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ALGORITHMS WITHOUT ACCURACY SATURATION FOR EVOLUTION EQUATIONS IN HILBERT AND BANACH SPACES

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Abstract. We consider the Cauchy problem for the first and the second order differential equations in Banach and Hilbert spaces with an operator coefficient $A(t)$ depending on the parameter t. We develop discretization methods with high parallelism level and without accuracy saturation; i.e., the accuracy adapts automatically to the smoothness of the solution. For analytical solutions the rate of convergence is exponential. These results can be viewed as a development of parallel approximations of the operator exponential e^{-tA} and of the operator cosine family $\cos\sqrt{A}t$ with a constant operator A possessing exponential accuracy and based on the Sinc-quadrature approximations of the corresponding Dunford-Cauchy integral representations of solutions or the solution operators.

1. Introduction

We consider the evolution problems

$$\frac{du}{dt} + A(t)u = f(t), \quad t \in (0,T]; \quad u(0) = u_0, \tag{1.1}$$

and

$$\frac{d^2u}{dt^2} + A(t)u = f(t), \quad t \in (0,T]; \quad u(0) = u_0, \quad u'(0) = u_{01} \tag{1.2}$$

where $A(t)$ is a densely defined closed (unbounded) operator with the domain $D(A)$ independent of t in a Banach space X, $u_0,\ u_{01}$ are given vectors and $f(t)$ is a given vector-valued function. We suppose the operator $A(t)$ to be strongly positive; i.e., there exists a positive constant M_R independent of t such that on the rays and outside a sector $\Sigma_\theta = \{z \in \mathbb{C} : 0 \le arg(z) \le \theta, \theta \in (0,\pi/2)\}$ the following resolvent estimate holds:

$$\|(zI - A(t))^{-1}\| \le \frac{M_R}{1+|z|}. \tag{1.3}$$

This assumption implies that there exists a positive constant c_κ such that (see [7], p. 103)

$$\|A^\kappa(t)e^{-sA(t)}\| \le c_\kappa s^{-\kappa}, \quad s > 0, \quad \kappa \ge 0. \tag{1.4}$$

Our further assumption is that there exists a real positive ω such that

$$\|e^{-sA(t)}\| \le e^{-\omega s} \quad \forall s,\ t \in [0,T] \tag{1.5}$$

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(see [23], Corollary 3.8, p. 12, for corresponding assumptions on $A(t)$). Let us also assume that the conditions

$$\|[A(t)-A(s)]A^{-\gamma}(t)\| \le \tilde{L}_{1,\gamma}|t-s| \quad \forall t,\ s,\ 0\le\gamma\le 1, \tag{1.6}$$

$$\|A^{\beta}(t)A^{-\beta}(s)-I\| \le \tilde{L}_{\beta}|t-s| \quad \forall t,\ s\in[0,T] \tag{1.7}$$

hold. Note that efficient approximations without accuracy saturation or with the exponential accuracy for the solution operators of equations (1.1), (1.2) with an unbounded operator A independent of t were proposed in [8, 9, 13, 14]. Using the Cayley transform, one can represent the exact solution (or the solution operator) by a series. The truncated sum of N series terms represents a numerical algorithm without accuracy saturation, i.e., with the convergence rate automatically adapting to the smoothness of the input data [8, 14]. Other approximations [9, 13] use representations of the exact solution or the solution operators through the resolvent of the spatial operator by the Dunford-Cauchy integral along a path in the complex plane enveloping the spectrum of the spatial operator. After parametrization this integral can be translated into an improper integral over the real axis and the last one is then approximated by a suitable Sinc-quadrature [26] possessing an exponential convergence rate. In this way we get an approximation to the solution operator as a short sum of resolvents of the spatial operator.

Sinc approximations can also be used to get exponentially convergent approximations to the solutions of various partial differential equations (PDE) with a known Green function or a fundamental solution which allow one to represent the solution by an integral [27]. Note that, in general, it is not the case for arbitrary equations of the type (1.1), (1.2) considered in the present paper.

The aim of this paper is to get algorithms without accuracy saturation and exponentially convergent algorithms for the solution of equations (1.1), (1.2). We use a piecewise constant approximation of the operator $A(t)$ and an exact integral corollary of these equations on the Chebyshev grid which is then approximated by the collocation method. The operator exponential (for equation (1.1)) and the operator cosine function (for equation (1.2)) with stationary operators involved in the algorithms can be computed by the Sinc approximations from [9, 13].

We begin with an example which shows the practical relevance for the assumptions above.

Example 1.1. Let $q(t)\ge q_0>0,\ t\in[0,T]$, be a given function from the Hölder class with the exponent $\alpha\in(0,1]$. We consider the operator $A(t)$ defined by

$$\begin{gathered} D(A(t))=\{u(x)\in H^4(0,1): u(0)=u''(0)=u(1)=u''(1)=0\},\\ A(t)u=\left[\frac{d^2}{dx^2}-q(t)\right]^2 u=\frac{d^4u}{dx^4}-2q(t)\frac{d^2u}{dx^2}+q^2(t)u \quad \forall u\in D\big(A(t)\big) \end{gathered} \tag{1.8}$$

with the domain independent of t. It is easy to show that

$$\begin{gathered} D(A^{1/2}(t))=\{u(x)\in H^2(0,1)\colon u(0)=u(1)=0\},\\ A^{1/2}(t)=-\frac{d^2u}{dx^2}+q(t)u \quad \forall u\in D\left(A^{1/2}(t)\right),\\ A^{-1/2}(t)=\int_0^1 G(x,\xi;t)v(\xi)\,d\xi, \end{gathered} \tag{1.9}$$

where the Green function is given by

(1.10)
$$G(x,\xi;t)=\frac{1}{\sqrt{q(t)}\sinh\sqrt{q(t)}}\begin{cases}\sinh\left(\sqrt{q(t)}x\right)\sinh\left(\sqrt{q(t)}(1-\xi)\right), & \text{if } x\le\xi,\\ \sinh\left(\sqrt{q(t)}\xi\right)\sinh\left(\sqrt{q(t)}(1-x)\right), & \text{if } \xi\le x.\end{cases}$$

Then we have the relation

(1.11)
$$\begin{aligned}[A(t)-A(s)]A^{-1/2}(t)v&=[q(t)-q(s)]\left\{-2\frac{d^2}{dx^2}+[q(t)+q(s)]\right\}\int_0^1 G(x,\xi;t)v(\xi)\,d\xi\\&=[q(t)-q(s)]\left\{2v(x)-[q(t)-q(s)]\int_0^1 G(x,\xi;t)v(\xi)\,d\xi\right\},\end{aligned}$$

which leads to the estimate

(1.12)
$$\begin{aligned}&\|[A(t)-A(s)]A^{-1/2}(t)v(\cdot)\|_{C[0,1]\to C[0,1]}\\&\quad\le L|t-s|^\alpha\left\{2\|v\|_{C[0,1]}+L|t-s|^\alpha\frac{1}{2\sqrt{q(t)}}\tanh\left(\sqrt{q(t)}/2\right)\|v\|_{C[0,1]}\right\},\end{aligned}$$

where L is the Hölder constant. This inequality yields

(1.13)
$$\begin{aligned}&\left\|[A(t)-A(s)]A^{-1/2}(t)\right\|_{C[0,1]\to C[0,1]}\\&\quad\le L\left\{2+LT^\alpha\tanh\left(\sqrt{q(t)}/2\right)/\left(2\sqrt{q(t)}\right)\right\}|t-s|^\alpha;\end{aligned}$$

i.e., condition (1.6) is fulfilled with $\gamma=1/2$ provided that $\alpha=1$. Let us prove condition (1.7). We have

(1.14)
$$\begin{aligned}\left[A^{1/2}(t)A^{-1/2}(s)-I\right]v&=\left[-\frac{d^2}{dx^2}+q(t)\right]\int_0^1 G(x,\xi;s)v(\xi)\,d\xi-v(x)\\&=[q(t)-q(s)]\int_0^1 G(x,\xi;s)v(\xi)\,d\xi,\end{aligned}$$

from which it follows that

(1.15)
$$\left\|A^{1/2}(t)A^{-1/2}(s)-I\right\|_{C[0,1]\to C[0,1]}\le L\frac{\tanh\left(\sqrt{q(t)}/2\right)}{2\sqrt{q(t)}}|t-s|^\alpha;$$

i.e., condition (1.7) is fulfilled with $\beta=1/2,\delta=\alpha=1$.

Remark 1.2. It is clear that, in general, for elliptic operators inequalities (1.6) and (1.7) hold true with $\gamma=1,\beta=1$.

Remark 1.3. Assumption (4.1) is not restrictive due to stability results from [15]. The two initial value problems

(1.16)
$$\frac{du}{dt}+A(t)u=f(t),\quad u(0)=u_0$$

and

(1.17)
$$\frac{dv}{dt}+B(t)v=g(t),\quad v(0)=v_0$$

with densely defined, closed operators $A(t),B(t)$ having a common domain $D\big(A(t)\big)=D\big(B(t)\big)$ independent of t were considered. The following assumptions were made.

(1) There exist bounded inverse operators $A^{-1}(t),\ B^{-1}(t)$ and for the resolvents $R_{A(t)}(z) = (z - A(t))^{-1}, R_{B(t)}(z) = (z - B(t))^{-1}$ we have

$$\|R_{A(t)}(z)\| \le \frac{1}{1+|z|},\ \|R_{B(t)}(z)\| \le \frac{1}{1+|z|} \qquad (\theta + \epsilon \le |\arg z| \le \pi) \tag{1.18}$$

for all $\theta \in (0, \pi/2), \epsilon > 0$ uniformly in $t \in [0, T]$.

(2) The operators $A(t), B(t)$ are strongly differentiable on $D\big(A(t)\big) = D\big(B(t)\big)$.

(3) There exists a constant M such that

$$\big\|A^{\beta}(s)B^{-\beta}(s)\big\| \le M. \tag{1.19}$$

(4) For the evolution operators $U_A(t,s), U_B(t,s)$ we have

$$\|A(t)U_A(t,s)\| \le \frac{C}{t-s}, \qquad \|B(t)U_B(t,s)\| \le \frac{C}{t-s}. \tag{1.20}$$

(5) There exist positive constants $C,\ C_\beta$ such that

$$\big\|A^{\rho}(t)A^{-\rho}(s) - I\big\| \le C|t-s|^{\alpha} \tag{1.21}$$

and

$$\|A^{\beta}(t)U_A(t,s)\| \le \frac{C_\beta}{|t-s|^{\beta}},\ \|B^{\beta}(t)U_B(t,s)\| \le \frac{C_\beta}{|t-s|^{\beta}} \tag{1.22}$$

for $0 \le \beta < \alpha + \beta$.

The following stability result for Banach spaces was proved in [15] under these assumptions:

$$\begin{aligned}\|A^{\beta}(t)z(t)\| &= \|A^{\beta}(t)(u(t) - v(t))\| \\ &\le M\|A^{\beta}(0)z(0)\| + c_\beta M \max_{0\le s\le T}\big\|[B(s) - A(s)]A^{-\beta}(s)\big\| \\ &\quad \times \frac{t^{1-\beta}}{1-\beta}\left\{\|B^{\beta}(0)v(0)\| + \int_0^t \|B^{\beta}(s)g(s)\|\,ds\right\} \\ &\quad + M\int_0^t \|B^{\beta}(s)g(s)\|\,ds.\end{aligned} \tag{1.23}$$

It is possible to avoid the restriction $\beta < 1$ if we consider equations (1.16), (1.17) in a Hilbert space. In this case we assume that there exists an operator $C = C^* \ge c_0 I$ such that

(1)

$$\big\|[A(s) - B(s)]C^{-1}\big\| \le \delta, \tag{1.24}$$

(2)

$$\begin{aligned}&\big(A(s)y, Cy\big) \ge c_0\|Cy\|^2, \\ &\big(B(s)y, Cy\big) \ge c_0\|Cy\|^2 \quad \forall s \in [0,T], c_0 > 0.\end{aligned} \tag{1.25}$$

Then the following stability estimate is fulfilled [15]:

$$\begin{aligned}\frac{1}{2}(Cz(t),z(t))+(c_0-\epsilon-\epsilon_1)\int_0^t\|Cz(s)\|^2ds\\ \le\max_{0\le s\le T}\|[A(s)-B(s)]C^{-1}\|^2\frac{(c_0-\epsilon_2)^{-1}}{2\epsilon}\\ \times\left[\frac{1}{4\epsilon_2}\int_0^t\|g(s)\|^2ds+\frac{1}{2}(Cv_0,v_0)\right]\\ +\frac{1}{2\epsilon_1}\int_0^t\|f(s)-g(s)\|^2ds+\frac{1}{2}\big(C(u_0-v_0),u_0-v_0\big),\end{aligned}\tag{1.26}$$

with arbitrary positive numbers $\epsilon,\epsilon_1,\epsilon_2$ such that $\epsilon+\epsilon_1<c_0,\epsilon_2<c_0$ which stand for the stability with respect to the right-hand side, the initial condition and the coefficient stability. Note that an analogous estimate in the case of a finite dimensional Hilbert spaces and of a constant operator A was proved in [25, p. 62].

Example 1.4. Let $\Omega\subset\mathbb{R}^2$ be a polygon and let

$$\mathcal{L}(x,t,D)=-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}a_{i,j}(x,t)\frac{\partial}{\partial x_j}+\sum_{j=1}^{2}b_j(x,t)\frac{\partial}{\partial x_j}+c(x,t)\tag{1.27}$$

be a second order elliptic operator with time-dependent real smooth coefficients satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^{2}a_{ij}(x,t)\xi_i\xi_j\ge\delta_1|\xi|^2\qquad(\xi=(\xi_1,\xi_2)\in\mathbb{R})\tag{1.28}$$

with a positive constant δ_1. Taking $X=L^2(\Omega)$ and $V=H_0^1(\Omega)$ or $V=H^1(\Omega)$ accordingly to the boundary condition

$$u=0\quad\text{on}\quad\partial\Omega\times(0,T)\tag{1.29}$$

or

$$\frac{\partial u}{\partial\nu_{\mathcal{L}}}+\sigma u=0\quad\text{on}\quad\partial\Omega\times(0,T),\tag{1.30}$$

we set

$$\begin{aligned}\mathcal{A}_t(u,v)=\sum_{i,j=1}^{2}\int_\Omega a_{i,j}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx\\ +\sum_{j=1}^{2}\int_\Omega b_j(x,t)\frac{\partial u}{\partial x_j}v\,dx+\int_\Omega c(x,t)uv\,dx+\int_{\partial\Omega}\sigma(x,t)uv\,dS\end{aligned}\tag{1.31}$$

for $u,v\in V$. An m-sectorial operator $A(t)$ in X can be defined through the relation

$$\mathcal{A}_t(u,v)=\big(A(t)u,v\big),\tag{1.32}$$

where $u\in D(A(t))\subset V$ and $v\in V$. The relation

$$D(A(t))=H^2(\Omega)\cap H_0^1(\Omega)\tag{1.33}$$

follows for $V=H_0^1(\Omega)$ and

$$D\big(A(t)\big)=\left\{v\in H^2(\Omega)\mid\frac{\partial v}{\partial\nu_{\mathcal{L}}}\text{ on }\partial\Omega\right\}\tag{1.34}$$

for $V=H^1(\Omega)$, if $\partial\Omega$ is smooth for instance.

It was proven in [7, pp. 95–101] that all the assumptions above hold for such an operator $A(t)$.

As we will see below, the parameter γ from (1.6) plays an essential role for the construction and the analysis of discrete approximations and algorithms for problems (1.1), (1.2).

2. Discrete first order problem in the case $\gamma < 1$

For the sake of simplicity we consider problem (1.1) on the interval $[-1, 1]$ (if it is not the case, one can reduce problem (1.1) to this interval by the variable transform $t = 2t'/T - 1, t \in [-1, 1], t' \in [0, T]$). We choose a mesh ω_n of n various points $\omega_n = \{t_k = \cos\frac{(2k-1)\pi}{2n}, k = 1, \dots, n\}$ on $[-1, 1]$ and set $\tau_k = t_k - t_{k-1}$,

$$\overline{A}(t) = A_k = A(t_k), \qquad t \in (t_{k-1}, t_k], \tag{2.1}$$

where t_k are zeros of Chebyshev orthogonal polynomial of the first kind $T_n(t) = \cos(n \arccos t)$. Let $t_\nu = \cos\theta_\nu, 0 < \theta_\nu < \pi, \nu = 1, 2, \dots, n$, be zeros of the Chebyshev orthogonal polynomial $T_n(t)$ taken in decreasing order. Then it is well known that (see [28], Ch.6, Th.6.11.12, [29], p. 123)

$$\begin{gathered} t_{\nu+1} - t_\nu < \frac{\pi}{n}, \qquad \nu = 1, \dots, n, \\ \tau_{\max} = \max_{1 \le k \le n} \tau_k < \frac{\pi}{n}. \end{gathered} \tag{2.2}$$

Let us rewrite problem (1.1) in the form

$$\begin{gathered} \frac{du}{dt} + \overline{A}(t)u = [\overline{A}(t) - A(t)]u(t) + f(t), \\ u(0) = u_0 \end{gathered} \tag{2.3}$$

from which we deduce

$$\begin{aligned} u(t) &= e^{-A_k(t-t_{k-1})}u(t_{k-1}) \\ &\quad + \int_{t_{k-1}}^{t} e^{-A_k(t-\eta)}\{[A_k - A(\eta)]u(\eta) + f(\eta)\}d\eta, \quad t \in [t_{k-1}, t_k]. \end{aligned} \tag{2.4}$$

Since A_{k-1} and $e^{-A_{k-1}\tau_k}$ commute, assumption (1.7) yields

$$\|A_k^\beta A_p^{-\beta}\| \le 1 + \|A_k^\beta A_p^{-\beta} - I\| \le 1 + \tilde{L}_\beta |t_k - t_p| \le 1 + \tilde{L}_\beta T. \tag{2.5}$$

Let

$$P_{n-1}(t; u) = P_{n-1}u = \sum_{p=1}^{n} u(t_p) L_{p,n-1}(t) \tag{2.6}$$

be the interpolation polynomial for the function $u(t)$ on the mesh ω_n, let $y = (y_1, \dots, y_n)$, $y_i \in X$ be a given vector, and let

$$P_{n-1}(t; y) = P_{n-1}y = \sum_{p=1}^{n} y_p L_{p,n-1}(t) \tag{2.7}$$

be the polynomial that interpolates y, where $L_{p,n-1} = \frac{T_n(t)}{T_n'(t_p)(t-t_p)}, p = 1, \dots, n$, are the Lagrange fundamental polynomials.

Substituting $P_n(\eta; y)$ for $u(\eta)$ and y_k for $u(t_k)$ in (2.4), we arrive at the following system of linear equations with respect to the unknowns y_k:

$$y_k = e^{-A_k\tau_k} y_{k-1} + \sum_{p=0}^{n} \alpha_{kp} y_p + \phi_k, \qquad k = 1, \dots, n, \tag{2.8}$$

where

$$\begin{aligned} \alpha_{kp} &= \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}[A_k - A(\eta)] L_{p,n-1}(\eta) d\eta, \\ \phi_k &= \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)} f(\eta) d\eta. \end{aligned} \tag{2.9}$$

Remark 2.1. In order to compute α_{kp} and ϕ_k efficiently, we replace $A(t), f(t)$ by their interpolation polynomials (it is possible due to stability results (1.23), (1.26); see also [15]) and then calculate the integrals analytically. We have

$$\begin{aligned} A(t) &= \sum_{l=1}^{n} \frac{A_l}{t - t_l} \frac{T_n(t)}{T_n'(t_l)}, \\ f(t) &= \sum_{l=1}^{n} \frac{f_l}{t - t_l} \frac{T_n(t)}{T_n'(t_l)}, \qquad f_l = f(t_l), \end{aligned} \tag{2.10}$$

so that

$$\begin{aligned} \alpha_{kp} &= \frac{1}{T_n'(t_p)} \sum_{l=1}^{n} \frac{1}{T_n'(t_l)} \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)} \frac{T_n^2(\eta)}{(\eta - t_l)(\eta - t_p)} d\eta [A_k - A_l], \\ \phi_k &= \sum_{l=1}^{n} \frac{f_k}{T_n'(t_l)} \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)} \frac{T_n(\eta)}{\eta - t_l} d\eta. \end{aligned} \tag{2.11}$$

Using the relation $2T_n^2(\eta) = 1 + 2T_{2n}(\eta)$, the polynomial $p_{2n-2}^{(l,p)} = \frac{T_n^2(\eta)}{(\eta - t_l)(\eta - t_p)}$ can be represented as (see [2])

$$\begin{aligned} p_{2n-2}^{(l,p)} &= \frac{2T_{2n}(\eta) + 1}{2(\eta - t_l)(\eta - t_p)} \\ &= \frac{1}{2(\eta - t_l)(\eta - t_p)} \left[2n \sum_{m=0}^{n} \frac{(-1)^m (2n - m - 1)!}{m!(2n - 2m)!} (2\eta)^{2n-2m} + 1 \right] \\ &= \sum_{i=0}^{2n-2} q_i(l,p) \eta^{2n-2-i}, \end{aligned} \tag{2.12}$$

where the coefficients $q_i(l,p)$ can be calculated, for example, by the Horner scheme. Given $q_i(l,p)$, we furthermore find that

$$\alpha_{kp} = \frac{1}{T_n'(t_p)} \sum_{l=1}^{n} \frac{1}{T_n'(t_l)} \sum_{i=0}^{2n-2} q_i(l,p) I_{k,i} [A_k - A_l], \tag{2.13}$$

where

$$
\begin{aligned}
I_{k,i} &= \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}\eta^{2n-2-i}d\eta \\
&= \sum_{s=0}^{2n-2-i} (-1)^s(2n-2-i)(2n-3-i) \\
&\quad \cdots(2n-2-i-s+1)A_k^{-s-1}t_k^{2n-2-i-s} \\
&\quad - \sum_{s=0}^{2n-2-i} (-1)^s(2n-2-i)(2n-3-i) \\
&\quad \cdots(2n-2-i-s+1)A_k^{-s-1}t_{k-1}^{2n-2-i-s}e^{-A_k\tau_k}.
\end{aligned} \tag{2.14}
$$

Analogously one can also calculate ϕ_k.

For the error $z=(z_0,z_1,\dots,z_n)$, $z_k=u(t_k)-y_k$ we have the relations

$$
z_k = e^{-A_k\tau_k}z_{k-1} + \sum_{p=0}^{n}\alpha_{kp}z_p + \psi_k, \qquad k=1,\dots,n, \tag{2.15}
$$

where

$$
\psi_k = \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}[A_k - A(\eta)][u(\eta)-P_n(\eta;u)]d\eta. \tag{2.16}
$$

We introduce the matrix

$$
S=\{s_{i,k}\}_{i,k=1}^n = \begin{pmatrix}
I & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
-e^{-A_1\tau_1} & I & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & -e^{-A_2\tau_2} & I & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & \cdot & \cdot & -e^{-A_{n-1}\tau_{n-1}} & I
\end{pmatrix}, \tag{2.17}
$$

the matrix $C=\{\tilde\alpha_{k,p}\}_{k,p=1}^n$ with $\tilde\alpha_{k,p}=A_k^\gamma\alpha_{k,p}A_p^{-\gamma}$ and the vectors

$$
y=\begin{pmatrix} A_1^\gamma y_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n^\gamma y_n \end{pmatrix}, f=\begin{pmatrix} A_1^\gamma\phi_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n^\gamma\phi_n \end{pmatrix}, \tilde f=\begin{pmatrix} A_1^\gamma e^{-A_1\tau_1}u_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \psi=\begin{pmatrix} A_1^\gamma\psi_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n^\gamma\psi_n \end{pmatrix}. \tag{2.18}
$$

It is easy to see that for

$$
\begin{aligned}
S^{-1} &= \{s_{i,k}^{-1}\}_{i,k=1}^n \\
&= \begin{pmatrix}
I & 0 & \cdots & 0 & 0 \\
e^{-A_1\tau_1} & I & \cdots & 0 & 0 \\
e^{-A_2\tau_2}e^{-A_1\tau_1} & e^{-A_2\tau_2} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdots & \cdot & \cdot \\
e^{-A_{n-1}\tau_{n-1}}\cdots e^{-A_1\tau_1} & e^{-A_{n-1}\tau_{n-1}}\cdots e^{-A_2\tau_2} & \cdots & e^{-A_{n-1}\tau_{n-1}} & I
\end{pmatrix}
\end{aligned} \tag{2.19}
$$

we have

$$
S^{-1}S = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & I \end{pmatrix}. \tag{2.20}
$$

Remark 2.2. Using results of [9, 13], one can get a parallel and sparse approximation with an exponential convergence rate of operator exponentials in S^{-1} and as a consequence a parallel and sparse approximation of S^{-1}.

We get from (2.8), (2.15)

$$\begin{aligned} A_k^\gamma y_k &= e^{-A_k\tau_k} A_k^\gamma y_{k-1} + \sum_{p=0}^{n} \tilde{\alpha}_{kp} A_p^\gamma y_p + A_k^\gamma \phi_k, \\ A_k^\gamma z_k &= e^{-A_k\tau_k} A_k^\gamma z_{k-1} + \sum_{p=0}^{n} \tilde{\alpha}_{kp} A_p^\gamma z_p + A_k^\gamma \psi_k, \qquad k = 1, \dots, n, \end{aligned} \tag{2.21}$$

or in matrix form

$$\begin{aligned} Sy &= Cy + f - \tilde{f}, \\ Sz &= Cz + \psi \end{aligned} \tag{2.22}$$

with

$$z = \begin{pmatrix} A_1^\gamma z_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n^\gamma z_n \end{pmatrix}. \tag{2.23}$$

Next, for a vector $v = (v_1, v_2, \dots, v_n)^T$ and a block operator matrix $A = \{a_{ij}\}_{i,j=1}^n$ we introduce the vector norm

$$|||v||| \equiv |||v|||_\infty = \max_{1 \le k \le n} \|v_k\| \tag{2.24}$$

and the consistent matrix norm

$$|||A||| \equiv |||A|||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} \|a_{i,j}\|. \tag{2.25}$$

Due to (1.5) we get

$$|||S^{-1}||| \le n. \tag{2.26}$$

For further analysis we need the following auxiliary result.

Lemma 2.3. *The estimates*

$$|||C||| \le c(1 + \tilde{L}_\gamma T) n^{\gamma-2} \ln n, \tag{2.27}$$

$$|||S^{-1}C||| \le c(1 + \tilde{L}_\gamma T) n^{\gamma-1} \ln n \tag{2.28}$$

with a positive constant c independent of n hold true.

Proof. Assumption (1.6) together with (2.5) implies

$$\begin{aligned} \|\tilde{\alpha}_{kp}\| &= \|A_k^\gamma \alpha_{kp} A_p^{-\gamma}\| \\ &= \| \int_{t_{k-1}}^{t_k} A_k^\gamma e^{-A_k(t_k-\eta)} [A_k - A(\eta)] A_p^{-\gamma} L_{p,n-1}(\eta) d\eta \| \\ &\le \left(1 + \tilde{L}_\gamma T\right) \tau_{\max}^{1-\gamma} \int_{t_{k-1}}^{t_k} |L_{p,n-1}(\eta)| d\eta, \qquad T = 2. \end{aligned} \tag{2.29}$$

Using the well-known estimate for the Lebesgue constant Λ_n related to the Chebyshev interpolation nodes (see, e.g., [28, 29])

$$\Lambda_n = \max_{\eta\in[-1,1]} \sum_{p=1}^{n} |L_{p,n-1}(\eta)| \le c \ln n \tag{2.30}$$

and (2.29), we have

$$\begin{aligned} |||C||| &\le \max_{1\le k\le n} \sum_{p=1}^{n} \|\tilde{\alpha}_{kp}\| \\ &\le (1+\tilde{L}_\gamma T)\tau_{\max}^{2-\gamma} \max_{\eta\in[-1,1]} \sum_{p=1}^{n} |L_{p,n-1}(\eta)| \\ &\le (1+\tilde{L}_\gamma T)\tau_{\max}^{2-\gamma}\Lambda_n \le c(1+\tilde{L}_\gamma T)\tau_{\max}^{2-\gamma} \ln n \\ &\le c(1+\tilde{L}_\gamma T)n^{\gamma-2} \ln n \end{aligned} \tag{2.31}$$

with some positive constant c independent of n. This estimate together with (2.26) implies

$$|||S^{-1}C||| \le c(1+\tilde{L}_\gamma T)n^{\gamma-1} \ln n \to 0 \tag{2.32}$$

as $n \to \infty$ provided that $\gamma < 1$. □

Remark 2.4. We have reduced the interval length to $T = 2$ but we write T explicitly in order to underline the dependence of constants involved on T in the general case.

Let Π_{n-1} be the set of all polynomials in t with vector coefficients of degree less than or equal to $n-1$. Then the Lebesgue inequality

$$\begin{aligned} &\|u(\eta) - P_{n-1}(\eta;u)\|_{C[-1,1]} \\ &\quad \equiv \max_{\eta\in[-1,1]} \|u(\eta) - P_{n-1}(\eta;u)\| \le (1+\Lambda_n)E_n(u) \end{aligned} \tag{2.33}$$

can be proved for vector-valued functions in complete analogy with [1, 28, 29] and with the error of the best approximation of u by polynomials of degree not greater than $n-1$

$$E_n(u) = \inf_{p\in\Pi_{n-1}} \max_{\eta\in[-1,1]} \|u(\eta) - p(\eta)\|. \tag{2.34}$$

Now, we can go over to the main result of this section.

Theorem 2.5. *Let assumptions* (1.3)–(1.7) *with* $\gamma < 1$ *hold. Then there exists a positive constant* c *such that the following hold.*

(1) *For* n *large enough it holds that*

$$|||z||| \equiv |||y-u||| \le cn^{\gamma-1} \ln n E_n(A_0^\gamma u), \tag{2.35}$$

where u *is the solution of* (1.1).

(2) *The system of linear algebraic equations*

$$Sy = Cy + f \tag{2.36}$$

with respect to the approximate solution y *can be solved by the fixed-point iteration*

$$y^{(k+1)} = S^{-1}Cy^{(k)} + S^{-1}(f - \tilde{f}), \qquad k = 0, 1, \ldots; \quad y^{(0)} \text{ arbitrary} \tag{2.37}$$

with the convergence rate of a geometrical progression with the denominator $q \le cn^{\gamma-1}\ln n < 1$ *for* n *large enough.*

Proof. From the second equation in (2.22) we get

$$z = S^{-1}Cz + S^{-1}\psi \tag{2.38}$$

from which due to Lemma 2.3 and (2.26) we get

$$|||z||| \le cn|||\psi||| \tag{2.39}$$

for n large enough. The last norm can be estimated in the following way:

$$\begin{aligned}
|||\psi||| &= \max_{1\le k\le n} \| \int_{t_{k-1}}^{t_k} \Big[A_k^{\gamma} e^{-A_k(t_k-\eta)} \, [A_k - A(\eta)] \\
&\qquad\qquad \times A_k^{-\gamma}(A_k^{\gamma}A_0^{-\gamma})(A_0^{\gamma}u(\eta) - P_n(\eta; A_0^{\gamma}u)) \Big] d\eta \| \\
&\le (1+\tilde{L}_{\gamma}T) \max_{1\le k\le n} \int_{t_{k-1}}^{t_k} |t_k-\eta|^{-\gamma}|t_k-\eta| \\
&\qquad\qquad \times \|A_0^{\gamma}u(\eta) - P_n(\eta; A_0^{\gamma}u)\| d\eta \\
&\le (1+\tilde{L}_{\gamma}T)\tau_{\max}^{2-\gamma} \|A_0^{\gamma}u(\cdot) - P_{n-1}(\cdot; A_0^{\gamma}u)\|_{C[-1,1]} \\
&\le (1+\tilde{L}_{\gamma}T)\tau_{\max}^{2-\gamma}(1+\Lambda_n) E_n(A_0^{\gamma}u) \le cn^{\gamma-2}\ln n E_n(A_0^{\gamma}u),
\end{aligned} \tag{2.40}$$

and taking into account (2.39), we get the statement of the theorem. □

3. Discrete first order problem in the case $\gamma \le 1$

In this section we construct a new discrete approximation of problem (1.1) which is a little more complicated than approximation (2.8) of the previous section but possesses a higher convergence order and allows the case $\gamma = 1$.

Applying transform (2.4) once more (i.e., substituting $u(t)$ recursively), we get

$$\begin{aligned}
u(t) &= \left\{ e^{-A_k(t-t_{k-1})} + \int_{t_{k-1}}^{t} e^{-A_k(t-\eta)} \, [A_k - A(\eta)] \, e^{-A_k(\eta-t_{k-1})} d\eta \right\} u(t_{k-1}) \\
&\quad + \int_{t_{k-1}}^{t} e^{-A_k(t-\eta)} [A_k - A(\eta)] \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)} [A_k - A(s)] u(s) ds d\eta \\
&\quad + \int_{t_{k-1}}^{t} e^{-A_k(t-\eta)} \left\{ [A_k - A(\eta)] \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)} f(s) ds + f(\eta) \right\} d\eta.
\end{aligned} \tag{3.1}$$

Setting $t = t_k$, we arrive at the relation

$$\begin{aligned}
u(t_k) &= S_{k,k-1}u(t_{k-1}) \\
&\quad + \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)} [A_k - A(\eta)] \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)} [A_k - A(s)] u(s) ds d\eta + \phi_k,
\end{aligned} \tag{3.2}$$

where

$$S_{k,k-1} = e^{-A_k\tau_k} + \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}[A_k - A(\eta)]e^{-A_k(\eta-t_{k-1})}d\eta,$$

$$\phi_k = \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}f(\eta)d\eta + \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}[A_k - A(\eta)]\int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)}f(s)dsd\eta. \tag{3.3}$$

Substituting the interpolation polynomial $P_{n-1}(\eta; y)$ from the previous section for $u(\eta)$ and y_k for $u(t_k)$ in (3.2), we arrive at the following system of linear equations with respect to the unknowns y_k:

$$y_k = S_{k,k-1}y_{k-1} + \sum_{p=1}^{n}\alpha_{kp}y_p + \phi_k, \qquad k = 1, \dots, n, \tag{3.4}$$

where

$$\alpha_{kp} = \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}[A_k - A(\eta)] \times \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)}[A_k - A(s)]L_{p,n-1}(s)dsd\eta. \tag{3.5}$$

Remark 3.1. Due to stability results (1.23), (1.26) (see also [15]) one can approximate the initial problems by problems with polynomials $\tilde{A}(t), \tilde{f}(t)$, for example, as interpolation polynomials for $A(t), f(t)$.

With the aim of getting a computational algorithm for α_{kp}, we write down formula (3.5) in the form

$$\alpha_{kp} = \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{\eta} e^{-A_k(t_k-\eta)} \times [A_k - A(\eta)]e^{-A_k(\eta-s)}d\eta[A_k - A(s)]L_{p,n-1}(s)ds. \tag{3.6}$$

In order to calculate the inner integral, we represent

$$A(\eta) = A_k + (t_k - \eta)B_{1,k} + \cdots + (t_k - \eta)^{n-1}B_{n-1,k}. \tag{3.7}$$

Then we have to calculate integrals of the type

$$\tilde{\alpha}_{kp} = \int_{t_{k-1}}^{\eta} e^{-A_k(t_k-\eta)}(t_k - \eta)^p B_{p,k}e^{-A_k(\eta-s)}d\eta. \tag{3.8}$$

Analogously to [15], using the representation by the Dunford-Cauchy integrals and the residue theorem under assumption of the strong P-positiveness [8, 13, 9] of the operator $A(t)$, one can get

$$\tilde{\alpha}_{kp} = \frac{p!}{2\pi i}\int_{\Gamma_I} e^{-z(t_k-\eta)}(A_k - zI)^{-p-1}B_{p,k}(zI - A_k)^{-1}dz, \tag{3.9}$$

where Γ_I is an integration parabola enveloping the spectral parabola of the strongly P-positive operator $A(t)$. Now, using (3.7), (3.9), formula (3.6) can be written down

as

$$\begin{aligned}\alpha_{kp} = -\frac{1}{2\pi i}\int_{\Gamma_I}\sum_{p=1}^{n-1} p!(A_k - zI)^{-p-1}B_{p,k}(zI - A_k)^{-1} \\ \times \int_{t_{k-1}}^{t_k} e^{-z(t_k-s)}[A_k - A(s)]L_{p,n-1}(s)dsdz.\end{aligned} \tag{3.10}$$

The inner integral in this formula can be calculated analogously to (2.13), and the integral along Γ_I can be calculated explicitly using the residue theorem.

For the error $z = (z_1, \dots, z_n), z_k = u(t_k) - y_k$ we have the relations

$$z_k = S_{k,k-1}z_{k-1} + \sum_{p=0}^{n}\alpha_{kp}z_p + \psi_k, \qquad k = 1, \dots, n, \tag{3.11}$$

where

$$\psi_k = \int_{t_{k-1}}^{t_k} e^{-A_k(t_k-\eta)}\int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)}[A_k - A(s)][u(s) - P_{n-1}(s;u)]dsd\eta. \tag{3.12}$$

We introduce the matrix

$$\tilde{S} = \{\tilde{s}_{i,k}\}_{i,k=1}^{n} = \begin{pmatrix} I & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ -\tilde{S}_{21} & I & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & -\tilde{S}_{32} & I & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & -\tilde{S}_{n,n-1} & I \end{pmatrix}, \tag{3.13}$$

with $\tilde{S}_{k,k-1} = A_k^{\gamma}S_{k,k-1}A_{k-1}^{-\gamma}$, the matrix $C = \{\tilde{\alpha}_{k,p}\}_{k,p=1}^{n}$ with $\tilde{\alpha}_{k,p} = A_k^{\gamma}\alpha_{k,p}A_p^{-\gamma}$ and the vectors

$$y = \begin{pmatrix} A_1^{\gamma}y_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n^{\gamma}y_n \end{pmatrix}, \quad f = \begin{pmatrix} A_1^{\gamma}\phi_1 \\ \cdot \\ \cdot \\ \cdot \\ a_n^{\gamma}\phi_n \end{pmatrix}, \quad \psi = \begin{pmatrix} A_1^{\gamma}\psi_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n^{\gamma}\psi_n \end{pmatrix}, \quad \tilde{f} = \begin{pmatrix} A_1^{\gamma}S_{21}u_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}. \tag{3.14}$$

It is easy to check that for

$$\begin{aligned}\tilde{S}^{-1} &= \{\tilde{s}_{i,k}^{-1}\}_{i,k=1}^{n} \\ &= \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ \tilde{S}_{21} & I & 0 & \cdots & 0 & 0 \\ \tilde{S}_{32}\tilde{S}_{21} & \tilde{S}_{32} & I & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \tilde{S}_{n,n-1}\cdots\tilde{S}_{21} & \tilde{S}_{n,n-1}\cdots\tilde{S}_{32} & \tilde{S}_{n,n-1}\cdots\tilde{S}_{43} & \cdots & \tilde{S}_{n,n-1} & I \end{pmatrix}\end{aligned} \tag{3.15}$$

we have that

$$\tilde{S}^{-1}\tilde{S} = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & I \end{pmatrix}. \tag{3.16}$$

Remark 3.2. Using results of [9], one can get a parallel and sparse approximation of operator exponentials in $\tilde{S}^{-1}$ and as a consequence a parallel and sparse approximation of $\tilde{S}^{-1}$.

We get from (3.4), (3.11)

$$\begin{aligned} A_k^\gamma y_k &= \tilde{S}_{k,k-1} A_{k-1}^\gamma y_{k-1} + \sum_{p=0}^{n} \tilde{\alpha}_{kp} A_p^\gamma y_p + A_k^\gamma \phi_k, \\ A_k^\gamma z_k &= \tilde{S}_{k,k-1} A_{k-1}^\gamma z_{k-1} + \sum_{p=0}^{n} \tilde{\alpha}_{kp} A_p^\gamma z_p + A_k^\gamma \psi_k, \end{aligned} \tag{3.17}$$

or in matrix form

$$\begin{aligned} \tilde{S} y &= C y + f + \tilde{f}, \\ \tilde{S} z &= C z + \psi \end{aligned} \tag{3.18}$$

with

$$z = \begin{pmatrix} A_1^\gamma z_1 \\ \cdot \\ \cdot \\ \cdot \\ A_n^\gamma z_n \end{pmatrix}. \tag{3.19}$$

In the next lemma we estimate the norms of C and $\tilde{S}^{-1}C$.

Lemma 3.3. *The estimates*

$$|||C||| \le c(\gamma, T) n^{2\gamma-4} \ln n, \tag{3.20}$$

$$|||\tilde{S}^{-1} C||| \le c(\gamma, T) n^{2\gamma-3} \ln n \tag{3.21}$$

with a positive constant $c = c(T, \gamma)$ depending on γ and the interval length T but independent of n and such that $c = c(T, \gamma) \to \infty$ as $\gamma \to 1$ hold true.

Proof. Assumption (1.6) together with (1.4), (2.5), (2.2), (2.30) imply

(3.22)

$$\begin{aligned} \|\tilde{\alpha}_{kp}\| &= \|A_k^\gamma \alpha_{kp} A_p^{-\gamma}\| \\ &= \| \int_{t_{k-1}}^{t_k} A_k^\gamma e^{-A_k(t_k-\eta)} [A_k - A(\eta)] \\ &\qquad \times A_k^{-\gamma} \int_{t_{k-1}}^{\eta} A_k^\gamma e^{-A_k(\eta-s)} [A_k - A(s)] A_p^{-\gamma} L_{p,n-1}(\eta) d\eta \| \\ &\le \Big(1 + \tilde{L}_\gamma T\Big) \Big(c_\gamma \tilde{L}_{1,\gamma}\Big)^2 \int_{t_{k-1}}^{t_k} |t_k - \eta|^{1-\gamma} \int_{t_{k-1}}^{\eta} |\eta - s|^{-\gamma} |t_k - s| |L_{p,n-1}(s)| ds d\eta. \end{aligned}$$

Due to (3.22) we have

$$
\begin{aligned}
|||C||| &= \max_{1\le k\le n}\sum_{p=1}^{n}\|\tilde\alpha_{kp}\| \\
&\le \left(1+\tilde L_\gamma T\right)\left(c_\gamma\tilde L_{1,\gamma}\right)^2\Lambda_n \max_{1\le k\le n} \\
&\qquad\times\int_{t_{k-1}}^{t_k}|t_k-\eta|^{1-\gamma}\int_{t_{k-1}}^{\eta}|\eta-s|^{-\gamma}|t_k-s|ds d\eta \\
&\le \left(1+\tilde L_\gamma T\right)\left(c_\gamma\tilde L_{1,\gamma}\right)^2\Lambda_n \max_{1\le k\le n}\frac{\tau_k}{1-\gamma} \\
&\qquad\times\int_{t_{k-1}}^{t_k}|t_k-\eta|^{1-\gamma}|\eta-t_{k-1}|^{1-\gamma}d\eta \\
&\le \left(1+\tilde L_\gamma T\right)\left(c_\gamma\tilde L_{1,\gamma}\right)^2\Lambda_n \max_{1\le k\le n}\frac{\tau_k^{2-\gamma}}{1-\gamma}\int_{t_{k-1}}^{t_k}|t_k-\eta|^{1-\gamma}d\eta \\
&\le c(\gamma,T)\Lambda_n\tau_{\max}^{4-2\gamma} \\
&\le c(\gamma,T)n^{2\gamma-4}\ln n,
\end{aligned}
\tag{3.23}
$$

where $c(\gamma,T)=c\frac{\left(1+\tilde L_\gamma T\right)\left(c_\gamma\tilde L_{1,\gamma}\right)^2}{(1-\gamma)(2-\gamma)}$, c is a constant independent of n,γ and (3.20) is proved.

Furthermore, the inequalities (1.4), (1.6), (1.7) imply

$$
\begin{aligned}
\|\tilde S_{k,k-1}\| &\le e^{-\omega\tau_k}+c_\gamma\tilde L_{1,\gamma}(1+\tilde L_\gamma\tau_k) \\
&\qquad\times\int_{t_{k-1}}^{t_k}|t_k-\eta|^{-\gamma}|t_k-\eta|e^{-\omega(\eta-t_{k-1})}d\eta \\
&\le e^{-\omega\tau_k}\left[1+\frac{c_\gamma\tilde L_{1,\gamma}(1+\tilde L_\gamma\tau_k)}{2-\gamma}\tau_k^{2-\gamma}\right]
\end{aligned}
\tag{3.24}
$$

which yields

$$
\|\tilde S^{-1}\|\le\sum_{p=0}^{n-1}q^p=\frac{q^n-1}{q-1} \tag{3.25}
$$

with

$$
q=\left\{e^{-\omega\tau_{\max}}\left[1+\frac{c_\gamma\tilde L_{1,\gamma}(1+\tilde L_\gamma\tau_{\max})}{2-\gamma}\tau_{\max}^{2-\gamma}\right]\right\}\to 1
$$

as $\tau_{\max}\to 0$. This means that there exists a constant $C=C(\gamma,c_\gamma,\tilde L_\gamma,\tilde L_{1,\gamma})$ such that

$$
\|\tilde S^{-1}\|\le Cn \tag{3.26}
$$

(it is easy to see that $C\le 1$ provided that $-\omega+\frac{c_\gamma\tilde L_{1,\gamma}(1+\tilde L_\gamma\tau_{\max})}{2-\gamma}\tau_{\max}^{1-\gamma}\le 0$). This estimate together with (3.23) implies (3.21). The proof is complete. □

Now, we can go to the first main result of this section.

Theorem 3.4. *Let assumptions* (1.3)–(1.7) *with* $\gamma<1$ *hold. Then there exists a positive constant* c *such that the following hold.*

(1) *For n large enough it holds that*

$$|||z||| \equiv |||y-u||| \le cn^{2\gamma-3}\ln n E_n(A_0^\gamma u), \qquad \gamma\in[0,1), \tag{3.27}$$

where u is the solution of (1.1) and $E_n(A_0^\gamma u)$ is the best approximation of $A_0^\gamma u$ by polynomials of degree not greater then $n-1$.

(2) *The system of linear algebraic equations*

$$Sy = Cy + f \tag{3.28}$$

from (3.18) with respect to the approximate solution y can be solved by the fixed-point iteration

$$y^{(k+1)} = S^{-1}Cy^{(k)} + S^{-1}(f-\tilde f), \qquad k=0,1,\ldots; \quad y^{(0)} \text{ arbitrary} \tag{3.29}$$

converging at least as a geometrical progression with the denominator $q = c(\gamma,T)n^{2\gamma-3}\ln n < 1, \gamma\in[0,1)$ for n large enough.

Proof. From the second equation in (3.18) we get

$$z = S^{-1}Cz + S^{-1}\psi \tag{3.30}$$

from which due to Lemma 2.3 and (3.25) we get

$$|||z||| \le cn|||\psi|||. \tag{3.31}$$

Let Π_{n-1} be the set of all polynomials in t with vector coefficients of degree less than or equal to $n-1$. Using the Lebesgue inequality (2.33), the last norm can be estimated as

(3.32)

$$\begin{aligned}
|||\psi||| &= \max_{1\le k\le n} \Big\| \int_{t_{k-1}}^{t_k} A_k^\gamma e^{-A_k(t_k-\eta)}\,[A_k - A(\eta)] \\
&\qquad \times \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)}\,[A_k - A(s)]\,A_0^{-\gamma}(A_0^\gamma u(s) - P_{n-1}(s;A_0^\gamma u))dsd\eta\Big\| \\
&\le (1+\tilde L_\gamma T)(c_\gamma L_{1,\gamma})^2 \max_{1\le k\le n}\int_{t_{k-1}}^{t_k} |t_k-\eta|^{-\gamma}|t_k-\eta| \\
&\qquad \times \int_{t_{k-1}}^{\eta} |\eta-s|^{-\gamma}|t_k-s|\|A_0^\gamma u(\eta) - P_{n-1}(s;A_0^\gamma u)\|dsd\eta \\
&\le (1+\tilde L_\gamma T)(c_\gamma L_{1,\gamma})^2(1+\Lambda_n)E_n(A_0^\gamma u) \\
&\qquad \times \max_{1\le k\le n}\left\{\int_{t_{k-1}}^{t_k} |t_k-\eta|^{-\gamma}|t_k-\eta| \int_{t_{k-1}}^{\eta} |\eta-s|^{-\gamma}|t_k-s|dsd\eta\right\} \\
&\le cc(\gamma,T)E_n(A_0^\gamma u)n^{2\gamma-4}\ln n,
\end{aligned}$$

and taking into account (3.31), we get the first assertion of the theorem.

The second assertion is a simple consequence of (3.28) and (3.21), which completes the proof of the theorem. □

Under somewhat stronger assumptions on the operator $A(t)$ one can improve the error estimate for our method in the case $0\le\gamma\le 1$. In order to do it we need the following lemma.

Lemma 3.5. *Let $L_{\nu,n-1}(t)$ be the Lagrange fundamental polynomials related to the Chebyshev interpolation nodes (zeros of the Chebyshev polynomial of the first kind $T_n(t)$). Then*

$$\sum_{\nu=1}^{n} |L'_{\nu,n-1}(t)| \le \frac{1}{\sqrt{1-x^2}} \sqrt{2/3} n^{3/2}. \tag{3.33}$$

Proof. Let $x \in [-1,1]$ be an arbitrary point and let $\epsilon_\nu = \operatorname{sign}(L'_{\nu,n-1}(x))$. We consider the polynomial of t

$$\rho(t;x) = \sum_{\nu=1}^{n} \epsilon_\nu(x) L_{\nu,n-1}(t) = \sum_{\nu=1}^{n-1} c_\nu(x) T_\nu(t). \tag{3.34}$$

Since $\rho^2(t;x)$ is the polynomial of degree $2n-2$, then using the Gauß-Chebyshev quadrature rule and the property $L_{k,n-1}(t_\nu) = \delta_{k,\nu}$ of the fundamental Lagrange polynomials ($\delta_{k,\nu}$ is the Kronecker symbol), we get

$$\begin{aligned} \int_{-1}^{1} \frac{\rho^2(t;x)}{\sqrt{1-t^2}} dt &= \sum_{\nu=0}^{n-1} c_\nu^2(x) \frac{\pi}{2} = \sum_{\nu=1}^{n} \lambda_\nu \rho^2(t_\nu;x) \\ &= \sum_{\nu=1}^{n} \lambda_\nu \epsilon_\nu^2 = \sum_{\nu=1}^{n} \lambda_\nu = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} dt = \pi \end{aligned} \tag{3.35}$$

with the quadrature coefficients λ_ν which yields

$$\sum_{\nu=0}^{n-1} c_\nu^2(x) = 2. \tag{3.36}$$

The next estimate

$$\begin{aligned} \rho'(x) &= \sum_{\nu=1}^{n} |L'_{\nu,n-1}(x)| \le \sum_{\nu=0}^{n-1} |c_\nu| |T'_\nu(x)| \\ &= \sum_{\nu=0}^{n-1} |c_\nu| \frac{\nu}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{1-x^2}} \left(\sum_{\nu=0}^{n-1} c_\nu^2 \right)^{1/2} \left(\sum_{\nu=1}^{n-1} \nu^2 \right)^{1/2} \\ &\le \frac{1}{\sqrt{1-x^2}} \left(\sum_{\nu=0}^{n-1} (c_\nu)^2 \right)^{1/2} \sqrt{n^3/3} \end{aligned} \tag{3.37}$$

together with (3.36) proves the lemma. □

Now we are in the position to prove the following important result of this section.

Lemma 3.6. *Let the operator $A(t)$ be strongly continuous differentiable on $[0,T]$ (see* [18]*, Ch.* 2*,* §1*, p.* 218*,* [19]*), satisfy condition* (1.6)*, and let $A'(s)A^{-\gamma}(0)$ be bounded for all $s \in [0,T]$ and $\gamma \in [0,1]$ by a constant c'. Then for n large enough the following estimates hold true:*

$$|||C||| \le c n^{\gamma - 5/2}, \quad \gamma \in [0,1], \tag{3.38}$$

$$|||\tilde{S}^{-1} C||| \le c n^{\gamma - 3/2}, \quad \gamma \in [0,1] \tag{3.39}$$

with some positive constant c independent of n, γ.

Proof. Opposite to the proof of Lemma 2.3 (see (3.22)), we estimate $\tilde{\alpha}_{kp}$ as

$$
\begin{aligned}
\|\tilde{\alpha}_{kp}\| &= \|A_k^{\gamma}\alpha_{kp}A_p^{-\gamma}\| \\
&= \| \int_{t_{k-1}}^{t_k} A_k^{\gamma} e^{-A_k(t_k-\eta)}[A_k - A(\eta)] \\
&\quad \times A_k^{-1} \int_{t_{k-1}}^{\eta} \frac{de^{-A_k(\eta-s)}}{ds}[A_k - A(s)]A_p^{-\gamma}L_{p,n-1}(\eta)d\eta\| \\
&= \| \int_{t_{k-1}}^{t_k} A_k^{\gamma} e^{-A_k(t_k-\eta)}[A_k - A(\eta)]A_k^{-1}\Big\{[A_k - A(\eta)]A_p^{-\gamma}L_{p,n-1}(\eta) \\
&\quad - e^{-A_k(\eta-t_{k-1})}[A_k - A_{k-1}]A_p^{-\gamma}L_{p,n-1}(t_{k-1}) \\
&\quad + \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)}A'(s)A_p^{-\gamma}L_{p,n-1}(s)ds \\
&\quad - \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)}[A_k - A(s)]A_p^{-\gamma}L'_{p,n-1}(s)ds\Big\}d\eta\| \\
&\le \int_{t_{k-1}}^{t_k} c_\gamma \tilde{L}_{1,1}(t_k-\eta)^{-\gamma}(t_k-\eta)\Big\{\tilde{L}_{1,\gamma}(t_k-\eta)(1+\tilde{L}_\gamma T)|L_{p,n-1}(\eta)| \\
&\quad + \tilde{L}_{1,\gamma}\tau_k(1+\tilde{L}_\gamma T)\delta_{p,k-1} + c'(1+\tilde{L}_\gamma T)\int_{t_{k-1}}^{\eta}|L_{p,n-1}(s)|ds \\
&\quad + \int_{t_{k-1}}^{\eta}\tilde{L}_{1,\gamma}(t_k-s)(1+\tilde{L}_\gamma T)|L'_{p,n-1}(s)|ds\Big\}d\eta \\
&\le \int_{t_{k-1}}^{t_k} c_\gamma \tilde{L}_{1,1}\Big\{\tilde{L}_{1,\gamma}(t_k-\eta)^{2-\gamma}|L_{p,n-1}(\eta)| \\
&\quad + \tilde{L}_{1,\gamma}\tau_k(1+\tilde{L}_\gamma T)(t_k-\eta)^{1-\gamma}\delta_{p,k-1} \\
&\quad + c'(1+\tilde{L}_\gamma T)(t_k-\eta)^{1-\gamma}\int_{t_{k-1}}^{\eta}|L_{p,n-1}(s)|ds \\
&\quad + \tilde{L}_{1,\gamma}\tau_k(1+\tilde{L}_\gamma T)(t_k-\eta)^{1-\gamma}\int_{t_{k-1}}^{\eta}|L'_{p,n-1}(s)|ds\Big\}d\eta.
\end{aligned}
\tag{3.40}
$$

Using this inequality together with (3.33), (3.25) and the relations $\arcsin t = \pi/2 - \arccos t, t_k = \cos\frac{2k-1}{2n}\pi, k = 1,\dots,n$, we arrive at the estimates

$$
\begin{aligned}
|||C||| &= \max_{1\le k\le n}\sum_{p=1}^{n}\|A_k^{\gamma}\alpha_{k,p}A_p^{-\gamma}\| \\
&\le M\left\{n^{\gamma-3}\ln n + n^{\gamma-3} + n^{\gamma-3/2}\max_{1\le k\le n}\int_{t_{k-1}}^{t_k}\frac{1}{\sqrt{1-s^2}}ds\right\} \\
&\le M\left\{n^{\gamma-3}\ln n + n^{\gamma-3} + n^{\gamma-3/2}\max_{1\le k\le n}(\arcsin t_k - \arcsin t_{k-1})\right\} \\
&\le Mn^{\gamma-5/2},
\end{aligned}
\tag{3.41}
$$

$$
|||\tilde{S}^{-1}C||| \le Mn^{\gamma-3/2}
$$

with a constant M independent of n. The proof is complete. □

Remark 3.7. If an operator $A(t)$ is strongly continuous differentiable, then condition (1.6) holds true with $\gamma = 1$ and the operator $A(t)A^{-1}(0)$ is uniformly bounded (see [18], Ch. 2, §1, p. 219, [19]).

Now, we can go to the second main result of this section.

Theorem 3.8. *Let the assumptions of Lemma 3.6 and conditions* (1.3)–(1.7) *hold. Then there exists a positive constant* c *such that the following hold.*

(1) *For* $\gamma \in [0, 1)$ *and* n *large enough it holds that*

$$|||z||| \equiv |||y - u||| \leq cn^{2\gamma-3} \ln n E_n(A_0^\gamma u), \tag{3.42}$$

where u *is the solution of* (1.1) *and* $E_n(A_0^\gamma u)$ *is the best approximation of* $A_0^\gamma u$ *by polynomials of degree not greater then* $n - 1$.

(2) *The system of linear algebraic equations* (3.28) *with respect to the approximate solution* y *can be solved by the fixed-point iteration*

$$y^{(k+1)} = S^{-1}Cy^{(k)} + S^{-1}f, \qquad k = 0, 1, \ldots; \quad y^{(0)} arbitrary \tag{3.43}$$

converging at least as a geometrical progression with the denominator $q = c(\gamma, T)n^{\gamma-3/2} < 1$ *for* n *large enough.*

Proof. Proceeding analogously as in the proof of Theorem 3.4 and using Lemma 3.6 and (3.25), we get

$$|||z||| \leq cn|||\psi|||. \tag{3.44}$$

For the norm $|||\psi|||$ we have (see (3.32))

$$|||\psi||| \leq c(1 + \tilde{L}_\gamma T)E_n(A_0^\gamma u)n^{2\gamma-4} \ln n, \quad \gamma \in [0, 1) \tag{3.45}$$

which together with (3.44) leads to the estimate (3.42) and to the first assertion of the theorem.

The second assertion is a consequence of (3.41). The proof is complete. □

Remark 3.9. A simple generalization of Bernstein's theorem (see [20, 22, 21]) to vector-valued functions gives the estimate

$$E_n(A_0^\gamma u) \leq \rho_0^{-n} \tag{3.46}$$

for the value of the best polynomial approximation provided that $A_0^\gamma u$ can be analytically extended from $[-1, 1]$ into an ellipse with the focus at the points $+1, -1$ and with the sum of semi-axes $\rho_0 > 1$.

If $A_0^\gamma u$ is p times continuously differentiable, then a generalization of Jackson's theorem (see [20, 22, 21]) gives

$$E_n(A_0^\gamma u) \leq c_p n^{-p} \omega\left(\frac{d^p A_0^\gamma u}{dt^p}; n^{-1}\right) \tag{3.47}$$

with the continuity modulus ω.

Further generalizations for Sobolev spaces of vector-valued functions can be proven analogously [3], Ch. 9. Let us define the weighted Banach space of vector-valued functions $L_w^p(-1, 1)$, $1 \leq p \leq +\infty$, with the norm

$$\|u\|_{L_w^p(-1,1)} = \left(\int_{-1}^{1} \|u(t)\|^p w(t)dt\right)^{1/p} \tag{3.48}$$

for $1 \le p < \infty$ and

$$\|u\|_{L^\infty(-1,1)} = \sup_{t\in(-1,1)} \|u(t)\| \tag{3.49}$$

for $p = \infty$. The weighted Sobolev space is defined by

$$H_w^m(-1,1) = \Big\{ v \in L_w^2(-1,1) : \text{for } 0 \le k \le m, \\ \text{the derivative } \frac{d^k v}{dt^k} \text{ belongs to } L_w^2(-1,1) \Big\}$$

with the norm

$$\|u\|_{H_w^m(-1,1)} = \left(\sum_{k=0}^{m} \left\| \frac{d^k v}{dt^k} \right\|^2_{L_w^2(-1,1)} \right)^{1/2}. \tag{3.50}$$

Then one gets for the Chebyshev weight $w(t) = \frac{1}{\sqrt{1-t^2}}$ (see [3], pp. 295–298), for the polynomial of the best approximation $B_n(t)$ and for the interpolation polynomial $P_n(t)$ with the Gauss (roots of the Chebyshev polynomial $T_{n+1}(t)$), Gauss-Radau (roots of the polynomial $T_{n+1}(t) - \frac{T_{n+1}(-1)}{T_n(1)} T_n(t)$) or the Gauss-Lobatto (roots of the polynomial $p(t) = T_{n+1}(t) + aT_n(t) + bT_{n-1}(t)$ with a, b such that $p(-1) = p(1) = 0$) nodes

$$\begin{gathered} E_n(u) \equiv \|u - B_n u\|_{L^\infty(-1,1)} \le c n^{1/2-m} \|u\|_{H_w^m(-1,1)}, \\ \|u - B_n u\|_{L_w^2(-1,1)} \le c n^{-m} \|u\|_{H_w^m(-1,1)}, \\ \|u - P_n u\|_{L_w^2(-1,1)} \le c n^{-m} \|u\|_{H_w^m(-1,1)}, \\ \|u' - (P_n u)'\|_{L_w^2(-1,1)} \le c n^{2-m} \|u\|_{H_w^m(-1,1)}. \end{gathered} \tag{3.51}$$

When the function u is analytic in $[-1,1]$ and has a regularity ellipse whose sum of semi-axes equals e^{η_0}, then

$$\|u' - (P_n u)'\|_{L_w^2(-1,1)} \le c(\eta) n^2 e^{-n\eta} \quad \forall \eta \in (0, \eta_0). \tag{3.52}$$

For the Legendre weight $w(t) = 1$ one has (see [3], pp. 289–294)

$$\begin{gathered} \|u - B_n u\|_{L^p(-1,1)} \le c n^{-m} \|u\|_{H^m(-1,1)}, \qquad 2 < p \le \infty, \\ \|u - B_n u\|_{H^l(-1,1)} \le c n^{2l-m+1/2} \|u\|_{H^m(-1,1)}, \qquad 1 \le l \le m, \\ \|u - P_n u\|_{H^l(-1,1)} \le c n^{2l-m+1/2} \|u\|_{H^m(-1,1)}, \qquad 1 \le l \le m, \\ \|u' - (P_n u)'\|_{L^2(-1,1)} \le c n^{5/2-m} \|u\|_{H_w^m(-1,1)}, \end{gathered} \tag{3.53}$$

where the interpolation polynomial $P_n(t)$ can be taken with the Gauss (roots of the Legendre polynomial $L_{n+1}(t)$), Gauss-Radau (roots of the polynomial $L_{n+1}(t) - \frac{L_{n+1}(-1)}{L_n(1)} L_n(t)$) or the Gauss-Lobatto (roots of the polynomial $p(t) = L_{n+1}(t) + aL_n(t) + bL_{n-1}(t)$ with a, b such that $p(-1) = p(1) = 0$) nodes.

Note that the restriction $\gamma \ne 1$ in Theorem 3.8 is only due to the estimate (3.32). Below we show how this restriction can be removed.

Using (2.30), (3.51), we estimate the norm $|||\psi|||_2$ for $\gamma = 1$ as

(3.54)

$$
\begin{aligned}
|||\psi||| &= \max_{1\le k\le n} \| \int_{t_{k-1}}^{t_k} A_k^{\gamma} e^{-A_k(t_k-\eta)} (A_k - A(\eta)) \\
&\quad \times \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)} (A_k - A(s)) A_k^{-\gamma} (A_k^{\gamma} A_0^{-\gamma})(A_0^{\gamma} u(s) - P_{n-1}(s; A_0^{\gamma} u)) ds d\eta \| \\
&= \max_{1\le k\le n} \| \int_{t_{k-1}}^{t_k} A_k e^{-A_k(t_k-\eta)} (A_k - A(\eta)) A_k^{-1} \\
&\quad \times \int_{t_{k-1}}^{\eta} \frac{de^{-A_k(\eta-s)}}{ds} (A_k - A(s)) A_k^{-1} (A_k A_0^{-1})(A_0 u(s) - P_{n-1}(s; A_0 u)) ds d\eta \| \\
&= \max_{1\le k\le n} \| \int_{t_{k-1}}^{t_k} A_k e^{-A_k(t_k-\eta)} (A_k - A(\eta)) A_k^{-1} \\
&\quad \times \Bigg[(A_k - A(\eta)) A_k^{-1} (A_k A_0^{-1})(A_0 u(\eta) - P_{n-1}(\eta; A_0 u)) \\
&\qquad + \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)} A'(s) A_k^{-1} (A_k A_0^{-1})(A_0 u(s) - P_{n-1}(s; A_0 u)) ds \\
&\qquad - \int_{t_{k-1}}^{\eta} e^{-A_k(\eta-s)} (A_k - A(s)) \\
&\qquad\quad \times A_k^{-1} (A_k A_0^{-1})(A_0 u'(s) - (P_{n-1}(s; A_0 u))') ds \Bigg] d\eta \| \\
&\le c_1 \tilde{L}_{1,1} \max_{1\le k\le n} \int_{t_{k-1}}^{t_k} |t_k - \eta|^{-1} |t_k - \eta| \\
&\quad \times \Bigg[|t_k - \eta| (1 + \tilde{L}_1 T) \max_{\eta\in[-1,1]} \|(A_0 u(\eta) - P_{n-1}(\eta; A_0 u))\| \\
&\qquad + c'(1 + \tilde{L}_1 T) \int_{t_{k-1}}^{\eta} \|(A_0 u(s) - P_{n-1}(s; A_0 u))\| ds \\
&\qquad + (1 + \tilde{L}_1 T) \int_{t_{k-1}}^{\eta} |t_k - s| \|(A_0 u'(s) - (P_{n-1}(s; A_0 u))')\| ds \Bigg] d\eta \\
&\le c_1 \tilde{L}_{1,1} \max_{1\le k\le n} \Bigg[\tau_k^2 (1 + \tilde{L}_1 T)(1 + \Lambda_n) E_n(A_0 u) \\
&\qquad\qquad + c'(1 + \tilde{L}_1 T) \tau_k^2 (1 + \Lambda_n) E_n(A_0 u) \\
&\qquad\qquad + (1 + \tilde{L}_1 T) \tau_k^2 \int_{t_{k-1}}^{t_k} \|(A_0 u'(s) - (P_{n-1}(s; A_0 u))')\| ds \Bigg] \\
&\le c_1 \tilde{L}_{1,1} \max_{1\le k\le n} \Bigg[\tau_k^2 (1 + \tilde{L}_1 T)(1 + \Lambda_n) E_n(A_0 u) \\
&\qquad\qquad + c'(1 + \tilde{L}_1 T) \tau_k^2 (1 + \Lambda_n) E_n(A_0 u) \\
&\qquad\qquad + (1 + \tilde{L}_1 T) \tau_k^2 \left(\int_{t_{k-1}}^{t_k} \frac{1}{\sqrt{1 - s^2}} ds \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{t_{k-1}}^{t_k} \frac{1}{\sqrt{1 - s^2}} \|(A_0 u'(s) - (P_{n-1}(s; A_0 u))')\| ds \right)^{1/2} \Bigg] \\
&\le c \Big[n^{-2} \ln n E_n(A_0 u) + n^{-5/2} \|u' - (P_{n-1} u)'\|_{L_w^2(-1,1)} \Big] \\
&\le c(n^{-(m+2)} \ln n + n^{-(m+1/2)}) \|u\|_{H_w^m(-1,1)} \le c n^{-(m+1/2)} \|u\|_{H_w^m(-1,1)}
\end{aligned}
$$

provided that the solution u of problem (1.1) belongs to the Sobolev class $H_w^m(-1,1)$. If u is analytic in $[-1,1]$ and has a regularity ellipse with the sum of the semi-axes equal to $e^{\eta_0} > 1$, then using (3.52), we get

$$|||\psi||| \le c(\eta_0) n^2 e^{-n\eta_0}. \tag{3.55}$$

Now, Lemma 3.6 together with the last estimates for $|||\psi|||$ yields the following third main result of this section.

Theorem 3.10. *Let the assumptions of Lemma 3.6 and conditions* (1.3)–(1.7) *with* $\gamma = 1$ *hold. Then there exists a positive constant* c *such that the following hold.*

(1) *For* $\gamma = 1$ *and* n *large enough we have*

$$|||z||| \equiv |||y - u||| \le c n^{-m} \|u\|_{H_w^m(-1,1)} \tag{3.56}$$

provided that the solution u *of problem* (1.1) *belongs to the class* $H_w^m(-1,1)$ *with* $w(t) = \frac{1}{\sqrt{1-t^2}}$.

(2) *For* $\gamma = 1$ *and* n *large enough it holds that*

$$|||z||| \equiv |||y - u||| \le c(\eta_0) n^{3/2} e^{-n\eta_0} \tag{3.57}$$

provided that u *is analytic in* $[-1,1]$ *and has a regularity ellipse with the sum of the semi-axes equal to* $e^{\eta_0} > 1$.

(3) *The system of linear algebraic equations* (3.28) *with respect to the approximate solution* y *can be solved by the fixed-point iteration*

$$y^{(k+1)} = S^{-1} C y^{(k)} + S^{-1} f, \qquad k = 0, 1, \ldots; \quad y^{(0)} \text{ arbitrary} \tag{3.58}$$

converging at least as a geometrical progression with the denominator $q = cn^{-1/2} < 1$ *for* n *large enough.*

Remark 3.11. Using estimates (3.53), one can analogously construct a discrete scheme on the Gauss, the Gauss-Radau or the Gauss-Lobatto grids relative to $w(t) = 1$ (i.e., connected with the Legendre orthogonal polynomials) and get the corresponding estimates in the $L^2(-1,1)$-norm.

4. Discrete second order problem

In this section we consider problem (1.2) in a Hilbert space H with the scalar product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Let assumption (1.7) related to $A(t)$ holds. In addition we assume in this section that

$$A(t) = \sum_{k=0}^{m_A} A_k t^k, \qquad f(t) = \sum_{k=0}^{m_f} f_k t^k \tag{4.1}$$

and

$$\|A^{\gamma-1/2}(0)(A(t) - A(s))A^{-\gamma}(0)\| \le \tilde{L}_{2,\gamma} |t-s|^\alpha, \qquad \gamma, \alpha \in [0,1]. \tag{4.2}$$

Remark 4.1. Condition (4.2) coincides with (1.6) for $\gamma = 1/2,\ \alpha = 1$.

Let $\overline{A}(t)$ be a piecewise constant operator defined as in the previous sections. We consider the auxiliary problem

$$\begin{aligned} &\frac{d^2u}{dt^2} + \overline{A}(t)u = [\overline{A}(t) - A(t)] + f(t), \\ &u(-1) = u_0, \qquad u'(1) = u_0', \end{aligned} \tag{4.3}$$

from which we get the following relations for the interval $[t_{k-1}, t_k]$:

$$
\begin{aligned}
u(t) &= \cos[\sqrt{A_k}(t-t_{k-1})]u(t_{t_{k-1}}) + A_k^{-1/2}\sin[\sqrt{A_k}(t-t_{k-1})]u'(t_{t_{k-1}}) \\
&\quad + \int_{t_{k-1}}^{t_k} A_k^{-1/2}\sin[\sqrt{A_k}(t-\eta)]\{[\overline{A}(\eta)-A(\eta)]u(\eta)+f(\eta)\}d\eta, \\
u'(t) &= -\sqrt{A_k}\sin[\sqrt{A_k}(t-t_{k-1})]u(t_{t_{k-1}}) + \cos[\sqrt{A_k}(t-t_{k-1})]u'(t_{t_{k-1}}) \\
&\quad + \int_{t_{k-1}}^{t_k} \cos[\sqrt{A_k}(t-\eta)]\{[\overline{A}(\eta)-A(\eta)]u(\eta)+f(\eta)\}d\eta.
\end{aligned} \tag{4.4}
$$

We chose a grid ω_n of n Chebyshev points as above and substitute in (4.4) the interpolation polynomial (2.7) instead of u. Then by collocation we arrive at the following system of linear algebraic equations with respect to the unknowns y_k, y'_k which approximate $u(t_k)$ and $u'(t_k)$, respectively:

$$
\begin{aligned}
y_k &= \cos[\sqrt{A_k}\tau_k]y_{k-1} + A_k^{-1/2}\sin[\sqrt{A_k}\tau_k]y'_{k-1} \\
&\quad + \sum_{i=1}^{n}\alpha_{k,i}y_i + \phi_k^{(1)}, \\
y'_k &= -\sqrt{A_k}\sin[\sqrt{A_k}\tau_k]y_{k-1} + \cos[\sqrt{A_k}\tau_k]y'_{k-1} \\
&\quad + \sum_{i=1}^{n}\beta_{k,i}y_i + \phi_k^{(2)}, \qquad k=1,2,\dots,n, \\
&\qquad y_0 = u_0, \qquad y'_0 = u'_0,
\end{aligned} \tag{4.5}
$$

where

$$
\begin{aligned}
\alpha_{k,i} &= \int_{t_{k-1}}^{t_k} A_k^{-1/2}\sin[\sqrt{A_k}(t_k-\eta)][A_k - A(\eta)]L_{i,n-1}(\eta)d\eta, \\
\beta_{k,i} &= \int_{t_{k-1}}^{t_k} \cos[\sqrt{A_k}(t_k-\eta)][A_k - A(\eta)]L_{i,n-1}(\eta)d\eta, \\
\phi_k^{(1)} &= \int_{t_{k-1}}^{t_k} A_k^{-1/2}\sin[\sqrt{A_k}(t_k-\eta)]f(\eta)d\eta, \\
\phi_k^{(2)} &= \int_{t_{k-1}}^{t_k} \cos[\sqrt{A_k}(t_k-\eta)]f(\eta)d\eta, \qquad k=1,2,\dots,n.
\end{aligned} \tag{4.6}
$$

The errors $z_k = u(t_k) - y_k, z'_k = u'(t_k) - y'_k$ satisfy the equations

$$
\begin{aligned}
z_k &= \cos[\sqrt{A_k}\tau_k]z_{k-1} + A_k^{-1/2}\sin[\sqrt{A_k}\tau_k]z'_{k-1} \\
&\quad + \sum_{i=1}^{n}\alpha_{k,i}z_i + \psi_k^{(1)}, \\
z'_k &= -\sqrt{A_k}\sin[\sqrt{A_k}\tau_k]z_{k-1} + \cos[\sqrt{A_k}\tau_k]z'_{k-1} \\
&\quad + \sum_{i=1}^{n}\beta_{k,i}z_i + \psi_k^{(2)}, \qquad k=1,2,\dots,n, \\
&\qquad z_0 = 0, \qquad z'_0 = 0,
\end{aligned} \tag{4.7}
$$

where

$$\psi_k^{(1)} = \int_{t_{k-1}}^{t_k} A_k^{-1/2} \sin\left[\sqrt{A_k}(t_k-\eta)\right][A_k - A(\eta)][u(\eta) - P_{n-1}(\eta;u)]d\eta, \tag{4.8}$$

$$\psi_k^{(2)} = \int_{t_{k-1}}^{t_k} \cos\left[\sqrt{A_k}(t_k-\eta)\right][A_k - A(\eta)][u(\eta) - P_{n-1}(\eta;u)] d\eta, k = 1,2,\ldots,n.$$

Let us denote $\tilde{y}_k = A_k^{\gamma} y_k,\ \tilde{y}'_k = A_k^{\gamma-1/2} y'_k,\ \tilde{z}_k = A_k^{\gamma} z_k,\ \tilde{z}'_k = A_k^{\gamma-1/2} z'_k$ and rewrite (4.7) in the form

(4.9)

$$\begin{aligned}
\tilde{z}_k &= \cos\left[\sqrt{A_k}\tau_k\right](A_k^{\gamma}A_{k-1}^{-\gamma})\tilde{z}_{k-1} \\
&\quad + \sin\left[\sqrt{A_k}\tau_k\right](A_k^{\gamma-1/2}A_{k-1}^{-(\gamma-1/2)})\tilde{z}'_{k-1} + \sum_{i=1}^{n}\tilde{\alpha}_{k,i}\tilde{z}_i + \tilde{\psi}_k^{(1)}, \\
\tilde{z}'_k &= -\sin\left[\sqrt{A_k}\tau_k\right](A_k^{\gamma}A_{k-1}^{-\gamma})\tilde{z}_{k-1} \\
&\quad + \cos\left[\sqrt{A_k}\tau_k\right](A_k^{\gamma-1/2}A_{k-1}^{-(\gamma-1/2)})\tilde{z}'_{k-1} + \sum_{i=1}^{n}\tilde{\beta}_{k,i}\tilde{z}_i + \tilde{\psi}_k^{(2)}, \qquad k = 1,2,\ldots,n, \\
&\qquad z_0 = 0, \qquad \tilde{z}'_0 = 0,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\alpha}_{k,i} &= A_k^{\gamma}\alpha_{k,i}A_i^{-\gamma} \\
&= \int_{t_{k-1}}^{t_k} A_k^{\gamma-1/2}\sin\left[\sqrt{A_k}(t_k-\eta)\right][A_k - A(\eta)]A_i^{-\gamma}L_{i,n-1}(\eta)d\eta, \\
\tilde{\beta}_{k,i} &= A_k^{\gamma-1/2}\beta_{k,i}A_i^{-\gamma} \\
&= \int_{t_{k-1}}^{t_k} A_k^{\gamma-1/2}\cos\left[\sqrt{A_k}(t_k-\eta)\right][A_k - A(\eta)]A_i^{-\gamma}L_{i,n-1}(\eta)d\eta, \\
\tilde{\psi}_k^{(1)} &= A_k^{\gamma}\phi_k^{(1)} \\
&= \int_{t_{k-1}}^{t_k} A_k^{\gamma-1/2}\sin\left[\sqrt{A_k}(t_k-\eta)\right]A_k^{-\gamma}[A_k^{\gamma}(u(\eta) - P_{n-1}(\eta;u))]d\eta, \\
\tilde{\psi}_k^{(2)} &= A_k^{\gamma-1/2}\phi_k^{(2)} \\
&= \int_{t_{k-1}}^{t_k} A_k^{\gamma-1/2}\cos\left[\sqrt{A_k}(t_k-\eta)\right]A_k^{-\gamma}[A_k^{\gamma}(u(\eta) - P_{n-1}(\eta;u))]d\eta.
\end{aligned} \tag{4.10}$$

We introduce the 2×2-block matrices

$$E = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad B_k(t_k-\eta) = \begin{pmatrix} \cos\left(\sqrt{A_k}\right)(t_k-\eta) & \sin\left(\sqrt{A_k}\right)(t_k-\eta) \\ -\sin\left(\sqrt{A_k}\right)(t_k-\eta) & \cos\left(\sqrt{A_k}\right)(t_k-\eta) \end{pmatrix},$$

$$D_k = \begin{pmatrix} A_k^{\gamma}A_{k-1}^{-\gamma} & 0 \\ 0 & A_k^{\gamma-1/2}A_{k-1}^{-(\gamma-1/2)} \end{pmatrix}, \quad F_i(\eta) = \begin{pmatrix} A_k^{\gamma-1/2}[A_i - A(\eta)]A_i^{-\gamma} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$S = \begin{pmatrix} E & 0 & 0 & \cdots & 0 & 0 & 0 \\ -B_2D_2 & E & 0 & \cdots & 0 & 0 & 0 \\ 0 & -B_3D_3 & E & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & -B_{n-1}D_{n-1} & E & 0 \\ 0 & 0 & 0 & \cdots & 0 & -B_nD_n & E \end{pmatrix},$$

$$C \equiv \{c_{i,j}\}_{i,j=1}^{n} = \begin{pmatrix} \tilde{\alpha}_{11} & 0 & \tilde{\alpha}_{12} & 0 & \cdots & \tilde{\alpha}_{1n} & 0 \\ \tilde{\beta}_{11} & 0 & \tilde{\beta}_{12} & 0 & \cdots & \tilde{\beta}_{1n} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \tilde{\alpha}_{n1} & 0 & \tilde{\alpha}_{n2} & 0 & \cdots & \tilde{\alpha}_{nn} & 0 \\ \tilde{\beta}_{n1} & 0 & \tilde{\beta}_{n2} & 0 & \cdots & \tilde{\beta}_{nn} & 0 \end{pmatrix} \tag{4.11}$$

with $B_k = B_k(t_k - t_{k-1}) = B_k(\tau_k)$ and the 2×2-operator blocks

$$c_{i,j} = \begin{pmatrix} \tilde{\alpha}_{i,j} & 0 \\ \tilde{\beta}_{i,j} & 0 \end{pmatrix}.$$

These blocks can also be represented as

$$c_{i,j} = \int_{t_{i-1}}^{t_i} L_{j,n}(\eta) D_i^* B_i^*(t_i - \eta) F_i(\eta) d\eta. \tag{4.12}$$

Using the integral representation of functions of self-adjoint operators by the corresponding spectral family, one can easily show that

$$\begin{gathered} B_k B_k^* = B_k^* B_k = E, \\ B_k(t_k - \eta) B_k(t_k - \eta)^* = B_k(t_k - \eta)^* B_k(t_k - \eta) = E. \end{gathered} \tag{4.13}$$

Analogously, as in the previous section we get

(4.14)

$$S^{-1} \equiv \{s_{i,k}^{(-1)}\}_{i,k=1}^{n} = \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ B_2D_2 & E & 0 & \cdots & 0 \\ B_3D_3B_2D_2 & B_3D_3 & E & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ B_nD_n \cdots B_2D_2 & B_nD_n \cdots B_3D_3 & B_nD_n \cdots B_4D_4 & \cdots & E \end{pmatrix}$$

with 2×2-operator block-elements $s_{i,k}^{(-1)}$. We introduce the vectors

$$\begin{gathered} z = (w_1, w_2, \ldots, w_n) = (\tilde{z_1}, z_1', \ldots, \tilde{z_n}, z_n'), \\ w_i = (z_i, z_i'), z_i, z_i' \in H, \\ \psi = (\Psi_1, \Psi_2, \ldots, \Psi_n) = (\tilde{\psi}_1^{(1)}, \psi_1^{(2)}, \ldots, \tilde{\psi}_n^{(1)}, \psi_n^{(2)}), \\ \Psi_i = (\tilde{\psi}_i^{(i)}, \psi_i^{(2)}), \tilde{\psi}_i^{(i)}, \psi_i^{(2)} \in H. \end{gathered} \tag{4.15}$$

Then equations (4.9) can be written in block matrix form as

$$z = S^{-1}Cz + S^{-1}\psi. \tag{4.16}$$

Note that the block vectors Ψ_i can be written as

$$\Psi_i = \int_{t_{i-1}}^{t_i} \tilde{B}_i^*(t_i - \eta) D_\psi(\eta) d\eta \tag{4.17}$$

with the block vectors

$$D_\psi(\eta) = \begin{pmatrix} 0 \\ A^{\gamma-1/2}[A_i - A(\eta)]A_i^{-\gamma}[A_i^\gamma(u(\eta) - P_n(\eta;u))] \end{pmatrix}. \tag{4.18}$$

The blocks E, B_i act in the space of two-dimensional block vectors $v = (v_1, v_2)$, $v_1, v_2 \in H$. In this space we define the new scalar product by

$$((u,v)) = (u_1, v_1) + (u_2, v_2), \tag{4.19}$$

the corresponding block-vector norm by

$$|||v|||_b = \sqrt{((v,v))} = \left(\|v_1\|^2 + \|v_2\|^2\right)^{1/2} \tag{4.20}$$

and the consistent norm for a block operator matrix $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ by

$$|||G|||_b = \sup_{y\neq 0} \frac{\sqrt{((Gv, Gv))}}{|||v|||}. \tag{4.21}$$

In the space of n-dimensional block vectors we define the block-vector norm by

$$|||y||| = \max_{1\le k\le n} |||v_k|||_b \tag{4.22}$$

and the consistent matrix norm

$$|||C||| \equiv |||\{c_{i,j}\}_{i,j=1}^n||| = \sup_{y\neq 0} \frac{|||Cy|||}{|||y|||} = \max_{1\le k\le n} \sum_{p=1}^n |||c_{kp}|||_b. \tag{4.23}$$

It is easy to see that

$$\begin{aligned} |||B_i|||_b = |||B_i^*|||_b &= \sup_{v\neq 0} \frac{|||B_i v|||}{|||v|||} = \sup_{v\neq 0} \frac{\sqrt{((B_i v, B_i v))}}{|||v|||} \\ &= \sup_{v\neq 0} \frac{\sqrt{((B_i B_i^* v, v))}}{|||v|||} = 1 \end{aligned} \tag{4.24}$$

and due to (1.7)

$$\begin{aligned} |||D_k|||_b &= \sup_{v\neq 0} \frac{|||D_k v|||}{|||v|||} = \sup_{v\neq 0} \frac{\sqrt{((D_k v, D_k v))}}{|||v|||} \\ &= \sup_{v\neq 0} \frac{\sqrt{((D_k D_i k^* v, v))}}{|||v|||} \le c_D, \\ |||D_k^*|||_b &\le c_D \end{aligned} \tag{4.25}$$

with

$$c_D = \sqrt{(1 + \tilde{L}_\gamma T)^2 + (1 + \tilde{L}_{\gamma-1/2} T)^2}.$$

Let us estimate $|||S^{-1}|||$. Due to (4.24), (4.25), (4.23) and (4.14) we have

$$|||S^{-1}||| \le c_D n. \tag{4.26}$$

Using assumption (4.2) and (4.13), (4.26), we get

$$\begin{aligned}
|||c_{i,j}|||_b &\le \int_{t_{i-1}}^{t_i} |||D_i^*|||||B_i(t_i-\eta)|||||F_i(\eta)|||L_{j,n}(\eta)|d\eta \\
&\le c_D \tilde{L}_{2,\gamma} \tau_{\max}^{\alpha} \int_{t_{i-1}}^{t_i} |L_{j,n-1}(\eta)|^2 d\eta, \\
|||C||| &\le \max_{1\le k\le n} \sum_{p=1}^{n} |||c_{k,p}|||_b \qquad (4.27) \\
&\le c\tau_{\max}^{\alpha} \left(\int_{t_{k-1}}^{t_k} \sum_{j=1}^{n} |L_{j,n}(\eta)| d\eta \right) \le c\Lambda_n \tau_{\max}^{1+\alpha} \le cn^{-1-\alpha} \ln n, \\
|||S^{-1}C||| &\le n^{-\alpha} \ln n,
\end{aligned}$$

with some positive constant c independent of n.

Now we are in a position to prove the main result of this section.

Theorem 4.2. *Let assumptions* (1.3)–(4.1), (4.2), (1.7) *hold. Then there exists a positive constant c such that the following hold.*

(1) *For n large enough it holds that*

$$|||z||| \equiv |||y-u||| \le cn^{-\alpha} \ln n E_n(A_0^{\gamma} u), \tag{4.28}$$

where u is the solution of (1.2) *and $E_n(A_0^{\gamma}u)$ is the best approximation of $A_0^{\gamma}u$ by polynomials of degree not greater then $n-1$.*

(2) *The system of linear algebraic equations*

$$Sy = Cy + f \tag{4.29}$$

with respect to the approximate solution y can be solved by the fixed-point iteration

$$y^{(k+1)} = S^{-1}Cy^{(k)} + S^{-1}f, \qquad k = 0, 1, \ldots; \quad y^{(0)} \text{ arbitrary}, \tag{4.30}$$

which converges as a geometric progression with the denominator $q = cn^{-\alpha} \ln n < 1$ for n large enough.

Proof. Due to (4.16), (4.26) for $\tau_{\max}$ small enough (or for n large enough) there exists a bounded norm $|||(E - S^{-1}S)^{-1}|||$ and we get

$$|||z||| \le cn|||\psi|||. \tag{4.31}$$

It remains to estimate $|||\psi|||$. Using (4.20), (4.17), (4.18) and (4.13), we have

$$\begin{aligned}
|||\psi||| &= \max_{1\le k\le n} |||\Psi_k|||_b \\
&= \max_{1\le k\le n} |||\int_{t_{k-1}}^{t_k} \tilde{B}_k^*(t_k-\eta) D_{\psi}(\eta) d\eta||| \\
&\le \max_{1\le k\le n} \int_{t_{k-1}}^{t_k} |t_k-\eta|^{\alpha} \|A_k^{\gamma} A_0^{-\gamma}\| \|[A_0^{\gamma}(u(\eta) - P_n(\eta;u))]\| d\eta \qquad (4.32) \\
&\le (1 + \tilde{L}_{\gamma} T) \tau_{\max}^{1+\alpha} (1+\Lambda_n) E_n(A_0^{\gamma} u) \\
&\le c(1+\tilde{L}_{\gamma}T) n^{-1-\alpha} \ln n E_n(A_0^{\gamma} u).
\end{aligned}$$

This inequality together with (4.31) completes the proof of the first assertion. The second one can be proved analogously as in Theorem 3.4. □

Remark 4.3. We arrive at an exponential accuracy for piecewise analytical solutions if we apply the methods described above successively on each subinterval of the analyticity.

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