1998.

[HSWW01] Wolfgang Hoffmann, Alfred H. Schatz, Lars B. Wahlbin, and Gabriel Wittum, *Asymptotically exact a posteriori estimators for the pointwise gradient error on each element in irregular meshes. I. A smooth problem and globally quasi-uniform meshes.*, Math. Comp. **70** (2001), no. 235, 897–909. MR **2002a:**65178

[HTB95] Kenneth H. Huebner, Earl A. Thornton, and Ted G. Byrom, *The finite element method for engineers*, 3rd ed., John Wiley and Sons, Inc., New York, 1995.

[Kra69] Ju. P. Krasovskii, *Properties of Green's functions and generalized solutions of elliptic boundary value problems*, Soviet Math. Dokl. **10** (1969), 54–58.

[Sch83] Reinhard Scholz, *Optimal L^∞-estimates for a mixed finite element method for second order elliptic and parabolic problems*, Calcolo **20** (1983), 355–377. MR **86j:**65164

[Sch98] Alfred H. Schatz, *Pointwise error estimates and asymptotic error expansion inequalities for the finite element method on irregular grids. I. Global estimates.*, Math. Comp. **67** (1998), no. 223, 877–899. MR **98j:**65082

[SF73] Gilbert Strang and George J. Fix, *An analysis of the finite element method*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1973. MR **56:**1747

[SW95] Alfred H. Schatz and Lars B. Wahlbin, *Interior maximum-norm estimates for finite element methods, Part II*, Math. Comp **64** (1995), no. 211, 907–928. MR **95j:**65143

[Wah91] Lars B. Wahlbin, *Local behavior in finite element methods*, Finite Element Methods. Part 1 (P. G. Ciarlet and J.-L. Lions, eds.), Handbook of Numerical Analysis, vol. II, North-Holland, Amsterdam, 1991, pp. 353–522.

Abteilung für Angewandte Mathematik, Hermann-Herder-Str. 10, 79104 Freiburg, Germany

E-mail address: demlow@mathematik.uni-freiburg.de