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PARAMETER-UNIFORM FINITE DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED PARABOLIC DIFFUSION-CONVECTION-REACTION PROBLEMS

E. O'RIORDAN, M. L. PICKETT, AND G. I. SHISHKIN

ABSTRACT. In this paper, parameter-uniform numerical methods for a class of singularly perturbed parabolic partial differential equations with two small parameters on a rectangular domain are studied. Parameter-explicit theoretical bounds on the derivatives of the solutions are derived. The solution is decomposed into a sum of regular and singular components. A numerical algorithm based on an upwind finite difference operator and an appropriate piecewise uniform mesh is constructed. Parameter-uniform error bounds for the numerical approximations are established. Numerical results are given to illustrate the parameter-uniform convergence of the numerical approximations.

1. INTRODUCTION

Consider the following class of singularly perturbed parabolic problems posed on the domain $G = \Omega \times (0, T]$, $\quad \Omega = (0, 1)$, $\quad \Gamma = \bar{G} \setminus G$:

$$L_{\varepsilon,\mu} u \equiv \varepsilon u_{xx} + \mu a u_x - bu - du_t = f(x,t) \qquad \text{in } G, \tag{1.1a}$$

$$u = s(x) \qquad \text{on } \Gamma_b, \tag{1.1b}$$

$$u = q(t) \qquad \text{on } \Gamma_l \cup \Gamma_r, \tag{1.1c}$$

$$a(x,t) \geq \alpha > 0, \quad b(x,t) \geq \beta > 0, \quad d(x,t) \geq \delta > 0, \tag{1.1d}$$

where $\Gamma_b = \{(x,0) \mid 0 \leq x \leq 1\}$, $\Gamma_l = \{(0,t) \mid 0 \leq t \leq T\}$, and $\Gamma_r = \{(1,t) \mid 0 \leq t \leq T\}$. We note that $0 < \varepsilon \leq 1$ and $0 \leq \mu \leq 1$ are perturbation parameters. We assume sufficient regularity and compatibility at the corners so that the solution and its regular component are sufficiently smooth for our analysis. Our interest lies in constructing parameter-uniform numerical methods [1] for this class of singularly perturbed problems. By this we mean numerical methods whose solutions converge uniformly with respect to the singular perturbation parameters.

When the parameter $\mu = 1$, the problem is the well-studied parabolic convection-diffusion problem [2, 10, 15] and in this case a boundary layer of width $O(\varepsilon)$ appears in the neighbourhood of the edge $x = 0$. When $\mu = 0$ we have a parabolic

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reaction-diffusion problem [6] and boundary layers of width $O(\sqrt{\varepsilon})$ appear in the neighbourhood of both $x = 0$ and $x = 1$.

Fitted operator methods based on exponentially fitted finite difference operators have been developed for the steady-state version of (1.1) when $\mu = 0$ and $\mu = 1$. Earlier results of Shishkin [13, 16] dealt with fitted operator methods for (1.1). In the time dependent problem, when $\mu = 1$, fitted operator methods were derived in [15]. However, Shishkin [14] established that in order to obtain a parameter-uniform numerical method, it is necessary to fit the mesh when parabolic layers are present. This implies that we cannot use fitted operators on a uniform mesh to obtain parameter-uniform convergence in the case of (1.1).

The asymptotic structure of the solutions to the steady-state version of (1.1) was examined by O'Malley [7, 8], where the ratio of μ to $\sqrt{\varepsilon}$ was identified as significant. Vulanović [17] considers finite difference methods in the case of $\mu = \varepsilon^{\frac{1}{2}+\lambda}$, $\lambda > 0$. More recently, parameter-uniform numerical methods for the steady-state version of (1.1) were examined by Linß and Roos [4], Roos and Uzelac [11] and O'Riordan et al. [9]. Both [4] and [9] are concerned with finite difference methods and apply standard finite difference operators on special piecewise uniform meshes. In [11] the problem is solved using the streamline-diffusion finite element method on a piecewise uniform mesh.

Parameter-uniform numerical methods composed of standard finite difference operators and piecewise uniform meshes have been established [2, 10] for both the steady-state and the time dependent versions of (1.1) in the two special cases of $\mu = 0$ and $\mu = 1$. We will apply an upwind finite difference operator on a piecewise uniform mesh in the construction of our numerical algorithm to solve (1.1) for all values of the parameters in the range $\mu \in [0, 1]$ and $\varepsilon \in (0, 1]$. The analysis in this paper naturally splits into the two cases of $\mu^2 \le C\varepsilon$ and $\mu^2 \ge C\varepsilon$. In the first case the analysis follows closely when $\mu = 0$; however, in the second case the analysis is more intricate. In the case of $\mu^2 \le C\varepsilon$ an $O(\sqrt{\varepsilon})$ layer appears in the neighbourhood of $x = 0$ and $x = 1$. In the other case of $\mu^2 \le C\varepsilon$ a layer of width $O(\frac{\varepsilon}{\mu})$ appears in the neighbourhood of $x = 0$ and a layer of width $O(\mu)$ appears near $x = 1$.

The analysis in this paper is based on the principles laid down in [12] and in the books [1] and [5] for a single parameter singularly perturbed problem. We apply similar analytical techniques to those used in [9] for a singularly perturbed ordinary differential equation with two small parameters. The argument consists of establishing a maximum principle, a decomposition of the solution into regular and layer components and deriving sharp parameter-explicit bounds on these components and their derivatives. The discrete solution is decomposed into analogous components and the numerical error between the discrete and continuous components are analysed separately using discrete maximum principle, truncation error analysis, and appropriate barrier functions. In [9], the piecewise uniform mesh constructed consisted of the two transition points

$$\sigma_1 = \min\{\frac{1}{4}, \frac{2\ln N}{\eta_1}\} \quad \text{and} \quad \sigma_2 = \min\{\frac{1}{4}, \frac{2\ln N}{\eta_2}\}, \tag{1.2}$$

where η_1 is the positive root of the quadratic equation $\varepsilon\eta_1^2 - \mu\alpha\eta_1 - \beta = 0$ and similarly η_2 is the positive root of the quadratic equation $\varepsilon\eta_2^2 + \mu\|a\|\eta_2 - \beta = 0$. However, in this paper the choice of transition points in (4.1b) is simpler then those given in (1.2).

Notation. We define the zero order, first order, and second order differential operators L_0, L_μ, and $L_{\varepsilon,\mu}$ as

$$L_0 z = -bz - dz_t, \tag{1.3a}$$

$$L_\mu z = a\mu z_x + L_0 z, \tag{1.3b}$$

$$L_{\varepsilon,\mu} z = \varepsilon z_{xx} + L_\mu z. \tag{1.3c}$$

We let $\gamma = \min_{\bar{G}}\{\frac{b}{a}\}$ and we also adopt the notation

$$\|u\|_{\bar{G}} = \max_{\bar{G}} |u(x,t)|.$$

If the norm is not subscripted, then $\|.\| = \|.\|_{\bar{G}}$. Throughout this paper C (sometimes subscripted) denotes a constant that is independent of the parameters ε, μ, N, and M (number of mesh elements used in the space (N) and time (M) direction).

2. Bounds on the solution u and its derivatives

We will establish *a priori* bounds on the solution of (1.1) and its derivatives. These bounds will be needed in the error analysis in later sections. We start by stating a continuous minimum principle for the differential operator in (1.1), whose proof is standard.

Minimum principle. If $w \in C^2(G) \cap C^0(\bar{G})$ such that $L_{\varepsilon,\mu} w \mid_G \leq 0$ and $w \mid_\Gamma \geq 0$, then $w \mid_{\bar{G}} \geq 0$.

The lemma below follows immediately from the minimum principle above.

Lemma 2.1. *The solution u of problem* (1.1), *satisfies the bound*

$$\|u\| \leq \|s\|_{\Gamma_b} + \|q\|_{\Gamma_l \cup \Gamma_r} + \frac{1}{\beta}\|f\|.$$

Lemma 2.2. *The derivatives of the solution u of* (1.1) *satisfy the bounds for all nonnegative integers k,m, such that $1 \leq k + 2m \leq 3$. If $\mu^2 \leq C\varepsilon$, then*

$$\left\|\frac{\partial^{k+m}u}{\partial x^k \partial t^m}\right\| \leq \frac{C}{(\sqrt{\varepsilon})^k} \max\left\{\|u\|, \sum_{i+2j=0}^{2} (\sqrt{\varepsilon})^i \left\|\frac{\partial^{i+j} f}{\partial x^i \partial t^j}\right\|, \sum_{i=0}^{4} \left\|\frac{d^{\,i} s}{d\, x^i}\right\|_{\Gamma_b} + \left\|\frac{d^{\,i} q}{d\, t^i}\right\|_{\Gamma_l \cup \Gamma_r}\right\},$$

and if $\mu^2 \geq C\varepsilon$, then

$$\left\|\frac{\partial^{k+m}u}{\partial x^k \partial t^m}\right\| \leq C\Big(\frac{\mu}{\varepsilon}\Big)^k \Big(\frac{\mu^2}{\varepsilon}\Big)^m \times \max\left\{\|u\|, \sum_{i+2j=0}^{2} \Big(\frac{\varepsilon}{\mu}\Big)^i \Big(\frac{\varepsilon}{\mu^2}\Big)^{j+1} \left\|\frac{\partial^{i+j} f}{\partial x^i \partial t^j}\right\|, \sum_{i=0}^{4} \left\|\frac{d^{\,i} s}{d\, x^i}\right\|_{\Gamma_b} + \left\|\frac{d^{\,i} q}{d\, t^i}\right\|_{\Gamma_l \cup \Gamma_r}\right\},$$

where C depends only on the coefficients a, b, d and their derivatives.

Proof. The argument splits into two cases: $\mu^2 \leq C\varepsilon$ and $\mu^2 \geq C\varepsilon$. If $\mu^2 \leq C\varepsilon$, consider the transformation $\xi = \frac{x}{\sqrt{\varepsilon}}$. Our transformed domain is given by $\tilde{G} = \left(0, \frac{1}{\sqrt{\varepsilon}}\right) \times (0, T]$. Also we have $\tilde{u}(\xi, t) = u(x, t)$ with $\tilde{a}$, $\tilde{b}$, $\tilde{d}$, and $\tilde{f}$ defined similarly. Applying this transformation to (1.1) we obtain

$$\tilde{u}_{\xi\xi} + \frac{\mu}{\sqrt{\varepsilon}}\tilde{a}\tilde{u}_\xi - \tilde{b}\tilde{u} - \tilde{d}\tilde{u}_t = \tilde{f}, \text{ on } \tilde{G}.$$

Then for every $\zeta \in \left(0, \frac{1}{\sqrt{\varepsilon}}\right)$ and $\delta > 0$, we denote the rectangle $((\zeta - \delta, \zeta + \delta) \times (0, T]) \cap \tilde{G}$ by $R_{\zeta,\delta}$. The closure of $R_{\zeta,\delta}$ is denoted $\bar{R}_{\zeta,\delta}$. For each $(\zeta, t) \in \tilde{G}$, we use [3] to obtain the following bounds for $1 \leq k + 2m \leq 3$:

$$\begin{aligned} &\left\| \frac{\partial^{k+m}\tilde{u}}{\partial\xi^k \partial t^m} \right\|_{\bar{R}_{\zeta,\delta}} \\ &\quad \leq C \max\left\{ \|\tilde{u}\|, \sum_{i+2j=0}^{2} \left\| \frac{\partial^{i+j}\tilde{f}}{\partial\xi^i \partial t^j} \right\|, \sum_{i=0}^{4} \left\| \frac{d^{\,i}\tilde{s}}{d\xi^i} \right\|_{\Gamma_b'} + \left\| \frac{d^{\,i}\tilde{q}}{dt^i} \right\|_{\Gamma_l' \cup \Gamma_r'} \right\}, \end{aligned}$$

where $\Gamma_b' = \bar{R}_{\zeta,2\delta} \cap \Gamma_b$, $\Gamma_l' = \bar{R}_{\zeta,2\delta} \cap \Gamma_l$, $\Gamma_r' = \bar{R}_{\zeta,2\delta} \cap \Gamma_r$, and C is independent of the rectangle $R_{\zeta,\delta}$. These bounds hold for any point $(\zeta, t) \in \tilde{G}$. Transforming back to the original (x, t) variables gives us the required result. If $\mu^2 \geq C\varepsilon$, then we are required to stretch in time also. Introduce the transformation $\varrho = \frac{\mu x}{\varepsilon}, \tau = \frac{\mu^2 t}{\varepsilon}$. Applying this transformation to (1.1) we obtain for $\hat{u}(\varrho, \tau) = u(x, t)$

$$\hat{u}_{\varrho\varrho} + \hat{a}\hat{u}_\varrho - \frac{\varepsilon}{\mu^2}\hat{b}\hat{u} - \hat{d}\hat{u}_\tau = \frac{\varepsilon}{\mu^2}\hat{f}, \text{ on } \hat{G}.$$

Our transformed domain is given by $\hat{G} = \left(0, \frac{\mu}{\varepsilon}\right) \times \left(0, \frac{\mu^2 T}{\varepsilon}\right]$. Repeat the argument for the previous case to obtain the result. $\square$

Corollary 2.2.1. *Assuming sufficient smoothness of the data, the second order time derivative of the solution of* (1.1) *satisfies the bound*

$$\|u_{tt}\| \leq \begin{cases} C, & \text{if } \mu^2 \leq C\varepsilon, \\ C\mu^4\varepsilon^{-2}, & \text{if } \mu^2 \geq C\varepsilon. \end{cases}$$

Proof. This follows using the same argument as in Lemma 2.2. $\square$

3. Decomposition of the solution

In order to obtain parameter-uniform error estimates we decompose the solution of (1.1) into regular and singular components. The regular component will be constructed so that the first two space derivatives of this component will be bounded independently of the small parameters. Consider the differential equation

$$L_{\varepsilon,\mu} v = f \text{ on } G. \tag{3.1}$$

In the case of $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, we decompose v as

$$v(x, t; \varepsilon, \mu) = v_0(x, t) + \sqrt{\varepsilon}v_1(x, t; \varepsilon, \mu) + \varepsilon v_2(x, t; \varepsilon, \mu), \tag{3.2a}$$

where

$$L_0 v_0 = f, \qquad v_0(x, 0) = u(x, 0), \tag{3.2b}$$

$$\sqrt{\varepsilon}L_0 v_1 = (L_0 - L_{\varepsilon,\mu})v_0, \qquad v_1(x, 0; \varepsilon, \mu) = 0, \tag{3.2c}$$

$$\varepsilon L_{\varepsilon,\mu} v_2 = \sqrt{\varepsilon}(L_0 - L_{\varepsilon,\mu})v_1, \qquad v_2|_\Gamma = 0. \tag{3.2d}$$

We see that $v(0,t;\varepsilon,\mu) = v_0(0,t) + \sqrt{\varepsilon}v_1(0,t;\varepsilon,\mu)$ and $v(1,t;\varepsilon,\mu) = v_0(1,t) + \sqrt{\varepsilon}v_1(1,t;\varepsilon,\mu)$. Assuming sufficient smoothness of the data, and noting that $\alpha\mu^2 \leq \gamma\varepsilon$, we see that v_0 and its derivatives with respect to x and t up to sixth order and v_1 and its derivatives with respect to x and t up to fourth order are bounded independently of ε and μ.

Since v_2 satisfies a similar equation to u we can apply Lemmas 2.1 and 2.2 to problem (3.2d). We obtain for $0 \leq k + 2m \leq 3$,

$$\left\| \frac{\partial^{k+m} v_2}{\partial x^k \partial t^m} \right\| \leq C\left(\frac{1}{\sqrt{\varepsilon}}\right)^k.$$

We conclude that when $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, there exists a function v satisfying (3.1) where the boundary conditions of v can be chosen so that it satisfies the following bounds for $0 \leq k + 2m \leq 3$,

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \leq C\big(1 + \varepsilon^{\frac{2-k}{2}}\big).$$

From Corollary 2.2.1 we deduce that

$$\|v_{tt}\| \leq C, \quad \text{if } \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}.$$

Now consider the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$. Again we consider the differential equation (3.1); however, we decompose v as

$$v(x,t;\varepsilon,\mu) = v_0(x,t;\mu) + \varepsilon v_1(x,t;\mu) + \varepsilon^2 v_2(x,t;\varepsilon,\mu), \tag{3.3a}$$

where

$$\begin{aligned}
&(3.3\text{b}) \quad & L_\mu v_0 &= f, & v_0(x,0;\mu) = u(x,0),\ \ & v_0(1,t;\mu) \text{ chosen in (3.6)},\\
&(3.3\text{c}) \quad & \varepsilon L_\mu v_1 &= (L_\mu - L_{\varepsilon,\mu})v_0, & & v_1(x,0;\mu) = v_1(1,t;\mu) = 0,\\
&(3.3\text{d}) \quad & \varepsilon^2 L_{\varepsilon,\mu} v_2 &= \varepsilon(L_\mu - L_{\varepsilon,\mu})v_1, & & v_2(x,t;\varepsilon,\mu)|_\Gamma = 0.
\end{aligned}$$

We can establish the following for the differential operator L_μ by considering the transformation $w = e^{\beta_1 T} z$ $\left(\beta_1 < \frac{b}{d}\right)$ and using a proof-by-contradiction argument:

$$\text{If } L_\mu z\Big|_{G_1} \leq 0 \qquad \text{and } z\Big|_{\Gamma_1} \geq 0, \qquad \text{then } z\Big|_{\bar{G_1}} \geq 0, \tag{3.4}$$

where $L_\mu z = a\mu z_x - bz - dz_t = f$, $\Gamma_1 = \Gamma_b \cup \Gamma_r$, and $G_1 = [0,1) \times (0,T]$. We should note that the proof only requires that a and d are strictly positive.

Lemma 3.1. *Suppose $z(x,t)$ satisfies the first order initial-boundary value problem*

$$L_\mu z = f \quad (x,t) \in [0,1) \times [0,T], \quad z(x,0) = g_1(x), \quad z(1,t) = g_2(t), \tag{3.5}$$

where $a > 0$, $d > 0$, and $b \geq \beta > 0$. Then

$$\|z\| \leq \frac{1}{\beta}\|f\| + \|g_1\|_{\Gamma_b} + \|g_2\|_{\Gamma_r}.$$

Proof. Consider $\Psi^{\pm}(x,t) = \frac{1}{\beta}\|f\| + \|g_1\|_{\Gamma_b} + \|g_2\|_{\Gamma_r} \pm z(x,t)$. We see that the functions $\Psi^{\pm}$ are nonnegative for $(x,t) \in \Gamma_1$ and

$$L_\mu \Psi^{\pm}(x,t) = -b(\frac{1}{\beta}\|f\| + \|g_1\|_{\Gamma_b} + \|g_2\|_{\Gamma_r}) \pm f \leq 0. \qquad \square$$

We now state and prove the following technical lemma that is needed when examining how v_0 and v_1 depend on μ.

Lemma 3.2. *Suppose $z(x,t) \in C^{k+m}(\bar{G}_1)$ satisfies the problem (3.5). Then its derivatives satisfy the bounds*

$$\left\| \frac{\partial^{k+m} z}{\partial x^k \partial t^m} \right\| \leq \frac{C}{\mu^k} \left(\left\| \frac{\partial^{k+m} f}{\partial t^{k+m}} \right\| + \sum_{r+s=0}^{k+m-1} \mu^r \left\| \frac{\partial^{r+s} f}{\partial x^r \partial t^s} \right\| \right.$$
$$\left. + \sum_{j=0}^{k+m} \left\| \frac{d^{\,j} g_1}{d\, x^j} \right\| + \sum_{j=0}^{k+m} \left\| \frac{d^{\,j} g_2}{d\, t^j} \right\| + \|z\| \right) e^{-(k+m)AT},$$

where $A = \min\{0, (\frac{a}{d}) \left(\frac{d}{a}\right)_t\}$ and the constant C depends only on the coefficients a, b, d and their derivatives.

Proof. Differentiate (3.5) with respect to time to obtain

$$L_\mu^{[1]} z_t = \mu z_{tx} - \left(\frac{b}{a} + \left(\frac{d}{a}\right)_t\right) z_t - \frac{d}{a} z_{tt} = \left(\frac{f}{a}\right)_t + \left(\frac{b}{a}\right)_t z,$$
$$z_t(1,t) = g_2'(t),\ z_t(x,0) = \phi_1(x),$$

where $\phi_1(x)$ can be expressed in terms of g_1, g_1', f and the coefficients of (3.5). Consider the barrier functions

$$\Psi_1^{\pm}(x,t) = C(\|f\| + \|f_t\| + \|g_1\| + \|g_1'\| + \|g_2'\| + \|z\|)e^{-At} \pm z_t$$

with A as above. For C large enough the functions $\Psi_1^{\pm}$ are nonnegative for $(x,t) \in \Gamma_1$. Also

$$L_\mu^{[1]} \Psi_1^{\pm}(x,t) = -C\left(\frac{b}{a} + \left(\frac{d}{a}\right)_t - \frac{d}{a}A\right)(\|f\| + \|f_t\| + \|g_1\| + \|g_1'\| + \|g_2'\| + \|z\|)e^{-At}$$
$$\pm \left(\left(\frac{f}{a}\right)_t + \left(\frac{b}{a}\right)_t z\right),$$

and we see that for C chosen correctly, we have $L_\mu^{[1]} \Psi_1^{\pm} \leq 0$. Therefore using (3.4) we obtain

$$\|z_t\| \leq C(\|f\| + \|f_t\| + \|g_1\| + \|g_1'\| + \|g_2'\| + \|z\|)e^{-AT},$$

and using (3.5) we have that

$$\|z_x\| \leq \frac{C}{\mu}(\|f\| + \|f_t\| + \|g_1\| + \|g_1'\| + \|g_2'\| + \|z\|)e^{-AT}.$$

Proceed by induction. Assume the statement is true for $0 \le k+m \le l$. Differentiate (3.5) $l+1$ times with respect to t to obtain

$$L_\mu^{[l+1]}\frac{\partial^{l+1}z}{\partial t^{l+1}} = \mu\left(\frac{\partial^{l+1}z}{\partial t^{l+1}}\right)_x - \left(\frac{b}{a} + (l+1)\left(\frac{d}{a}\right)_t\right)\left(\frac{\partial^{l+1}z}{\partial t^{l+1}}\right) - \frac{d}{a}\left(\frac{\partial^{l+1}z}{\partial t^{l+1}}\right)_t$$
$$= \rho(x,t),$$
$$\frac{\partial^{l+1}z}{\partial t^{l+1}}(1,t) = \frac{d^{l+1}g_2}{dt^{l+1}}, \qquad \frac{\partial^{l+1}z}{\partial t^{l+1}}(x,0) = \phi_{l+1}(x).$$

The expression $\rho(x,t)$ involves z and its t derivatives up to order l, f and its t derivatives up to order $l+1$ and the coefficients and their derivatives. The function $\phi_{l+1}(x)$ involves g_1 and all its derivatives up to order $l+1$, the derivatives of f of the form $\mu^r\frac{\partial^{r+s}f}{\partial x^r\partial t^s}$ up to order l and the coefficients and their derivatives. Consider the barrier functions

$$\Psi_{l+1}^{\pm}(x,t) = C\left(\left\|\frac{\partial^{l+1}f}{\partial t^{l+1}}\right\| + \sum_{r+s=0}^{l}\mu^r\left\|\frac{\partial^{r+s}f}{\partial x^r\partial t^s}\right\|\right.$$
$$\left. + \sum_{j=0}^{l+1}\left\|\frac{d^{\,j}g_1}{d\,x^j}\right\| + \sum_{j=0}^{l+1}\left\|\frac{d^{\,j}g_2}{d\,t^j}\right\| + \|z\|\right)e^{-(k+m)At} \pm \frac{\partial^{l+1}z}{\partial t^{l+1}}.$$

We see that for C large enough $\Psi_{l+1}^{\pm}(x,t)$ is nonnegative for $(x,t)\in\Gamma_1$. Also for C chosen correctly we see that $L_\mu^{[l+1]}\Psi^{\pm} \le 0$. Therefore, using (3.4), we obtain

$$\left\|\frac{\partial^{l+1}z}{\partial t^{l+1}}\right\| \le C\left(\left\|\frac{\partial^{l+1}f}{\partial t^{l+1}}\right\| + \sum_{r+s=0}^{l}\mu^r\left\|\frac{\partial^{r+s}f}{\partial x^r\partial t^s}\right\|\right.$$
$$\left. + \sum_{j=0}^{l+1}\left\|\frac{d^{\,j}g_1}{d\,x^j}\right\| + \left\|\frac{d^{\,j}g_2}{d\,t^j}\right\| + \|z\|\right)e^{-(k+m)AT}.$$

Differentiate (3.5) appropriately to obtain the required result for $k+m = l+1$. $\square$

We now continue with our analysis of v_0 and v_1. The following two lemmas establish that when the boundary condition $v_0(1,t;\mu)$ is chosen correctly, the first two derivatives of $v_0(x,t;\mu)$ are bounded independent of μ and the derivatives of $v_1(x,t;\mu)$ are bounded by inverse powers of μ.

Lemma 3.3. *If v_0 satisfies the first order problem* (3.3b) *then there exists a value for $v_0(1,t;\mu)$ such that the following bounds hold for $0 \le k+m \le 6$:*

$$\left\|\frac{\partial^{k+m}v_0}{\partial x^k\partial t^m}\right\| \le C(1+\mu^{2-k}).$$

Proof. We further decompose $v_0(x,t;\mu)$ as

$$v_0(x,t;\mu) = s_0(x,t) + \mu s_1(x,t) + \mu^2 s_2(x,t;\mu), \tag{3.6a}$$

where

$$L_0 s_0 = f, \qquad s_0(x,0) = u(x,0), \tag{3.6b}$$
$$\mu L_0 s_1 = (L_0 - L_\mu)s_0, \qquad s_1(x,0) = 0, \tag{3.6c}$$
$$\mu^2 L_\mu s_2 = \mu(L_0 - L_\mu)s_1, \qquad s_2|_{\Gamma_1} = 0. \tag{3.6d}$$

We see that $v_0(1,t;\mu) = s_0(1,t) + \mu s_1(1,t)$, and if a, b, d, $f \in C^7(G)$ and $u(x,0) \in C^7(\Gamma_b)$, we have

$$\left\| \frac{\partial^{k+m} s_0}{\partial x^k \partial t^m} \right\| \leq C \qquad \text{for} \quad 0 \leq k+m \leq 7, \tag{3.7}$$

$$\left\| \frac{\partial^{k+m} s_1}{\partial x^k \partial t^m} \right\| \leq C \qquad \text{for} \quad 0 \leq k+m \leq 6 \qquad \text{and} \qquad \left\| \frac{\partial^7 s_1}{\partial x \partial t^6} \right\| \leq C. \tag{3.8}$$

Next we apply Lemmas 3.1 and 3.2 to obtain for $0 \leq k+m \leq 6$

$$\left\| \frac{\partial^{k+m} s_2}{\partial x^k \partial t^m} \right\| \leq \frac{C}{\mu^k} e^{-(k+m)AT}, \tag{3.9}$$

where $A = \min\left\{0, \frac{a}{d}\left(\frac{d}{a}\right)_t\right\}$. Using the decomposition (3.6) and the bounds (3.7), (3.8) and (3.9), we obtain the required result. □

Lemma 3.4. *If v_1 satisfies the first order problem* (3.3c) *then the following bounds hold for $0 \leq k+m \leq 4$:*

$$\left\| \frac{\partial^{k+m} v_1}{\partial x^k \partial t^m} \right\| \leq \frac{C}{\mu^k}.$$

Proof. We simply apply Lemmas 3.1 and 3.2 to (3.3c). □

Lemma 3.5. *If $v_2(x,t;\varepsilon,\mu)$ satisfies the parabolic problem* (3.2d)*, then the following bounds hold for $0 \leq k+2m \leq 3$:*

$$\left\| \frac{\partial^{k+m} v_2}{\partial x^k \partial t^m} \right\| \leq C\mu^{m-2}\left(\frac{\mu}{\varepsilon}\right)^{k+m}, \quad \textit{if} \;\; \mu^2 \geq C\varepsilon.$$

Proof. Since v_2 satisfies an equation similar to u, we can use Lemma 2.1 and

$$\|v_2(x,t;\varepsilon,\mu)\| \leq \|v_2\|_\Gamma + \frac{1}{\beta}\|v_{1xx}\|.$$

Applying the bounds in Lemma 3.4, we have $\|v_2\| \leq C\mu^{-2}$. Noting that v_2 has zero boundary conditions, we use Lemma 2.2, the bounds for v_1, and the fact that

$$\left(\frac{\varepsilon}{\mu}\right)^k \left\| \frac{\partial^{k+m} v_{1xx}}{\partial x^k \partial t^m} \right\| \leq C\mu^{-2}\left(\frac{\varepsilon}{\mu^2}\right)^k \leq C\mu^{-2}$$

to obtain the required result. □

Substituting all of these bounds for $v_0(x,t;\mu)$, $v_1(x,t;\mu)$, and $v_2(x,t;\varepsilon,\mu)$ into equation (3.3) and noting that $\mu^2 \geq C\varepsilon$, we can conclude that in this case there exists a function v satisfying (3.1) where the boundary conditions of v can be chosen so that the following bounds hold for $0 \leq k+2m \leq 3$:

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \leq C\left(1 + \left(\frac{\mu}{\varepsilon}\right)^{k-2}\right).$$

Assuming sufficient smoothness of the data, from Corollary 2.2.1 and extending the argument in the previous lemma to the case of $k+2m=4$, we deduce that

$$\|v_{tt}\| \leq C(1 + \varepsilon^2\mu^{-2}\mu^4\varepsilon^{-2}) \leq C, \quad \text{if } \mu^2 \geq \frac{\gamma\varepsilon}{\alpha}.$$

In both cases we now have the following decomposition of the solution u. Let

$$u(x,t) = v(x,t) + w_L(x,t) + w_R(x,t), \tag{3.10a}$$

where w_L and w_R satisfy homogeneous differential equations and

$$\begin{aligned}
&(3.10\text{b}) && L_{\varepsilon,\mu} v = f,\ v(0,t) \text{ and } v(1,t) \text{ chosen in (3.2) or (3.3)},\\
& && v(x,0) = u(x,0),\\
&(3.10\text{c}) && L_{\varepsilon,\mu} w_L = 0,\ w_L(x,0) = w_L(1,t) = 0,\\
& && w_L(0,t) = u(0,t) - v(0,t) - w_R(0,t),\\
&(3.10\text{d}) && L_{\varepsilon,\mu} w_R = 0,\ w_R(x,0) = 0,\ w_R(1,t) = u(1,t) - v(1,t),\\
& && \text{if } \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}, w_R(0,t) = 0 \text{ else } w_R(0,t) \text{ is chosen in (3.12).}
\end{aligned}$$

The boundary conditions of v are chosen in (3.2) or (3.3) so that the regular component satisfies the bounds

$$\left\| \frac{\partial^{k+m} v}{\partial x^k \partial t^m} \right\| \leq C\big(1 + \varepsilon^{2-k}\big), \quad \text{for} \quad 0 \leq k + 2m \leq 3, \quad \|v_{tt}\| \leq C. \tag{3.11}$$

When $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, the singular components w_L and w_R satisfy the derivative bounds given in Lemma 2.2 and Corollary 2.2.1. When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, the value for $w_R(0,t)$ is taken from the decomposition

$$w_R(x,t;\varepsilon,\mu) = w_0(x,t;\mu) + \varepsilon w_1(x,t;\mu) + \varepsilon^2 w_2(x,t;\varepsilon,\mu), \tag{3.12a}$$

where $v(1,t) = v_0(1,t)$ is given in (3.6) and

$$\begin{aligned}
&(3.12\text{b}) & L_\mu w_0 &= 0, & w_0(x,0;\mu) &= 0,\ \ w_0(1,t;\mu) = u(1,t) - v_0(1,t),\\
&(3.12\text{c}) & \varepsilon L_\mu w_1 &= (L_\mu - L_{\varepsilon,\mu}) w_0, & & w_1(x,0;\mu) = w_1(1,t;\mu) = 0,\\
&(3.12\text{d}) & \varepsilon^2 L_{\varepsilon,\mu} w_2 &= \varepsilon(L_\mu - L_{\varepsilon,\mu}) w_1, & & w_2(x,t;\varepsilon,\mu)|_\Gamma = 0.
\end{aligned}$$

Lemma 3.6. *When $w_R(x,t)$ is decomposed as in* (3.12)*, the following bound holds:*

$$|w_R(0,t)| \leq e^{-2Bt} e^{\frac{-\gamma}{\mu}},$$

where $B < A = \min\left\{0, \frac{a}{d}(\frac{d}{a})_t\right\}$.

Proof. We only need to consider the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$. Using the decomposition (3.12), we see that $w_R(0,t) = w_0(0,t) + \varepsilon w_1(0,t)$. We start by analysing $w_0(x,t)$. Consider the barrier functions $\psi^\pm(x,t) = Ce^{-\frac{\gamma}{\mu}(1-x)} \pm w_0(x,t)$. We can show that for C large enough, $\psi^\pm\big|_{\Gamma_l \cup \Gamma_r} \geq 0$ and

$$L_\mu \Psi^\pm(x,t) = C(a\gamma - b)e^{-\frac{\gamma}{\mu}(1-x)} \leq 0.$$

We can therefore apply (3.4) in order to obtain

$$|w_0(x,t)| \leq Ce^{\frac{-\gamma}{\mu}(1-x)}. \tag{3.13}$$

In order to analyse $w_1(x,t)$, we first obtain sharp bounds on $w_{0xx}(x,t)$. Differentiate (3.12b) with respect to t to obtain

$$\begin{gathered}
L_\mu^{[1]}(w_{0t}) = \mu(w_{0t})_x - \left(\frac{b}{a} + \left(\frac{d}{a}\right)_t\right) w_{0t} - \frac{d}{a}(w_{0t})_t = \left(\frac{b}{a}\right)_t w_0,\\
w_{0t}(x,0) = 0, \quad w_{0t}(1,t) = (w_R(1,t))_t.
\end{gathered}$$

Consider the barrier functions $\psi_1^{\pm}(x,t) = Ce^{-Bt}e^{-\frac{\gamma}{\mu}(1-x)} \pm w_{0t}(x,t)$, where $B<0$ is as defined. We can show that for C large enough $\psi_1^{\pm}\big|_{\Gamma_l\cup\Gamma_r} \geq 0$ and $L_\mu^{[1]}\Psi_1^{\pm}(x,t) \leq 0$. Apply (3.4) in order to obtain

$$|w_{0t}(x,t)| \leq Ce^{-Bt}e^{-\frac{\gamma}{\mu}(1-x)}.$$

Using (3.12b) this implies that

$$|w_{0x}(x,t)| \leq \frac{C}{\mu}e^{-Bt}e^{-\frac{\gamma}{\mu}(1-x)}.$$

If we differentiate (3.12b) twice with respect to t and apply the same argument, we obtain

$$|w_{0tt}(x,t)| \leq Ce^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)}.$$

Using (3.12b), this implies that

$$|w_{0xt}(x,t)| \leq \frac{C}{\mu}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)} \qquad \text{and} \qquad |w_{0xx}(x,t)| \leq \frac{C}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)}.$$

We now examine $w_1(x,t)$. Consider the barrier functions

$$\psi_2^{\pm}(x,t) = \frac{C}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)} \pm w_1(x,t).$$

Note that $\psi_2^{\pm}(x,t)\big|_{\Gamma_l\cup\Gamma_r} \geq 0$, also for C large enough

$$L_\mu\psi_2^{\pm}(x,t) = C\big[\gamma a - b + Bd\big]\frac{1}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)} \pm w_{0xx} \leq 0.$$

Therefore, using (3.4) we obtain

$$|w_1(x,t)| \leq \frac{C}{\mu^2}e^{-2Bt}e^{-\frac{\gamma}{\mu}(1-x)}. \tag{3.14}$$

Since $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, we can use (3.13) and (3.14) to complete the proof. ☐

Lemma 3.7. *When the solution of* (1.1) *is decomposed as in* (3.10a)*, the singular components w_L and w_R satisfy the bounds*

$$|w_L(x,t)| \leq Ce^{-\theta_1 x} \quad \text{and} \quad |w_R(x,t)| \leq Ce^{-\theta_2(1-x)},$$

where

$$\theta_1 = \begin{cases} \frac{\sqrt{\gamma\alpha}}{\sqrt{\varepsilon}}, & \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}, \\ \frac{\alpha\mu}{\varepsilon}, & \mu^2 \geq \frac{\gamma\varepsilon}{\alpha}, \end{cases} \qquad \theta_2 = \begin{cases} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}, & \mu^2 \leq \frac{\gamma\varepsilon}{\alpha}, \\ \frac{\gamma}{2\mu}, & \mu^2 \geq \frac{\gamma\varepsilon}{\alpha}. \end{cases}$$

Proof. Use the barrier functions

$$\psi^{\pm}(x,t) = Ce^{-\theta_1 x} \pm w_L(x,t)$$

to obtain the required bound on $|w_L(x,t)|$. When $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, the proof in the case of w_R is similar. Consider the barrier functions $\psi^{\pm}(x,t) = Ce^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \pm w_R(x,t)$. Note that

$$\begin{aligned} L_{\varepsilon,\mu}\psi^{\pm}(x,t) &= C\left(\frac{\gamma\alpha}{4} + \frac{\mu a\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}} - b\right)e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \\ &\leq C\left(\frac{\gamma a}{4} + \frac{\gamma a}{2} - b\right)e^{-\frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}(1-x)} \leq 0. \end{aligned}$$

Since $w_R(0,t) \neq 0$ in the case of $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, we have to be more careful. Consider the barrier functions

$$\psi_1^{\pm}(x,t) = Ce^{-2At}e^{\frac{-\gamma}{2\mu}(1-x)} \pm w_R(x,t),$$

where A is defined as before. Using the previous lemma, we have that $\psi_1^{\pm}(x,t)\big|_{\Gamma} \geq 0$ for C large enough and $L_{\varepsilon,\mu}\psi_1^{\pm}(x,t) \leq 0$. Use the minimum principle and the fact that $t \in (0,T]$ to obtain the required bound. ☐

Lemma 3.8. *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, w_R the solution of* (3.10d)*, satisfies the bounds*

$$\left\|\frac{\partial^k w_R}{\partial x^k}\right\| \leq C\big(\mu^{-k} + \mu^{-1}\varepsilon^{2-k}\big),\ 1 \leq k \leq 3 \quad \text{and} \quad \left\|\frac{\partial^m w_R}{\partial t^m}\right\| \leq C.\ m = 1,2.$$

Proof. Consider the decomposition (3.12). We start by analysing $w_0(x,t)$. Using the same method as used for v_1 in Lemma 3.4 we obtain for $0 \leq k+m \leq 6$ that

$$\left|\frac{\partial^{k+m} w_0}{\partial x^k \partial t^m}\right| \leq \frac{C}{\mu^k}, \tag{3.15}$$

where A is defined as before. Using this method again for $w_1(x,t)$, we obtain for $0 \leq k+m \leq 4$ that

$$\left|\frac{\partial^{k+m} w_1}{\partial x^k \partial t^m}\right| \leq \frac{C}{\mu^{k+2}}. \tag{3.16}$$

We can apply Lemma 2.1 to obtain

$$\|w_2\|_{\bar{G}} \leq \|w_2\|_{\Gamma} + \frac{1}{\beta}\|w_{1xx}\|_{\bar{G}} \leq \frac{C}{\mu^4}.$$

Finally from Lemma 2.2 we obtain for $1 \leq k+2m \leq 3$ that

$$\left\|\frac{\partial^{k+m} w_2}{\partial x^k \partial t^m}\right\|_{\bar{G}} \leq C\mu^{-4}\Big(\frac{\mu}{\varepsilon}\Big)^{k+m}\mu^m,$$

and by Corollary 2.2.1

$$\left\|\frac{\partial^2 w_2}{\partial t^2}\right\|_{\bar{G}} \leq C\mu^{-4}\mu^4\varepsilon^{-2}.$$

Using (3.12) and $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ gives us the required result. ☐

Lemma 3.9. *When $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, w_L the solution of* (3.10c)*, satisfies the bounds*

$$\left\|\frac{\partial^k w_L}{\partial x^k}\right\| \leq C\big(\frac{\mu}{\varepsilon}\big)^k,\ 1 \leq k \leq 3 \quad \text{and} \quad \left\|\frac{\partial^2 w_L}{\partial t^2}\right\| \leq C(1 + \mu^2\varepsilon^{-1}).$$

Proof. The bounds on the derivatives of the space derivatives follow from Lemma 2.2 and the fact that

$$w_L(0,t) = (u - v_0 - w_0)(0,t) - \varepsilon(v_1 + w_1)(0,t).$$

To obtain the bound on the time derivative we introduce the decomposition

$$w_L(x,t) = w_L(0,t)\phi(x,t) + \varepsilon\mu^{-2}R(x,t),$$

where the function ϕ is the solution of the boundary value problem

$$\varepsilon\phi_{xx} + \mu a(0,t)\phi_x = 0,\ x \in (0,1), \quad \phi(0,t) = 1,\ \phi(1,t) = 0.$$

Note that, by using $z^n e^{-z} \le Ce^{-z/2},\ n \ge 1,\ z \ge 0$, we have

$$\left|\frac{\partial^{k+m}\phi}{\partial x^k \partial t^m}\right| \le C\big(\frac{\mu}{\varepsilon}\big)^k e^{-\frac{\mu\alpha x}{(m+1)\varepsilon}}.$$

Note that $R = 0$ on Γ and

$$\mu^{-2}\varepsilon L_{\varepsilon,\mu}R = w_L(0,t)(\mu(a(0,t) - a(x,t))\phi_x + b\phi) + d(w_L(0,t)\phi)_t.$$

Thus, using

$$|L_{\varepsilon,\mu}R(x,t)| \le \frac{C\mu^2}{\varepsilon}\big(1 + \frac{\mu^2 x}{\varepsilon}\big)e^{-\frac{\mu\alpha x}{\varepsilon}} + C\frac{\mu^2}{\varepsilon}e^{-\frac{\mu\alpha x}{2\varepsilon}} \le \frac{C\mu^2}{\varepsilon}e^{-\frac{\mu\alpha x}{2\varepsilon}}$$

one can deduce that

$$|R(x,t)| \le Ce^{-\frac{\mu\alpha x}{2\varepsilon}}.$$

Finally, note that for $1 \le k + 2m \le 3$,

$$\left\|\frac{\partial^{k+m}(L_{\varepsilon,\mu}R)}{\partial x^k \partial t^m}\right\|_{\bar{G}} \le C\frac{\mu^2}{\varepsilon}\big(\frac{\mu}{\varepsilon}\big)^k.$$

Using Lemma 2.2 (extended to the case of $k + 2m = 4$) and noting the exponent of $(m + 1)$, implies that

$$\left\|\frac{\partial^2 R}{\partial t^2}\right\| \le C\varepsilon^{-2}\mu^4. \qquad \square$$

4. Discrete problem

We discretize (1.1) using a numerical method that is composed of a fully implicit upwinded finite difference operator $L^{N,M}$ on a tensor product mesh $\bar{G}^{N,M} = \{(x_i,t_j)\}_{i=0,j=0}^{N,M}$, which is piecewise uniform in space and uniform in time. We have the discrete problem

$$\begin{aligned} L^{N,M}U(x_i,t_j) &= \varepsilon\delta^2 U + \mu a D_x^+ U - bU - dD_t^- U = f, \qquad (x_i,t_j) \in G^{N,M} \\ &\qquad U = u, \qquad (x_i,t_j) \in \Gamma^{N,M} = \bar{G}^{N,M} \cap \Gamma, \end{aligned} \tag{4.1a}$$

where the finite difference operators D_x^+, D_t^-, and δ_x^2 are

$$D_x^+ U(x_i,t_j) = \frac{U(x_{i+1},t_j) - U(x_i,t_j)}{x_{i+1} - x_i},$$
$$D_x^- U(x_i,t_j) = \frac{U(x_i,t_j) - U(x_{i-1},t_j)}{x_i - x_{i-1}},$$
$$D_t^- U(x_i,t_j) = \frac{U(x_i,t_j) - U(x_i,t_{j-1})}{t_j - t_{j-1}},$$

and

$$\delta_x^2 U(x_i,t_j) = \frac{D_x^+ U(x_i,t_j) - D_x^- U(x_i,t_j)}{(x_{i+1} - x_{i-1})/2}.$$

The piecewise uniform mesh in space Ω^N consists of two transition points:

$$\sigma_1 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}\ln N\right\}, & \mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \\ \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\mu\alpha}\ln N\right\}, & \mu^2 \ge \frac{\gamma\varepsilon}{\alpha}, \end{cases}$$

(4.1b)

$$\sigma_2 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}\ln N\right\}, & \mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \\ \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma}\ln N\right\}, & \mu^2 \ge \frac{\gamma\varepsilon}{\alpha}. \end{cases}$$

More specifically,

$$\Omega^N = \left\{ x_i | x_i = \left\{ \begin{array}{ll} \frac{4\sigma_1 i}{N}, & i \le \frac{N}{4} \\ \sigma_1 + (i - \frac{N}{4})H, & \frac{N}{4} \le i \le \frac{3N}{4} \\ 1 - \sigma_2 + (i - \frac{3N}{4})\frac{4\sigma_2}{N}, & \frac{3N}{4} \le i \le N \end{array} \right. \right\}, \tag{4.1c}$$

where $NH = 2(1-\sigma_1-\sigma_2)$ and the mesh in time is taken to be uniform with $t_j = \frac{j}{M}$, $j = 0, \dots M$. We now state a discrete comparison principle for the finite difference operator in (4.1a), whose proof is standard.

Discrete minimum principle. If Z is any mesh function and $L^{N,M}Z\mid_{G^{N,M}} \le 0$ and $Z\mid_{\Gamma^{N,M}} \ge 0$, then $Z\mid_{\bar{G}^{N,M}} \ge 0$.

A standard corollary to this is that for any mesh function Z,

$$\|Z\| \le C\|L^{N,M}Z\| + \|Z\|_{\Gamma^{N,M}}. \tag{4.2}$$

The discrete solution can be decomposed into the sum

$$U = V + W_L + W_R, \tag{4.3a}$$

where the components are the solutions to the following:

$$L^{N,M}V = f, \qquad V|_{\Gamma^{N,M}} = v|_{\Gamma^{N,M}}, \tag{4.3b}$$

$$L^{N,M}W_L = 0, \qquad W_L|_{\Gamma^{N,M}} = w_L|_{\Gamma^{N,M}}, \tag{4.3c}$$

$$L^{N,M}W_R = 0, \qquad W_R|_{\Gamma^{N,M}} = w_R|_{\Gamma^{N,M}}. \tag{4.3d}$$

Theorem 4.1. *We have the following bounds on W_L and W_R:*

$$|W_L(x_j,t_k)| \le C\prod_{i=1}^{j}(1+\theta_L h_i)^{-1} = \Psi_{L,j}, \qquad \Psi_{L,0} = C, \tag{4.4a}$$

$$|W_R(x_j,t_k)| \le C\prod_{i=j+1}^{N}(1+\theta_R h_i)^{-1} = \Psi_{R,j}, \qquad \Psi_{R,N} = C, \tag{4.4b}$$

where W_L and W_R are solutions of (4.3c) *and* (4.3d), *respectively, $h_i = x_i - x_{i-1}$ and the parameters θ_L and θ_R are defined as*

$$\theta_L = \left\{ \begin{array}{ll} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}, & \mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \\ \frac{\mu\alpha}{2\varepsilon}, & \mu^2 \ge \frac{\gamma\varepsilon}{\alpha}, \end{array} \right. \qquad \theta_R = \left\{ \begin{array}{ll} \frac{\sqrt{\gamma\alpha}}{2\sqrt{\varepsilon}}, & \mu^2 \le \frac{\gamma\varepsilon}{\alpha}, \\ \frac{\gamma}{2\mu}, & \mu^2 \ge \frac{\gamma\varepsilon}{\alpha}. \end{array} \right. \tag{4.4c}$$

Proof. We start with W_L. Consider $\Phi_L^{\pm}(x_j,t_k) = \Psi_{L,j} \pm W_L(x_j,t_k)$. We have $L^{N,M}\Phi_L^{\pm}(x_j,t_k) = \varepsilon\delta_x^2\Psi_{L,j} + \mu a D_x^+\Psi_{L,j} - b\Psi_{L,j}$. Using the properties

$$\Psi_{L,j} > 0, \quad D_x^+\Psi_{L,j} = -\theta_L\Psi_{L,j+1} < 0, \quad \text{and} \quad \delta^2\Psi_{L,j} = {\theta_L}^2\Psi_{L,j+1}\frac{h_{j+1}}{\bar{h_j}} > 0,$$

we obtain

$$L^{N,M}\Phi_L^{\pm}(x_j,t_k) = \varepsilon{\theta_L}^2\Psi_{L,j+1}\frac{h_{j+1}}{\bar{h_j}} - \mu a\theta_L\Psi_{L,j+1} - b\Psi_{L,j},$$

where $\bar{h_j} = \frac{h_{j+1}+h_j}{2}$. Rewriting the right hand side of this equation, we have

$$L^{N,M}\Phi_{L,j}^{\pm} \le \Psi_{L,j+1}\left(2\varepsilon{\theta_L}^2\left(\frac{h_{j+1}}{2\bar{h_j}} - 1\right) + (2\varepsilon{\theta_L}^2 - \mu a\theta_L - b) - \beta\theta_L h_{j+1}\right).$$

Using this expression, we can show that for both values of θ_L, $L^{N,M}\Phi_{L,j}^{\pm} \le 0$. Now using the discrete minimum principle we obtain the required bound (4.4a).

The same idea is applied to W_R. Consider $\Phi_R^{\pm}(x_j,t_k) = \Psi_{R,j}^{\pm} \pm W_R(x_j,t_k)$. If $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, it is easy to see that $\Phi_R(0,t_k) \ge 0$, $\Phi_R(1,t_k) \ge 0$, and $\Phi_R(x_j,0) \ge 0$. However, in the other case we need to look at $\Phi_R(0,t_k)$ in more detail. We know that

$$\Phi_R^{\pm}(0,t_k) = C\prod_{i=1}^{N}(1+\frac{\gamma}{2\mu}h_i)^{-1} \pm W_R(0,t_k). \tag{4.5}$$

However, given that $e^{-\frac{\gamma}{\mu}h_i} \le (1+\frac{\gamma}{2\mu}h_i)^{-1}$ and $e^{-\frac{\gamma}{\mu}} = e^{-\frac{\gamma}{\mu}\sum_{i=1}^N h_i} = \prod_{i=1}^N e^{-\frac{\gamma}{\mu}hi}$, we see using Lemma 3.6 that $\Phi_R^{\pm}(0,t_k) \ge 0$.

Considering both cases together again,

$$L^{N,M}\Phi_R^{\pm}(x_j,t_k) = \varepsilon\delta_x^2\Psi_{R,j} + \mu a D_x^+\Psi_{R,j} - b\Psi_{R,j},$$

and using

$$\Psi_{R,j} \le \Psi_{R,j+1},\ \Psi_{R,j} > 0,\ D_x^+\Psi_{R,j} = \theta_R\Psi_{R,j},\ \text{and } \delta^2\Psi_{R,j} = \frac{{\theta_R}^2}{(1+\theta_R h_j)}\Psi_{R,j}\frac{h_j}{\bar{h}_j},$$

we obtain

$$\begin{aligned} &L^{N,M}\Phi^{\pm}(x_j,t_k) \\ &\quad \le \frac{\Psi_{R,j}}{(1+\theta_R h_j)}\left(2\varepsilon{\theta_R}^2(\frac{h_j}{2\bar{h}_j}-1) + (2\varepsilon\theta_R^2 + \mu a\theta_R - b)(1+\theta_R h_j) - 2\varepsilon\theta_R^3 h_j\right). \end{aligned}$$

Again, we can see that for both values of θ_R, that $L^{N,M}\Phi_R^{\pm}(x_j,t_k) \le 0$. Therefore, we apply the discrete minimum principle to obtain the required bound (4.4b). $\square$

5. Error analysis

In this section, we analyse the error between the continuous solution of (1.1) and the discrete solution of (4.1)

Lemma 5.1. *At each mesh point $(x_i,t_j) \in \bar{G}^{N,M}$ the regular component of the error satisfies the estimate*

$$|(V-v)(x_i,t_j)| \le C(N^{-1}+M^{-1}),$$

where v is the solution of (3.10b) *and V is the solution of* (4.3b).

Proof. Using the usual truncation error argument and (3.11), we have

$$\begin{aligned} &|L^{N,M}(V-v)(x_i,t_j)| \le C_1N^{-1}\left(\varepsilon\|v_{xxx}\| + \mu\|v_{xx}\|\right) + C_2M^{-1}\|v_{tt}\| \\ &\qquad \le C(N^{-1}+M^{-1}), \end{aligned}$$

and we apply (4.2) to obtain the required result. $\square$

Lemma 5.2. *At each mesh point $(x_i,t_j) \in \bar{G}^{N,M}$, the left singular component of the error satisfies the estimate*

$$|(W_L-w_L)(x_i,t_j)| \le \begin{cases} C(N^{-1}(\ln N)+M^{-1}), & \text{if } \mu^2 \le C\varepsilon, \\ C(N^{-1}(\ln N)^2+M^{-1}\ln N), & \text{if } \mu^2 \ge C\varepsilon, \end{cases}$$

where w_L is the solution of (3.10c) *and W_L is the solution of* (4.3c).

Proof. We use a classical argument in order to obtain the truncation error bounds

$$\begin{aligned} &|L^{N,M}(W_L - w_L)(x_i,t_j)| \\ &\quad \le C_1(h_{i+1}+h_i)\left(\varepsilon\|w_{Lxxx}\| + \mu\|w_{Lxx}\|\right) + C_2M^{-1}\|w_{Ltt}\|. \end{aligned} \tag{5.1}$$

The proof splits into the two cases of (**a**) $\sigma_1 < \frac{1}{4}$ and (**b**) $\sigma_1 = \frac{1}{4}$.

(**a**) We consider the case of $\sigma_1 < \frac{1}{4}$. In this case the mesh is piecewise uniform. First we analyse the error in the region $[\sigma_1, 1) \times (0, T]$ and then we proceed to analyse the fine mesh on $(0, \sigma_1) \times (0, T]$. To obtain the required error bounds in $[\sigma_1, 1) \times (0, T]$, we will use Lemma 3.7 and (4.4a) instead of the usual truncation error argument. From (4.4a) we have

$$|W_L(x_{\frac{N}{4}}, t_j)| \le C\Big(1 + \theta_L \frac{4\sigma_1}{N}\Big)^{-\frac{N}{4}},$$

where θ_1 and σ_1 depend on the ratio of μ^2 to ε and are given in (4.4c) and (4.1b), respectively. For both these choices of θ_L and σ_1 we can show that

$$|W_L(x_{\frac{N}{4}}, t_j)| \le C(1 + 4N^{-1}\ln N)^{-\frac{N}{4}}.$$

Using the inequality $\ln(1+t) > t(1-\frac{t}{2})$ and letting $t = 4N^{-1}\ln N$, it follows that $(1+4N^{-1}\ln N)^{-\frac{N}{4}} \le 4N^{-1}$. Therefore,

$$|W_L(x_i,t_j)| \le CN^{-1}, \qquad (x_i,t_j) \in [\sigma_1, 1) \times (0,T].$$

Looking at the continuous solution in this region, we have from Lemma 3.7 that

$$|w_L(x,t)| \le Ce^{-\theta_1 x} \le Ce^{\theta_1\sigma_1} \le CN^{-2},$$

for both choices of σ_1 and θ_1. Combining these two results we obtain the following error bounds in the region $[\sigma_1, 1) \times (0, T]$ when $\sigma_1 < \frac{1}{4}$:

$$|(W_L - w_L)(x_i,t_j)| \ \le CN^{-1}.$$

We now consider the fine mesh region $(0, \sigma_1) \times (0, T]$. We start with the case $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$. In this case the truncation error bound is

$$|L^{N,M}(W_L - w_L)(x_i)| \le \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1}+h_i) + C_2M^{-1}. \tag{5.2}$$

Since $\sigma_1 < \frac{1}{4}$, we know that $h_{i+1} = h_i = \frac{8\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}N^{-1}\ln N$, and therefore we obtain

$$|L^{N,M}(W_L - w_L)(x_i,t_j)| \le C_1(N^{-1}\ln N + M^{-1}).$$

We use (4.2) to obtain the required error bound. Next we consider the case of $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$. Here we know that $h_{i+1} = h_i = \frac{8\varepsilon}{\mu\alpha}N^{-1}\ln N$. The bound on the truncation error given in (5.1) reduces to

$$|L^{N,M}(W_L - w_L)(x_i,t_j)| \le C_1N^{-1}\ln N + C_2N^{-1}\frac{\mu^2}{\varepsilon}\ln N + C_4M^{-1}(1+\mu^2\varepsilon^{-1}).$$

If we choose

$$\begin{aligned} \psi^{\pm}(x_i,t_j) = C\Big(&N^{-1}\ln N + M^{-1} + \Big((\sigma_1 - x_i)\frac{\mu}{\varepsilon}\Big)(N^{-1}\ln N + M^{-1})\Big) \\ &\pm (W_L - w_L)(x_i,t_j) \end{aligned}$$

as our barrier functions, we find that we can choose C large enough, that both functions are nonnegative at all points in $G^{N,M}$ of the form $(0,t_j)$, $(x_{\frac{N}{4}}, t_j)$ and

$(x_i, 0)$ and $L^{N,M}\psi^{\pm} \le 0$. Therefore, by applying the discrete minimum principle, we obtain

$$|(W_L - w_L)(x_i, t_j)| \ \le C\Big(N^{-1}\ln N + M^{-1} + \Big((\sigma_1 - x_i)\frac{\mu}{\varepsilon}\Big)\,(N^{-1}\ln N + M^{-1})\Big).$$

Finally, using $\sigma_1 = \frac{2\varepsilon}{\mu\alpha}\ln N$, we have

$$|(W_L - w_L)(x_i, t_j)| \ \le C(N^{-1}(\ln N)^2 + M^{-1}\ln N). \tag{5.3}$$

(**b**) If $\sigma_1 = \frac{1}{4}$ and $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$, then $\sqrt{\frac{\gamma\alpha}{\varepsilon}} \le 8\ln N$. Since (5.2) still holds, we obtain

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \le C_1(N^{-1}\ln N + M^{-1}).$$

When $\mu^2 \ge \frac{\gamma\varepsilon}{\alpha}$ and $\sigma_1 = \frac{1}{4}$, we have $\frac{\mu\alpha}{\varepsilon} \le 8\ln N$ and then

$$|L^{N,M}(W_L - w_L)(x_i, t_j)| \le C(N^{-1}(\ln N)^2 + M^{-1}\ln N).$$

In both cases we use (4.2) to finish. $\square$

Lemma 5.3. *At each mesh point $(x_i, t_j) \in \bar{G}^{N,M}$, the right singular component of the error satisfies the estimate*

$$|(W_R - w_R)(x_i, t_j)| \le C(N^{-1}\ln N + M^{-1}), \tag{5.4}$$

where w_R is the solution of (3.10d) *and W_R is the solution of* (4.3d).

Proof. (**a**) We consider the case of $\sigma_2 < \frac{1}{4}$. We will start by examining the region $(0, 1 - \sigma_2] \times (0, T]$. Using (4.4b) we have

$$|W_R(x_{\frac{3N}{4}}, t_j)| \le C\Big(1 + \theta_R\frac{4\sigma_2}{N}\Big)^{-\frac{N}{4}},$$

where θ_R and σ_2 depend on the ratio of μ^2 to ε and are given in (4.4c) and (4.1b), respectively. We can show that for both choices of θ_R and σ_2, we have

$$|W_R(x_{\frac{3N}{4}}, t_j)| \le C\Big(1 + 4N^{-1}\ln N\Big)^{-\frac{N}{4}}$$

and, using the same argument as with W_L, we conclude that if

$$(x_i, t_j) \in (0, 1 - \sigma_2] \times (0, T],$$

then

$$|W_R(x_i, t_j)| \le CN^{-1}.$$

Next, looking at the continuous solution in this region we have

$$|w_R(x, t)| \le Ce^{-\theta_2(1-x)} \le Ce^{-\theta_2\sigma_2} \le CN^{-1}$$

for both choices of σ_2 and θ_2. Therefore, we obtain the following bounds on the error in the region $(0, 1 - \sigma_2] \times (0, T]$ when $\sigma_2 < \frac{1}{4}$:

$$|(W_R - w_R)(x_i, t_j)| \ \le CN^{-1}. \tag{5.5}$$

We now consider the fine mesh region $(1 - \sigma_2, 1) \times (0, T]$, where W_R satisfies a similar truncation error bound to that of W_L in (5.1). We start with the case of $\mu^2 \le \frac{\gamma\varepsilon}{\alpha}$. As with W_L, (5.1) simplifies to

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \le \frac{C_1}{\sqrt{\varepsilon}}(h_{i+1} + h_i) + C_2M^{-1}. \tag{5.6}$$

Since we are in the fine mesh region, we have $h_{i+1} = h_i = \frac{8\sqrt{\varepsilon}}{\sqrt{\alpha\gamma}} N^{-1} \ln N$, and using (5.6) we now obtain

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}.$$

If $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$, using classical analysis we can obtain the following truncation error bounds

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1(h_{i+1} + h_i)\left(\varepsilon\|w_{Rxxx}\| + \mu\|w_{Rxx}\|\right) + C_2 M^{-1}\|w_{Rtt}\|.$$

Using the bounds on w_R in Lemma 3.8 and Corollary 2.2.1, we find that this simplifies to

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq \frac{C_1}{\mu}(h_{i+1} + h_i) + C_2 M^{-1}. \tag{5.7}$$

Since we are in the fine mesh region, we have $h_{i+1} = h_i = \frac{8\mu}{\gamma} N^{-1} \ln N$. Using (5.7) we now obtain

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}.$$

Use (4.2) to finish in both cases.

(**b**) If $\sigma_2 = \frac{1}{4}$ and $\mu^2 \leq \frac{\gamma\varepsilon}{\alpha}$, then $\sqrt{\frac{\gamma\alpha}{\varepsilon}} \leq 8 \ln N$, and since (5.2) holds, we have

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}.$$

If $\mu^2 \geq \frac{\gamma\varepsilon}{\alpha}$ and $\sigma_2 = \frac{1}{4}$, then $\frac{\gamma}{\mu} \leq 8 \ln N$, and using (5.7), we obtain

$$|L^{N,M}(W_R - w_R)(x_i, t_j)| \leq C_1 N^{-1} \ln N + C_2 M^{-1}.$$

In both cases, we use (4.2) to complete the proof. □

Theorem 5.4. *At each mesh point $(x_i, t_j) \in \bar{G}^{N,M}$ the maximum pointwise error satisfies the following parameter-uniform error bound*

$$\|U - u\|_{G^{N,M}} \leq \begin{cases} C(N^{-1}(\ln N) + M^{-1}), & \text{if } \mu^2 \leq C\varepsilon, \\ C(N^{-1}(\ln N)^2 + M^{-1}\ln N), & \text{if } \mu^2 \geq C\varepsilon, \end{cases} \tag{5.8}$$

where u is the solution of (1.1) *and U is the solution of* (4.1).

Proof. The proof follows from Lemmas 5.1, 5.2, and 5.3. □

Remark 5.5. It is worth noting that the error bound (5.8) extends to the case of $-1 \leq \mu \leq 1$, where the discrete problem is defined to be

$$L^{N,M}U(x_i, t_j) = \varepsilon\delta^2 U + \mu a D_x U - bU - dD_t^- U = f, \qquad (x_i, t_j) \in G^{N,M}, \tag{5.9a}$$

$$D_x = \begin{cases} D_x^- & \mu < 0, \\ D_x^+ & \mu \geq 0, \end{cases}$$

and the transition points in the piecewise uniform mesh in space are taken to be

$$\sigma_1 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2|\mu|}{\gamma}\ln N\right\}, & \mu \leq -\sqrt{\frac{\gamma\varepsilon}{\mu}}, \\ \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}\ln N\right\}, & |\mu| \leq \sqrt{\frac{\gamma\varepsilon}{\alpha}}, \\ \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\mu\alpha}\ln N\right\}, & \mu \geq \sqrt{\frac{\gamma\varepsilon}{\alpha}}, \end{cases} \tag{5.9b}$$

$$\sigma_2 = \begin{cases} \min\left\{\frac{1}{4}, \frac{2\varepsilon}{|\mu|\alpha}\ln N\right\}, & \mu \leq -\sqrt{\frac{\gamma\varepsilon}{\alpha}}, \\ \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\sqrt{\gamma\alpha}}\ln N\right\}, & |\mu| \leq \sqrt{\frac{\gamma\varepsilon}{\alpha}}, \\ \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma}\ln N\right\}, & \mu \geq \sqrt{\frac{\gamma\varepsilon}{\alpha}}. \end{cases} \tag{5.9c}$$

6. Numerical results

The numerical method (4.1), has been applied to the particular problem,

$$(\varepsilon u_{xx} + \mu(1+x)u_x - u - u_t)(x,t) \\ = 16x^2(1-x)^2,\ (x,t) \in (0,1) \times (0,1]; \quad u|_\Gamma = 0. \tag{6.1}$$

In the numerical experiments we have taken $N = M$. We define the maximum pointwise two-mesh differences to be

$$D^N_{\varepsilon,\mu} = \|U^N_{\varepsilon,\mu} - \overline{U}^{2N}_{\varepsilon,\mu}\|_{G^{N,M}},$$

where $\overline{U}^N_{\varepsilon,\mu}$ are the piecewise linear interpolants of the numerical solutions $U^N_{\varepsilon,\mu}$. From these values one can compute the ε-uniform maximum pointwise two-mesh differences D^N_μ and the (ε,μ)-uniform maximum pointwise two-mesh differences D^N, which are defined by

$$D^N_\mu = \max_{\varepsilon \in R_\varepsilon} D^N_{\varepsilon,\mu}, \qquad D^N = \max_{\mu \in R_\mu} \max_{\varepsilon \in R_\varepsilon} D^N_{\varepsilon,\mu},$$

where $R_\varepsilon = [2^{-26}, 1]$ and $R_\mu = [2^{-22}, 1]$. Approximations for the order of local convergence $p^N_{\varepsilon.\mu}$, the ε-uniform order of local convergence p^N_μ and the (ε,μ)-uniform order of convergence p^N are computed from

$$p^N_{\varepsilon,\mu} = \log_2 \frac{D^N_{\varepsilon.\mu}}{D^{2N}_{\varepsilon.\mu}}, \quad p^N_\mu = \log_2 \frac{D^N_\mu}{D^{2N}_\mu}, \qquad \text{and} \quad p^N = \log_2 \frac{D^N}{D^{2N}}.$$

The numerical results presented in Tables 1, 2, and 3 are in agreement with the theoretical asymptotic error bound (5.8).

Table 1. The orders of local convergence $p^N_{\varepsilon,\mu}$ and the ε-uniform orders of local convergence p^N_μ generated by the upwind finite difference operator (4.1a) and the mesh (4.1c) applied to problem (6.1) for $\mu = \mathbf{2^{-2}}$ and for various values of ε and $N(=M)$.

	Number of intervals $N(=M)$					
ε	8	16	32	64	128	256
2^0	0.62	0.76	0.87	0.93	0.96	0.98
2^{-2}	0.76	0.89	0.95	0.97	0.99	0.99
2^{-4}	0.80	0.90	0.95	0.97	0.99	0.99
2^{-6}	0.78	0.85	0.92	0.95	0.98	0.99
2^{-8}	0.68	0.76	0.90	0.97	1.00	1.02
2^{-10}	0.65	0.76	0.86	0.93	0.97	0.99
2^{-12}	0.61	0.75	0.86	0.93	0.97	0.98
2^{-14}	0.60	0.75	0.86	0.93	0.96	0.98
2^{-16}	0.59	0.75	0.86	0.93	0.96	0.98
2^{-18}	0.59	0.75	0.86	0.93	0.96	0.98
2^{-20}	0.59	0.75	0.86	0.93	0.96	0.98
2^{-22}	0.59	0.75	0.86	0.93	0.96	0.98
2^{-24}	0.59	0.75	0.86	0.93	0.96	0.98
2^{-26}	0.59	0.75	0.86	0.93	0.96	0.98
$p^N_{\mu=2^{-2}}$	0.59	0.75	0.86	0.93	0.96	0.98

TABLE 2. The orders of local convergence $p^N_{\varepsilon,\mu}$ and the ε-uniform orders of local convergence p^N_μ generated by the upwind finite difference operator (4.1a) and the mesh (4.1c) applied to problem (6.1) for $\mu = \mathbf{2^{-10}}$ and for various values of ε and $N(= M)$.

ε	Number of intervals $N(= M)$					
	8	16	32	64	128	256
2^0	0.61	0.75	0.87	0.93	0.96	0.98
2^{-2}	0.75	0.88	0.94	0.97	0.98	0.99
2^{-4}	0.80	0.90	0.95	0.98	0.99	0.99
2^{-6}	0.86	0.93	0.97	0.98	0.99	1.00
2^{-8}	0.92	0.96	0.98	0.99	0.99	1.00
2^{-10}	0.93	0.97	0.99	0.99	1.00	1.00
2^{-12}	0.94	0.97	0.99	0.99	1.00	1.00
2^{-14}	0.94	0.97	0.99	0.99	1.00	1.00
2^{-16}	0.94	0.97	0.99	0.99	1.00	1.00
2^{-18}	0.94	0.97	0.99	0.99	1.00	1.00
2^{-20}	0.94	0.97	0.99	0.99	1.00	1.00
2^{-22}	0.94	0.97	0.99	0.99	0.99	0.99
2^{-24}	0.94	0.97	0.98	0.99	0.99	0.99
2^{-26}	0.94	0.97	0.98	0.99	0.99	0.99
$p^N_{\mu=2^{-10}}$	0.94	0.97	0.99	0.99	1.00	1.00

TABLE 3. The orders of ε-uniform local convergence p^N_μ and the (ε,μ)-uniform orders of local convergence p^N generated by the upwind finite difference operator (4.1a) and the mesh (4.1c) applied to problem (6.1) for various values of ε, μ and $N(= M)$.

μ	Number of intervals $N(= M)$					
	8	16	32	64	128	256
2^0	0.41	0.46	0.58	0.66	0.71	0.80
2^{-2}	0.59	0.75	0.86	0.93	0.96	0.98
2^{-4}	0.85	0.91	0.97	0.98	0.99	1.00
2^{-6}	0.89	0.98	0.97	0.98	1.01	1.00
2^{-10}	0.94	0.97	0.99	0.99	1.00	1.00
2^{-14}	0.95	0.97	0.99	0.99	1.00	1.00
2^{-18}	0.95	0.97	0.99	0.99	1.00	1.00
2^{-22}	0.95	0.97	0.99	0.99	1.00	1.00
p^N	0.95	0.97	0.99	0.99	1.00	1.00

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School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland

E-mail address: eugene.oriordan@dcu.ie

School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland

E-mail address: maria.pickett2@mail.dcu.ie

Institute for Mathematics and Mechanics, Russian Academy of Sciences, Ekaterinburg, Russia

E-mail address: shishkin@imm.uran.ru