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OPTIMAL C^2 TWO-DIMENSIONAL INTERPOLATORY TERNARY SUBDIVISION SCHEMES WITH TWO-RING STENCILS

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ABSTRACT. For any interpolatory ternary subdivision scheme with two-ring stencils for a regular triangular or quadrilateral mesh, we show that the critical Hölder smoothness exponent of its basis function cannot exceed $\log_3 11(\approx 2.18266)$, where the critical Hölder smoothness exponent of a function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ is defined to be

$$\nu_\infty(f) := \sup\{\nu \ : \ f \in \operatorname{Lip}\nu\}.$$

On the other hand, for both regular triangular and quadrilateral meshes, we present several examples of interpolatory ternary subdivision schemes with two-ring stencils such that the critical Hölder smoothness exponents of their basis functions do achieve the optimal smoothness upper bound $\log_3 11$. Consequently, we obtain optimal smoothest C^2 interpolatory ternary subdivision schemes with two-ring stencils for the regular triangular and quadrilateral meshes. Our computation and analysis of optimal multidimensional subdivision schemes are based on the projection method and the ℓ_p-norm joint spectral radius.

1. INTRODUCTION AND MOTIVATION

Subdivision schemes have proved to be a useful way of generating surfaces in CAGD ([1, 6]). In general, one first constructs a stationary subdivision scheme with certain desired properties on a regular mesh. Then for a given initial mesh of arbitrary topology, one applies such a subdivision rule for regular vertices and handles extraordinary vertices of a mesh by modified special subdivision rules (see the course note *Subdivision for modeling and animation* by P. Schröder *et al.*, 1998). Since the number of extraordinary vertices in all subdivision levels remains the same, construction of stationary subdivision schemes on a regular mesh with certain desired properties, such as good smoothness and small subdivision stencils, is important in CAGD.

In order to have good visual quality of the generated subdivision surfaces, one is generally interested in subdivision schemes whose basis functions are at least C^2 so that the curvature of the generated subdivision surfaces is continuous. On the other hand, in order to reduce the number of special subdivision rules for

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extraordinary vertices, from the point of view of implementation and computation, one prefers in CAGD that the associated subdivision stencils have no more than two-ring neighboring vertices, which is almost equivalent to saying that its mask should have a very short support. But it is well known that high smoothness of a basis function in a subdivision scheme and the shortness of the support size of its mask are two mutually conflicting requirements; that is, it is well known that in order to have a smoother subdivision scheme, it is necessary to enlarge the support of its mask. For example, it was proved in [9, Corollary 4.3] that for any dimension s, there is no C^2 s-dimensional interpolatory dyadic subdivision scheme whose mask can be supported on $[-3,3]^s$ (that is, it has two-ring stencils); therefore, the well-known butterfly scheme proposed by Dyn, Levin and Gregory [7] and several other examples of interpolatory dyadic subdivision schemes in [17, 26], which are two-dimensional interpolatory dyadic subdivision schemes with two-ring stencils, cannot be C^2 schemes. On the other hand, as we shall see in Section 3, for any dimension s, there is no C^2 s-dimensional interpolatory m-adic ($m \in \mathbb{Z}$ with $|m| > 1$) subdivision scheme whose mask can be supported on $[-|m|, |m|]^s$.

In order to obtain subdivision schemes with various desired properties beyond the traditional dyadic schemes, subdivision schemes with other possible refinements of a mesh have recently been studied in the literature. For example, interpolatory quincunx subdivision schemes have been studied in [17], and the construction in [17] has been generalized to interpolatory $\sqrt{3}$ subdivision schemes in Jiang, Oswald and Riemenschneider [23]. See [17, 23, 27] and many references therein for more detail.

In order to achieve continuity of the curvature in a subdivision surface, very recently there has been a growing interest in investigating interpolatory ternary subdivision schemes, due to some of their interesting properties. One-dimensional C^2 interpolatory ternary subdivision schemes with two-ring stencils have been obtained in [19]. Some examples of two-dimensional interpolatory ternary subdivision schemes have been proposed in [4, 15]. For some desired properties of ternary subdivision schemes, the reader is referred to the work [4, 19, 24] for more detail. It is the purpose of this paper to investigate the smoothest optimal interpolatory ternary subdivision schemes with two-ring stencils in one and two dimensions.

The following is the structure of this paper. In Section 2, we shall recall the notion of subdivision triplets in [15] and subdivision stencils. Then we shall discuss and review some results on convergence and smoothness of multidimensional subdivision schemes by using ℓ_p-norm joint spectral radius. We shall explicitly discuss the connection between the cascade algorithms in the function setting and the subdivision schemes in the discrete sequence setting. Some results on estimating the critical Hölder smoothness exponent of a basis function in a subdivision scheme will be given.

In Section 3, we shall discuss one-dimensional smoothest interpolatory ternary subdivision schemes with two-ring stencils. Then we shall investigate optimal multidimensional interpolatory ternary subdivision schemes via the projection method. We show that for any interpolatory ternary subdivision schemes with two-ring stencils in any dimension, the critical Hölder smoothness exponent of its basis function cannot exceed $\log_3 11$. Moreover, we prove that there is a unique one-dimensional interpolatory ternary subdivision scheme with two-ring stencils such that its basis function has the optimal critical Hölder smoothness exponent $\log_3 11$. Since subdivision schemes with one-ring stencils are of particular interest in CAGD, one may wonder whether there is a smooth interpolatory m-adic subdivision scheme with one-ring stencils. In Section 3, we show that there is no C^2 s-dimensional interpolatory m-adic subdivision scheme whose mask is supported on $[-|m|, |m|]^s$.

In Section 4, we shall present some examples of optimal C^2 two-dimensional interpolatory ternary subdivision schemes with two-ring stencils for the regular triangular and quadrilateral meshes. Our computation and analysis of all the examples in Section 4 are based on the projection method in [11] and the ℓ_p-norm joint spectral radius.

In order to apply the subdivision schemes constructed in this paper to free-form subdivision surfaces in CAGD, we have to design the special subdivision rules for extraordinary vertices, which we shall discuss elsewhere.

2. Some properties of subdivision triplets

In this section, we shall recall some properties of a general subdivision scheme in any dimension.

We say that G is a *symmetry group* on $\mathbb{Z}^s$ if each element $E \in G$ is an integer matrix with $|\det E| = 1$ and G forms a group under matrix multiplication. A subdivision scheme is completely determined by a triplet (a, M, G), where G is a symmetry group on $\mathbb{Z}^s$ distinguishing the type of a mesh, M is a dilation matrix determining the refinement of the mesh and a is a mask yielding all the subdivision stencils. The quadrilateral mesh and the triangular mesh are invariant under the symmetry groups D_4 and D_6, respectively, which are defined to be

$$D_4 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

and

$$D_6 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \right\}.$$

A finitely supported sequence $a : \mathbb{Z}^s \mapsto \mathbb{R}$ is called a *mask*. A quincunx (also called $\sqrt{2}$) subdivision scheme is given by a triplet $(a, M_{\sqrt{2}}, D_4)$, a $\sqrt{3}$ subdivision scheme is given by $(a, M_{\sqrt{3}}, D_6)$, and a ternary subdivision scheme is either $(a, 3I_2, D_4)$ or $(a, 3I_2, D_6)$, where a is a mask and

$$M_{\sqrt{2}} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad M_{\sqrt{3}} := \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad I_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Unlike the lattice $\mathbb{Z}^s$ which is a set of discrete points without connectivity, it is difficult to have a global coordinate system on a general mesh. In order to overcome such a difficulty, symmetry is required in a subdivision scheme. That is, a subdivision scheme should be given by a *subdivision triplet* (a, M, G) (see [15]) which satisfies

(1) The mask a is G-*symmetric*: $a(E\beta) = a(\beta)$ for all $\beta \in \mathbb{Z}^s$ and $E \in G$.
(2) The symmetry group G is compatible with the dilation matrix M ([13]): $MEM^{-1} \in G$ for all $E \in G$.

In this paper, for simplicity of presentation, we only consider the dilation matrices $M = mI_s$, where $m \in \mathbb{Z}$ with $|m| > 1$; that is, we only consider m-adic subdivision schemes. The basis function ϕ of a subdivision triplet (a, mI_s, G) is a unique solution to the refinement equation

$$\phi = \sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi(m\cdot -\beta) \qquad \text{with} \qquad \hat{\phi}(0) = 1, \tag{2.1}$$

where the Fourier transform is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}\,dx, \xi \in \mathbb{R}^s$. Since (a, mI_s, G) is a subdivision triplet, it is easy to see that $\phi(E\cdot) = \phi$ for all $E \in G$. In fact, ϕ is given by $\hat{\phi}(\xi) = \prod_{j=1}^{\infty}[\hat{a}(m^{-j}\xi)/|m|]$, where $\hat{a}$ is the *Fourier series* of the sequence a and is defined to be

$$\hat{a}(\xi) := \sum_{\beta\in\mathbb{Z}^s} a(\beta)e^{-i\beta\cdot\xi}, \qquad \xi \in \mathbb{R}^s. \tag{2.2}$$

By $\ell_0(\mathbb{Z}^s)$ we denote the space of all finitely supported sequences on $\mathbb{Z}^s$. For a subdivision triplet (a, mI_s, G), the *subdivision operator* $S_{a,mI_s} : \ell_0(\mathbb{Z}^s) \mapsto \ell_0(\mathbb{Z}^s)$ is defined to be

$$[S_{a,mI_s}u](\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - m\beta)u(\beta), \qquad \alpha \in \mathbb{Z}^s,\ u \in \ell_0(\mathbb{Z}^s). \tag{2.3}$$

Since (a, mI_s, G) is a subdivision triplet, it is easy to check that if $u \in \ell_0(\mathbb{Z}^s)$ is G-symmetric, then $S_{a,mI_s}u$ is also G-symmetric. The subdivision operator S_{a,mI_s} plays an important role in CAGD. Let Π_k denote the space of all polynomials of total degree at most k. In general, one requires that a subdivision scheme can reproduce some polynomial space Π_k for some integer k. In other words, the mask a satisfies the *sum rules* of order $k + 1$ ([3, 21]) with respect to the lattice $m\mathbb{Z}^s$, that is,

$$\sum_{\beta\in m\mathbb{Z}^s} a(\alpha + \beta)p(\alpha + \beta) = \sum_{\beta\in m\mathbb{Z}^s} a(\beta)p(\beta) \qquad \forall\ \alpha \in \mathbb{Z}^s, p \in \Pi_k. \tag{2.4}$$

We say that (a, mI_s, G) is an *interpolatory subdivision triplet* if (a, mI_s, G) is a subdivision triplet and a is an *interpolatory mask* with respect to the lattice $m\mathbb{Z}^s$, that is,

$$a(0) = 1 \quad \text{and} \quad a(m\beta) = 0 \qquad \forall\ \beta \in \mathbb{Z}^s\backslash\{0\}. \tag{2.5}$$

If (a, mI_s, G) is an interpolatory subdivision triplet, then $[S_{a,mI_s}u](m\beta) = u(\beta)$ for all $\beta \in \mathbb{Z}^s$ and $u \in \ell_0(\mathbb{Z}^s)$. The subdivision stencils are derived from a subdivision triplet (a, mI_s, G). Let $(f^0(\beta))_{\beta\in\mathbb{Z}^s}$ be an initial given data. Attaching the number $[S_{a,mI_s}f^0](\gamma)$ to the point $m^{-1}\gamma$, for the next level refined data f^1 on $m^{-1}\mathbb{Z}^s$, we have

$$f^1(m^{-1}\gamma) = [S_{a,mI_s}f^0](\gamma) = \sum_{\beta\in\mathbb{Z}^s} a(\gamma - m\beta)f^0(\beta) = \sum_{\beta\in\mathbb{Z}^s} a(-m(\beta - m^{-1}\gamma))f^0(\beta).$$

In order to compute the value of f^1 at the point $m^{-1}\gamma$, the stencil is given by $(a(\gamma-m\beta))_{\beta\in\mathbb{Z}^s}$; that is, $(a_*(\beta-m^{-1}\gamma))_{\beta\in\mathbb{Z}^s}$, where $a_*(\beta) := a(-m\beta), \beta \in m^{-1}\mathbb{Z}^s$.

In the rest of this section, we shall discuss convergence and smoothness properties of subdivision triplets, in particular, of interpolatory subdivision triplets. In order to do so, let us first introduce some notation. We denote by $\ell_p(\mathbb{Z}^s)$ the linear space of all sequences u on $\mathbb{Z}^s$ such that $\|u\|^p_{\ell_p(\mathbb{Z}^s)} := \sum_{\beta\in\mathbb{Z}^s} |u(\beta)|^p < \infty$.

For any $\alpha \in \mathbb{Z}^s$, let δ_α denote the sequence on $\mathbb{Z}^s$ such that $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for all $\beta \in \mathbb{Z}^s\backslash\{\alpha\}$. In particular, we denote $\delta := \delta_0$. The convolution of two sequences is defined to be

$$[u * v](\alpha) := \sum_{\beta\in\mathbb{Z}^s} u(\beta)v(\alpha-\beta), \qquad u, v \in \ell_0(\mathbb{Z}^s).$$

Clearly, $\widehat{u * v} = \hat{u}\hat{v}$. For a finitely supported sequence a on $\mathbb{Z}^s$, we define the following quantity:

$$\rho(a, mI_s, p, u) := \lim_{n\to\infty} \|u * [S^n_{a,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)}, \qquad 1 \leqslant p \leqslant \infty, u \in \ell_0(\mathbb{Z}^s). \tag{2.6}$$

For $\alpha \in \mathbb{Z}^s$ and $t \in \mathbb{R}^s$, we define

$$\nabla_\alpha v := v - v(\cdot - \alpha), \quad \nabla_t f := f - f(\cdot - t), \qquad v \in \ell_0(\mathbb{Z}^s), f \in L_p(\mathbb{R}^s). \tag{2.7}$$

Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, $|\mu| = \mu_1 + \cdots + \mu_s$ and $\nabla^\mu := \nabla^{\mu_1}_{e_1} \cdots \nabla^{\mu_s}_{e_s}$, where e_j is the jth coordinate unit vector in $\mathbb{R}^s$. Note that $\nabla^\mu v = [\nabla^\mu \delta] * v$ and

$$\nabla^\mu f = [\nabla^\mu\delta] * f := \sum_{\beta\in\mathbb{Z}^s} [\nabla^\mu\delta](\beta) f(\cdot - \beta).$$

The partial derivative of a differentiable function f with respect to the jth coordinate is denoted by $\partial_j f$. For $\mu = (\mu_1, \dots, \mu_s)$, we denote $\partial^\mu := \partial_1^{\mu_1} \cdots \partial_s^{\mu_s}$.

In the frequency domain, a mask a satisfies the sum rules of order $k+1$ with respect to the lattice $m\mathbb{Z}^s$ if and only if

$$\partial^\mu \hat{a}(2\pi\gamma/m) = 0 \qquad \forall\, |\mu| \leqslant k \quad \text{and} \quad \gamma \in [0, |m|-1]^s \cap \mathbb{Z}^s \backslash \{0\}.$$

If a mask a satisfies the sum rules of order k but not $k+1$, then we define (see [13, 14])

$$\nu_p(a, mI_s) := s/p - \log_{|m|} \max\{\rho(a, mI_s, p, \nabla^\mu\delta) \;:\; |\mu| = k\}, \quad 1 \leqslant p \leqslant \infty. \tag{2.8}$$

For a positive integer k, the B-spline function h_k of order k is defined to be $\chi_{[0,1]} * \cdots * \chi_{[0,1]}$ with k-copies of the characteristic function $\chi_{[0,1]}$ of the interval $[0,1]$. For any $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, the multivariate spline function h_μ of order μ is defined to be $h_\mu(x_1, \dots, x_s) := \prod_{j=1}^s h_{\mu_j}(x_j)$, $x_1, \dots, x_s \in \mathbb{R}$.

It is well known that for any $\nu = (\nu_1, \dots, \nu_s) \in \mathbb{N}_0^s$ such that $\nu_j \leqslant \mu_j$ for all $j = 1, \dots, s$, one has $\partial^\nu h_\mu = \nabla^\nu h_{\mu-\nu}$.

The following result is essentially known in the literature in various forms, and we shall provide a self-contained proof here. For simplicity, from now on we assume $m > 1$.

Theorem 2.1. *Let (a, mI_s, G) be a subdivision triplet and let ϕ denote its basis function. Then for any nonnegative integer k, the following statements are equivalent:*

(1) $\nu_\infty(a, mI_s) > k$.

(2) *For every compactly supported function $f \in C^k(\mathbb{R}^s)$ such that*

$$\hat{f}(0) = 1 \quad \textit{and} \quad \partial^\mu \hat{f}(2\pi\beta) = 0 \qquad \forall\, |\mu| \leqslant k, \beta \in \mathbb{Z}^s \backslash \{0\}, \tag{2.9}$$

the cascade sequence $Q^n_{a,mI_s} f$ is a Cauchy sequence in $C^k(\mathbb{R}^s)$ (as a matter of fact, one has $\lim_{n\to\infty} \|Q^n_{a,mI_s} f - \phi\|_{C^k(\mathbb{R}^s)} = 0$), where the cascade operator $Q_{a,mI_s} : C(\mathbb{R}^s) \mapsto C(\mathbb{R}^s)$ is defined to be

$$Q_{a,mI_s} f := \sum_{\beta\in\mathbb{Z}^s} a(\beta) f(m\cdot - \beta), \qquad f \in C(\mathbb{R}^s). \tag{2.10}$$

(3) *The basis function* $\phi \in C^k(\mathbb{R}^s)$ *and*

$$\lim_{n\to\infty} \|m^{kn}\nabla^\mu[S^n_{a,mI_s}\delta](\cdot) - [\partial^\mu\phi](m^{-n}\cdot)\|_{\ell_\infty(\mathbb{Z}^s)} = 0 \qquad \forall\, |\mu| = k. \tag{2.11}$$

(4) *For every sequence* $u \in \ell_\infty(\mathbb{Z}^s)$*, there exists a function* $g \in C^k(\mathbb{R}^s)$ *such that*

$$\lim_{n\to\infty} \|m^{n|\mu|}[\nabla^\mu S^n_{a,mI_s}u](\cdot) - [\partial^\mu g](m^{-n}\cdot)\|_{\ell_\infty(\mathbb{Z}^s)} = 0 \qquad \forall\, |\mu| \leqslant k. \tag{2.12}$$

Proof. The equivalence between (1) and (2) has been established in [2, 14] and many references therein. In the following, we show that (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and (3) $\Leftrightarrow$ (4).

Take $f = h_{(k+2,\dots,k+2)}$ to be the spline function of order $(k+2,\dots,k+2)$. Then $f \in C^k(\mathbb{R}^s)$, and (2.9) holds. By assumption in item (2), we must have $\phi \in C^k(\mathbb{R}^s)$.

Let $\mu = (\mu_1,\dots,\mu_s) \in \mathbb{N}_0^s$ such that $|\mu| = k$. Since

$$\partial^\mu f = \partial^\mu h_{(k+2,\dots,k+2)} = \nabla^\mu h_{(k+2-\mu_1,\dots,k+2-\mu_s)},$$

by induction on n, we have

$$\begin{aligned} f_{n,\mu} &:= \partial^\mu[Q^n_{a,mI_s}f] \\ &= \sum_{\beta\in\mathbb{Z}^s} m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\beta) h_{(k+2-\mu_1,\dots,k+2-\mu_s)}(m^n\cdot-\beta). \end{aligned} \tag{2.13}$$

Denote $h := h_{(2,\dots,2)}(\cdot - (1,\dots,1))$, which is the multivariate tensor product hat function such that $h(\beta) = \delta(\beta)$ for all $\beta \in \mathbb{Z}^s$. Define

$$g_{n,\mu} := \sum_{\beta\in\mathbb{Z}^s} m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\beta) h(m^n\cdot-\beta).$$

It follows directly from the above identity that

$$g_{n,\mu}(m^{-n}\alpha) = m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\alpha) \qquad \forall\, \alpha \in \mathbb{Z}^s,\ n \in \mathbb{N},$$

since $h(\beta) = \delta(\beta)$ for all $\beta \in \mathbb{Z}^s$. Consequently, for every $\alpha \in \mathbb{Z}^s$, we have

$$\begin{aligned} m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\alpha) - \partial^\mu\phi(m^{-n}\alpha) &= [g_{n,\mu}(m^{-n}\alpha) - f_{n,\mu}(m^{-n}\alpha)] \\ &\quad + [f_{n,\mu}(m^{-n}\alpha) - \partial^\mu\phi(m^{-n}\alpha)]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\|m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\cdot) - [\partial^\mu\phi](m^{-n}\cdot)\|_{\ell_\infty(\mathbb{Z}^s)} \\ &\qquad \leqslant \|g_{n,\mu} - f_{n,\mu}\|_{L_\infty(\mathbb{R}^s)} + \|f_{n,\mu} - \partial^\mu\phi\|_{L_\infty(\mathbb{R}^s)} \qquad \forall\, n \in \mathbb{N}. \end{aligned}$$

By assumption in item (2), we have $\lim_{n\to\infty}\|f_{n,\mu} - \partial^\mu\phi\|_{L_\infty(\mathbb{R}^s)} = 0$. In order to show (2) $\Rightarrow$ (3), it now suffices to show that $\lim_{n\to\infty}\|g_{n,\mu} - f_{n,\mu}\|_{L_\infty(\mathbb{R}^s)} = 0$.

Let $\eta := h - h_{(k+2-\mu_1,\dots,k+2-\mu_s)}$. It is simple to verify that $\hat{\eta}(2\pi\beta) = 0$ for all $\beta \in \mathbb{Z}^s$; that is, $\sum_{\beta\in\mathbb{Z}^s}\eta(\cdot+\beta) = 0$. By [14, Theorem 3.6], there exist compactly supported functions $\eta_j \in L_\infty(\mathbb{R}^s)$, $j = 1,\dots,s$, such that $\eta = \sum_{j=1}^s \nabla_{e_j}\eta_j$. By the definition of $f_{n,\mu}$ and $g_{n,\mu}$, we have

$$g_{n,\mu} - f_{n,\mu} = m^{kn}\sum_{j=1}^s\sum_{\beta\in\mathbb{Z}^s}[\nabla_{e_j}\nabla^\mu S^n_{a,mI_s}\delta](\beta)\eta_j(m^n\cdot-\beta).$$

Since all $\eta_j, j = 1, \dots, s$, are compactly supported functions in $L_\infty(\mathbb{R}^s)$, there exists a positive constant C, depending only on $\eta_j, j = 1, \dots, s$, such that

$$\|g_{n,\mu} - f_{n,\mu}\|_{L_\infty(\mathbb{R}^s)} \leqslant Cm^{kn} \sum_{j=1}^{s} \|\nabla_{e_j} \nabla^\mu S^n_{a,mI_s}\delta\|_{\ell_\infty(\mathbb{Z}^s)} \qquad \forall\, n \in \mathbb{N}.$$

Note that $\nabla_{e_j}\nabla^\mu = \nabla^{\mu+e_j}$ and $\nabla^{\mu+e_j} S^n_{a,mI_s}\delta = [\nabla^{\mu+e_j}\delta] * S^n_{a,mI_s}\delta$. Since (1) $\Leftrightarrow$ (2), by $\nu_\infty(a, mI_s) > k$ and $|\mu + e_j| = k+1$, it follows from [14, Theorem 4.3] that

$$\lim_{n\to\infty} m^{kn} \|\nabla^{\mu+e_j} S^n_{a,mI_s}\delta\|_{\ell_\infty(\mathbb{Z}^s)} = 0.$$

Therefore, we must have $\lim_{n\to\infty} \|g_{n,\mu} - f_{n,\mu}\|_{L_\infty(\mathbb{R}^s)} = 0$, which completes the proof of (2) $\Rightarrow$ (3).

Now we demonstrate that (3) $\Rightarrow$ (1). Let $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$ such that $|\mu| = k$. Then (2.11) holds and $\phi \in C^k(\mathbb{R}^s)$. Note that

$$\begin{aligned} m^{kn}[\nabla_{e_j}\nabla^\mu S^n_{a,mI_s}\delta](\alpha) = &\left\{ m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\alpha) - \partial^\mu \phi(m^{-n}\alpha) \right\} \\ &- \left\{ m^{kn}[\nabla^\mu S^n_{a,mI_s}\delta](\alpha - e_j) - \partial^\mu\phi(m^{-n}(\alpha - e_j)) \right\} \\ &+ \left\{ \partial^\mu\phi(m^{-n}\alpha) - \partial^\mu\phi(m^{-n}\alpha - m^{-n}e_j) \right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} m^{kn}\|\nabla^{\mu+e_j} S^n_{a,mI_s}\delta\|_{\ell_\infty(\mathbb{Z}^s)} \leqslant\ & 2\|m^{kn}\nabla^\mu S^n_{a,mI_s}\delta - [\partial^\mu\phi](m^{-n}\cdot)\|_{\ell_\infty(\mathbb{Z}^s)} \\ &+ \|\partial^\mu\phi - \partial^\mu\phi(\cdot - m^{-n}e_j)\|_{L_\infty(\mathbb{R}^s)}. \end{aligned}$$

Since $\phi \in C^k(\mathbb{R}^s)$ and $|\mu| = k$, $\partial^\mu\phi \in C(\mathbb{R}^s)$, and so,

$$\lim_{n\to\infty} \|\partial^\mu\phi - \partial^\mu\phi(\cdot - m^{-n}e_j)\|_{L_\infty(\mathbb{R}^s)} = 0.$$

By (2.11), it follows from the above inequality that

$$\lim_{n\to\infty} m^{kn}\|\nabla^{\mu+e_j} S^n_{a,mI_s}\delta\|_{\ell_\infty(\mathbb{Z}^s)} = 0 \qquad \forall\, |\mu| = k,\ j = 1, \dots, s.$$

That is, $\lim_{n\to\infty} m^{kn}\|\nabla^\mu S^n_{a,mI_s}\delta\|_{\ell_\infty(\mathbb{Z}^s)} = 0$ for all $|\mu| = k+1$. Now by [14, Theorem 4.3], we conclude that $\nu_\infty(a, mI_s) > k$. So, (3) $\Rightarrow$ (1).

Obviously, (4) $\Rightarrow$ (3) by taking $u = \delta$ and $g = \phi$. Suppose that (3) holds. Then (2.11) holds for all $|\mu| \leqslant k$ since (1) implies that $\nu(a, mI_s) > j$ for all $j = 0, \dots, k$. Let $u \in \ell_\infty(\mathbb{Z}^s)$ and $g = u * \phi = \sum_{\beta\in\mathbb{Z}^s} u(\beta)\phi(\cdot - \beta)$. By a simple calculation, we observe that

$$[\nabla^\mu S^n_{a,mI_s}u](\alpha) = \sum_{\beta\in\mathbb{Z}^s} u(\beta)[\nabla^\mu S^n_{a,mI_s}\delta](\alpha - m^n\beta).$$

Therefore, we deduce that

$$\begin{aligned} &m^{kn}[\nabla^\mu S^n_{a,mI_s}u](\alpha) - \partial^\mu g(m^{-n}\alpha) \\ &\quad = \sum_{\beta\in\mathbb{Z}^s} u(\beta)[m^{kn}\nabla^\mu S^n_{a,mI_s}\delta(\alpha - m^n\beta) - \partial^\mu\phi(m^{-n}(\alpha - m^n\beta))]. \end{aligned}$$

Since both the mask a and the basis function ϕ are compactly supported, now (2.12) follows easily from the above identity. $\square$

For $0<\nu\leqslant 1$ and a function $f\in L_p(\mathbb{R}^s)$, we say that f belongs to the *Lipschitz space* $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ if there exists a positive constant C such that

$$\|f-f(\cdot-t)\|_{L_p(\mathbb{R}^s)}\leqslant C\|t\|^{\nu}$$

for all $t\in\mathbb{R}^s$. The L_p smoothness of a function $f\in L_p(\mathbb{R}^s)$ is measured by its L_p *critical smoothness exponent* $\nu_p(f)$ which is defined by

$$\nu_p(f):=\sup\{n+\nu \quad : \quad \partial^{\mu}f\in \mathrm{Lip}(\nu, L_p(\mathbb{R}^s)) \quad \forall\, |\mu|=n\}.$$

For the basis function ϕ of any subdivision triplet (a, mI_s, G), one always has $\nu_p(\phi)\geqslant \nu_p(a, mI_s)$. For more detail on L_p smoothness of refinable functions, see [2, 3, 9, 14, 22, 25] and many references therein. A function f is an *interpolatory function* if f is a continuous function such that $f(\beta)=\delta(\beta)$ for all $\beta\in\mathbb{Z}^s$.

The following result is known in the literature (e.g., see [13, 14, 16]).

Theorem 2.2. *Let (a, mI_s, G) be an interpolatory subdivision triplet and ϕ denote its basis function. Then ϕ is an interpolatory function if and only if $\nu_\infty(a, mI_s)>0$. Moreover, if $\nu_\infty(a, mI_s)>0$, then $\nu_p(\phi)=\nu_p(a, mI_s)$ for all $1\leqslant p\leqslant\infty$.*

The quantity $\rho(a, mI_s, p, u)$ in (2.6) can be rewritten using the ℓ_p-norm joint spectral radius. Let $\mathcal{T}$ be a finite collection of linear operators acting on a finite-dimensional normed vector space V. For a positive integer n, we define for $1\leqslant p<\infty$,

$$\|\mathcal{T}^n\|_p^p:=\sum_{T_1,\ldots,T_n\in\mathcal{T}}\|T_1\cdots T_n\|^p$$

and

$$\|\mathcal{T}^n\|_\infty:=\max\{\|T_1\cdots T_n\| \ : \ T_1,\ldots,T_n\in\mathcal{T}\},$$

where $\|\cdot\|$ denotes some operator norm. For $1\leqslant p\leqslant\infty$, the *$\ell_p$-norm joint spectral radius* of $\mathcal{T}$ is defined to be (see [8, 16, 20, 22, 28] and references therein)

$$\rho_p(\mathcal{T}):=\lim_{n\to\infty}\|\mathcal{T}^n\|_p^{1/n}=\inf_{n\geqslant 1}\|\mathcal{T}^n\|_p^{1/n}. \tag{2.14}$$

Let $\Gamma:=[0, |m|-1]^s\cap\mathbb{Z}^s$. To relate the quantity $\rho(a, mI_s, p, u)$ to the ℓ_p-norm joint spectral radius, we introduce the linear operators $T_{a,\gamma}, \gamma\in\Gamma$ on $\ell_0(\mathbb{Z}^s)$ by

$$T_{a,\gamma}v(\alpha):=\sum_{\beta\in\mathbb{Z}^s}a(m\alpha-\beta+\gamma)v(\beta), \qquad v\in\ell_0(\mathbb{Z}^s), \alpha\in\mathbb{Z}^s. \tag{2.15}$$

It was proved in [16, Lemma 2.3] that if a is finitely supported, then for any finitely supported sequence u on $\mathbb{Z}^s$, there exists a finite-dimensional subspace $V(u)$ of $\ell_0(\mathbb{Z}^s)$ such that $V(u)$ contains u and $V(u)$ is the smallest subspace of $\ell_0(\mathbb{Z}^s)$ which is invariant under the operators $T_{a,\gamma}$, $\gamma\in\Gamma$. We call such $V(u)$ the minimal $\{T_{a,\gamma} \ : \ \gamma\in\Gamma\}$-invariant subspace generated by u.

Let $\mathcal{T}:=\{T_{a,\gamma}|_{V(u)} \ : \ \gamma\in\Gamma\}$, where $V(u)$ is the minimal $\{T_{a,\gamma} \ : \ \gamma\in\Gamma\}$-invariant subspace generated by u. Then it is known ([16]) that

$$\rho(a, mI_s, p, u)=\lim_{n\to\infty}\|u*[S^n_{a,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)}=\rho_p(\mathcal{T})=\inf_{n\geqslant 1}\|\mathcal{T}^n\|_p^{1/n}. \tag{2.16}$$

For a sequence $c\in\ell_0(\mathbb{Z}^s)$ and a nonnegative integer m, we define a new sequence $c(m^{-1}\cdot)$ on $\mathbb{Z}^s$ by $\widehat{c(m^{-1}\cdot)}(\xi):=\hat{c}(m\xi)$.

The following result is useful in calculating the quantity $\rho(a, mI_s, p, u)$ in (2.6) and appeared in [1, 8] for the special case $c=\nabla^{\mu}\delta$ and $m=2$.

Theorem 2.3. *Let a be a finitely supported mask on $\mathbb{Z}^s$. If*

$$\hat{a}(\xi) = \frac{\hat{c}(m\xi)}{\hat{c}(\xi)}\hat{b}(\xi) \quad \text{for finitely supported sequences } b \text{ and } c \text{ on } \mathbb{Z}^s, \tag{2.17}$$

such that $\hat{c}(m\xi)/\hat{c}(\xi)$ is a 2π-periodic trigonometric polynomial, then for any $1 \leqslant p \leqslant \infty$ and $u \in \ell_0(\mathbb{Z}^s)$,

$$\begin{aligned} \rho(a, mI_s, p, u * c) &:= \lim_{n\to\infty} \|u * c * [S^n_{a,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)} \\ &= \lim_{n\to\infty} \|u * [S^n_{b,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)} =: \rho(b, mI_s, p, u). \end{aligned} \tag{2.18}$$

Proof. By induction and (2.17), we have $c * \widehat{S^n_{a,mI_s}}\delta(\xi) = \hat{c}(m^n\xi)\widehat{S^n_{b,mI_s}}\delta(\xi)$. That is, we have

$$u * c * [S^n_{a,mI_s}\delta] = c(m^{-n}\cdot) * (u * [S^n_{b,mI_s}\delta]). \tag{2.19}$$

Consequently, by Young's inequality and $\|c(m^{-n}\cdot)\|_{\ell_1(\mathbb{Z}^s)} = \|c\|_{\ell_1(\mathbb{Z}^s)}$, we have

$$\begin{aligned} \|u * c * [S^n_{a,mI_s}\delta]\|_{\ell_p(\mathbb{Z}^s)} &= \|u * c(m^{-n}\cdot) * [S^n_{b,mI_s}\delta]\|_{\ell_p(\mathbb{Z}^s)} \\ &\leqslant \|c\|_{\ell_1(\mathbb{Z}^s)}\|u * [S^n_{b,mI_s}\delta]\|_{\ell_p(\mathbb{Z}^s)} \qquad \forall\, n \in \mathbb{N}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \rho(a, mI_s, p, u * c) &:= \lim_{n\to\infty} \|u * c * [S^n_{a,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)} \\ &\leqslant \lim_{n\to\infty} \|u * [S^n_{b,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)} =: \rho(b, mI_s, p, u). \end{aligned} \tag{2.20}$$

Since b and u are finitely supported sequences on $\mathbb{Z}^s$, there exists a positive integer N such that the support of the sequence $u * [S^n_{b,MmI_s}\delta]$ is contained in the set $[-m^{n+N}, m^{n+N}]^s$ for all $n \in \mathbb{N}$.

Define

$$\hat{d}(\xi) := \hat{c}(m^{n+N+1}\xi)/\hat{c}(m^n\xi) = \prod_{j=0}^{N}[\hat{c}(m^{n+j+1}\xi)/\hat{c}(m^{n+j}\xi)].$$

Since $\hat{c}(m\xi)/\hat{c}(\xi)$ is a 2π-periodic trigonometric polynomial, so is $\hat{d}$, and therefore, d is a finitely supported sequence.

By (2.19), we have

$$\hat{u}(\xi)\hat{c}(\xi)\widehat{S^n_{a,mI_s}}\delta(\xi)\hat{d}(\xi) = \hat{u}(\xi)\hat{c}(m^n\xi)\widehat{S^n_{b,mI_s}}\delta(\xi)\hat{d}(\xi) = \hat{c}(m^{n+N}\xi)\hat{u}(\xi)\widehat{S^n_{b,mI_s}}\delta(\xi).$$

That is,

$$c(m^{-n-N-1}\cdot) * [u * S^n_{b,mI_s}\delta] = [u * c * S^n_{a,mI_s}\delta] * d.$$

Since the sequence $u * S^n_{b,mI_s}\delta$ vanishes outside the set $[-m^{n+N}, m^{n+N}]^s$, we conclude from the above identity that

$$\begin{aligned} \|c\|_{\ell_p(\mathbb{Z}^s)}\|u * S^n_{b,mI_s}\delta\|_{\ell_p(\mathbb{Z}^s)} &= \|c(m^{-n-N-1}\cdot) * [u * S^n_{b,mI_s}\delta]\|_{\ell_p(\mathbb{Z}^s)} \\ &\leqslant \|d\|_{\ell_1(\mathbb{Z}^s)}\|u * c * S^n_{a,mI_s}\delta\|_{\ell_p(\mathbb{Z}^s)}. \end{aligned}$$

The above inequality yields that

$$\begin{aligned} \rho(b, mI_s, p, u) &:= \lim_{n\to\infty} \|u * [S^n_{b,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)} \\ &\leqslant \lim_{n\to\infty} \|u * c * [S^n_{a,mI_s}\delta]\|^{1/n}_{\ell_p(\mathbb{Z}^s)} =: \rho(a, mI_s, p, u * c). \end{aligned} \tag{2.21}$$

Now putting (2.20) and (2.21) together, we conclude that (2.18) is true. $\square$

The following result provides a convenient way for estimating the quantity $\rho(b, mI_s, \infty, \delta)$.

Theorem 2.4 (see also [8])**.** *Let b be a finitely supported sequence on $\mathbb{Z}^s$. Then*

$$\begin{aligned}\rho(b, mI_s, \infty, \delta) &:= \lim_{n\to\infty} \|S^n_{b,mI_s}\delta\|^{1/n}_{\ell_\infty(\mathbb{Z}^s)} \\ &= \inf_{n\in\mathbb{N}} \Big(\max_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} |S^n_{b,mI_s}\delta(\alpha + m^n\beta)|\Big)^{1/n}.\end{aligned} \tag{2.22}$$

Proof. Denote $b_n := S^n_{b,mI_s}\delta$ and $\rho_n := \max_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} |b_n(\alpha + m^n\beta)|$. By induction, we have $b_{j+k} = b_j * [b_k(m^{-k}\cdot)]$ and

$$b_{j+k}(\alpha + m^{j+k}\beta) = \sum_{\gamma\in\Gamma_{m^k}} \sum_{\eta\in\mathbb{Z}^s} b_j(\alpha - m^j(m^k\eta + \gamma))b_k(\gamma + m^k(\beta + \eta)),$$

where $\Gamma_k := [0, |m|^k - 1]^s \cap \mathbb{Z}^s$. Hence,

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^s} |b_{j+k}(\alpha + m^{j+k}\beta)| &\leqslant \sum_{\gamma\in\Gamma_k}\sum_{\eta\in\mathbb{Z}^s} |b_j(\alpha - m^j(m^k\eta + \gamma))| \sum_{\beta\in\mathbb{Z}^s} |b_k(\gamma + m^k(\beta + \eta))| \\ &\leqslant \rho_j\rho_k.\end{aligned}$$

So, we conclude that $\rho_{j+k} \leqslant \rho_j\rho_k$ for all $j, k \in \mathbb{N}$, which implies $\lim_{n\to\infty} \rho_n^{1/n} = \inf_{n\in\mathbb{N}} \rho_n^{1/n}$.

Since the sequence b is finitely supported, there must exist a positive constant C depending only on the support of b such that the number of elements in the set $\{\beta \in \mathbb{Z}^s \,:\, b_n(\alpha + m^n\beta) \neq 0\}$ is no more than C. Therefore, we see that

$$\|b_n\|_{\ell_\infty(\mathbb{Z}^s)} \leqslant \rho_n = \max_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} |b_n(\alpha + m^n\beta)| \leqslant C\|b_n\|_{\ell_\infty(\mathbb{Z}^s)} \qquad \forall\, n \in \mathbb{N}.$$

So, (2.22) must hold. □

3. Optimal one-dimensional interpolatory ternary subdivision schemes and the projection method

In this section, we shall investigate optimal one-dimensional interpolatory ternary subdivision triplets with two-ring stencils. Then we shall discuss the projection method and the optimal multidimensional interpolatory ternary subdivision schemes with one-ring or two-ring stencils.

For one-dimensional interpolatory ternary subdivision schemes with two-ring stencils, we have the following result.

Theorem 3.1. *Let $(a, 3, \{-1, 1\})$ be a one-dimensional interpolatory ternary subdivision triplet such that the real-valued mask a is supported on $[-5, 5]$ (that is, all its subdivision stencils have two-ring neighboring vertices). Then $\nu_\infty(a, 3) \leqslant \log_3 11$. Moreover, $\nu_\infty(a, 3) = \log_3 11$ if and only if a is the unique mask a^{best} given by*

$$\widehat{a^{best}}(\xi) = \frac{1}{99}(e^{-i\xi} + 1 + e^{i\xi})^3[9 + 10\cos(\xi) - 8\cos(2\xi)], \tag{3.1}$$

or equivalently, the mask a^{best} is supported on $[-5, 5]$ and given by

$$\Big[-\frac{4}{99},\ -\frac{7}{99},\ 0,\ \frac{34}{99},\ \frac{76}{99},\ \mathbf{1},\ \frac{76}{99},\ \frac{34}{99},\ 0,\ -\frac{7}{99},\ -\frac{4}{99}\Big].$$

Proof. Suppose that $\nu_\infty(a,3) > \log_3 11 \approx 2.18266$. Then the mask a must satisfy the sum rules of order 3 with respect to the lattice $3\mathbb{Z}$ (see [14, Theorem 4.3]). Solving the system of linear equations given by (2.4) with $k = 2$ for a $\{-1,1\}$-symmetric interpolatory mask a with support $[-5,5]$, we see that the interpolatory mask a must take the following form:

$$\hat{a}(\xi) = (e^{-i\xi} + 1 + e^{i\xi})^3 \hat{b}(\xi) \tag{3.2}$$

with

$$\hat{b}(\xi) := t\, e^{-2i\xi} - (4t+1/9)e^{-i\xi} + (6t+1/3) - (4t+1/9)e^{i\xi} + te^{2i\xi}, \qquad t \in \mathbb{R}. \tag{3.3}$$

Let $T_{b,\gamma}(\gamma = -1,0,1)$ be the linear operators defined in (2.15) with $m = 3$. It is easy to see that the linear space $\ell([-1,1])$ is $T_{b,\gamma}$-invariant for $\gamma = -1,0,1$. Their matrix representations H_γ, under the standard basis $\{\delta_{-1},\delta_0,\delta_1\}$ of $\ell([-1,1])$, are $H_\gamma = (b(3k - j + \gamma))_{-1\leqslant j,k\leqslant 1}$ for $\gamma = -1,0,1$. So, $\rho(b,3,\infty,\delta) = \rho_\infty(\{H_{-1},H_0,H_1\})$, where

$$\begin{aligned} H_{-1} &= \begin{bmatrix} 0 & 6t+1/3 & 0 \\ 0 & -4t-1/9 & t \\ 0 & t & -4t-1/9 \end{bmatrix}, \qquad H_0 = \begin{bmatrix} t & -4t-1/9 & 0 \\ 0 & 6t+1/3 & 0 \\ 0 & -4t-1/9 & t \end{bmatrix}, \\ H_1 &= \begin{bmatrix} -4t-1/9 & t & 0 \\ t & -4t-1/9 & 0 \\ 0 & 6t+1/3 & 0 \end{bmatrix}. \end{aligned} \tag{3.4}$$

It is easy to see that $6t + 1/3$ and $-5t - 1/9$ are eigenvalues of H_0 and H_1, respectively. Consequently,

$$\begin{aligned} \rho(b,3,\infty,\delta) = \rho_\infty(\{H_{-1},H_0,H_1\}) &\geqslant \max\{\rho(H_0),\rho(H_1)\} \\ &\geqslant \max\{|6t+1/3|, |5t+1/9|\} \geqslant 1/11, \end{aligned}$$

where the equal sign in the last inequality holds if and only if $t = -4/99$. By Theorem 2.3, we have $\rho(a,3,\infty,\nabla^3_{e_1}\delta) = \rho(b,3,\infty,\delta)$. Therefore, we conclude that

$$\nu_\infty(a,3) = -\log_3 \rho(a,3,\infty,\nabla^3_{e_1}\delta) = -\log_3 \rho(b,3,\infty,\delta) \leqslant \log_3 11.$$

On the other hand, by Theorem 2.4, we have

$$\rho(b,3,\infty,\delta) \leqslant \max_{\alpha\in\mathbb{Z}} \sum_{\beta\in\mathbb{Z}} |b(\alpha+3\beta)| \leqslant \max\{|6t+1/3|, |4t+1/9|+|t|\}.$$

Therefore, we have

$$\max\{|6t+1/3|, |5t+1/9|\} \leqslant \rho(b,3,\infty,\delta) \le \max\{|6t+1/3|, |4t+1/9|+|t|\}. \tag{3.5}$$

When $t = -4/99$, the above inequalities yield that $\rho(b,3,\infty,\delta) = 1/11$. Therefore, we conclude that $\nu_\infty(a^{best},3) = -\log_3 \rho(b,3,\infty,\delta) = \log_3 11$. □

More precisely, we have the following result.

Corollary 3.2. *Let $(a,3,\{-1,1\})$ be an interpolatory subdivision triplet such that the real-valued mask a is supported on $[-5,5]$ and satisfies the sum rules of order 3. Then the mask a must be given by (3.2) and (3.3). Moreover, we have*

$$\nu_\infty(a,3) = \begin{cases} -\log_3(-5t-1/9), & \text{if } t \leqslant -4/99, \\ -\log_3(6t+1/3), & \text{if } t > -4/99. \end{cases}$$

In particular, the subdivision triplet is C^2 if and only if $-2/45 < t < -1/27$.

Proof. By the proof of Theorem 3.1, we see that (3.5) holds. By a simple calculation, we observe that

$$\max\{|6t+1/3|, |5t+1/9|\} = \max\{|6t+1/3|, |4t+1/9|+|t|\} = \begin{cases} -5t-1/9, & \text{if } t \leqslant -4/99, \\ 6t+1/3, & \text{if } t > -4/99. \end{cases}$$

So, the claim follows directly from (3.5) and $\nu_\infty(a,3) = -\log_3 \rho(b,3,\infty,\delta)$. □

For a sequence a on $\mathbb{Z}^s$, we define a new sequence Pa via the projection operator $P : \ell_0(\mathbb{Z}^s) \mapsto \ell_0(\mathbb{Z})$ (see [11, 12]) as follows:

$$[Pa](j) := \sum_{\beta \in \mathbb{Z}^{s-1}} a(j,\beta), \qquad j \in \mathbb{Z}. \tag{3.6}$$

Now we have the following result on optimal multidimensional interpolatory ternary subdivision triplets with two-ring stencils.

Theorem 3.3. *Let $(a, 3I_s, \{I_s, -I_s\})$ be an interpolatory subdivision triplet such that the real-valued mask a is supported on $[-5,5]^s$. Then $\nu_\infty(a,3) \leqslant \log_3 11$. Moreover, if $\nu_\infty(a,3I_s) = \log_3 11$, then the projected mask $3^{1-s}Pa$ must be the unique mask a^{best} defined in* (3.1).

Proof. Suppose that $\nu_\infty(a,3I_s) > \log_3 11$. Then a must satisfy the sum rules of order at least 3. Let Pa be the one-dimensional sequence defined in (3.6). Then by [11, Lemma 2.1] or [12, Theorem 3.2], Pa must satisfy the sum rules of order at least 3. Moreover, since $a(-\beta) = a(\beta)$ for all $\beta \in \mathbb{Z}^s$, it is easy to see that $[Pa](-j) = [Pa](j)$ for all $j \in \mathbb{Z}$. Therefore, $(3^{1-s}Pa, 3, \{1,-1\})$ is a subdivision triplet.

Since a is an interpolatory mask such that a is supported on $[-5,5]^s$ and satisfies the sum rules of order 3, by [11, Theorem 3.2] we see that $3^{1-s}Pa$ must be an interpolatory mask; that is, $(3^{1-s}Pa, 3, \{1,-1\})$ is an interpolatory subdivision triplet and Pa is supported on $[-5,5]$. Now by [11, Theorem 2.5] and Theorem 3.1, we must have $\nu_\infty(a,3I_s) \le \nu_\infty(3^{1-s}Pa,3) \leqslant \log_3 11$. When $\nu_\infty(a,3I_s) = \log_3 11$, we must have $\nu_\infty(3^{1-s}Pa,3) = \log_3 11$. Therefore, by the uniqueness of the mask a^{best} in (3.1), we conclude that $3^{1-s}Pa = a^{best}$. □

Since subdivision schemes with one-ring stencils are of particular interest in CAGD, we have the following result on interpolatory subdivision schemes with one-ring stencils.

Theorem 3.4. *Let m be an integer satisfying $|m| > 1$ and let $(a, mI_s, \{I_s\})$ be an interpolatory subdivision triplet.*

(i) *If a is supported on $[1-|m|, |m|-1]^s$, then $\nu_\infty(a, mI_s) \leqslant 1$.*
(ii) *If a is supported on $[-|m|, |m|]^s$ (that is, the subdivision scheme has one-ring stencils), then a can satisfy the sum rules of order at most 2 with respect to the lattice $m\mathbb{Z}^s$, and therefore $\nu_\infty(a, mI_s) \leqslant 2$.*

Proof. Suppose that a is supported on $[1-|m|, |m|-1]^s$ and $\nu_\infty(a, mI_s) > 1$. Then a must satisfy the sum rules of order at least 2. Consequently, by [12, Theorem 3.2], the projected mask $|m|^{1-s}Pa$ must satisfy the sum rules of order at least 2. Since a is supported on $[1-|m|, |m|-1]^s$, then $|m|^{1-s}Pa$ is a univariate mask supported on $[1-|m|, |m|-1]$ and satisfying the sum rules of order 2; therefore, we must have

$|m|^{1-s}\widehat{Pa}(\xi) = |1+e^{-i\xi}+\cdots+e^{-i(|m|-1)\xi}|^2/|m|$. This gives us $\nu_\infty(|m|^{1-s}Pa, m) = 1$. Now by [11, Theorem 2.5], we conclude that $\nu_\infty(a, mI_s) \leqslant \nu_\infty(|m|^{1-s}Pa, m) = 1$. So, (i) holds.

It is easy to see that a univariate interpolatory mask a with support $[-|m|, |m|]$ must be supported on $[1-|m|, |m|-1]$. Therefore, a can satisfy the sum rules of order at most 2 with respect to the lattice $m\mathbb{Z}$. Now by a similar argument as in [17, Theorem 2.2] or [12, Theorem 3.2], we see that any interpolatory mask a with support $[-|m|, |m|]^s$ can satisfy the sum rules of order at most 2 with respect to the lattice $m\mathbb{Z}^s$. Consequently, we have $\nu_\infty(a, mI_s) \leqslant 2$. So, there is no C^2 interpolatory m-adic subdivision scheme with one-ring stencils. $\square$

In fact, it has been proved in [10, Theorem 2.7] that there is no C^1 interpolatory dyadic subdivision scheme for any subdivision triplet $(a, 2I_2, D_4)$ such that a is supported inside $[-2, 2]^2$. By a similar complicated argument as in [10, Theorem 2.7], we conjecture that there is no C^1 interpolatory m-adic subdivision scheme such that its interpolatory mask is supported inside the set $[-|m|, |m|]^s$.

4. Optimal C^2 two-dimensional interpolatory ternary subdivision schemes with two-ring stencils

In this section, we shall present some examples of interpolatory subdivision triplets $(a, 3I_2, D_4)$ and $(a, 3I_2, D_6)$ such that their masks a are supported on $[-5, 5]^2$ and $\nu_\infty(a, 3I_2) = \log_3 11$. Therefore, they are the smoothest interpolatory ternary subdivision schemes with two-ring stencils, according to Theorem 3.3.

Let us first consider subdivision triplets $(a, 3I_2, D_6)$ for the regular triangular mesh. In order to facilitate our analysis, we require that the mask a should take the following form:

$$\hat{a}(\xi_1, \xi_2) = (e^{-i\xi_1}+1+e^{i\xi_1})(e^{-i\xi_2}+1+e^{i\xi_2})(e^{-i(\xi_1+\xi_2)}+1+e^{i(\xi_1+\xi_2)})\hat{b}(\xi_1, \xi_2), \tag{4.1}$$

where the sequence b is supported on $[-3, 3]^2$ and is D_6-symmetric. Consider a system of linear equations, which are induced by the following requirements:

(i) The mask a is interpolatory with respect to the lattice $3\mathbb{Z}^2$.
(ii) The mask a satisfies the sum rules of order 3 with respect to the lattice $3\mathbb{Z}^2$.
(iii) The projected mask $3^{-1}Pa$ must be the unique mask a^{best} in (3.1).

Solving the system of linear equations, we see that the sequence b, which is supported on $[-3, 3]^2$ and is D_6-symmetric, must take the following form:

$$\frac{1}{99}\begin{bmatrix} 0 & 0 & 0 & t_2 & -2-t_2 & -2-t_2 & t_2 \\ 0 & 0 & -2-t_2 & t_1 & 5-2t_1+2t_2 & t_1 & -2-t_2 \\ 0 & -2-t_2 & 5-2t_1+2t_2 & 2+2t_1-t_2 & 2+2t_1-t_2 & 5-2t_1+2t_2 & -2-t_2 \\ t_2 & t_1 & 2+2t_1-t_2 & 15-6t_1 & 2+2t_1-t_2 & t_1 & t_2 \\ -2-t_2 & 5-2t_1+2t_2 & 2+2t_1-t_2 & 2+2t_1-t_2 & 5-2t_1+2t_2 & -2-t_2 & 0 \\ -2-t_2 & t_1 & 5-2t_1+2t_2 & t_1 & -2-t_2 & 0 & 0 \\ t_2 & -2-t_2 & -2-t_2 & t_2 & 0 & 0 & 0 \end{bmatrix}. \tag{4.2}$$

Now we have the following result on subdivision triplets $(a, 3I_2, D_6)$ with two-ring stencils.

Theorem 4.1. *Let $(a, 3I_2, D_6)$ be an interpolatory subdivision triplet, where the mask a is given by (4.1) and the sequence b is given in (4.2). Then*

$$\nu_\infty(a, 3I_2) = -\log_3 \max\{1/11, \rho(b, 3I_2, \infty, \delta)\}.$$

In particular, $\nu_\infty(a, 3I_2) = \log_3 11$ *if and only if* $\rho(b, 3I_2, \infty, \delta) \leqslant 1/11$. *For example, if* $t_1 = 1$ *and* $t_2 = 0$*, then* $\rho(b, 3I_2, \infty, \delta) \leqslant 1/11$ *and* $\nu_\infty(a, 3I_2) = \log_3 11$. *Therefore, the subdivision triplet is the smoothest two-dimensional interpolatory ternary subdivision scheme with two-ring stencils for the regular triangular mesh.*

Proof. Since a satisfies the sum rules of order 3, in order to calculate $\nu_\infty(a, 3I_2)$, we have to calculate

$$\begin{aligned}\rho(a, 3I_2, \infty, \nabla^\mu\delta) &:= \lim_{n\to\infty} \|\nabla^\mu\delta * [S^n_{a,3I_2}\delta]\|_{\ell_\infty(\mathbb{Z}^2)},\\ \mu &= (3,0), (2,1), (1,2), (0,3).\end{aligned} \tag{4.3}$$

Since the mask a is D_6-symmetric, the sequence $S^n_{a,3I_2}\delta$ is also D_6-symmetric, and it is easy to see that

$$\rho(a, 3I_2, \infty, \nabla^3_{e_2}\delta) = \rho(a, 3I_2, \infty, \nabla^3_{e_1}\delta)$$

and

$$\rho(a, 3I_2, \infty, \nabla_{e_1}\nabla^2_{e_2}\delta) = \rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_2}\delta).$$

So it suffices to calculate $\rho(a, 3I_2, \infty, \nabla^3_{e_1}\delta)$ and $\rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_2}\delta)$. Note that

$$\nabla_{e_1}\delta = \delta - \delta_{e_1} = [\delta_{-e_2} - \delta_{e_1}] + [\delta - \delta_{-e_2}] = [\nabla_{e_1+e_2}\delta](\cdot + e_2) - [\nabla_{e_2}\delta](\cdot + e_2). \tag{4.4}$$

For any $\beta \in \mathbb{Z}^s$, it is easy to see that $u(\cdot - \beta) * [S^n_{a,M}\delta] = (u * [S^n_{a,M}\delta])(\cdot - \beta)$. Therefore,

$$\rho(a, M, p, u(\cdot - \beta)) = \rho(a, M, p, u) \qquad \forall\, \beta \in \mathbb{Z}^s.$$

Consequently, by the definition of $\rho(a, 3I_2, \infty, \nabla^\mu\delta)$ in (4.3), it follows directly from (4.4) that

$$\begin{aligned}&\rho(a, 3I_2, \infty, \nabla^3_{e_1}\delta)\\ &\qquad \leqslant \max\{\rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_1+e_2}\delta), \rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_2}\delta)\}.\end{aligned} \tag{4.5}$$

Let $E := \left[\begin{smallmatrix} 1 & -1 \\ 0 & -1 \end{smallmatrix}\right] \in D_6$. It is easy to check that

$$[\nabla^2_{e_1}\nabla_{e_2}\delta](E\cdot) = \nabla^2_{E^{-1}e_1}\nabla_{E^{-1}e_2}\delta = \nabla^2_{e_1}\nabla_{-e_1-e_2}\delta.$$

Since $[S^n_{a,3I_2}\delta](E\cdot) = S^n_{a,3I_2}\delta$ by $E \in D_6$, we must have

$$[(\nabla^2_{e_1}\nabla_{e_2}\delta) * (S^n_{a,3I_2}\delta)](E\cdot) = (\nabla^2_{e_1}\nabla_{-e_1-e_2}\delta) * (S^n_{a,3I_2}\delta).$$

Therefore, it follows that

$$\rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_2}\delta) = \rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{-e_1-e_2}\delta) = \rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_1+e_2}\delta).$$

Since $\nabla^{(2,1)}\delta = \nabla^2_{e_1}\nabla_{e_2}\delta$, in order to calculate $\rho(a, 3I_2, \infty, \nabla^\mu\delta)$ in (4.3), by (4.5) and the above identity, we see that it suffices to calculate the quantity

$$\rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_1+e_2}\delta).$$

Note that

$$\begin{aligned}\frac{(\nabla_{e_1}\nabla_{e_1+e_2}\delta)\widehat{\ }(3\xi_1, 3\xi_2)}{(\nabla_{e_1}\nabla_{e_1+e_2}\delta)\widehat{\ }(\xi_1, \xi_2)} &= \frac{1 - e^{-3i\xi_1}}{1 - e^{-i\xi_1}}\,\frac{1 - e^{-3i(\xi_1+\xi_2)}}{1 - e^{-i(\xi_1+\xi_2)}}\\ &= (1 + e^{-i\xi_1} + e^{-2i\xi_1})(1 + e^{-i(\xi_1+\xi_2)} + e^{-2i(\xi_1+\xi_2)}).\end{aligned}$$

By Theorem 2.3, it follows from $\nabla^2_{e_1}\nabla_{e_1+e_2}\delta = [\nabla_{e_1}\delta] * [\nabla_{e_1}\nabla_{e_1+e_2}\delta]$ that

$$\rho(a, 3I_2, \infty, \nabla^2_{e_1}\nabla_{e_1+e_2}\delta) = \rho(h_1, 3I_2, \infty, \nabla_{e_1}\delta) = \rho(h, 3I_2, \infty, \nabla_{e_1}\delta), \tag{4.6}$$

where $\widehat{h_1}(\xi_1, \xi_2) := e^{i\xi_1} e^{i(\xi_1+\xi_2)}\widehat{h}(\xi_1, \xi_2)$ and

$$\hat{h}(\xi_1, \xi_2) := (e^{-i\xi_2} + 1 + e^{i\xi_2})\hat{b}(\xi_1, \xi_2). \tag{4.7}$$

It is easy to check that h satisfies the sum rules of order 1. Define $\Gamma := [-1, 1]^2 \cap \mathbb{Z}^2$. Then Γ is a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^2/3\mathbb{Z}^2$. Denote

$$K := \{(j, k) \in \mathbb{Z}^2 \ : \ |j| \leqslant 1, |k| \leqslant 2\} \tag{4.8}$$

and define the linear space U by

$$U := \Big\{u \in \ell_0(\mathbb{Z}^2) \ : \ u(\beta) = 0 \quad \forall\, \beta \in \mathbb{Z}^s \backslash K \quad \text{and} \quad \sum_{\beta \in \mathbb{Z}^2} u(\beta) = 0\Big\}. \tag{4.9}$$

Then it is easy to check that $[(\mathrm{supp} h - \Gamma + K)/3] \cap \mathbb{Z}^2 \subseteq K$. Since h satisfies the sum rules of order 1, we see that $T_{h,\gamma}U \subseteq U$ for all $\gamma \in \Gamma$. Set

$$\begin{aligned} \mathcal{A} &:= \{\delta_{(0,0)} - \delta_{(-1,0)}, \delta_{(1,0)} - \delta_{(0,0)}\}, \\ \mathcal{B} &:= \{\delta_{(j,k+1)} - \delta_{(j,k)} \ : \ j = -1, 0, 1; k = -2, -1, 0, 1\}. \end{aligned} \tag{4.10}$$

Since $\hat{h}(\xi_1, \xi_2) = (e^{-i\xi_2} + 1 + e^{i\xi_2})\hat{b}(\xi_1, \xi_2)$, we see that $W := \mathrm{span}\mathcal{B}$ is invariant under all the operators $T_{h,\gamma}, \gamma \in \Gamma$. Therefore, by [22], we have

$$\begin{aligned} &\rho_\infty(\{T_{h,\gamma}|_U \ : \ \gamma \in \Gamma\}) \\ &\qquad = \max\{\rho_\infty(\{T_{h,\gamma}|_W \ : \ \gamma \in \Gamma\}),\ \rho_\infty(\{T_{h,\gamma}|_{U/W} \ : \ \gamma \in \Gamma\})\}. \end{aligned} \tag{4.11}$$

Since all the elements in $\mathcal{B}$ take the form $[\nabla_{e_2}\delta](\cdot - \beta)$ for some $\beta \in \mathbb{Z}^2$, by Theorem 2.3, we have

$$\rho_\infty(\{T_{h,\gamma}|_W \ : \ \gamma \in \Gamma\}) = \rho_\infty(h, 3I_2, \infty, \nabla_{e_2}\delta) = \rho(b, 3I_2, \infty, \delta). \tag{4.12}$$

For any $u \in U$, we denote by $[u]$ its equivalence class in U/W. The representation matrices of $T_{h,\gamma}|_{U/W}$, denoted by H_γ, under the basis $\{[u] \ : \ u \in \mathcal{A}\} = \{[\delta_{(0,0)} - \delta_{(-1,0)}], [\delta_{(1,0)} - \delta_{(0,0)}]\}$, are given by

$$\begin{aligned} H_{(-1,1)} = H_{(-1,0)} = H_{(-1,-1)} &= \frac{1}{99}\begin{bmatrix} -4 & 5 \\ 0 & 9 \end{bmatrix}, \\ H_{(0,1)} = H_{(0,0)} = H_{(0,-1)} &= \frac{1}{99}\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}, \\ H_{(1,1)} = H_{(1,0)} = H_{(1,-1)} &= \frac{1}{99}\begin{bmatrix} 9 & 0 \\ 5 & -4 \end{bmatrix}. \end{aligned} \tag{4.13}$$

By a simple calculation, we have

$$\begin{aligned} \rho_\infty(\{T_{h,\gamma}|_{U/W} \ : \ \gamma \in \Gamma\}) &= \rho_\infty(\{H_\gamma \ : \ \gamma \in \Gamma\}) \\ &\leqslant \max\{\|H_\gamma\|_{\ell_{1,\infty}} \ : \ \gamma \in \Gamma\} = 1/11, \end{aligned} \tag{4.14}$$

where $\|\cdot\|_{\ell_{1,\infty}}$ is a matrix norm which is defined to be

$$\|(t_{ij})_{1 \leqslant i \leqslant I, 1 \leqslant j \leqslant J}\|_{\ell_{1,\infty}} := \max_{1 \leqslant i \leqslant I} \sum_{j=1}^{J} |t_{ij}|.$$

Since $\nabla_{e_1}\delta \in U$, by (4.11) and (4.12), we conclude that

$$\begin{aligned} \max\{\rho(a, 3I_2, \infty, \nabla^\mu\delta) \ : \ |\mu| = 3\} \leq \rho(h, 3I_2, \infty, \nabla_{e_1}\delta) &\leqslant \rho_\infty(\{T_{h,\gamma}|_U \ : \ \gamma \in \Gamma\}) \\ &\leqslant \max\{1/11, \rho(b, 3I_2, \infty, \delta)\}. \end{aligned}$$

On the other hand, by (4.4), we have $\nabla_{e_1+e_2}\delta = \nabla_{e_2}\delta + [\nabla_{e_1}\delta](\cdot - e_2)$, and it is not difficult to see that

$$\begin{aligned}\rho(b, 3I_2, \infty, \delta) &= \rho(a, 3I_2, \infty, \nabla_{e_1}\nabla_{e_2}\nabla_{e_1+e_2}\delta)\\ &\leqslant \max\{\rho(a, 3I_2, \infty, \nabla_{e_1}^2\nabla_{e_2}\delta), \rho(a, 3I_2, \infty, \nabla_{e_1}\nabla_{e_2}^2\delta)\}.\end{aligned}$$

By Theorem 3.1, we conclude that

$$\max\{\rho(a, 3I_2, \infty, \nabla^\mu\delta) \ : \ |\mu| = 3\} = \max\{1/11, \rho(b, 3I_2, \infty, \delta)\}. \tag{4.15}$$

In the following, we estimate $\rho(b, 3I_2, \infty, \delta)$. By Theorem 2.4, we have

$$\begin{aligned}\rho(b, 3I_2, \infty, \delta) &\leqslant \max_{\alpha\in\mathbb{Z}^2}\sum_{\beta\in\mathbb{Z}^2}|b(\alpha + 3\beta)|\\ &= \frac{1}{99}\max\{|6t_1 - 15| + 6|t_2|, |t_1| + 2|t_2 + 2| + |2 + 2t_1 - t_2|, 3|5 - 2t_1 + 2t_2|\}.\end{aligned}$$

When $t_1 = 1$ and $t_2 = 0$, it follows from the above inequality that $\rho(b, 3I_2, \infty, \delta) \leqslant 1/11$. Therefore, the claim in this theorem follows directly from (4.15). □

The stencils of the subdivision triplets in Theorem 4.1 are given in Figure 1. See Figure 2 for the graph of the basis function in the subdivision triplet in Theorem 4.1 with the choice $t_1 = 1$ and $t_2 = 0$. Note that the support of the basis function is contained in $[-5/2, 5/2]^2$, while the basis function of the butterfly scheme is supported on $[-3, 3]^2$. The parameters $w_1, \dots, w_7$ in Figure 1 are given by

$$\begin{aligned}&w_1 := 72 - t_2, \quad w_2 := 31 - t_1, \quad w_3 := 7 + t_1 + t_2, \quad w_4 := -3 - t_1,\\ &w_5 := -4 + t_1, \quad w_6 := -4 - t_2, \quad w_7 := t_2.\end{aligned} \tag{4.16}$$

For quadrilateral meshes, we can use the tensor product of the one-dimensional interpolatory subdivision triplet $(a^{best}, 3, \{-1, 1\})$ to get an optimal two-dimensional interpolatory ternary subdivision scheme. In the following, let us present some other examples of subdivision triplets $(a, 3I_2, D_4)$ with better time localization of their basis functions for the quadrilateral meshes.

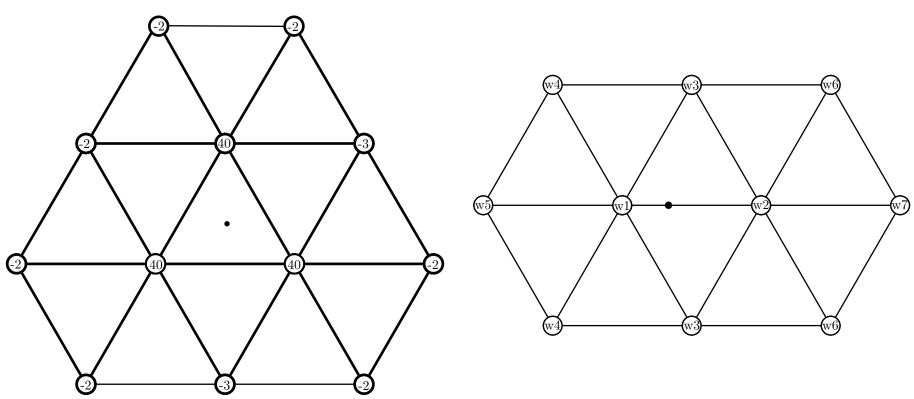

FIGURE 1. The stencils of the subdivision triplets in Theorem 4.1, where the parameters $w_1, \dots, w_7$ are given in (4.16). All the numbers in the above stencils should be divided by 99.

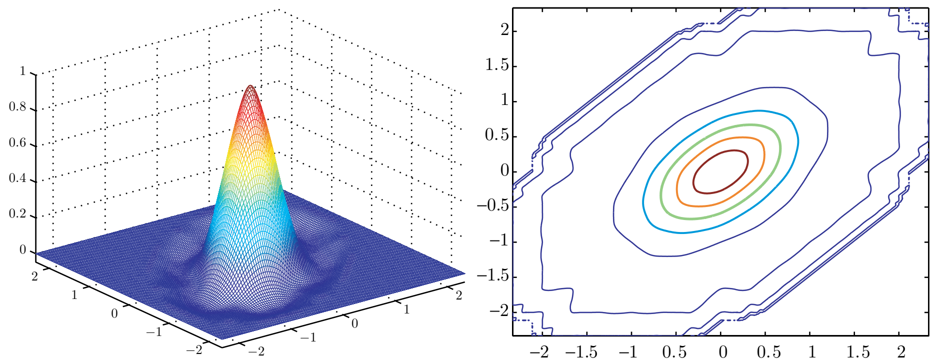

FIGURE 2. The graph of the basis function ϕ for the subdivision triplet in Theorem 4.1 with $t_1 = 1$ and $t_2 = 0$. We have $\nu_\infty(\phi) = \log_3 11$ and therefore, $\phi \in C^2(\mathbb{R}^2)$. Moreover, the support of the interpolatory function ϕ is contained in the set $[-5/2, 5/2]^2$.

In order to facilitate our analysis, we require that the mask a should take the following form:

$$\hat{a}(\xi_1, \xi_2) = (e^{-i\xi_1} + 1 + e^{i\xi_1})^2 (e^{-i\xi_2} + 1 + e^{i\xi_2})^2 \hat{b}(\xi_1, \xi_2), \tag{4.17}$$

where the sequence b is supported on $[-3, 3]^2$ and is D_4-symmetric. By solving a system of linear equations, which are induced by the same three conditions (i), (ii), (iii) on a as for the symmetry group D_6, we see that the sequence b, which is supported on $[-3, 3]^2$ and is D_4-symmetric, must take the following form:

$$\frac{1}{297}\begin{bmatrix} t_5 & t_4 & t_3 & t_{10} & t_3 & t_4 & t_5 \\ t_4 & t_2 & t_1 & t_9 & t_1 & t_2 & t_4 \\ t_3 & t_1 & t_8 & t_7 & t_8 & t_1 & t_3 \\ t_{10} & t_9 & t_7 & t_6 & t_7 & t_9 & t_{10} \\ t_3 & t_1 & t_8 & t_7 & t_8 & t_1 & t_3 \\ t_4 & t_2 & t_1 & t_9 & t_1 & t_2 & t_4 \\ t_5 & t_4 & t_3 & t_{10} & t_3 & t_4 & t_5 \end{bmatrix}, \tag{4.18}$$

where t_1, t_2, t_3, t_4, t_5 are free parameters and $t_6, t_7, t_8, t_9, t_{10}$ are given by

$$\begin{aligned} t_6 &:= 5 - 8t_1 - 12t_2 - 16t_3 - 40t_4 - 32t_5, \\ t_7 &:= 10 + 6t_1 + 8t_2 + 10t_3 + 24t_4 + 18t_5, \\ t_8 &:= -4t_1 - 4t_2 - 6t_3 - 12t_4 - 9t_5, \\ t_9 &:= 1 - 2t_1 - 2t_2 - 2t_4, \\ t_{10} &:= -4 - 2t_3 - 2t_4 - 2t_5. \end{aligned} \tag{4.19}$$

Now we have the following result on subdivision triplets $(a, 3I_2, D_4)$ with two-ring stencils.

Theorem 4.2. *Let $(a, 3I_2, D_4)$ be an interpolatory subdivision triplet, where the mask a is given by* (4.17) *and the sequence b is given in* (4.18)*. Then*

$$\nu_\infty(a, 3I_2) = -\log_3 \max\{1/11, \rho(b, 3I_2, \infty, \delta)\}.$$

In particular, $\nu_\infty(a, 3I_2) = \log_3 11$ if and only if $\rho(b, 3I_2, \infty, \delta) \leqslant 1/11$. Moreover, if $-3/4 < t_1 < 2$ and $t_2 = t_3 = t_4 = t_5 = 0$, then $\rho(b, 3I_2, \infty, \delta) \leqslant 1/11$

and $\nu_\infty(a, 3I_2) = \log_3 11$. *Therefore, the subdivision triplet is the smoothest two-dimensional interpolatory ternary subdivision scheme with two-ring stencils for the regular quadrilateral mesh.*

Proof. By symmetry on the mask a, it suffices to compute the two quantities $\rho(a, 3I_2, \infty, \nabla_{e_1}^3\delta)$ and $\rho(a, 3I_2, \infty, \nabla_{e_1}^2\nabla_{e_2}\delta)$. Note that

$$\frac{\widehat{\nabla_{e_1}^2\delta}(3\xi_1, 3\xi_2)}{\widehat{\nabla_{e_1}^2\delta}(\xi_1, \xi_2)} = \frac{(1-e^{-3i\xi_1})^2}{(1-e^{-i\xi_1})^2} = (1+e^{-i\xi_1}+e^{-2i\xi_1})^2.$$

By Theorem 2.3, we see that

$$\rho(a, 3I_2, \infty, \nabla_{e_1}^3\delta) = \rho(h, 3I_2, \infty, \nabla_{e_1}\delta)$$

and

$$\rho(a, 3I_2, \infty, \nabla_{e_1}^2\nabla_{e_2}\delta) = \rho(h, 3I_2, \infty, \nabla_{e_2}\delta),$$

where

$$\hat{h}(\xi_1, \xi_2) := (e^{-i\xi_2} + 1 + e^{i\xi_2})^2\hat{b}(\xi_1, \xi_2). \tag{4.20}$$

It is easy to verify that h satisfies the sum rules of order 1. Denote $\Gamma := [-1, 1]^2 \cap \mathbb{Z}^2$. Let K and U be defined in (4.8) and (4.9), respectively. Since h satisfies the sum rules of order 1, we have $T_{h,\gamma}U \subseteq U$ for all $\gamma \in \Gamma$. Set

$$\begin{aligned}
&\mathcal{A} := \{\delta_{(0,0)} - \delta_{(-1,0)}, \delta_{(1,0)} - \delta_{(0,0)}\},\\
&\mathcal{B} := \{\delta_{(-1,1)} - \delta_{(-1,0)}, \delta_{(0,1)} - \delta_{(0,0)}, \delta_{(1,1)} - \delta_{(1,0)}\},\\
&\mathcal{C} := \{\delta_{(j,k+2)} - 2\delta_{(j,k+1)} + \delta_{(j,k)} \ : \ j = -1, 0, 1; k = -2, -1, 0\}.
\end{aligned} \tag{4.21}$$

Define $W := \operatorname{span}(\mathcal{B} \cup \mathcal{C})$ and $V := \operatorname{span}\mathcal{C}$. Since $\hat{h}(\xi_1, \xi_2) = (e^{-i\xi_2} + 1 + e^{i\xi_2})^2\hat{b}(\xi_1, \xi_2)$, we see that $T_{h,\gamma}W \subseteq W$ and $T_{h,\gamma}V \subseteq V$ for all $\gamma \in \Gamma$.

For any $u \in U$, we denote by $[u]$ its equivalence class in U/W. The representation matrices of $T_{h,\gamma}|_{U/W}$, denoted by H_γ, under the basis $\{[u] \ : \ u \in \mathcal{A}\}$, are given in (4.13). Therefore, by what has been proved, (4.14) holds.

For any $u \in W$, we denote by $[u]$ its equivalence class in W/V. The representation matrices of $T_{h,\gamma}|_{W/V}$, denoted by $H_{2,\gamma}$, under the basis $\{[u] \ : \ u \in \mathcal{B}\}$, are given by

$$H_{2,(-1,1)} = H_{2,(-1,0)} = H_{2,(-1,-1)} = \frac{1}{297}\begin{bmatrix} -4 & 19 & -4 \\ 0 & 10 & 1 \\ 0 & 1 & 10 \end{bmatrix},$$

$$H_{2,(0,1)} = H_{2,(0,0)} = H_{2,(0,-1)} = \frac{1}{297}\begin{bmatrix} 1 & 10 & 0 \\ -4 & 19 & -4 \\ 0 & 10 & 1 \end{bmatrix},$$

$$H_{2,(1,1)} = H_{2,(1,0)} = H_{2,(1,-1)} = \frac{1}{297}\begin{bmatrix} 10 & 1 & 0 \\ 1 & 10 & 0 \\ -4 & 19 & -4 \end{bmatrix}.$$

Therefore, we have

$$\begin{aligned}
\rho_\infty(\{T_{h,\gamma}|_{W/V} \ : \ \gamma \in \Gamma\}) &= \rho_\infty(\{H_{2,\gamma} \ : \ \gamma \in \Gamma\})\\
&\leqslant \max\{\|H_{2,\gamma}\|_{\ell_{1,\infty}} \ : \ \gamma \in \Gamma\} = 1/11.
\end{aligned} \tag{4.22}$$

Note that every element in $\mathcal{C}$ takes the form $\nabla_{e_2}^2\delta(\cdot-\beta)$ for some $\beta\in\mathbb{Z}^2$. By Theorem 2.3, we have

$$\rho_\infty(\{T_{h,\gamma}|_{\mathcal{C}}\ :\ \gamma\in\Gamma\})=\rho(h,3I_2,\infty,\nabla_{e_2}^2\delta)=\rho(b,3I_2,\infty,\delta). \tag{4.23}$$

Now by Theorem 2.4, we have

$$\begin{aligned}\rho(b,3I_2,\infty,\delta)&\leqslant\max_{\alpha\in\mathbb{Z}^2}\sum_{\beta\in\mathbb{Z}^2}|b(\alpha+3\beta)|\\&\leqslant\frac{1}{297}\max\{|t_6|+4|t_{10}|+4|t_5|,|t_7|+|t_9|+2|t_3|+2|t_4|,|t_8|+2|t_1|+|t_2|\},\end{aligned}$$

where $t_j, j=6,\dots,10$ are defined in (4.19). When $t_2=t_3=t_4=t_5=0$, the above inequality becomes

$$\rho(b,3I_2,\infty,\delta)\leqslant\frac{1}{297}\max\{16+|5-8t_1|,|1-2t_1|+|10+6t_1|,6|t_1|\}.$$

It follows easily from the above inequality that if $-3/4<t_1<2$ and $t_2=t_3=t_4=t_5=0$, then $\rho(b,3I_2,\infty,\delta)\leqslant 1/11$. Since the elements in $\mathcal{A}$ take the form $\nabla_{e_1}\delta(\cdot-\beta)$ and the elements in $\mathcal{B}$ take the form $\nabla_{e_2}\delta(\cdot-\beta)$, by (4.14) and (4.23), we see that

$$\begin{aligned}\max\{\rho(a,3I_2,\infty,\nabla_{e_1}^3\delta),&\rho(a,3I_2,\infty,\nabla_{e_1}^2\nabla_{e_2}\delta)\}\\&=\max\{\rho(h,3I_2,\infty,\nabla_{e_1}\delta),\rho(h,3I_2,\infty,\nabla_{e_2}\delta)\}\\&=\rho_\infty(\{T_{h,\gamma}|_U\ :\ \gamma\in\Gamma\})\\&=\max\{1/11,\rho(b,3I_2,\infty,\delta)\},\end{aligned}$$

which completes the proof. □

The stencils of the subdivision triplets in Theorem 4.2 are given in Figure 3. See Figure 4 for the graph of the basis function in the subdivision triplet in Theorem 4.2 with $t_1=t_2=t_3=t_4=t_5=0$, and the corresponding stencils for the case $t_1=t_2=t_3=t_4=t_5=0$ are given in Figure 5. The parameters $w_0,\dots,w_9,u_0,\dots,u_7$ in Figure 3 are given by

$$\begin{aligned}&w_0:=172+4t_2+12t_4+9t_5,\quad w_1:=78-2t_2-3t_4,\\&w_2:=-14-2t_2-7t_4-6t_5,\quad w_3:=35+t_2,\\&w_4:=t_2+4t_4+4t_5,\quad w_5:=-7+t_2+2t_4,\quad w_6:=t_4+2t_5,\\&w_7:=-8-2t_4-3t_5,\quad w_8:=-4-t_4,\quad w_9=t_5,\\&u_0:=228+4t_1+8t_2+6t_3+24t_4+18t_5,\\&u_1:=102-2t_1-4t_2-6t_4,\\&u_2:=-2t_1-4t_2-3t_3-12t_4-9t_5,\\&u_3:=-21-2t_1-4t_2-4t_3-14t_4-12t_5,\\&u_4:=t_1+2t_2+2t_3+7t_4+6t_5,\\&u_5:=t_1+2t_2+3t_4,\\&u_6:=-12-2t_3-4t_4-6t_5,\quad u_7:=t_3+2t_4+3t_5.\end{aligned}\tag{4.24}$$

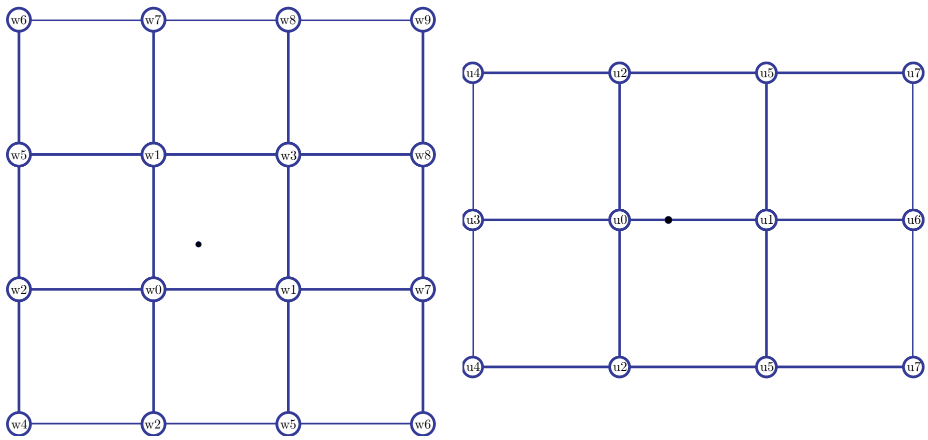

FIGURE 3. The stencils of the subdivision triplets in Theorem 4.2, where all the parameters are given in (4.24). All the numbers in the above stencils should be divided by 297.

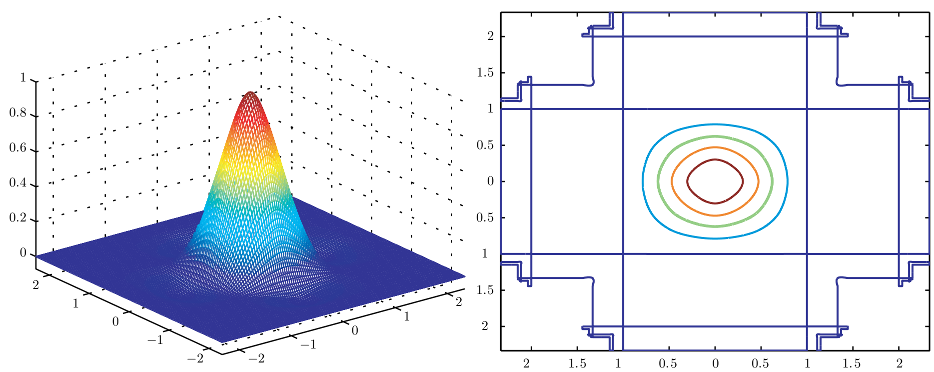

FIGURE 4. The graph of the basis function ϕ for the subdivision triplet in Theorem 4.2 with $t_1 = t_2 = t_3 = t_4 = t_5 = 0$. We have $\nu_\infty(\phi) = \log_3 11$ and therefore, $\phi \in C^2(\mathbb{R}^2)$. The corresponding stencils are given in Figure 5.

When $t_1 = t_2 = t_3 = t_4 = t_5 = 0$, the parameters in (4.24) become

$$
\begin{aligned}
&w_0 = 78, \quad w_1 = -78, \quad w_2 = -14, \quad w_3 = 35,\\
&w_5 = -7, \quad w_7 = -8, \quad w_8 = -4,\\
&w_4 = w_6 = w_9 = 0,\\
&u_0 = 228, \quad u_1 = 102, \quad u_3 = -21, \quad u_6 = -12,\\
&u_2 = u_4 = u_5 = u_7 = 0.
\end{aligned} \tag{4.25}
$$

Finally, we mention that by using the same technique as in Theorem 4.2, we have $\nu_\infty(g_2, 2I_2) = 2$, where g_2 is the D_4-symmetric interpolatory mask given in [17] with support $[-3, 3]^2$.

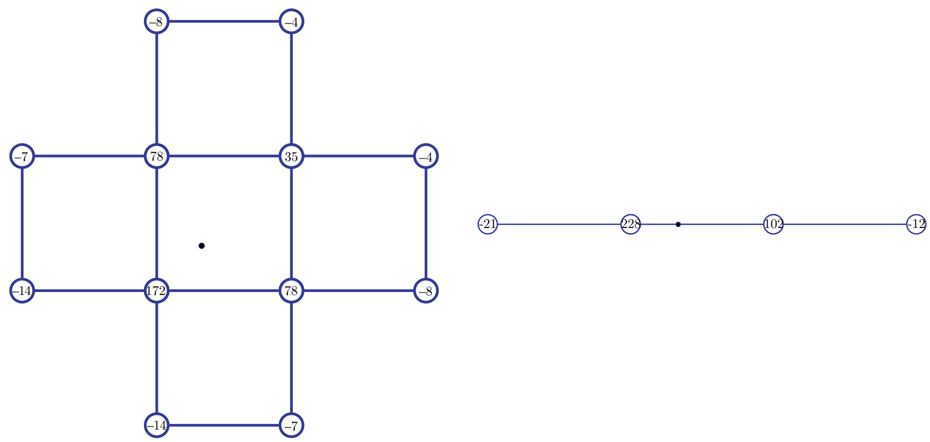

FIGURE 5. The stencils of the subdivision triplets in Theorem 4.2 with the choice $t_1 = t_2 = t_3 = t_4 = t_5 = 0$. All the numbers in the above stencils should be divided by 297. The stencil on the left-hand side reduces to be the one-dimensional stencil for the mask a^{best} given in (3.1).

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