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WEAK COUPLING OF SOLUTIONS OF FIRST-ORDER LEAST-SQUARES METHOD

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ABSTRACT. A theoretical analysis of a first-order least-squares finite element method for second-order self-adjoint elliptic problems is presented. We investigate the coupling effect of the approximate solutions u_h for the primary function u and $\boldsymbol{\sigma}_h$ for the flux $\boldsymbol{\sigma} = -\mathcal{A}\nabla u$. We prove that the accuracy of the approximate solution u_h for the primary function u is weakly affected by the flux $\boldsymbol{\sigma} = -\mathcal{A}\nabla u$. That is, the bound for $\|u - u_h\|_1$ is dependent on $\boldsymbol{\sigma}$, but only through the best approximation for $\boldsymbol{\sigma}$ multiplied by a factor of meshsize h. Similarly, we provide that the bound for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(div)}$ is dependent on u, but only through the best approximation for u multiplied by a factor of the meshsize h. This weak coupling is not true for the non-selfadjoint case. We provide the numerical experiment supporting the theorems in this paper.

1. Introduction

The purpose of this paper is to analyze the least-squares finite element method for second order self-adjoint elliptic partial differential equations proposed by Cai et al. [7] and Pehlivanov et al. [11]. They introduced a new variable $\boldsymbol{\sigma}$ that transforms the corresponding second-order problem into a system of first-order. Mixed Galerkin methods applied to the system lead to a saddle-point problem and finite element spaces should satisfy the *inf-sup* condition of Ladyzhenskaya-Babuska-Brezzi [1, 5]. Although there has been substantial progress, it may still be difficult and expensive to solve saddle-point problems. It is well known that least-squares type methods applied to the system lead to a minimization problem solving symmetric and positive definite system. The approximation spaces do not require the *inf-sup* condition and any conforming finite element spaces including piecewise continuous polynomial spaces can be used as approximation spaces.

The main theoretical result in this paper shows the dependence of $\|u - u_h\|_1$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(div)}$. Our interest stems from the the following observation. For the problem $-div\mathcal{A}\nabla u + u = f$, let $(u_h^1, \boldsymbol{\sigma}_h^1)$ be the approximate solution for $(u, -\mathcal{A}\nabla u)$ obtained by the least-squares method and let u_h^G be the approximate solution for u obtained by the standard Galerkin method. Then, $u_h^1 = u_h^G$. We will provide a proof in Section 4. Hence, the bound for $\|u - u_h\|_1$ is independent of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$. This is not true in general. However, for selfadjoint elliptic problems, we are able to

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show that the effect of $\boldsymbol{\sigma}$ and its approximate space is weak in the following sense,

$$\|u-u_h\|_1 \leq C\left(\inf_{\chi\in V_h}\|u-\chi\|_1 + h\inf_{\mathbf{q}_h\in\mathbf{W}_h}\|\boldsymbol{\sigma}-\mathbf{q}_h\|_{H(div)}\right), \tag{1.1}$$

where h is the meshsize. That is, the bound for $\|u-u_h\|_1$ is dependent on the best approximation for $\boldsymbol{\sigma}$ multiplied by a factor of the meshsize h. Similarly, the effect of u and its approximate space is weak on $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}$ in the following sense,

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)} \leq C\left(\inf_{\mathbf{q}_h\in\mathbf{W}_h}\|\boldsymbol{\sigma}-\mathbf{q}_h\|_{H(div)} + h\inf_{\chi\in V_h}\|u-\chi\|_1\right). \tag{1.2}$$

As a consequence of (1.1) and (1.2), we obtain improved error estimates for the primary function u and flux $\boldsymbol{\sigma}$. In the non-selfadjoint case, i.e. in the presence of first-order term in the differential equations, (1.1) and (1.2) are no longer true and we provide the numerical experiment confirming this.

Various error estimates are obtained for least-squares methods similar to the one we discuss in this paper; see, e.g. Manteuffel et al. [10], and Pehlivanov et al. [12]. Their error estimate $\|u-u_h\|_1$ for the primary function u is $O(h)$ for $u\in H^3(\Omega)$. Our estimate provides $O(h^2)$ for $u\in H^3(\Omega)$. Brandts et al. [4] recently obtained an error estimate for a least-squares method by comparing the least-squares solution with the standard Galerkin solution and mixed Galerkin solution. They showed that those solutions are superclose. Here, we use the duality argument to show the relationship between the least-squares solution in (1.1). Moreover, any conforming finite element spaces including the standard piecewise continuous polynomial spaces can be used as approximate spaces for our analysis. This is not the case for [4]. For other least-squares type methods, we refer to [2, 7, 10, 12] and the references therein.

This paper is organized as follows: Section 2 formulates the problem. For a detailed presentation, we refer to [7, 11]. The finite element approximation is presented with a known error estimate for the method in Section 3. Section 4 deals with the case when the solutions for the least-squares method are decoupled. In Section 5, we provide a proof for (1.1) and (1.2) and state the resulting error estimates. We present the confirming numerical experiment in Section 6.

2. Problem formulation

Let Ω be a convex bounded domain in $\mathbb{R}^n, n=2,3$, with Lipschitz boundary $\Gamma=\partial\Omega$. We consider the Dirichlet boundary value problem

$$\begin{aligned} -div\mathcal{A}\nabla u + c\,u &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{2.1}$$

for $f\in L^2(\Omega)$ and $\mathcal{A}=(a_{ij}(x))_{i,j=1}^n, x\in\Omega$. The coefficients a_{ij} and c are smooth and the matrix $\mathcal{A}$ is symmetric and uniformly positive definite, i.e., there exist positive constants α_0 and α_1 such that

$$\alpha_0\zeta^T\zeta \leq \zeta^T\mathcal{A}\zeta \leq \alpha_1\zeta^T\zeta, \tag{2.2}$$

for all $\zeta\in\mathbb{R}^n$ and all $x\in\Omega$.

We assume that there exists a unique solution to (2.1). Also, we assume the following *a priori* estimate for u satisfying (2.1): there exists a positive constant C independent of f satisfying

$$\|u\|_2 \leq C\,\|f\|_0. \tag{2.3}$$

By introducing a new variable $\boldsymbol{\sigma} = -\mathcal{A}\nabla u$, we transform the original problem into a system of first-order:

$$
\begin{aligned}
\boldsymbol{\sigma} + \mathcal{A}\nabla u &= 0 \qquad \text{in } \Omega, \\
div\boldsymbol{\sigma} + c\,u &= f \qquad \text{in } \Omega, \\
u &= 0 \qquad \text{on } \Gamma.
\end{aligned} \tag{2.4}
$$

Let $H^s(\Omega)$ denote the Sobolev space of order s defined on Ω. The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. For $s = 0, H^s(\Omega)$ coincides with $L^2(\Omega)$. We shall use the spaces

$$
\begin{aligned}
V &= \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}, \\
\mathbf{W} &= H(div) \equiv \{\boldsymbol{\sigma} \in (L^2(\Omega))^n : div\boldsymbol{\sigma} \in L^2(\Omega)\},
\end{aligned}
$$

with norms

$$
\begin{aligned}
\|u\|_1^2 &= (u, u) + (\nabla u\,,\, \nabla u), \\
\|\boldsymbol{\sigma}\|_{H(div)}^2 &= (div\boldsymbol{\sigma}\,,\, div\boldsymbol{\sigma}) + (\boldsymbol{\sigma}\,,\, \boldsymbol{\sigma}).
\end{aligned}
$$

Also, we introduce a new norm

$$
\|\boldsymbol{\sigma}\|_{H(div,\mathcal{A})}^2 = (div\boldsymbol{\sigma}\,,\, div\boldsymbol{\sigma}) + \left(\mathcal{A}^{-1}\boldsymbol{\sigma}\,,\, \boldsymbol{\sigma}\right),
$$

for $\boldsymbol{\sigma} \in \mathbf{W}$. Since $\mathcal{A}$ is uniformly positive definite, we have

$$
c\|\boldsymbol{\sigma}\|_{H(div)} \leq \|\boldsymbol{\sigma}\|_{H(div,\mathcal{A})} \leq C\|\boldsymbol{\sigma}\|_{H(div)} \text{ for all } \boldsymbol{\sigma} \in \mathbf{W}. \tag{2.5}
$$

The space $H^{-1}(\Omega)$ is defined by duality and consists of the functional v for which the norm

$$
\|v\|_{-1} = \sup_{\phi \in V} \frac{(v, \phi)}{\|\phi\|_1} \tag{2.6}
$$

is finite, where (v, ϕ) is the value of the functional at ϕ.

Then, the least-squares method for the first-order system (2.4) is: Find $u \in V, \boldsymbol{\sigma} \in \mathbf{W}$ such that

$$
\begin{aligned}
b(u, \boldsymbol{\sigma}; v, \mathbf{q}) &\equiv (div\,\boldsymbol{\sigma} + c\,u, div\,\mathbf{q} + c\,v) + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \mathbf{q} + \mathcal{A}\nabla v\right) \\
&= (f, div\,\mathbf{q} + c\,v),
\end{aligned} \tag{2.7}
$$

for all $v \in V, \mathbf{q} \in \mathbf{W}$.

3. Finite element approximation

Let τ_h be a regular triangulation of Ω (see [8]) with triangular/tetrahedra elements of size $h = \max\{diam(K); K \in \tau_h\}$. Let $P_k(K)$ be the space of polynomials of degree k on triangle K and denote the local Raviart-Thomas space of order k on K:

$$
RT_k(K) = P_k(K)^n + \mathbf{x}\, P_k(K)
$$

with $\mathbf{x} = (x_1, \ldots, x_n)$. Then the standard (conforming) continuous piecewise polynomials of degree k and the standard $H(div)$ conforming Raviart-Thomas space of index r [13] are defined, respectively, by

$$
\begin{aligned}
V_h &= \{v \in V : v|_K \in P_k(K) \text{ for all K } \in \tau_h\}, \\
\mathbf{W}_h &= \{\boldsymbol{\tau} \in \mathbf{W} : \boldsymbol{\tau}|_K \in RT_r(K) \text{ for all } K \in \tau_h\}.
\end{aligned}
$$

It is well known (see [8]) that V_h has the following approximation property: let $k \geq 1$ be an integer and let $l \in [1, k+1]$,

$$\inf_{v_h \in V_h} \|v - v_h\|_1 \leq Ch^l \|v\|_{l+1}, \tag{3.1}$$

for $u \in H^{l+1}(\Omega)$. It is also well-known (see [13]) that $\mathbf{W}_h$ has the following approximate property: let $r \geq 0$ be an integer and let $l \in [1, r+1]$,

$$\inf_{\mathbf{q}_h \in \mathbf{W}_h} \|\mathbf{q} - \mathbf{q}_h\|_{H(div)} \leq Ch^l (\|\mathbf{q}\|_l + \|div\, \mathbf{q}\|_l) \tag{3.2}$$

for $\mathbf{q} \in H^l(\Omega)^n$ with $div\, \mathbf{q} \in H^l(\Omega)$.

Remark 3.1. The analysis in this paper can be generalized to any conforming finite element spaces. We choose the conforming Raviert-Thomas family of elements as an approximate space for $\mathbf{W}$ since it does not require $\mathbf{q} \in H^{l+1}(\Omega)^n$ but only requires $div\, \mathbf{q} \in H^l(\Omega)$ to have the approximate property of (3.2). If the standard conforming piecewise continuous polynomial spaces are used for $\mathbf{W}$, then higher regularity, i.e. H^3, is required for the solution of the adjoint problem to obtain (1.1) and (1.2).

The finite element approximation to (2.7) is: Find $u_h \in V_h$ and $\boldsymbol{\sigma}_h \in \mathbf{W}_h$ such that

$$b(u_h, \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = (f, div\, \mathbf{q}_h + c\, v_h), \tag{3.3}$$

for all $v_h \in V_h, \mathbf{q}_h \in \mathbf{W}_h$. It is well-known that (3.3) has a unique solution. Moreover, the error has the orthogonal property

$$b(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = 0, \text{ for all } v_h \in V_h, \mathbf{q}_h \in \mathbf{W}_h. \tag{3.4}$$

The following theorem is proved in [7].

Theorem 3.1. *Let $(u, \boldsymbol{\sigma})$ and $(u_h, \boldsymbol{\sigma}_h)$ satisfy the equations in (2.4) and (3.3) respectively. Then,*

$$\|u - u_h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(div)} \leq C \inf_{\chi \in V_h, \mathbf{q}_h \in \mathbf{W}_h} (\|u - \chi\|_1 + \|\boldsymbol{\sigma} - \mathbf{q}_h\|_{H(div)})$$

where the constant C is independent of h, u, or $\boldsymbol{\sigma}$.

Also, we need to have the following L^2-norm error estimate for $\|u - u_h\|_0$ proved in [6].

Theorem 3.2. *Let $(u, \boldsymbol{\sigma})$ and $(u_h, \boldsymbol{\sigma}_h)$ satisfy the equations in (2.4) and (3.3) respectively. Then,*

$$\|u - u_h\|_0 \leq C\, h \inf_{\chi \in V_h, \mathbf{q}_h \in \mathbf{W}_h} (\|u - \chi\|_1 + \|\boldsymbol{\sigma} - \mathbf{q}_h\|_{H(div)})$$

where the constant C is independent of h, u, or $\boldsymbol{\sigma}$.

4. Decoupling of the least-squares solution when $c = 1$

In this section, we provide a proof that the least-squares approximate solution u_h for the primary function u and $\boldsymbol{\sigma}_h$ for the dual function $\boldsymbol{\sigma}$ is independent when $c = 1$. We denote $(u_h^1, \boldsymbol{\sigma}_h^1)$ for the least-squares approximate solution when $c = 1$. Then, u_h^1 is the standard Galerkin solution (see Lemma 4.1) and $\boldsymbol{\sigma}_h^1$ is the $H(div, \mathcal{A})$ projection; see Lemma 4.2.

Lemma 4.1. *Let $c\,u = u$ in (2.1). Let $(u, \boldsymbol{\sigma})$ and $(u_h^1, \boldsymbol{\sigma}_h^1)$ satisfy the equations in (2.4) and (3.3) respectively. Let u_h^G be the approximate solution obtained by the standard Galerkin method. Then $u_h^1 = u_h^G$.*

Proof. By the orthogonal property of the least-squares solution (3.4), the definition in (2.7), and the integration by parts, we have

$$\begin{aligned} 0 &= b(u-u_h^1, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1\,, v_h, 0) \\ &= (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1 + \mathcal{A}\nabla(u-u_h^1)\,, \nabla v_h) + (div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1) + u - u_h^1\,, v_h) \\ &= (\mathcal{A}\nabla(u-u_h^1)\,,\, \nabla v_h) + (u-u_h^1, v_h) \end{aligned}$$

for all $v_h \in V_h$. Now, the last equation implies that u_h^1 is the solution of the standard Galerkin method, i.e, $u_h^1 = u_h^G$ by the uniqueness of the approximate solution. This completes the proof. □

Lemma 4.2. *Let $c\,u = u$ in (2.1). Let $(u, \boldsymbol{\sigma})$ and $(u_h^1, \boldsymbol{\sigma}_h^1)$ satisfy the equations in (2.4) and (3.3) respectively. Then,*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1\|_{H(div)} \le C \inf_{\mathbf{q}\in\mathbf{W}_h} \|\boldsymbol{\sigma}-\mathbf{q}\|_{H(div)}.$$

Proof. Due to (2.5), it suffices to prove that

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1\|_{H(div,\mathcal{A})} = \inf_{\mathbf{q}\in\mathbf{W}_h} \|\boldsymbol{\sigma}-\mathbf{q}\|_{H(div,\mathcal{A})}.$$

In order to prove the above equality, it suffices to show that

$$(div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1), div\,\mathbf{q}_h) + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1), \mathbf{q}_h\right) = 0,$$

for all $\mathbf{q}_h \in \mathbf{W}_h$.

By the orthogonal property (3.4), the definition in (2.7), and the integration by parts, we have

$$\begin{aligned} 0 &= b(u-u_h^1, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1\,; 0, \mathbf{q}_h) \\ &= \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1) + \nabla(u-u_h^1)\,, \mathbf{q}_h\right) + (div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1) + u - u_h^1\,, div\,\mathbf{q}_h) \\ &= \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1), \mathbf{q}_h\right) + (div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^1), div\,\mathbf{q}_h) \end{aligned} \tag{4.1}$$

for all $\mathbf{q}_h \in \mathbf{W}_h$. This completes the proof. □

Remark 4.1. Since $(u_h^1, \boldsymbol{\sigma}_h^1)$ satisfies (4.1) the following equality holds:

$$\begin{aligned} &(div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + (u-u_h)\,,\, div\,\mathbf{q}_h) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + \nabla\,(u-u_h)\,,\, \mathbf{q}_h) \\ &= (div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) + (u_h^1-u_h),\, div\,\mathbf{q}_h) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) + \nabla\,(u_h^1-u_h), \mathbf{q}_h) \end{aligned} \tag{4.2}$$

for all $\mathbf{q}_h \in \mathbf{W}_h$.

5. Weak coupling of the least-squares solution

In this section, we provide a weak coupling of $u-u_h$ and $\boldsymbol{\sigma}-\boldsymbol{\sigma}_h$ with arbitrary smooth function c. The bound $\|u-u_h\|_1$ is dependent on $\boldsymbol{\sigma}$ via an approximate term $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}$ multiplied by the meshsize h. The bound $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}$ is dependent on $\boldsymbol{\sigma}$ via an approximate term $\|u-u_h\|_1$ multiplied by the meshsize h.

Throughout Section 5, we have

$$a(u,v) = (\mathcal{A}\nabla u, \nabla v) + (c\,u, v).$$

Also, as mentioned before $(u_h^1, \boldsymbol{\sigma}_h^1)$ denote the least-squares solution with $c = 1$.

Theorem 5.1. *Let* $(u, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy the equations in* 2.4 *and* (3.3) *respectively. Then,*

$$\|u - u_h\|_1 \le C \left(\inf_{\chi \in V_h} \|u - \chi\|_1 + h \inf_{\mathbf{q}_h \in \mathbf{W}_h} \|\boldsymbol{\sigma} - \mathbf{q}_h\|_{H(div)}) \right).$$

Proof. By Bramble et al. [3, Lemma 2.2], $C_0^\infty(\Omega)$ is dense in V, for $\psi \in V$,

$$C\|\psi\|_1 \le \sup_{v \in C_0^\infty(\Omega)} \frac{a(\psi, v)}{\|v\|_1}, \text{ where } C \text{ is independent of } \psi.$$

By taking $\psi = u - u_h$, we obtain

$$C\|u - u_h\|_1 \le \sup_{v \in C_0^\infty(\Omega)} \frac{a(u - u_h, v)}{\|v\|_1}. \tag{5.1}$$

By the definition of $a(\cdot, \cdot)$ and integration by parts, we have

$$\begin{aligned} a(u - u_h, v) &= (\mathcal{A}\nabla(u - u_h), \nabla v) + (c\,(u - u_h), v) \\ &= (div(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + c\,(u - u_h), v) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h + \mathcal{A}\nabla(u - u_h), \nabla v). \end{aligned} \tag{5.2}$$

For recovering the bilinear form $b(\cdot, \cdot\,; \cdot, \cdot)$, let w be such that

$$\begin{array}{rcll} -div\mathcal{A}\nabla w + c\,w &=& -div\mathcal{A}\nabla v + v & \text{in } \Omega, \\ w &=& 0 & \text{on } \Gamma. \end{array} \tag{5.3}$$

Since Ω is a convex polygonal domain and $v \in C_0^\infty(\Omega)$, we have $w \in H^2(\Omega)$. Also, by a well-known *a priori* estimate and the definition of negative norm (2.6), we have

$$\|w\|_1 \le C\| - div\mathcal{A}\nabla v + v\|_{-1} \le C\|v\|_1. \tag{5.4}$$

From (5.3) and $v \in C_0^\infty(\Omega)$, i.e. $v = 0$ on Γ, we have

$$div\mathcal{A}\nabla(v - w) = v - c\,w \text{ in } \Omega \quad \text{ and } \quad v - w = 0 \text{ on } \Gamma. \tag{5.5}$$

Set $\mathbf{d} = \mathcal{A}\nabla(v - w)$. Then, we have $div\,\mathbf{d} = v - c\,w$. For example,

$$v = div\,\mathbf{d} + c\,w. \tag{5.6}$$

Also, it is clear from $\mathbf{d} = \mathcal{A}\nabla(v - w)$ that

$$\mathcal{A}\nabla v = \mathbf{d} + \mathcal{A}\nabla w. \tag{5.7}$$

By using equalities (5.6), (5.7) in (5.2) and by the definition of $b(\cdot, \cdot\,; \cdot, \cdot)$ in (2.7), we obtain

$$\begin{aligned} a(u - u_h, v) &= (div(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + c(u - u_h), div\,\mathbf{d} + cw) \\ &\quad + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h + \mathcal{A}\nabla(u - u_h)), \mathbf{d} + \mathcal{A}\nabla w\right) \\ &= b(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\,; w, \mathbf{d}). \end{aligned} \tag{5.8}$$

To approximate w, let $S_h w \in V_h$ be the Ritz projection of w, i.e.,

$$(\mathcal{A}\nabla(w - S_h w), \nabla v_h) = 0 \text{ for all } v_h \in V. \tag{5.9}$$

It is well known that

$$\|w - S_h w\|_0 \le Ch\|w\|_1 \quad \text{and} \quad \|w - S_h w\|_1 \le C\|w\|_1. \tag{5.10}$$

To approximate $\mathbf{d} = \nabla(v - w)$, first note that $\mathbf{d} \in (H^1)^n$ and $div\mathbf{d} = v - c\,w \in H^1(\Omega)$ since $v \in C_0^\infty(\Omega)$ and $w \in H^2(\Omega)$. Using *a priori* estimate (2.3) and (5.4), we have

$$\|\mathbf{d}\|_1 \le C\|v - w\|_2 \le C\|v - c\,w\|_1 \le C\|w\|_1 + \|v\|_1 \le C\|v\|_1.$$

Also, by triangle inequality and (5.4), we have

$$\|div\mathbf{d}\|_1 = \|v - c\,w\|_1 \le C\|w\|_1 + \|v\|_1 \le C\|v\|_1.$$

Let $\mathbf{d}_I \in \mathbf{W}_h$ satisfy

$$\|\mathbf{d} - \mathbf{d}_I\|_{H(div)} \le Ch(\|\mathbf{d}\|_1 + \|div\,\mathbf{d}\|_1) \le C\,h\|v\|_1. \tag{5.11}$$

Using the orthogonal property (3.4) in (5.8) and the definition of $b(\cdot,\cdot;\cdot,\cdot)$ in (2.7), we have

$$\begin{aligned} a(u-u_h,v) &= b(u-u_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\,;\,w-S_hw,\mathbf{d}-\mathbf{d}_I) \\ &= (div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+c\,(u-u_h), div(\mathbf{d}-\mathbf{d}_I)+c(w-S_hw)) \\ &\quad + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h+\mathcal{A}\nabla(u-u_h))\,,\mathbf{d}-\mathbf{d}_I+\mathcal{A}\nabla(w-S_hw)\right) \\ &= (div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+c\,(u-u_h)\,, div(\mathbf{d}-\mathbf{d}_I)+c\,(w-S_hw)) \\ &\quad + (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h+\mathcal{A}\nabla(u-u_h)),\mathbf{d}-\mathbf{d}_I) \\ &\quad + (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathcal{A}\nabla(w-S_hw)) + (\nabla(u-u_h),\mathcal{A}\nabla(w-S_hw)) \\ &= I_1+I_2+I_3+I_4. \end{aligned} \tag{5.12}$$

By Cauchy-Schwartz inequality, (5.11), (5.10) and (5.4), it can be easily shown that

$$I_1 + I_2 \le C\,h\,(\|u-u_h\|_1 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)})\,\|v\|_1.$$

By integration by parts, Cauchy-Schwartz inequality, (5.10) and (5.4), we have

$$\begin{aligned} I_3 &= (\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\mathcal{A}\nabla(w-S_hw)) = (-div(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), w-S_hw) \\ &\le \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}\cdot\|w-S_hw\|_0 \le Ch\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}\|w\|_1 \\ &\le Ch\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}\|v\|_1 \le C\,h\,(\|u-u_h\|_1+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)})\,\|v\|_1. \end{aligned}$$

By (5.9), the second inequality in (5.10) and (5.4), for all $\chi \in V_h$,

$$\begin{aligned} I_4 &= (\nabla(u-u_h),\mathcal{A}\,\nabla(w-S_hw)) = (\nabla(u-\chi)\,,\,\mathcal{A}\nabla(w-S_hw)) \\ &\le C\inf_{\chi\in V_h}\|u-\chi\|_1\,\|w\|_1 \le C\inf_{\chi\in V_h}\|u-\chi\|_1\,\|v\|_1. \end{aligned}$$

By combining the above inequalities in (5.12) and then (5.1), we obtain

$$\|u-u_h\|_1 \le C\inf_{\chi\in V_h}\|u-\chi\|_1 + C\,h\,(\|u-u_h\|_1+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)}).$$

Applying Theorem 3.1, we obtain the desired inequality. □

Remark 5.1. Suppose that $u \in H^{2+\epsilon}(\Omega)$ for some $\epsilon \in [0,1]$. Let $k \ge 1+\epsilon$, $r \ge 1$. Then, the following is the immediate consequence of Theorem 5.1:

$$\|u-u_h\|_1 \le Ch^{1+\epsilon}\|u\|_{2+\epsilon}$$

where C is independent of h and u.

Theorem 5.2. *Let* $(u,\boldsymbol{\sigma})$ *and* $(u_h,\boldsymbol{\sigma}_h)$ *satisfy the equations in* (2.4) *and* (3.3) *respectively. Then,*

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)} \le C\left(\inf_{\mathbf{q}_h\in\mathbf{W}_h}\|\boldsymbol{\sigma}-\mathbf{q}_h\|_{H(div)} + h\inf_{\chi\in V_h}\|u-\chi\|_1\right).$$

Proof. Clearly, we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(div)} \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^1\|_{H(div)} + \|\boldsymbol{\sigma}_h^1 - \boldsymbol{\sigma}_h\|_{H(div)}. \tag{5.13}$$

Using the orthogonal property (3.4), the definition of $b(\cdot,\cdot;\,\cdot,\cdot)$ in (2.7) and (4.1), for all $\mathbf{q}_h \in \mathbf{W}_h$, we have

$$\begin{aligned}
0 &= b(u-u_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h;\, 0, \mathbf{q}_h) \\
&= (div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + c\,(u-u_h)\,,\, div\,\mathbf{q}_h) + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + \nabla\,(u-u_h),\, \mathbf{q}_h\right) \\
&= (div\,(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + (u-u_h)\,,\, div\,\mathbf{q}_h) + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + \nabla\,(u-u_h),\, \mathbf{q}_h\right) \\
&\qquad +((c-1)\,(u-u_h)\,,\, div\,\mathbf{q}_h) \\
&= (div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) + (u_h^1-u_h)\,,\, div\,\mathbf{q}_h) + \left(\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) + \nabla\,(u_h^1-u_h),\, \mathbf{q}_h\right) \\
&\qquad +((c-1)\,(u-u_h)\,,\, div\,\mathbf{q}_h).
\end{aligned}$$

By taking $\mathbf{q}_h = \boldsymbol{\sigma}_h^1 - \boldsymbol{\sigma}_h$ and integration by parts, we obtain

$$\begin{aligned}
&(div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h),\, div(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)\,,\, \boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) \\
&\qquad = -\left(u_h^1-u_h\,,\, div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)\right) - (\nabla\,(u_h^1-u_h),\, \boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) \\
&\qquad\quad - ((c-1)\,(u-u_h)\,,\, div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)) \\
&\qquad = -((c-1)\,(u-u_h)\,,\, div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)).
\end{aligned}$$

Thus, by standard arithmetics-geometric inequality, we have

$$(div(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h), div(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h), \boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) \leq C\|u-u_h\|_0^2. \tag{5.14}$$

Due to (2.5) and (5.14), we have

$$\begin{aligned}
\|\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h\|_{H(div)}^2 &\leq C\,(div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h), div\,(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)) + (\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h)\,,\, \boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h) \\
&\leq C\,\|u-u_h\|_0^2.
\end{aligned}$$

Using the above inequality and Theorem 3.2, we have

$$\begin{aligned}
\|\boldsymbol{\sigma}_h^1-\boldsymbol{\sigma}_h\|_{H(div)} &\leq C\,\|u-u_h\|_0 \\
&\leq C\left(h\inf_{\chi\in V}\|u-\chi\|_1 + h\inf_{\mathbf{q}_h\in\mathbf{W}_h}\|\boldsymbol{\sigma}-\mathbf{q}_h\|_{H(div)}\right).
\end{aligned}$$

Now, using the above inequality in (5.13) and Lemma 4.2, we have

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)} \leq C\left(\inf_{\mathbf{q}_h\in\mathbf{W}_h}\|\boldsymbol{\sigma}-\mathbf{q}_h\|_{H(div)} + h\inf_{\chi\in V_h}\|u-\chi\|_1\right). \qquad \square$$

Remark 5.2. Suppose that $u \in H^{3+\epsilon}(\Omega)$, i.e. $\boldsymbol{\sigma} \in (H^{2+\epsilon}(\Omega))^n$ for some $\epsilon \in [0,1]$. Let $k \geq 1$, $r \geq 2+\epsilon$. Then, the following is the immediate consequence of Theorem 5.2:

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{H(div)} \leq Ch^{1+\epsilon}\|\boldsymbol{\sigma}\|_{2+\epsilon}$$

where C is independent of h and $\boldsymbol{\sigma}$.

6. Numerical results

As a test for the accuracy and the convergence rate of the method, we consider the following boundary value problems in $\Omega = (0,1) \times (0,1)$:

$$\begin{aligned} -\triangle u + \mathbf{b} \cdot \nabla u + 5u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \Gamma, \end{aligned}$$

with $\mathbf{b} = (0,0)$ and $\mathbf{b} = (5,2)$. The exact solution for our problem is $u(x,y) = (x-x^2)(y-y^2)$ and $\boldsymbol{\sigma} = -\big((1-2x)(y-y^2), (x-x^2)(1-2y)\big)$. For approximation, we use uniform triangulation of Ω. A 13-point quadrature rule which integrates exactly polynomials of up to degree 7 was used [14, p. 184]. For approximate spaces for V and $\mathbf{W}$, we choose the standard piecewise continuous polynomial spaces of order k and r satisfying the following approximate properties:

$$\begin{aligned} \inf_{v_h \in V_h} \|v - v_h\|_1 &\leq Ch^k \|v\|_{k+1}, \\ \inf_{\mathbf{q}_h \in \mathbf{W}_h} \|\mathbf{q} - \mathbf{q}_h\|_{H(div)} &\leq Ch^r \|\mathbf{q}\|_{r+1}, \end{aligned}$$

where $k, r > 0$ are integers, $v \in H^{k+1}(\Omega)$, and $\mathbf{q} \in (H^{r+1}(\Omega))^n$.

Our goal is to test the rates of convergence of $\|u - u_h\|_1$ for the case $k = 2, r = 1$ and test the rate of convergence of $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(div)}$ for the case $k = 1, r = 2$. The results are listed in Table 6.1 and Table 6.2. In the presence of the first order term, i.e. $\mathbf{b} \neq 0$, our numerical experiment shows that Theorem 5.1 and Theorem 5.2 are no longer true. For the selfadjoint case, i.e. $\mathbf{b} = 0$, the experiments confirm our theoretical estimate with $\epsilon = 1$ in Remark 5.1 and Remark 5.2.

Table 6.1. Error in $\|u - u_h\|_1$ with different $\mathbf{b}$.

meshsize	$\mathbf{b} = (0,0)$	rate	$\mathbf{b} = (5,2)$	rate
$\frac{1}{8}$	0.0046	-	0.0200	-
$\frac{1}{16}$	0.0013	1.85	0.0118	0.75
$\frac{1}{32}$	0.00033	1.91	0.0065	0.86
$\frac{1}{64}$	0.000086	1.96	0.0034	0.92

Table 6.2. Error in $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(div)}$ with different $\mathbf{b}$.

meshsize	$\mathbf{b} = (0,0)$	rate	$\mathbf{b} = (5,2)$	rate
$\frac{1}{8}$	0.0051	-	0.1021	-
$\frac{1}{16}$	0.0013	1.85	0.0512	0.99
$\frac{1}{32}$	0.00032	1.91	0.0256	1.00
$\frac{1}{64}$	0.000079	1.96	0.0128	1.00

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