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NEW IRRATIONALITY MEASURES FOR q-LOGARITHMS

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ABSTRACT. The three main methods used in diophantine analysis of q-series are combined to obtain new upper bounds for irrationality measures of the values of the q-logarithm function

$$\ln_q(1-z) = \sum_{\nu=1}^{\infty} \frac{z^\nu q^\nu}{1-q^\nu}, \qquad |z| \leqslant 1,$$

when $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $z \in \mathbb{Q}$.

1. INTRODUCTION

The main purpose of this article is to improve the earlier irrationality measures of the values of the q-logarithm function

$$\ln_q(1-z) = \sum_{\nu=1}^{\infty} \frac{z^\nu q^\nu}{1-q^\nu}, \qquad |z| \leqslant 1. \tag{1}$$

In order to improve the earlier results we shall combine the following three major methods used in diophantine analysis of q-series:

(1) a general hypergeometric construction of rational approximations to the values of q-logarithms vs. the q-arithmetic approach ([Z1]);
(2) a continuous iteration procedure for additional optimization of analytic estimates ([Bo], [MV]);
(3) introducing the cyclotomic polynomials for sharpening least common multiples of the constructed linear forms in the case when z is a root of unity ([BV], [As], [MP]).

Also, some standard analytic tools (i.e., from [Ha]) for deducing irrationality measures will be required. We underline that in the corresponding arithmetic study of the values of the ordinary logarithm (cf. [Ru] for $\log 2$ and [Ha] for other logarithms) only feature (1) is mainly applied, but in particular feature (3) has no ordinary analogues. Thus the present q-problems invoke new attractions in arithmetic questions.

We present the bounds for irrationality measures by means of certain estimates for irrationality exponents. Recall that the *irrationality exponent* of a real irrational

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number γ is defined by the relation

$$\mu=\mu(\gamma)=\inf\{c\in\mathbb{R}:\text{the inequality } |\gamma-a/b|\leqslant|b|^{-c} \text{ has only finitely many solutions in } a,b\in\mathbb{Z}\}.$$

Our main results include the case of general rational z satisfying $|z|\leqslant 1$ as well as the case $z=-1$ of $\ln_q(2)$. Another special case, $z=1$ in (1), of the q-harmonic series, is considered in [Z2]. Our present methods do not allow us to sharpen the result in [Z2], where the arithmetic group structure approach (specific for $z=1$) is used.

Theorem 1. *Let $z\in\mathbb{Q}$ be such that $0<|z|\leqslant 1$. Then the irrationality exponent of $\ln_q(1-z)$ satisfies the estimate*

$$\mu(\ln_q(1-z))\leqslant 3.76338419\cdots,$$

where $q=p^{-1}$ and $p\in\mathbb{Z}\setminus\{0,\pm1\}$.

Theorem 2. *The irrationality exponent of $\ln_q(2)$ satisfies the estimate*

$$\mu(\ln_q(2))\leqslant 2.93832530\cdots,$$

where $q=p^{-1}$ and $p\in\mathbb{Z}\setminus\{0,\pm1\}$.

The estimate in Theorem 1 improves corresponding results of [BV], [MV]; the estimate in Theorem 2 sharpens results in [As], [Z1].

One important part in the proof of Theorem 2 is the precise knowledge of the least common multiple $D_n(x,z)$ of the polynomials $x-z,x^2-z,\dots,x^n-z$ at $z=-1$. This is a special case of a general algebraic result on $D_n(x,\omega)$ with a root of unity ω. The proof of this result, the following Theorem 3, seems to be an interesting application of cyclotomic polynomials.

Theorem 3. *Let ω denote a primitive r-th root of unity for some $r\geqslant 2$. Then in the polynomial ring $\mathbb{Z}[\omega][x]$ the following estimate is valid:*

$$\deg_x D_n(x,\omega)=\frac{3n^2}{\pi^2}\prod_{p|r}\frac{p^2}{p^2-1}\sum_l{}^{*}\,\frac{1}{l^2}+O(n\log^2 n)\qquad \text{as } n\to\infty, \tag{2}$$

where $\sum_l^*$ stands for summation over integers l in the interval $1\leqslant l\leqslant r$ and coprime with r.

To the end of Section 3, the integer p stands for $1/q$. We recall some standard q-notation:

$$(a;q)_n:=\prod_{\nu=1}^{n}(1-aq^{\nu-1}),$$

$$[n]_q!:=\frac{(q;q)_n}{(1-q)^n},\qquad \begin{bmatrix}n\\k\end{bmatrix}_q:=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}=\frac{(q;q)_n}{(q;q)_k\cdot(q;q)_{n-k}},$$

where $k=0,1,\dots,n$ and $n=0,1,2,\dots$.

2. Hypergeometric construction

Let n_0, n_1, n_2, and m be positive integers satisfying $n_1 \geqslant n_0$, $n_2 \geqslant n_0$. The additional condition $n_2 - n_0 \leqslant m \leqslant n_2$ will be required to further simplify the explanation (the choices $m < n_2 - n_0$ and $m > n_2$ do not correspond to nice approximations to the q-logarithm). First, consider the rational function

$$\begin{aligned}\widetilde{R}_q(T) &= \frac{\prod_{k=1}^{n_0}(1-q^kT)}{\prod_{k=1}^{n_0}(1-q^k)} \cdot \frac{\prod_{k=1}^{n_2}(1-q^k)}{\prod_{k=0}^{n_2}(1-q^{k+n_1+1}T)} \cdot T^{n_2-n_0} \\ &= \frac{(qT;q)_{n_0}}{(q;q)_{n_0}} \cdot \frac{(q;q)_{n_2}}{(q^{n_1+1}T;q)_{n_2+1}} \cdot T^{n_2-n_0},\end{aligned}$$

which is of order $O(T^{-1})$ as $T \to \infty$. This may be decomposed into the sum of partial fractions:

$$\widetilde{R}_q(T) = \sum_{k=0}^{n_2} \frac{A_k(q)}{1-q^{k+n_1+1}T},$$

where the standard procedure of determining coefficients gives us

$$\begin{aligned}A_k(q) &= (-1)^{n_0} q^{n_0(n_0+1)/2 - n_0(k+n_1+1)} \begin{bmatrix} k+n_1 \\ n_0 \end{bmatrix}_q \\ &\quad \times (-1)^k q^{k(k+1)/2} \begin{bmatrix} n_2 \\ k \end{bmatrix}_q \cdot q^{-(n_2-n_0)(k+n_1+1)} \\ &= (-1)^{k+n_0} p^{n_0(n_0+1)/2} \begin{bmatrix} k+n_1 \\ n_0 \end{bmatrix}_p \cdot p^{-n_2k+k(k-1)/2} \begin{bmatrix} n_2 \\ k \end{bmatrix}_p \cdot p^{(n_2-n_0)(k+n_1+1)}\end{aligned}$$

for $k = 0, 1, \ldots, n_2$. Setting $R_q(T) = \widetilde{R}_q(T) \cdot T^{m_0+1}$, where $m_0 = m - n_2 + n_0$, we introduce the quantity

$$I_q(z) = z^{n_1+1} \sum_{t=0}^{\infty} z^t R_q(T)\big|_{T=q^t}.$$

Since $R_q(T)$ has zeros at the points $T = q^{-1}, q^{-2}, \ldots, q^{-n_0}$, after reordering of the summation we may write

$$\begin{aligned}I_q(z) &= \sum_{k=0}^{n_2} A_k(q) q^{-(k+n_1+1)(m_0+1)} z^{-k} \sum_{t=-n_0}^{\infty} \frac{z^{t+k+n_1+1} q^{(t+k+n_1+1)(m_0+1)}}{1-q^{t+k+n_1+1}} \\ &= \sum_{k=0}^{n_2} A_k(q) p^{(k+n_1+1)(m_0+1)} z^{-k} \sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l q^{l(m_0+1)}}{1-q^l}.\end{aligned}$$

The last inner sum may be computed as follows:

$$\sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l q^{l(m_0+1)}}{1-q^l} = \sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l q^l}{1-q^l} - \sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l (q^l - q^{l(m_0+1)})}{1-q^l};$$

writing the first sum on the right-hand side as

$$\sum_{l=1}^{\infty} \frac{z^l q^l}{1-q^l} - \sum_{l=1}^{k+n_1-n_0} \frac{z^l q^l}{1-q^l} = \ln_q(1-z) - \sum_{l=1}^{k+n_1-n_0} \frac{z^l q^l}{1-q^l}$$

and the second sum as

$$\sum_{l=k+n_1-n_0+1}^{\infty} z^l \sum_{j=1}^{m_0} q^{jl} = \sum_{j=1}^{m_0} \sum_{l=k+n_1-n_0+1}^{\infty} (q^j z)^l = \sum_{j=1}^{m_0} \frac{(q^j z)^{k+n_1-n_0+1}}{1-q^j z},$$

we finally obtain

$$I_q(z) = A(p,z)\ln_q(1-z) + A'(p,z) + A''(p,z),$$

where

$$\begin{aligned}
A(p,z) &= \sum_{k=0}^{n_2} A_k(q) p^{(k+n_1+1)(m_0+1)} z^{-k} \\
&= (-1)^{n_0} p^{n_0(n_0+1)/2+(m+1)(n_1+1)} \\
&\quad \times \sum_{k=0}^{n_2} (-1)^k p^{-n_2 k+(m+1)k+k(k-1)/2} \begin{bmatrix} k+n_1 \\ n_0 \end{bmatrix}_p \begin{bmatrix} n_2 \\ k \end{bmatrix}_p z^{-k}, \\
A'(p,z) &= \sum_{k=0}^{n_2} A_k(q) p^{(k+n_1+1)(m_0+1)} z^{-k} \sum_{l=1}^{k+n_1-n_0} \frac{z^l}{p^l-1} \\
&= (-1)^{n_0} p^{n_0(n_0+1)/2+(m+1)(n_1+1)} \\
&\quad \times \sum_{k=0}^{n_2} (-1)^k p^{-n_2 k+(m+1)k+k(k-1)/2} \begin{bmatrix} k+n_1 \\ n_0 \end{bmatrix}_p \begin{bmatrix} n_2 \\ k \end{bmatrix}_p z^{-k} \sum_{l=1}^{k+n_1-n_0} \frac{z^l}{p^l-1}, \\
A''(p,z) &= \sum_{k=0}^{n_2} A_k(q) p^{(k+n_1+1)(m_0+1)} z^{n_1-n_0+1} \sum_{j=1}^{m_0} \frac{p^{-j(k+n_1-n_0)}}{p^j-z} \\
&= (-1)^{n_0} z^{n_1-n_0+1} p^{n_0(n_0+1)/2+(n_0+1)(m+1)} \sum_{j=1}^{m_0} \frac{1}{p^j-z} \\
&\quad \times \sum_{k=0}^{n_2} (-1)^k p^{-n_2 k+k(k-1)/2} \begin{bmatrix} k+n_1 \\ n_0 \end{bmatrix}_p \begin{bmatrix} n_2 \\ k \end{bmatrix}_p (p^{m+1-j})^{n_1-n_0+k} \\
&= z^{n_1-n_0+1} p^{n_0(n_0+1)/2+(n_0+1)(m+1)+(n_2+1)(n_1-n_0)} \sum_{j=1}^{m_0} \frac{1}{p^j-z} \\
&\quad \times \sum_{k=0}^{n_1} (-1)^k p^{(n_0-k)(n_0-k+1)/2} \begin{bmatrix} k+n_2 \\ n_0 \end{bmatrix}_p \begin{bmatrix} n_1 \\ k \end{bmatrix}_p (p^{m-j}; p^{-1})_{n_2-n_0+k}
\end{aligned}$$

(the last step uses Lemma 3 from [Z1]).

Since $m \leqslant n_2$, we have

$$\begin{aligned}
M_1 &= \frac{n_0(n_0+1)}{2} + (m+1)(n_1+1) + \min_{0\leqslant k\leqslant n_2}\left\{-n_2 k + (m+1)k + \frac{k(k-1)}{2}\right\} \\
&= \frac{n_0(n_0+1)}{2} + (m+1)(n_1+1) - \frac{(n_2-m)(n_2-m-1)}{2};
\end{aligned}$$

set also

$$M_2 = \frac{n_0(n_0+1)}{2} + (n_0+1)(m+1) + (n_2+1)(n_1-n_0),$$

and by $D_n(p,z)$ denote the least common multiple of the polynomials $p-z$, $p^2-z,\dots,p^n-z$. Then the above formulae yield the inclusions

$$p^{-M_1}z^{n_2}\cdot A(p,z)\in\mathbb{Z}[p,z], \qquad p^{-M_1}z^{n_2}D_{n_1+n_2-n_0}(p,1)\cdot A'(p,z)\in\mathbb{Z}[p,z],$$
$$p^{-M_2}D_{m_0}(p,z)\cdot A''(p,z)\in\mathbb{Z}[p,z]$$

(by noticing that $(p^{m-j};p^{-1})_{n_2-n_0+k}=0$ if $m-j-n_2+n_0-k\leqslant 0$); hence

$$p^{-M}\widehat{D}_{n_1+n_2-n_0,m_0}(p,z)\cdot I_q(z)\in\mathbb{Z}[p,z]\ln_q(1-z)+\mathbb{Z}[p,z], \tag{3}$$

where $M=\min\{M_1,M_2\}=M_1$ and $\widehat{D}_{n,m}(p,z)$ denotes a common multiple of the polynomials $D_n(p)=D_n(p,1)$ and $D_m(p,z)$. It is known [Ge] that the polynomial $D_n(p)$ is the product of the first n cyclotomic polynomials

$$\Phi_l(p)=\prod_{\substack{k=1\\(k,l)=1}}^{l}(p-e^{2\pi ik/l})\in\mathbb{Z}[p], \qquad l=1,2,3,\dots, \tag{4}$$

so that the usual choice of $\widehat{D}_{n,m}(p,z)$ is as follows:

$$\widehat{D}_{n,m}(p,z)=D_n(p)\cdot\prod_{j=1}^{m}(p^j-z). \tag{5}$$

However, if z is a root of unity, there is a better choice instead; we discuss this type of question in Sections 3 and 4 below.

Finally, we would like to mention that the quantity $I_q(z)$ is in fact the value of the Heine series,

$$I_q(z)=z^{n_1+1}\cdot\frac{(q;q)_{n_1}(q;q)_{n_2}}{(q;q)_{n_1+n_2+1}}\cdot{}_2\phi_1\left(\begin{matrix}q^{n_0+1}, & q^{n_1+1}\\ & q^{n_1+n_2+2}\end{matrix}\,\middle|\, q,q^{m+1}z\right)$$

(see [GR]), and that the construction in [MV] corresponds to the following choice of the parameters: $n_0=n_2=n$, $n_1=n+1$, and $m=K-1$.

3. Analytic and arithmetic valuation

Writing

$$A(p,z)=(-1)^{n_0}p^{-n_0(n_0+1)/2+(n_0+m+1)(n_1+1)}\times\sum_{k=0}^{n_2}(-1)^k p^{(n_0+m+1)k-k(k+1)/2}\begin{bmatrix}k+n_1\\ n_0\end{bmatrix}_q\begin{bmatrix}n_2\\ k\end{bmatrix}_q z^{-k}$$

and using

$$\max_{0\leqslant k\leqslant n_2}\left\{(n_0+m+1)k-\frac{k(k+1)}{2}\right\}=(n_0+m+1)n_2-\frac{n_2(n_2+1)}{2}$$

(since $n_0+m+1>n_2$), we conclude that

$$|A(p,z)|=|p|^{-n_0(n_0+1)/2-n_2(n_2+1)/2+(n_0+m+1)(n_1+n_2+1)+O(n_0+n_1+n_2+m)}, \tag{6}$$

where the constant in O depends on z only. Similarly,

$$|I_q(z)|=|p|^{O(n_0+n_1+n_2+m)}. \tag{7}$$

The general asymmetry of our construction yields the existence of a common divisor $\Pi(p)=\Pi_{n_0,n_1,n_2}(p)\in\mathbb{Z}[p]$ of the polynomials

$$\begin{bmatrix}k+n_1\\ n_0\end{bmatrix}_p\begin{bmatrix}n_2\\ k\end{bmatrix}_p,\quad k=0,1,\dots,n_2,\qquad \begin{bmatrix}k+n_2\\ n_0\end{bmatrix}_p\begin{bmatrix}n_1\\ k\end{bmatrix}_p,\quad k=0,1,\dots,n_1,$$

and hence of the coefficients $A(p,z)$, $A'(p,z)$, $A''(p,z)$ after multiplication by $p^{-M}\cdot\widehat{D}_{n_1+n_2-n_0,m_0}(p,z)$ in (3). Namely, using representations

$$\begin{bmatrix}k+n_1\\ n_0\end{bmatrix}_p\begin{bmatrix}n_2\\ k\end{bmatrix}_p=\frac{[n_1]_p!\,[n_2]_p!}{[n_0]_p!\,[n_1+n_2-n_0]_p!}\cdot\begin{bmatrix}k+n_1\\ k\end{bmatrix}_p\begin{bmatrix}n_1+n_2-n_0\\ n_2-k\end{bmatrix}_p,\quad k=0,1,\dots,n_2,$$

$$\begin{bmatrix}k+n_2\\ n_0\end{bmatrix}_p\begin{bmatrix}n_1\\ k\end{bmatrix}_p=\frac{[n_1]_p!\,[n_2]_p!}{[n_0]_p!\,[n_1+n_2-n_0]_p!}\cdot\begin{bmatrix}k+n_2\\ k\end{bmatrix}_p\begin{bmatrix}n_1+n_2-n_0\\ n_1-k\end{bmatrix}_p,\quad k=0,1,\dots,n_1,$$

and the knowledge that p-binomial coefficients are polynomials from $\mathbb{Z}[p]$ having only cyclotomic polynomials as irreducible factors, we may take

$$\Pi(p)=\prod_{l=1}^{n_1+n_2-n_0}\Phi_l(p)^{\varpi(l)},$$

where

$$\varpi(l)=\max\left\{0,\left\lfloor\frac{n_1}{l}\right\rfloor+\left\lfloor\frac{n_2}{l}\right\rfloor-\left\lfloor\frac{n_0}{l}\right\rfloor-\left\lfloor\frac{n_1+n_2-n_0}{l}\right\rfloor\right\}$$

and $\lfloor\cdot\rfloor$ denotes the integer part of a number (see [Z1], the proof of Lemma 5). These arguments allow us to sharpen the inclusions (3) as follows:

$$p^{-M}\widehat{D}_{n_1+n_2-n_0,m_0}(p,z)\cdot\Pi_{n_0,n_1,n_2}(p)^{-1}\cdot I_q(z)\in\mathbb{Z}[p,z]\ln_q(1-z)+\mathbb{Z}[p,z].$$

Finally, set

$$n_0=\alpha_0 n,\quad n_1=\alpha_1 n,\quad n_2=\alpha_2 n,\quad m=\lfloor\alpha n\rfloor,$$

where the parameter n tends to ∞. Then

$$\lim_{n\to\infty}\frac{\log|A(p,z)|}{n^2\log|p|}=C_1,\qquad \lim_{n\to\infty}\frac{\log|I_q(z)|}{n^2\log|p|}=0$$

by (6), (7), and

$$\lim_{n\to\infty}\frac{\log|p^M\widehat{D}_{n_1+n_2-n_0,m_0}(p,z)^{-1}\cdot\Pi_{n_0,n_1,n_2}(p)|}{n^2\log|p|}=C_0 \tag{8}$$

with the choice (5), where

(9)

$$C_1=C_1(\alpha)=-\frac{\alpha_0^2+\alpha_2^2}{2}+(\alpha_0+\alpha)(\alpha_1+\alpha_2),$$

$$C_0=C_0(\alpha)=\frac{\alpha_0^2}{2}+\alpha_1\alpha-\frac{(\alpha_2-\alpha)^2}{2}-\frac{3}{\pi^2}\left((\alpha_1+\alpha_2-\alpha_0)^2-\int_0^1\varpi_0(x)\mathrm{d}(-\psi'(x))\right)-\frac{(\alpha-\alpha_2+\alpha_0)^2}{2}$$

and

$$\varpi_0(x)=\max\{0,\lfloor\alpha_1x\rfloor+\lfloor\alpha_2x\rfloor-\lfloor\alpha_0x\rfloor-\lfloor(\alpha_1+\alpha_2-\alpha_0)x\rfloor\}.$$

Then $\mu(\ln_q(1-z)) \leqslant C_1(\alpha)/C_0(\alpha)$ provided that $\alpha_2-\alpha_0 \leqslant \alpha \leqslant \alpha_2$ and $C_0(\alpha)>0$. It is important that the parameters $\alpha_0, \alpha_1, \alpha_2$ should be positive integers to ensure validity of the above formula for $C_0(\alpha)$ (namely, its integration part due to [Z1], Lemma 1). Thus after making a suitable choice for these three parameters we can minimize the quantity $C_1(\alpha)/C_0(\alpha)$ with respect to the remaining parameter α, which may take any (even irrational) value in the interval $\alpha_2-\alpha_0 \leqslant \alpha \leqslant \alpha_2$. This idea comes from [MV], and, as in that work, there is no difficulty in mimimizing $C_1(\alpha)/C_0(\alpha)$ since $C_1(\alpha)$ depends linearly and $C_0(\alpha)$ quadratically on the parameter α.

Proof of Theorem 1. Taking $\alpha_0=6$, $\alpha_1=\alpha_2=7$, so that $\varpi_0(x)=1$ for $x\in[0,1)$ lying in the following set:

$$\left[\tfrac{1}{7},\tfrac{1}{6}\right)\cup\left[\tfrac{2}{7},\tfrac{1}{3}\right)\cup\left[\tfrac{3}{7},\tfrac{1}{2}\right)\cup\left[\tfrac{4}{7},\tfrac{5}{8}\right)\cup\left[\tfrac{5}{7},\tfrac{3}{4}\right)\cup\left[\tfrac{6}{7},\tfrac{7}{8}\right),$$

and then $\alpha=5.63997199\cdots$, we arrive at the estimate

$$\mu(\ln_q(1-z)) \leqslant 3.76338419\cdots$$

of the theorem. □

4. Cyclotomic background

We will agree from the beginning to deal with the cyclotomic polynomials $\Phi_l(x)$ and least common multiples $D_n(x,z)$ and $\widehat{D}_{n,m}(x,z)$ as polynomials in the variable x, and to keep the substitution $x=p\in\mathbb{Z}\setminus\{0,\pm1\}$ for final arithmetic results. As follows from definition (4), $\deg\Phi_l(x)=\varphi(l)$, Euler's totient function. Therefore, the degree of the polynomial $D_n(x)=D_n(x,1)=\prod_{l=1}^n\Phi_l(x)$ may be computed by application of Mertens' formula

$$\deg D_n(x)=\sum_{1\leqslant l\leqslant n}\varphi(l)=\frac{3}{\pi^2}n^2+O(n\log n) \qquad \text{as } n\to\infty; \tag{10}$$

hence

$$\lim_{n\to\infty}\frac{\log|D_n(p)|}{n^2\log|p|}=\frac{3}{\pi^2}.$$

This is the formula used in computing the right-hand side of (8). We will also require the following summation formulae for Euler's totient function:

$$\sum_{1\leqslant j\leqslant n}\varphi(2j)=\frac{4}{\pi^2}n^2+O(n\log n), \qquad \sum_{0\leqslant j\leqslant n}\varphi(2j+1)=\frac{8}{\pi^2}n^2+O(n\log n) \tag{11}$$

as $n\to\infty$ (for n real and not necessarily integral); see also the general formula (14) below.

Lemma 1. *In the polynomial ring* $\mathbb{Z}[x]$ *the following estimate is valid:*

$$\deg D_n(x,-1)=\frac{4}{\pi^2}n^2+O(n\log n) \qquad \textit{as } n\to\infty. \tag{12}$$

First proof. Since $x^k-1=\prod_{l|k}\Phi_l(x)$, we have

$$x^k+1=\frac{x^{2k}-1}{x^k-1}=\frac{\prod_{l|2k}\Phi_l(x)}{\prod_{l|k}\Phi_l(x)}=\prod_{\substack{l|2k\\ l\nmid k}}\Phi_l(x)=\prod_{\substack{l|k\\ k/l\text{ is odd}}}\Phi_{2l}(x), \qquad k=1,\dots,n.$$

Therefore, x^k+1 divides $\prod_{l=1}^n \Phi_{2l}(x)$ for $k=1,\dots,n$ and, clearly, $\Phi_{2l}(x)$ divides x^l+1 for $l=1,\dots,n$. Thus $D_n(x,-1)=\prod_{l=1}^n \Phi_{2l}(x)$ and application of the first formula in (11) leads to the desired result. □

Second proof. This proof follows the ideas of proving Lemma 2 in [MP]; we indicate it to make clear the ideas of proving Theorem 3 below.

For each $n>0$ (not necessarily integral!), denote by $L_n(x)$ the least common multiple of the polynomials x^k+1, where k runs over positive odd integers in the interval $1\leqslant k\leqslant n$. Since $x^k+1=-((-x)^k-1)=-\prod_{l|k}\Phi_l(-x)$ for k odd, we obtain

$$L_n(x)=\prod_{\substack{1\leqslant l\leqslant n\\ l \text{ is odd}}}\Phi_l(-x)=\prod_{j=0}^{\lfloor n/2\rfloor}\Phi_{2j+1}(-x);$$

hence

$$\deg L_n(x)=\frac{2}{\pi^2}n^2+O(n\log n)\qquad \text{as } n\to\infty, \tag{13}$$

by the second formula in (11). Clearly, $L_{n/2}(x^2)$ gives the least common multiple of the polynomials x^k+1, where k runs over positive even integers in the interval $1\leqslant k\leqslant n$ not divisible by 4; then $L_{n/4}(x^4)$ gives the least common multiple of the polynomials x^k+1, where $k\equiv 4 \pmod 8$ runs in the interval $1\leqslant k\leqslant n$, and so on. If exponents of 2 in the prime decompositions of the numbers k and j are different, then polynomials x^k+1 and x^j+1 have no common *complex* roots; hence they are coprime over $\mathbb{C}[x]$ and as a consequence over $\mathbb{Z}[x]$ as well. Therefore, we arrive at the formula

$$D_n(x,-1)=L_n(x)L_{n/2}(x^2)L_{n/4}(x^4)L_{n/8}(x^8)\cdots,$$

where the product on the right contains only a finite number $O(\log n)$ of factors, and the (almost desired) estimate for the degree of $D_n(x,-1)$,

$$\deg D_n(x,-1)=\frac{4}{\pi^2}n^2+O(n\log^2 n)\qquad \text{as } n\to\infty,$$

follows from an accurate substitution of formula (13). □

Corollary. *If $n/2\leqslant m\leqslant n$, then a common multiple $\widehat{D}_{n,m}(x,-1)$ (over $\mathbb{Z}[x]$) of the polynomials $D_n(x)$ and $D_m(x,-1)$ may be taken in such a way that*

$$\deg \widehat{D}_{n,m}(x,-1)=\frac{1}{\pi^2}(2n^2+4m^2)+O(n\log n)\qquad \textit{as } n\to\infty.$$

Proof. The polynomials x^k+1 for $1\leqslant k\leqslant n/2$ divide both $D_n(x)$ and $D_m(x,-1)$. Therefore we may take

$$\widehat{D}_{n,m}(x,-1)=\frac{D_n(x)D_m(x,-1)}{D_{\lfloor n/2\rfloor}(x,-1)},$$

and estimates (10), (12) give the desired result. □

Remark. The above choice of $\widehat{D}_{n,m}(x,-1)$ sharpens the choice in [Z1], Lemma 8.

Proof of Theorem 2. Using the above corollary of Lemma 1 we may replace the constant C_0 in (9) by

$$C_0' = C_0'(\alpha) = \frac{\alpha_0^2}{2} + \alpha_1\alpha - \frac{(\alpha_2-\alpha)^2}{2} - \frac{1}{\pi^2}\biggl(2(\alpha_1+\alpha_2-\alpha_0)^2 + 4(\alpha-\alpha_2+\alpha_0)^2 - 3\int_0^1 \varpi_0(x)\mathrm{d}(-\psi'(x))\biggr),$$

with the result $\mu(\ln_q(2)) \leqslant C_1/C_0' \leqslant 2.93832530\cdots$ obtained by using the values $\alpha_0 = 4$, $\alpha_1 = \alpha_2 = 5$, $\alpha = 4.09112737\cdots$. In this case, $\varpi_0(x) = 1$ for $x \in [0,1)$ belonging to the following set:

$$\bigl[\tfrac{1}{5},\tfrac{1}{4}\bigr) \cup \bigl[\tfrac{2}{5},\tfrac{1}{2}\bigr) \cup \bigl[\tfrac{3}{5},\tfrac{2}{3}\bigr) \cup \bigl[\tfrac{4}{5},\tfrac{5}{6}\bigr).$$

This proves Theorem 2. □

5. Common multiples involving cyclotomic polynomials

The number p will be used to denote a prime. We will require the asymptotic formula

$$\sum_{j=0}^{n}\varphi(rj+b) = \frac{3r}{\pi^2}n^2\prod_{p\mid r}\frac{p^2}{p^2-1} + O(n\log n) \qquad \text{as } n\to\infty, \tag{14}$$

where $1 \leqslant b \leqslant r$ and $(b,r)=1$ (see [Ba] and [MP]).

Proof of Theorem 3. For each $n > 0$ (not necessarily integral!) and any integer b satisfying $1 \leqslant b \leqslant r$ and $(b,r)=1$, denote by $L_{n,b}(x)$ the least common multiple of the polynomials $x^k-\omega$, where k runs over integers in the interval $1 \leqslant k \leqslant n$ satisfying $k \equiv b \pmod r$. The polynomials $x^k-\omega$ and $x^j-\omega$, where k and j are integers coprime with r and $k \not\equiv j \pmod r$, have no common roots; hence these polynomials are coprime over $\mathbb{C}[x]$. This, in particular, yields that the $\varphi(r)$ polynomials $L_{n,b}(x)$, $1 \leqslant b \leqslant r$, $(b,r)=1$, are pairwise coprime over $\mathbb{C}[x]$ and over $\mathbb{Z}[\omega][x] \subset \mathbb{C}[x]$ as well; hence

$$L_n(x) = \prod_{\substack{1\leqslant b\leqslant r\\(b,r)=1}} L_{n,b}(x) \tag{15}$$

is the least common multiple of the polynomials $x^k-\omega$, where k runs over integers satisfying $1 \leqslant k \leqslant n$ coprime with r. Having this common multiple and concluding as in the second proof of Lemma 1, we obtain

$$D_n(x,\omega) = \prod_{s_1=0}^{\infty}\cdots\prod_{s_m=0}^{\infty} L_{n/(p_1^{s_1}\cdots p_m^{s_m})}\bigl(x^{p_1^{s_1}\cdots p_m^{s_m}}\bigr), \tag{16}$$

where $p_1,\dots,p_m$ are all distinct prime divisors of the number r. Note that, in spite of infinite products in (16), only a finite number $[O(\log n)]$ of the factors differ from 1.

In order to compute the polynomials $L_{n,b}(x)$, we start by noting the formula

$$x^{rj+b}-\omega = \omega\bigl((\omega^a x)^{rj+b}-1\bigr) = \omega\prod_{d\mid rj+b}\Phi_d(\omega^a x),$$

where $ab \equiv -1 \pmod r$. Therefore, assigning the numbers b_l in the interval $1 \leqslant b_l \leqslant r$ to each l, $1 \leqslant l \leqslant r$, $(l,r)=1$, by the rule $lb_l \equiv b \pmod r$ (as in [MP]) we obtain

$$\prod_{\substack{1\leqslant l\leqslant r\\(l,r)=1}} \prod_{j=0}^{\lfloor n/(rl)\rfloor-1} \Phi_{rj+b_l}(\omega^a x) \,\Big|\, L_{n,b}(x) \,\Big|\, \prod_{\substack{1\leqslant l\leqslant r\\(l,r)=1}} \prod_{j=0}^{\lfloor n/(rl)\rfloor} \Phi_{rj+b_l}(\omega^a x)$$

(where "|" means "divides", as before); hence

$$\begin{aligned}\deg_x L_{n,b} &= \sum_l{}^{*} \biggl(\sum_{j=0}^{\lfloor n/(rl)\rfloor} \varphi(rj+b_l) + O(n\log n) \biggr) \\ &= \sum_l{}^{*} \biggl(\frac{3r}{\pi^2} \Bigl(\frac{n}{rl}\Bigr)^2 \prod_{p\mid r} \frac{p^2}{p^2-1} + O(n\log n) \biggr) \\ &= \frac{3n^2}{\pi^2 r} \prod_{p\mid r} \frac{p^2}{p^2-1} \sum_l{}^{*} \frac{1}{l^2} + O(n\log n) \qquad \text{as } n\to\infty,\end{aligned}$$

by (14). Using (15) we obtain

$$\deg_x L_n = \frac{3n^2\varphi(r)}{\pi^2 r} \prod_{p\mid r} \frac{p^2}{p^2-1} \sum_l{}^{*} \frac{1}{l^2} + O(n\log n) \qquad \text{as } n\to\infty.$$

Finally, computing the degree of the polynomial $D_n(x,\omega)$ in (16) with the help of the relation

$$\sum_{s_1=0}^{\infty} \cdots \sum_{s_m=0}^{\infty} \frac{1}{p_1^{s_1}\cdots p_m^{s_m}} = \Bigl(1-\frac{1}{p_1}\Bigr)^{-1} \cdots \Bigl(1-\frac{1}{p_m}\Bigr)^{-1} = \frac{r}{\varphi(r)}$$

gives the desired result (2). This proves Theorem 3. □

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References

[As] W. Van Assche, *Little q-Legendre polynomials and irrationality of certain Lambert series*, Ramanujan J. **5** (3) (2001), 295–310. MR1876702 (2002k:11124)

[Ba] E. Bavencoffe, *Plus petit commun multiple de suites de polynômes*, Ann. Fac. Sci. Toulouse Math. **1** (2) (1992), 147–168. MR1202069 (93m:11021)

[Bo] P. Borwein, *On the irrationality of* $\sum(1/(q^n+r))$, J. Number Theory **37** (1991), 253-259. MR1096442 (92b:11046)

[BV] P. Bundschuh and K. Väänänen, *Arithmetical investigations of a certain infinite product*, Compositio Math. **91** (1994), 175–199. MR1273648 (95e:11081)

[GR] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge Univ. Press, Cambridge, 1990. MR1052153 (91d:33034)

[Ge] A. O. Gel′fond, *On functions assuming integral values*, Mat. Zametki [Math. Notes] **1** (5) (1967), 509–513. MR0215795 (35:6630)

[Ha] M. Hata, *Legendre type polynomials and irrationality measures*, J. Reine Angew. Math. **407** (1) (1990), 99–125. MR1048530 (91i:11081)

[MP] T. Matala-aho and M. Prévost, *Quantitative irrationality for sums of reciprocals of Fibonacci and Lucas numbers*, Ramanujan J. **11** (2006).

[MV] T. Matala-aho and K. Väänänen, *On approximation measures of q-logarithms*, Bull. Austral. Math. Soc. **58** (1998), 15–31. MR1633740 (99e:11098)

[Ru] E. A. Rukhadze, *A lower bound for the approximation of* $\ln 2$ *by rational numbers*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.] (6) (1987), 25–29. MR0922879 (89b:11064)

[Z1] W. Zudilin, *Remarks on irrationality of q-harmonic series*, Manuscripta Math. **107** (4) (2002), 463–477. MR1906771 (2003f:11103)

[Z2] W. Zudilin, *Heine's basic transform and a permutation group for q-harmonic series*, Acta Arith. **111** (2) (2004), 153-164. MR2039419 (2005f:11148)

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