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ON STRONG TRACTABILITY OF WEIGHTED MULTIVARIATE INTEGRATION

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ABSTRACT. We prove that for every dimension s and every number n of points, there exists a point-set $\mathcal{P}_{n,s}$ whose $\boldsymbol{\gamma}$-*weighted unanchored* L_∞ *discrepancy* is bounded from above by $C(b)/n^{1/2-b}$ independently of s provided that the sequence $\boldsymbol{\gamma} = \{\gamma_k\}$ has $\sum_{k=1}^{\infty} \gamma_k^a < \infty$ for some (even arbitrarily large) a. Here b is a positive number that could be chosen arbitrarily close to zero and $C(b)$ depends on b but not on s or n. This result yields strong tractability of the corresponding integration problems including approximation of weighted integrals $\int_D f(\mathbf{x})\,\rho(\mathbf{x})\,d\mathbf{x}$ over unbounded domains such as $D = \mathbb{R}^s$. It also supplements the results that provide an upper bound of the form $C\sqrt{s/n}$ when $\gamma_k \equiv 1$.

1. INTRODUCTION

This article studies the unanchored L_∞ discrepancy and strong tractability of the corresponding integration problem. We begin the discussion with the integration problem. Consider approximating the following type of weighted integrals:

$$I_\rho(f) = \int_D f(\mathbf{x})\,\rho(\mathbf{x})\,d\mathbf{x}. \tag{1}$$

Here D is an s-dimensional box,

$$D = \overline{(a_1, b_1)} \times \cdots \times \overline{(a_s, b_s)} \subseteq \mathbb{R}^s, \tag{2}$$

with possibly infinite a_i and/or b_i. It is assumed that the *weight* function ρ has a tensor product form,

$$\rho(\mathbf{x}) = \prod_{k=1}^{s} \rho_k(x_k), \tag{3}$$

for nonnegative and Lebesgue integrable functions ρ_k. For simplicity, it is assumed that the ρ_k are probability densities on $\overline{(a_k, b_k)}$, i.e.,

$$\int_{a_k}^{b_k} \rho_k(x)\,dx = 1.$$

However, as explained in Section 6.1 in [3], it is sufficient to assume that the integrals of ρ_k are finite.

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Without loss of generality, the analysis can be restricted to *linear algorithms*

$$\mathcal{A}_{\mathcal{P}_{n,s}}(f) = \sum_{i=1}^{n} w_i\, f(\mathbf{x}^i),$$

where $\mathcal{P}_{n,s} = \{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$ is a set of points from D and the w_i are numbers. Important examples of such algorithms are provided by the *quasi–Monte Carlo* methods, denoted by $\mathcal{Q}_{\mathcal{P}_{n,s}}$,

$$\mathcal{Q}_{\mathcal{P}_{n,s}}(f) = n^{-1} \sum_{i=1}^{n} f(\mathbf{x}^i).$$

For a given normed space $\mathcal{F}$ of integrands, the error of $\mathcal{A}_{\mathcal{P}_{n,s}}$ is given by

$$\operatorname{error}(\mathcal{A}_{\mathcal{P}_{n,s}}; \mathcal{F}) := \sup_{\|f\| \le 1} |I_\rho(f) - \mathcal{A}_{\mathcal{P}_{n,s}}(f)| = \|I_\rho - \mathcal{A}_{\mathcal{P}_{n,s}}\|,$$

where $\|f\|$ is the norm of f in the space $\mathcal{F}$. The importance of this definition is that

$$|I_\rho(f) - \mathcal{A}_{\mathcal{P}_{n,s}}(f)| \le \|f\| \ \operatorname{error}(\mathcal{A}_{\mathcal{P}_{n,s}}; \mathcal{F}), \quad \forall\, f \in \mathcal{F}.$$

Let $\operatorname{error}(n; \mathcal{F})$ denote the *nth minimal error*, i.e., the minimal error among all algorithms that use at most n function evaluations,

$$\operatorname{error}(n; \mathcal{F}) := \inf_{\mathcal{A}_{\mathcal{P}_{n,s}}} \operatorname{error}(\mathcal{A}_{\mathcal{P}_{n,s}}; \mathcal{F}) \quad \text{and} \quad \operatorname{error}(0; \mathcal{F}) := \|I_\rho\|.$$

We say that the corresponding problem is *tractable* if there exist nonnegative constants C, q_1, and positive q_2 such that

$$\frac{\operatorname{error}(n; \mathcal{F})}{\operatorname{error}(0; \mathcal{F})} \le C\, s^{q_1}\, n^{-q_2}, \quad \forall\, n, s \ge 1. \tag{4}$$

Moreover, the problem is *strongly tractable* if (4) holds with $q_1 = 0$.

In an earlier paper [3], we considered this problem in the space $\mathcal{F} = \mathcal{F}_{1,1,s}$ (see the next section for a formal definition), whose norm is given by

$$\|f\|_{1,1,s} := |f(\mathbf{c})| + \sum_{U \ne \emptyset} \|f'_U\|_{L_1(D_U)}. \tag{5}$$

Here the point $\mathbf{c}$, called the *anchor*, is a fixed point from D. The summation is over nonempty subsets U of $\{1, \ldots, s\}$, and f'_U is obtained from $\left(\prod_{k \in U} \frac{\partial}{\partial x_k}\right) f$ by fixing the variables x_k for $k \notin U$ and replacing them by the kth coefficients of the anchor $\mathbf{c}$. Moreover, D_U is the Cartesian product $\prod_{k \in U} (a_k, b_k)$ and $\mathbf{x}_U$ is the projection of $\mathbf{x}$ onto D_U. Note that for $D = [0,1]^s$ and $\mathbf{c} = \mathbf{1}$, this is a classical norm often assumed in the theory of low discrepancy points and multidimensional integration. Using a proof technique similar to the one used in [2], we proved tractability of weighted integration for any ρ satisfying the above assumptions. That is, in particular, we showed that there exists a constant C such that

$$\operatorname{error}(n; \mathcal{F}_{1,1,s}) \le C \sqrt{\frac{s}{n}}, \quad \forall\, n, s \ge 1. \tag{6}$$

Since the initial error $\operatorname{error}(0; \mathcal{F}_{1,1,s})$ equals 1, this implies tractability of the weighted integration problem for the space $\mathcal{F}_{1,1,s}$.

Although we doubt that the upper bound (6) is sharp, it was also shown in [2] that the dependence on s cannot be eliminated. It is therefore natural to ask under what restrictions on the class $\mathcal{F}_{1,1,s}$ the corresponding nth minimal errors

can be bounded from above independently of s, i.e., when the integration problem is strongly tractable.

The main result of this paper provides a partial answer to this question by considering so-called weighted spaces introduced in [9] and then further studied in a number of papers. More specifically, let $\boldsymbol{\gamma} = \{\gamma_k\}_k$ be a sequence of positive numbers (called weights). For simplicity, we assume that $\gamma_k \le 1$. Then the space $\mathcal{F}_{1,\boldsymbol{\gamma},s}$ is a Banach space whose norm is given by

$$\|f\|_{1,\boldsymbol{\gamma},s} := |f(\mathbf{c})| + \sum_{U \neq \emptyset} \gamma_U^{-1} \, \|f'_U\|_{L_1(D_U)}, \tag{7}$$

where

$$\gamma_U := \prod_{k \in U} \gamma_k \quad \text{and} \quad \gamma_\emptyset := 1.$$

Note that for $\gamma_k \equiv 1$, the norms (5) and (7) coincide. Moreover, also for arbitrary $\boldsymbol{\gamma}$ we have $\mathrm{error}(0; \mathcal{F}_{1,\boldsymbol{\gamma},s}) = 1$.

We prove that the nth minimal errors are arbitrarily close to a constant times $n^{-1/2}$ (independently of s) provided that

$$\sum_{k=1}^{\infty} \gamma_k^a < \infty \tag{8}$$

for a positive (even arbitrarily large) number a. For instance, condition (8) is satisfied if $\gamma_k = O(1/k^\delta)$ for a positive (even arbitrarily small) number δ.

That is, we show that (8) implies the following: for every $b > 0$ there exists a constant $C(b)$ such that

$$\mathrm{error}(n; \mathcal{F}_{1,\boldsymbol{\gamma},s}) \le C(b)\, n^{-1/2+b}, \quad \forall\, n, s \ge 1. \tag{9}$$

This is proved by showing an existence of quasi–Monte Carlo algorithms $\mathcal{Q}_{\mathcal{P}_{n,s}}$ with errors bounded from above by $C(b)\, n^{-1/2+b}$.

Actually, we show an even stronger result by considering the so-called *unanchored L_∞ discrepancy* $\mathcal{D}_\infty(\mathcal{P}_{n,s}, \boldsymbol{\gamma})$. Its definition is presented in the next section; here we only mention that for every anchor $\mathbf{c}$, every sequence $\boldsymbol{\gamma}$, and every $\mathcal{P}_{n,s}$, we have

$$\mathrm{error}(\mathcal{Q}_{\mathcal{P}_{n,s}}; \mathcal{F}_{1,\boldsymbol{\gamma},s}) \le \mathcal{D}_\infty(\mathcal{P}_{n,s}, \boldsymbol{\gamma}). \tag{10}$$

Our main result is as follows.

Main Theorem. *Let* (8) *hold for some $a > 0$. Then for any positive b there exists a constant $C(b)$ such that for every n and s there is $\mathcal{P}_{n,s}$ with*

$$\mathcal{D}_\infty(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) \le C(b)\, n^{b-1/2}. \tag{11}$$

This and (10) imply that the weighted integration problem for the space $\mathcal{F}_{1,\boldsymbol{\gamma},s}$ is strongly tractable under a very mild assumption concerning the convergence of the γ_k's. This is in contrast to tractability and strong tractability results obtained for weighted classes with L_1-norm replaced by L_p-norm ($p > 1$). Indeed, let $\mathcal{F}_{p,\boldsymbol{\gamma},s}$ be the space equipped with the norm

$$\|f\|_{p,\boldsymbol{\gamma},s} := \left(\sum_U \gamma_U^{-p} \, \|f'_U\|_{L_p}^p \right)^{1/p} \qquad \text{for } p < \infty$$

and

$$\|f\|_{\infty,\boldsymbol{\gamma},s} := \max_U \gamma_U^{-1} \, \|f'_U\|_{L_\infty}.$$

Then for $p > 1$, a necessary and sufficient condition for tractability of the integration problem for the space $\mathcal{F} = \mathcal{F}_{p,\boldsymbol{\gamma},s}$ (with an arbitrary anchor) is that

$$\sup_{s\geq 2} \frac{\sum_{k=1}^{s} \gamma_k^{p^*}}{\ln(s)} < \infty,$$

where p^* is the conjugate to p, $1/p^* = 1-1/p$. For strong tractability, it is necessary and sufficient that

$$\sum_{k=1}^{\infty} \gamma_k^{p^*} < \infty;$$

see [4, 7]. The results for $p > 1$ suggest that condition (8) might be necessary for strong tractability as well.

For every point-set $\mathcal{P}_{n,s}$, its *unanchored discrepancy* is an upper bound on its *same-quadrant* and *star* discrepancies. Hence the upper bound (11) holds for these discrepancies as well. The star discrepancy (with $\gamma_k \equiv 1$), denoted by $\mathcal{D}^*_\infty(\mathcal{P}_{n,s})$, has been analyzed in many papers, and many deep and important results are available. In particular, it is known (see, e.g., [1, 5]) that there are point-sets $\mathcal{P}_{n,s}$ for which

$$\mathcal{D}^*_\infty(\mathcal{P}_{n,s}) \leq c(s)\, \frac{(\ln n)^{s-1}}{n},$$

where $c(s)$ depends on s but not on n. Such bounds are sharper than (6) and (11) when n is very large relative to s. Note, however, that $(\ln n)^{s-1}/n$ is an increasing function of n when $n \leq \exp(s-1)$ and that $\exp(s-1)$ could be very large even for relatively small s. Hence, for problems with not too small s and/or not too large n, the bounds (6) and (11) are sharper.

We now summarize the content of this paper. In Section 2, we provide basic definitions and facts. The proof of the main theorem is presented in Section 3.

2. Basic definitions and facts

In this section we briefly recall basic facts and definitions used in this paper. For more detailed discussion on discrepancies and the worst case setting, we refer the reader to [1, 5] and to [6, 10], respectively.

Recall that we are interested in approximating integrals I_ρ given by (1)–(3) in the worst case setting with respect to the Banach space $\mathcal{F} = \mathcal{F}_{1,\boldsymbol{\gamma},s}$ whose formal definition is presented now.

Let $\mathcal{H}^s$ be the linear space generated by the linear combinations of functions

$$f(\mathbf{x}) = \prod_{k=1}^{s} h_k(x_k),$$

where for every k, $h_k : \overline{(a_k, b_k)} \to \mathbb{R}$ is an absolutely continuous function with $h_k' \in L_1(\overline{(a_k, b_k)})$. Then, for a given sequence $\boldsymbol{\gamma}$, the number s of variables, and the anchor $\mathbf{c}$, the space $\mathcal{F}_{1,\boldsymbol{\gamma},s}$ is the completion of $\mathcal{H}^s$ with respect to the norm $\|\cdot\|_{1,\boldsymbol{\gamma},s}$ given by (7).

As shown in [3], the weighted integration problem I_ρ for the space $\mathcal{F}_{1,\boldsymbol{\gamma},s}$ has the same complexity as the complexity of unweighted integration on the unit cube for the space $\mathcal{F}$ defined as $\mathcal{F}_{1,\boldsymbol{\gamma},s}$, however, with D replaced by the unit cube and (in general) a different anchor. Moreover, for a quasi–Monte Carlo method for one problem, there exists a quasi–Monte Carlo method for the other problem, both with identical worst case errors. (Formally, this has been shown in [3] only for $\gamma_k \equiv 1$;

however, the proof extends trivially to any $\boldsymbol{\gamma}$.) Hence, from now on, we can assume that

$$\rho \equiv 1, \quad D = [0,1]^s, \quad \text{and} \quad \mathcal{P}_{n,s} \subset [0,1]^s.$$

As follows from [3], the worst case error of $\mathcal{Q}_{\mathcal{P}_{n,s}}$ in the space $\mathcal{F}_{1,\boldsymbol{\gamma},s}$ is equal to the so-called (*weighted*) L_∞ *same-quadrant discrepancy* of the point-set $\mathcal{P}_{n,s} = \{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$. That is,

$$\operatorname{error}(\mathcal{Q}_{\mathcal{P}_{n,s}}; \mathcal{F}_{1,\boldsymbol{\gamma},s}) = \mathcal{D}_{\infty,\mathbf{c}}(\mathcal{P}_{n,s}, \boldsymbol{\gamma}),$$

where

$$\mathcal{D}_{\infty,\mathbf{c}}(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) := \max_{U \neq \emptyset} \gamma_U \sup_{\mathbf{h} \in [0,1]^s} \left|\operatorname{disc}(B(\mathbf{h}, \mathbf{c}), U, \mathcal{P}_{n,s})\right|,$$

with

$$\operatorname{disc}(B(\mathbf{h}, \mathbf{c}), U, \mathcal{P}_{n,s}) := \operatorname{Vol}(B_U(\mathbf{h}_U, \mathbf{c}_U)) - n^{-1} \sum_{i=1}^{n} \mathbf{1}_{B_U(\mathbf{h}_U, \mathbf{c}_U)}(\mathbf{x}_U^i).$$

Here $B(\mathbf{h}, \mathbf{c})$ is the box with one corner at $\mathbf{h}$ and the opposite corner given by the unique vertex of $[0,1]^s$ that lies in the same-quadrant as $\mathbf{h}$ with respect to the anchor $\mathbf{c}$. Of course, by $B_U(\mathbf{h}_U, \mathbf{c}_U)$ we mean the projection of $B(\mathbf{h}, \mathbf{c})$ onto D_U. See [3] for the precise definition.

Recall that the classical L_∞-star discrepancy of $\mathcal{P}_{n,s}$ is given by

$$\begin{aligned} \mathcal{D}_\infty^*(\mathcal{P}_{n,s}) &:= \max_{U \neq \emptyset} \sup_{\mathbf{h} \in D} \left|\operatorname{disc}([\mathbf{0}, \mathbf{h}), U, \mathcal{P}_{n,s})\right| \\ &= \sup_{\mathbf{h} \in D} \left| \operatorname{Vol}([\mathbf{0}, \mathbf{h})) - n^{-1} \sum_{i=1}^{n} \mathbf{1}_{[\mathbf{0}, \mathbf{h})}(\mathbf{x}^i) \right|, \end{aligned}$$

and its $\boldsymbol{\gamma}$-*weighted* version is given by

$$\mathcal{D}_\infty^*(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) := \max_{U \neq \emptyset} \gamma_U \sup_{\mathbf{h} \in D} \left|\operatorname{disc}([\mathbf{0}, \mathbf{h}), U, \mathcal{P}_{n,s})\right|.$$

By $[\mathbf{0}, \mathbf{h})$ we mean a box with opposite corners $\mathbf{0}$ and $\mathbf{h}$.

Of course, for every $\mathcal{P}_{n,s}$ and $\boldsymbol{\gamma}$,

$$\mathcal{D}_\infty^*(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) = \mathcal{D}_{\infty,\mathbf{1}}(\mathcal{P}_{n,s}, \boldsymbol{\gamma});$$

i.e., the star discrepancy and the same-quadrant discrepancy coincide when $\mathbf{c} = \mathbf{1}$.

The (*weighted*) L_∞ *unanchored* discrepancy is defined by

$$\mathcal{D}_\infty(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) := \max_{U} \gamma_U \sup_{\mathbf{g}, \mathbf{h} \in [0,1]^s} \left|\operatorname{disc}([\mathbf{g}, \mathbf{h}), U, \mathcal{P}_{n,s})\right|,$$

where now $[\mathbf{g}, \mathbf{h})$ is the box with opposite corners $\mathbf{g}$ and $\mathbf{h}$.

It is easy to see that for every $\mathcal{P}_{n,s}$, every $\boldsymbol{\gamma}$, and every anchor $\mathbf{c}$,

$$\mathcal{D}_{\infty,\mathbf{c}}(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) \leq \mathcal{D}_\infty(\mathcal{P}_{n,s}, \boldsymbol{\gamma}), \quad \forall\, \mathbf{c} \in D$$

which implies (10).

3. Proof of the main result

We need the following three lemmas.

Lemma 1. *The function* $B(t) := t^{-2}\left((1+t)\ln(1+t) - t\right)$ $(t > 0)$ *is decreasing and* $\sup_{t>0} B(t) = 1/2$.

Proof. Note that $\lim_{t\downarrow 0} B(t) = 1/2$ and $B'(t) = g(t)/t^3$, where $g(t) = 2t - (2+t)\ln(1+t)$. Since $g'(t) = t/(1+t) - \ln(1+t)$ and $g''(t) = -t/(1+t)^2 < 0\ \forall t > 0$, it follows that $g'(t) \le g'(0) = 0\ \forall t > 0$, and so $g(t) \le g(0) = 0\ \forall t > 0$. The conclusion of this lemma follows. □

Lemma 2. *If* $c_a := \sum_{k=1}^{\infty} \gamma_k^a < \infty$, *then there exists* $\hat{\gamma} < \infty$ *such that*

$$\gamma_U \le (c_a/|U|)^{|U|/a} \le \hat{\gamma}, \qquad \forall\, U \ne \emptyset.$$

Proof. The proof of this inequality follows from the inequality relating the geometric mean to the arithmetic mean:

$$\gamma_U = \prod_{k\in U} \gamma_k = \left[\prod_{k\in U} \gamma_k^a\right]^{1/a} \le \left[\frac{1}{|U|}\sum_{k\in U} \gamma_k^a\right]^{|U|/a} \le \left(\frac{c_a}{|U|}\right)^{|U|/a}.$$

Taking the limit of $|U| \to \infty$ of the right-hand side gives zero, so γ_U is bounded above by some $\hat{\gamma}$. □

Lemma 3. *Under the conditions of Lemma 2 it follows that for* $\delta < 1$,

$$\delta < \gamma_U \Rightarrow |U| < c_a e - a\ln(\delta).$$

Proof. Under the assumption that $\delta < \gamma_U$ and by Lemma 2 it follows that

$$|U|\ln(|U|/c_a) \le -a\ln(\gamma_U) < -a\ln(\delta).$$

Either $|U| < c_a e$ or $|U| \ge c_a e$. The former case is contained in the conclusion of this lemma. In the latter case the inequality above implies that $|U| < -a\ln(\delta)$. □

We are ready to prove the theorem. The proof uses Bennett's inequality (see, e.g., [8]) and is somewhat similar to the proof of [2, Theorem 1]. Let $\delta = (\hat{c}/n)^b$ for some $b < 1/2$ and some constant $\hat{c}$ independent of n and s to be determined later. It will be shown that there exists a set $\mathcal{P}_{n,s}$ with

$$\mathcal{D}_\infty(\mathcal{P}_{n,s}, \boldsymbol{\gamma}) = \max_U \sup_{\mathbf{g},\mathbf{h}\in[0,1]^s} \gamma_U \left|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})\right| \le 3\delta. \tag{12}$$

For any $U \subseteq \{1,\dots,s\}$, let $\delta_U = \delta/\gamma_U$. Since the absolute value of the local discrepancy function is no larger than 1, it follows automatically that

$$\sup_{\mathbf{g},\mathbf{h}\in[0,1]^s} \gamma_U \left|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})\right| \le \gamma_U \le \delta \quad \text{for} \quad \delta_U \ge 1.$$

Therefore, to prove (12), it is only necessary to consider those U with $\delta_U < 1$. For U satisfying this condition, define $m_U = \lceil |U|/\delta_U \rceil$. Let Γ_U be the equidistant grid $\{1/m_U, \dots, (m_U-1)/m_U\}^{|U|}$ in D_U. Of course, the cardinality of this grid is $|\Gamma_U| = (m_U - 1)^{|U|}$. As in [2] it can be shown that for any nonempty U and any set $\mathcal{P}_{n,s}$,

$$\sup_{\mathbf{g},\mathbf{h}\in[0,1]^s} \left|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})\right| \le \frac{2|U|}{m_U} + \sup_{\mathbf{g},\mathbf{h}\in\Gamma_U} \left|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})\right|,$$

$$\sup_{\mathbf{g},\mathbf{h}\in[0,1]^s} \gamma_U \left|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})\right| \le 2\delta + \sup_{\mathbf{g},\mathbf{h}\in\Gamma_U} \gamma_U \left|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})\right|.$$

For any positive integers n and s, let $\mathcal{P}_{n,s} = \{\mathbf{x}^1, \ldots, \mathbf{x}^n\}$ be a set of independent and identically distributed random points uniformly distributed on $[0,1]^s$. The quantities $\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \{\mathbf{x}^i\})$ are also iid random variables indexed by $\mathbf{g}, \mathbf{h}$, and U. Each of these random variables has zero mean and is bounded by ± 1. Moreover,

$$\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s}) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{disc}([\mathbf{g},\mathbf{h}), U, \{\mathbf{x}^i\}).$$

To get an upper bound on the local discrepancy function, Bennett's inequality (see, e.g., [8]) is used. Let $Y_1, \ldots, Y_n$ be independent random variables with zero means and bounded ranges: $|Y_i| \le M$. Let σ_i^2 be variances of Y_i and let V be any number such that $V \ge \sum_{i=1}^n \sigma_i^2$. Then for each $\eta > 0$, Bennett's inequality states that

$$\mathrm{Prob}\left\{|Y_1 + \cdots + Y_n| \ge \eta\right\} \le 2\,\exp\left(-\frac{\eta^2}{V}\,B(M\eta/V)\right),$$

where $B(t) = t^{-2}\,((1+t)\,\ln(1+t) - t)$.

We shall use this inequality for $Y_i = \mathrm{disc}([\mathbf{g},\mathbf{h}), U, \{\mathbf{x}^i\})$, so we may take $M = 1$ and $V = n$. Taking $\eta = n\delta_U$, it follows from above that $\eta^2/V = n\delta_U^2$, $M\eta/V = \delta_U$, and

$$\mathrm{Prob}\left\{|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})| \ge \delta_U\right\} \le 2\,\exp\left(-n\delta_U^2 B(\delta_U)\right), \quad \forall \mathbf{g},\mathbf{h} \in \Gamma_U.$$

Since it is assumed above that $\delta_U < 1$, Lemma 1 implies that $B(\delta_U) \ge B(1) = 2\ln(2) - 1$. Since $\delta = (\hat{c}/n)^b$ for some $b < 1/2$, we have $n = \hat{c}/\delta^{1/b}$, and

(13)
$$\mathrm{Prob}\left\{|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})| \ge \delta_U\right\} \le 2\,\exp\left(-(\hat{c}B(1)/\gamma_U^2)\delta^{2-1/b}\right), \quad \forall \mathbf{g},\mathbf{h} \in \Gamma_U.$$

Note that $2 - 1/b < 0$.

For a given U, there are no more than

$$(m_U - 1)^{2|U|} \le (|U|/\delta_U)^{2|U|} = \exp\left(2|U|[\ln(|U|) - \ln(\delta) + \ln(\gamma_U)]\right)$$

possible values of $(\mathbf{g},\mathbf{h}) \in \Gamma_U \times \Gamma_U$ with $\mathbf{g} \le \mathbf{h}$. In view of Lemmas 2 and 3 one may then write

$$(m_U - 1)^{2|U|} \le \exp\left(c_1 - \ln(\delta)[c_2 - c_3\ln(\delta)]\right) \le \exp\left((c_4 B(1)/\hat{\gamma}^2)\delta^{2-1/b}\right) \tag{14}$$

for some positive constants c_1, c_2, c_3, and c_4 that are independent of U, n, and δ, although c_4 depends on b.

Combining (13) and (14), it follows that

$$\mathrm{Prob}\left\{\sup_{\mathbf{g},\mathbf{h}\in\Gamma_U} \gamma_U\,|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})| \ge \delta\right\} \le 2\exp\left((-\hat{c} + c_4)(B(1)/\gamma_U^2)\delta^{2-1/b}\right).$$

By choosing $\hat{c} > 2c_4$, one ensures that

(15)
$$\mathrm{Prob}\left\{\sup_{\mathbf{g},\mathbf{h}\in\Gamma_U} \gamma_U\,|\mathrm{disc}([\mathbf{g},\mathbf{h}), U, \mathcal{P}_{n,s})| \ge \delta\right\} \le 2\exp\left(-[\hat{c}B(1)/(2\gamma_U^2)]\delta^{2-1/b}\right).$$

To obtain a lower bound on the probability of having a low discrepancy set, one sums the above quantity over U:

$$\begin{aligned}
&\text{Prob}\left\{\mathcal{D}_\infty(\mathcal{P}_{n,s},\boldsymbol{\gamma}) \ge 3\delta\right\}\\
&\le \text{Prob}\left\{\sup_{U \text{ with } \delta_U \le 1} \sup_{\mathbf{g},\mathbf{h}\in\Gamma_U} \gamma_U\,|\text{disc}([\mathbf{g},\mathbf{h}),U,\mathcal{P}_{n,s})| \ge \delta\right\}\\
&\le \sum_U 2\exp\left(-[\hat{c}B(1)/(2\gamma_U^2)]\delta^{2-1/b}\right)\\
&\le \sum_U \frac{2}{\exp\left([\hat{c}B(1)/(2\gamma_U^2)]\delta^{2-1/b}\right)}\\
&\le \sum_U \frac{2^{1+a/2}(a/(2e))^{a/2}\gamma_U^a\delta^{a(1-2b)/(2b)}}{(\hat{c}B(1))^{a/2}}\\
&\le \frac{2^{1+a/2}(a/(2e))^{a/2}\delta^{a(1-2b)/(2b)}}{(\hat{c}B(1))^{a/2}}\sum_U \gamma_U^a\\
&\le \frac{2^{1+a/2}(a/(2e))^{a/2}\delta^{a(1-2b)/(2b)}}{(\hat{c}B(1))^{a/2}}\prod_{j=1}^\infty(1+\gamma_j^a).
\end{aligned}$$

The product involving γ_j^a is finite because of the summability condition on the γ_j^a. If $\delta > \hat{\gamma}$, then $\text{Prob}\{\mathcal{D}_\infty(\mathcal{P}_{n,s},\boldsymbol{\gamma}) \ge 3\delta\} = 0$ automatically. Otherwise, $\delta^{a(1-2b)/(2b)} \le \hat{\gamma}^{a(1-2b)/(2b)}$ and the last term above may be made as small as desired by choosing $\hat{c}$ large enough. Doing so forces $\text{Prob}\{\mathcal{D}_\infty(\mathcal{P}_{n,s},\boldsymbol{\gamma}) < 3\delta\}$ to be greater than zero, completing the proof. □

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