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COMPUTATION OF THE EIGENVALUES OF STURM-LIOUVILLE PROBLEMS WITH PARAMETER DEPENDENT BOUNDARY CONDITIONS USING THE REGULARIZED SAMPLING METHOD

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ABSTRACT. The purpose in this paper is to compute the eigenvalues of Sturm-Liouville problems with quite general separated boundary conditions nonlinear in the eigenvalue parameter using the *regularized sampling method*, an improvement on the method based on Shannon sampling theory, which does not involve any multiple integration and provides higher order estimates of the eigenvalues at a very low cost. A few examples shall be presented to illustrate the power of the method and a comparison made with the the exact eigenvalues obtained as squares of the zeros of the exact characteristic functions.

1. INTRODUCTION

In 1996, we introduced a method for computing the eigenvalues of regular Sturm-Liouville (SL) problems with Dirichlet boundary conditions [4]. The simple observation that the boundary (characteristic) function associated with the SL problem happens to be in a Paley-Wiener space led to the conclusion that it can be recovered from its samples at a countable number of points, using the well known Whitaker-Shannon-Kotel′nikov theorem. We generalized this idea to some classes of singular problems [5], [6]. Subsequently, we extended the scope of the method to include regular SL problems with general separated boundary conditions [7], SL problems with coupled self-adjoint boundary conditions [10], random SL problems [11] and regular fourth order SL problems [12]. In fact, we obtained much higher order estimates at the expense of having to subtract terms involving multiple integrals [8].

The purpose of this paper is twofold. First we shall consider a regularization avoiding any (multiple) integration and show that we can get higher order estimates of the eigenvalues at a very low cost. This will constitute a substantial improvement on the original method, and we will take this opportunity to call this method the *regularized sampling method*. Second, we tackle the computation of the eigenvalues of Sturm-Liouville problems with quite general separated parameter dependent boundary conditions although the known theory is for boundary conditions which are affine in the parameter [1], [2], [3], [13], [18].

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2. Main results

We are interested in finding the eigenvalues of the following SL problem with separated parameter dependent boundary conditions:

$$\left\{\begin{array}{c} -y''+qy=\mu^2 y,\ \ x\in[0,\gamma], \\ a_{11}(\mu)y(0)-a_{12}(\mu)y'(0)=0, \\ a_{21}(\mu)y(\gamma)+a_{22}(\mu)y'(\gamma)=0, \end{array}\right. \tag{2.1}$$

where a_{ij} are entire functions satisfying the growth conditions

$$|a_{ij}(\mu)|\le c_{ij}(1+|\mu|)^{m_0}\exp(L_{ij}|\mathrm{Im}\mu|)\ ,\ 1\le i,j\le 2,$$

$a_{11}^2(\mu)+a_{12}^2(\mu)\ne 0$, $a_{22}^2(\mu)+a_{21}^2(\mu)\ne 0$, m_0 is a nonnegative integer, and $q\in L^1([0,\gamma])$. Let $c_0=\max c_{ij}$, $L=\max L_{ij}$.

We shall assume further that the definiteness conditions

$$\left\{\begin{array}{c} a_{11}\mu\frac{da_{12}}{d\mu}-a_{12}\mu\frac{da_{11}}{d\mu}\ge 0, \\ a_{21}\mu\frac{da_{22}}{d\mu}-a_{22}\mu\frac{da_{21}}{d\mu}\le 0 \end{array}\right.$$

or

$$\left\{\begin{array}{c} a_{11}\mu\frac{da_{12}}{d\mu}-a_{12}\mu\frac{da_{11}}{d\mu}\le 0, \\ a_{21}\mu\frac{da_{22}}{d\mu}-a_{22}\mu\frac{da_{21}}{d\mu}\ge 0 \end{array}\right.$$

be satisfied (at least one inequality must be strict in any one of these conditions to make the miss distance function strictly monotone [16]). This is to ensure a monotone Prufer miss distance which guarantees at most one eigenvalue $\lambda_k=\mu_k^2$ of any index k having an eigenfunction with just k zeros in $(0,\gamma)$.

Consider the two base problems

$$\left\{\begin{array}{c} -y''+qy=\mu^2 y,\ \ x\in[0,\gamma], \\ y(0,\mu)=1,\ y'(0,\mu)=0 \end{array}\right. \tag{2.2}$$

and

$$\left\{\begin{array}{c} -y''+qy=\mu^2 y,\ \ x\in[0,\gamma], \\ y(0,\mu)=0,\ y'(0,\mu)=1, \end{array}\right. \tag{2.3}$$

whose solutions are denoted y_1 and y_2, respectively. The solution to the initial value problem

$$\left\{\begin{array}{c} -y''+qy=\mu^2 y,\ \ x\in[0,\gamma], \\ y(0,\mu)=a_{12}(\mu),\ y'(0,\mu)=a_{11}(\mu) \end{array}\right. \tag{2.4}$$

and its derivative are, therefore,

$$\left\{\begin{array}{l} y(x,\mu)=a_{12}(\mu)y_1(x,\mu)+a_{11}(\mu)y_2(x,\mu), \\ y'(x,\mu)=a_{12}(\mu)y_1'(x,\mu)+a_{11}(\mu)y_2'(x,\mu). \end{array}\right.$$

It is well known that $y_1(x,\mu)-\cos\mu x$, $y_1'(x,\mu)+\mu\sin\mu x$, $y_2(x,\mu)$ and $y_2'(x,\mu)-\cos\mu x$ are entire functions of μ for each $x\in(0,\gamma]$ and satisfy the same growth condition

$$\begin{gathered} \left|y_1(x,\mu)-\cos\mu x\right|,\quad \left|y_1'(x,\mu)+\mu\sin\mu x\right|,\quad \left|y_2(x,\mu)\right|, \\ \left|y_2'(x,\mu)-\cos\mu x\right|\le c_1\exp\left(x\left|\mathrm{Im}\mu\right|\right) \end{gathered}$$

for some constant c_1. In [8], we have shown that $y_1(x,\mu) - \cos\mu x$, $y_1'(x,\mu) + \mu\sin\mu x - \int_0^x q(t)\cos\mu(x-t)\cos\mu t dt$, $y_2(x,\mu) - \frac{\sin\mu x}{\mu x}$ and $y_2'(x,\mu) - \cos\mu x - \int_0^x q(t)\cos\mu(x-t)\frac{\sin\mu t}{\mu t}dt$ are entire functions of μ for each $x \in (0,\gamma]$ and satisfy the same growth condition

$$\begin{aligned} &\left|y_1(x,\mu) - \cos\mu x\right|, \ \left|y_1'(x,\mu) + \mu\sin\mu x - \int_0^x q(t)\cos\mu(x-t)\cos\mu t dt\right|, \\ &\left|y_2(x,\mu) - \frac{\sin\mu x}{\mu x}\right|, \ \left|y_2'(x,\mu) - \cos\mu x - \int_0^x q(t)\cos\mu(x-t)\frac{\sin\mu t}{\mu t}dt\right| \\ \leq \ & \frac{c_2}{1+\gamma|\mu|}\exp\left(x\left|\mathrm{Im}\mu\right|\right), \end{aligned}$$

which means that they belong to the Paley-Wiener space PW_x defined by

$$PW_x = \left\{ h(z) \text{ entire } / \ |h(z)| \leq C\exp\left\{x\,|z|\right\}, \ \int_{-\infty}^{\infty} \ |h(z)|^2\, dz < \infty \right\}.$$

In fact we have obtained much higher order estimates at the expense of having to subtract from y_i and y_i' terms involving multiple integrals. In this paper, we shall stick with the first estimate given, hence avoiding any (multiple) integration and show by the same token that we can get a higher order estimate at a very low cost. Let

$$B(x,\mu) = a_{21}(\mu)y(x,\mu) + a_{22}(\mu)y'(x,\mu).$$

The eigenvalues of the problem are seen as the square of the zeros of the boundary (characteristic) function $B(\gamma,\mu)$.

Replacing $y(x,\mu)$ and $y'(x,\mu)$ by their expressions, we obtain

$$\begin{aligned} B(x,\mu) = a_{21}(\mu)\left\{a_{12}(\mu)y_1(x,\mu) + a_{11}(\mu)y_2(x,\mu)\right\} \\ + a_{22}(\mu)\left\{a_{12}(\mu)y_1'(x,\mu) + a_{11}(\mu)y_2'(x,\mu)\right\} \end{aligned}$$

from which we get

$$B(x,\mu) = B_0(x,\mu) + B_1(x,\mu),$$

where

$$\left\{ \begin{array}{c} B_0(x,\mu) = a_{21}(\mu)\{a_{12}(\mu)\left[y_1(x,\mu) - \cos\mu x\right] + a_{11}(\mu)y_2(x,\mu)\} \\ +a_{22}(\mu)\{a_{12}(\mu)\left[y_1'(x,\mu) + \mu\sin\mu x\right] + a_{11}(\mu)\left[\ y_2'(x,\mu) - \cos\mu x\right]\}, \\ B_1(x,\mu) = a_{21}(\mu)a_{12}(\mu)\cos\mu x + a_{22}(\mu)a_{12}(\mu)\mu\sin\mu x + a_{22}(\mu)a_{11}(\mu)\cos\mu x. \end{array} \right.$$

Theorem 2.1. *B_0 is an entire function of μ for each $x \in (0,\gamma]$ and satisfies the growth condition*

$$|B_0(x,\mu)| \leq c_2(1+|\mu|)^{2m_0}\exp\left((2L+x)\left|\mathrm{Im}\,\mu\right|\right).$$

Proof. B_0 is an entire function of μ for each $x \in (0,\gamma]$ as a sum of products of entire functions. As for the estimate, we have

$$\begin{aligned} |B_0(x,\mu)| &\leq |a_{21}(\mu)|\left\{|a_{12}(\mu)|\left|y_1(x,\mu) - \cos\mu x\right| + |a_{11}(\mu)|\, y_2(x,\mu)\right\} \\ &\quad + |a_{22}(\mu)|\left\{|a_{12}(\mu)|\left|y_1'(x,\mu) + \mu\sin\mu x\right| + |a_{11}(\mu)|\left|\ y_2'(x,\mu) - \cos\mu x\right|\right\} \\ &\leq c_2(1+|\mu|)^{2m_0}\exp\left((2L+x)\left|\mathrm{Im}\mu\right|\right) \end{aligned}$$

with $c_2 = 4c_0^2c_1$. $\square$

Corollary 2.2. $\left(\frac{\sin\theta\mu}{\theta\mu}\right)^m B_0(\gamma,\mu)$ *is an entire function of* μ *and satisfies the estimate*

$$\left|\left(\frac{\sin\theta\mu}{\theta\mu}\right)^m B_0(\gamma,\mu)\right| \le \frac{c_3}{(1+\theta|\mu|)^{m-2m_0}}\exp\left((2L+\gamma+m\theta)\,|\mathrm{Im}\mu|\right)$$

for a positive integer $m \ge 2m_0+2$ *and positive constants* c_3 *and* ϑ.

Proof. It is enough to note that $\frac{\sin\theta\mu}{\theta\mu}$ is an entire function of μ and use the standard estimate $\left|\frac{\sin z}{z}\right| \le \frac{c_4}{1+|z|}\exp(|\mathrm{Im}z|)$, where $c_4 = 1.72$ (say), $c_3 = c_2c_4^m$, and the above theorem. □

Thus, $h(\mu) = \left(\frac{\sin\theta\mu}{\theta\mu}\right)^m B_0(\gamma,\mu)$ belongs to the Paley-Wiener space PW_σ with $\sigma = 2L+\gamma+m\theta$. Hence, h can be recovered from its values at the points $\mu_j = j\frac{\pi}{\sigma}$, $j\in\mathbb{Z}$, using the following celebrated theorem.

Theorem 2.3 (Whitaker-Shannon-Kotel′nikov)**.** *Let* $h\in PW_\sigma$, *then*

$$h(\mu) = \sum_{j=-\infty}^{\infty} h(\mu_j)\frac{\sin\sigma(\mu-\mu_j)}{\sigma(\mu-\mu_j)}$$

$\mu_j = j\frac{\pi}{\sigma}$. *The series converges absolutely and uniformly on compact subsets of* $\mathbb{C}$ *and in* $L^2_{d\mu}(\mathbb{R})$.

Now, since $\mu^{m-2m_0-1}h(\mu)\in L^2(-\infty,\infty)$, Jagerman's result (see [19], Theorem 3.21, p.90) is applicable and yields the following very sharp estimate.

Lemma 2.4 (Truncation error)**.** *Let* $h_N(\mu) = \sum_{j=-N}^{N} h(\mu_j)\frac{\sin\sigma(\mu-\mu_j)}{\sigma(\mu-\mu_j)}$ *denote the truncation of* $h(\mu)$. *Then, for* $|\mu| < N\pi/\sigma$,

$$\begin{aligned}&|h(\mu)-h_N(\mu)|\\ &\le \frac{|\sin\gamma\mu|c_5}{\pi(\pi/\sigma)^{m-2m_0-1}\sqrt{1-4^{-m+2m_0+1}}}\left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu}}+\frac{1}{\sqrt{(N\pi/\sigma)+\mu}}\right]\frac{1}{(N+1)^{m-2m_0-1}},\end{aligned}$$

where $c_5 = ||\mu^{m-2m_0-1}h(\mu)||_2$.

Let $\overline{\mu}^2$ be an exact eigenvalue and μ_N^2 the corresponding approximation obtained as a square of a zero of $\left[\left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m+2m_0} h_N(\mu)+B_1(\gamma,\mu)\right]$. Then from

$$\begin{aligned}&\left|B(\gamma,\mu)-\left[\left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m+2m_0} h_N(\mu)+B_1(\gamma,\mu)\right]\right|\\ &\quad=\left|\left(\frac{\sin\theta\mu}{\theta\mu}\right)^{-m+2m_0}[h(\mu)-h_N(\mu)]\right|\\ &\quad\le\left|\frac{\sin\theta\mu}{\theta\mu}\right|^{-m+2m_0}\frac{|\sin\gamma\mu|c_5}{\pi(\pi/\sigma)^{m-2m_51}\sqrt{1-4^{-m+2m_0+1}}}\\ &\quad\quad\times\left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu}}+\frac{1}{\sqrt{(N\pi/\sigma)+\mu}}\right]\frac{1}{(N+1)^{m-2m_0-1}}\end{aligned}$$

we get

$$\begin{aligned}
&|B(\gamma,\mu_N)-B(\gamma,\overline{\mu})| \\
&\quad = \left| B(\gamma,\mu_N) - \left[\left(\frac{\sin\theta\mu_N}{\theta\mu_N}\right)^{-m+2m_0} h_N(\mu_N) + B_1(\gamma,\mu_N)\right]\right| \\
&\quad \le \left|\frac{\sin\theta\mu_N}{\theta\mu_N}\right|^{-m+2m_0} \frac{|\sin\gamma\mu_N|c_5}{\pi(\pi/\sigma)^{m-2m_0-1}\sqrt{1-4^{-m+2m_0+1}}} \\
&\qquad \times \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu_N}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu_N}}\right] \frac{1}{(N+1)^{m-2m_0-1}},
\end{aligned}$$

but $|B(\gamma,\mu_N)-B(\gamma,\overline{\mu})| = |B_\mu(\gamma,\xi)|\,|\mu_N-\overline{\mu}|$ for some ξ in a small ball centered at μ_N with radius $|\mu_N-\overline{\mu}|$ and not containing a multiple of π/θ. Hence,

Theorem 2.5 (Error bounds)**.** *For $|\mu_N| < N\pi/\sigma$,*

$$\begin{aligned}
|\mu_N-\overline{\mu}| \quad \le \quad & \frac{1}{\inf|B_\mu(\gamma,\xi)|}\left|\frac{\sin\theta\mu_N}{\theta\mu_N}\right|^{-m+2m_0} \frac{|\sin\gamma\mu_N|c_5}{\pi(\pi/\sigma)^{m-2m_0-1}\sqrt{1-4^{-m+2m_0+1}}} \\
& \times \left[\frac{1}{\sqrt{(N\pi/\sigma)-\mu_N}} + \frac{1}{\sqrt{(N\pi/\sigma)+\mu_N}}\right] \frac{1}{(N+1)^{m-2m_0-1}},
\end{aligned}$$

where the inf *is taken over a ball centered at μ_N with radius $|\mu_N-\overline{\mu}|$ and does not contain a multiple of π/θ.*

3. Numerical examples

In this section, we shall present a few examples to illustrate our method. We have taken $\theta = (2L+\gamma)/(N-m)$ in order to avoid the first singularity of $\left(\frac{\sin\theta\mu_N}{\theta\mu_N}\right)^{-1}$. The sampling values were obtained using the Fehlberg 4-5 order Runge-Kutta method.

Example 3.1 (Taken from Binding and Browne [2])**.**

$$\begin{cases} -y''(x) = \lambda y(x),\ 0\le x\le 1, \\ -y(0) = (\lambda+d)y'(0), \\ y(1) = \lambda y'(1), \end{cases}$$

where $d=-4\pi^2$. We took $L=1$, $m=2$, $N=40$, and a precision of 10^{-10}. The computed eigenvalues together with the "exact" ones are displayed in Table 3.1. As pointed out in [2], the oscillation counts occur in the sequence $1,3,4,5,\dots$, the minimal count is 1 not 0, and there is count 2 missing. The exact characteristic function is

$$B_{\text{exact}}(\mu) = (1+4\pi^2\mu^4-\mu^6)\frac{\sin\mu}{\mu} - (2\mu^2-4\pi^2)\cos\mu,$$

where zero is not an eigenvalue.

Table 3.1. The first three eigenvalues in Example 3.1

Index	Exact Eigenvalues	Approximate Eigenvalues	Absolute Error
1	9.730886578213082033	9.730887696302056807	0.00000111808897477
2	88.76331625258976337	88.76323738197181406	0.00007887061794931
3	157.88411043863472059	157.88422274978466468	0.00011231114994409

TABLE 3.2a. The first three eigenvalues in Example 3.2

Index	Exact Eigenvalues	Approximate Eigenvalues	Absolute Error
1	9.929679054283188	0.929678898812778	$0.155470411 \times 10^{-6}$
2	9.9387434140	9.9387439016	$4.8758708700 \times 10^{-7}$
3	11.2738742105212	11.2738738490945	$-0.3614267 \times 10^{-6}$

TABLE 3.2b. Eigenvalues from different ranges for Example 3.2

Exact Eigenvalues	Approximate Eigenvalues	Relative Error	Absolute Error
1.0169749051×10^{6}	$1.0169749052 \times10^{6}$	$-3.882156763\times10^{-11}$	0.000136868
$1.00119488705\times10^{8}$	$1.00119488709\times10^{8}$	$3.882156763\times10^{-11}$	0.00388679
$1.000126382927378\times10^{10}$	$1.00012638292493\times10^{10}$	$2.44062879930\times10^{-12}$	0.024409

Example 3.2.

$$\begin{cases} -y''(x) = \lambda y(x),\ 0 \leq x \leq 1, \\ y(0) - 2y'(0) = 0, \\ (1+\sqrt{\lambda})y(1) + (1-\lambda)y'(1) = 0. \end{cases}$$

We took $L = 0$, $m_0 = 2$, $N = 40$, and a precision of 10^{-10}. The computed eigenvalues $\lambda = \mu^2$ together with the "exact" ones are displayed in Table 3.2a. The exact characteristic function is

$$B_{\text{exact}}(\mu) = (2\cos(\mu) + \sin(\mu)/\mu) + (1-\mu)(-2\mu\sin(\mu) + \cos(\mu)).$$

1 is not an eigenvalue.

It is appropriate to note how perfect the approximation is over different ranges by checking Table 3.2b, in which we include both the absolute and relative error.

Example 3.3 (Taken from Pryce [16], an "indefinite" case)**.**

$$\begin{cases} -y''(x) = \lambda y(x),\ 0 \leq x \leq \pi/2, \\ y'(0) = \lambda(\frac{3}{2}y(0) + y'(0)), \\ y'(\pi/2) = 0. \end{cases}$$

This example illustrates the case in which the definiteness condition is not satisfied. We have three eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1/4$, $\lambda_3 = 1$ to which correspond three eigenfunctions $y_1(x) = 1, y_2(x) = \sin\frac{x}{2} + \cos\frac{x}{2}, y_3(x) = \sin x$. We took $L = 1$, $m = 2$, $N = 30$, and a precision of 10^{-10}. The computed eigenvalues $\lambda = \mu^2$ together with the "exact" ones are displayed in Table 3.3. The exact characteristic function is

$$B_{\text{exact}}(\mu) = -(1-\mu^2)\mu\ \sin\left(\mu\frac{\pi}{2}\right) + \frac{3}{2}\mu^2\ \cos\left(\mu\frac{\pi}{2}\right).$$

TABLE 3.3. The first three eigenvalues in Example 3.3

Index	Exact Eigenvalues	Approximate Eigenvalues	Absolute Error
1	0	$1.2857880801746297 \times 10^{-10}$	$1.2857880801746297 \times 10^{-10}$
2	1/4	0.24999814884168953010	$1.8511583104698968\times10^{-6}$
3	1	0.9999881573635478785	$1.18426364521215 \times10^{-5}$

Example 3.4.

$$\begin{cases} -y''(x) + e^x y(x) = \lambda y(x),\ 0 \le x \le 1, \\ y(0) = 0, \\ -\sqrt{\lambda}\sin(\sqrt{\lambda})y(1) + \cos(\sqrt{\lambda})y'(1) = 0. \end{cases}$$

It is easy to check that the definiteness condition is satisfied. We took $L = 1$, $m_0 = 1$, $N = 40$, and a precision of 10^{-10}. The computed eigenvalues $\lambda = \mu^2$ together with the "exact" ones are displayed in Table 3.4. Here again we are in a position to derive the exact characteristic function which in fact can be expressed in terms of the modified Bessel functions of the first kind. Indeed, let $\lambda = \mu^2$ and consider the change of variables $t = 2\mathbf{e}^{x/2}$ and $\nu = 2\iota\mu$, where $\iota = \sqrt{-1}$. The differential equation becomes the modified Bessel equation of order ν given by

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} - (t^2 + \nu^2)z = 0$$

whose solution is

$$z(t) = c_1\mathcal{I}_\nu(t) + c_2\mathcal{I}_{-\nu}(t),$$

where $\mathcal{I}_\nu$ and $\mathcal{I}_{-\nu}$ are the modified Bessel functions of the first kind of order ν.

Returning to the original variables, we obtain

$$y(x) = c_1\mathcal{I}_{2\iota\mu}(2\mathbf{e}^{x/2}) + c_2\mathcal{I}_{-2\iota\mu}(2\mathbf{e}^{x/2}).$$

Taking into account the boundary conditions, we obtain the homogeneous system in $c_1$and c_2

$$\begin{cases} c_1\mathcal{I}_{2\iota\mu}(2) + c_2\mathcal{I}_{-2\iota\mu}(2) = 0 \\ c_1\left\{-\mu\mathcal{I}_{2\iota\mu}(2\sqrt{\mathbf{e}})\sin\mu + \sqrt{\mathbf{e}}\mathcal{I}'_{2\iota\mu}(2\sqrt{\mathbf{e}})\cos\mu\right\} \\ \quad + c_2\left\{-\mu\mathcal{I}_{-2\iota\mu}(2\sqrt{\mathbf{e}})\sin\mu + \sqrt{\mathbf{e}}\mathcal{I}'_{-2\iota\mu}(2\sqrt{\mathbf{e}})\cos\mu\right\} = 0 \end{cases}$$

In order to have a nontrivial solution, a necessary and sufficient condition is to have $B_{\text{exact}}(\mu) = 0$ where

$$B_{\text{exact}}(\mu) = \iota\begin{vmatrix} \mathcal{I}_{2\iota\mu}(2) & \mathcal{I}_{-2\iota\mu}(2) \\ -\mu\mathcal{I}_{2\iota\mu}(2\sqrt{\mathbf{e}})\sin\mu + \sqrt{\mathbf{e}}\mathcal{I}'_{2\iota\mu}(2\sqrt{\mathbf{e}})\cos\mu & -\mu\mathcal{I}_{-2\iota\mu}(2\sqrt{\mathbf{e}})\sin\mu + \sqrt{\mathbf{e}}\mathcal{I}'_{-2\iota\mu}(2\sqrt{\mathbf{e}})\cos\mu \end{vmatrix}$$

is the characteristic function. The ι in front of the determinant makes B_{exact} a real function. Now, using the well known result

$$\frac{d}{dx}\left[x^\nu\mathcal{I}_\nu(x)\right] = x^\nu\mathcal{I}_{\nu-1}(x),$$

TABLE 3.4. The first six eigenvalues in Example 3.4

Index	Exact Eigenvalues	Approximate Eigenvalues	Absolute Error
1	0.92906202857	0.9290620353	$6.75215627149 \times 10^{-9}$
2	6.7478811782	6.7478811410	$3.7173601254 \times 10^{-8}$
3	16.1245477258	16.1245478044	$7.8559790579 \times 10^{-8}$
4	31.2202765051	31.22027698028	$4.7517563634 \times 10^{-7}$
5	50.73392783919	50.73392843916	$5.9996614290 \times 10^{-7}$
6	75.5814691882	75.5814692597	$7.1549766566 \times 10^{-8}$

we obtain

$$
\begin{aligned}
&B_{\text{exact}}(\mu) \\
&= \iota\mathcal{I}_{2\iota\mu}(2)\left\{-\mu\mathcal{I}_{-2\iota\mu}(2\sqrt{\mathbf{e}})\sin\mu+\sqrt{\mathbf{e}}\left[\mathcal{I}_{-2\iota\mu-1}(2\sqrt{\mathbf{e}})+\frac{\iota\mu}{\sqrt{\mathbf{e}}}\mathcal{I}_{-2\iota\mu}(2\sqrt{\mathbf{e}})\right]\cos\mu\right\} \\
&\quad-\iota\mathcal{I}_{-2\iota\mu}(2)\left\{-\mu\mathcal{I}_{2\iota\mu}(2\sqrt{\mathbf{e}})\sin\mu+\sqrt{\mathbf{e}}\left[\mathcal{I}_{2\iota\mu-1}(2\sqrt{\mathbf{e}})-\frac{\iota\mu}{\sqrt{\mathbf{e}}}\mathcal{I}_{2\iota\mu}(2\sqrt{\mathbf{e}})\right]\cos\mu\right\}.
\end{aligned}
$$

4. Conclusion

In this paper, we have improved upon the method based on Shannon sampling theory introduced in [4] by considering a regularization avoiding any multiple integration and shown that we can get higher order estimates of the eigenvalues at a very low cost. We shall call this method the *regularized sampling method*. We have presented a few examples to illustrate a method and compared the computed eigenvalues with the exact ones obtained as squares of the zeros of the exact characteristic functions.

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