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SHORT EFFECTIVE INTERVALS CONTAINING PRIMES IN ARITHMETIC PROGRESSIONS AND THE SEVEN CUBES PROBLEM

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Abstract. For any $\epsilon > 0$ and any non-exceptional modulus $q \geq 3$, we prove that, for x large enough ($x \geq \alpha_\epsilon \log^2 q$), the interval $\left[e^x, e^{x+\epsilon}\right]$ contains a prime p in any of the arithmetic progressions modulo q. We apply this result to establish that every integer n larger than $\exp(71\,000)$ is a sum of seven cubes.

1. Introduction

Let $q \geq 3$ be a non-exceptional modulus, a a positive integer, $x > 0$ and $\epsilon > 0$ some real numbers. One way to establish that the interval $[e^x, e^{x+\epsilon}]$ contains a prime $p \equiv a \pmod q$ would be to determine a condition on x such that

$$\theta\left(e^{x+\epsilon}; q, a\right) - \theta\left(e^x; q, a\right) = \sum_{\substack{e^x < p \leq e^{x+\epsilon} \\ p \equiv a[q]}} \log p \tag{1.1}$$

is positive. It will be convenient to work with Von Mangoldt's function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, p \text{ prime}, \\ 0 & \text{otherwise.} \end{cases}$$

Showing that (1.1) is positive follows from showing that

$$\psi\left(e^{x+\epsilon}; q, a\right) - \psi\left(e^x; q, a\right) = \sum_{\substack{e^x < n \leq e^{x+\epsilon} \\ n \equiv a[q]}} \Lambda(n)$$

is larger than a positive constant times the error term between ψ and θ. In [8], following Rosser's method for $\psi(x)$ in [12], McCurley approximated $\psi\left(e^x; q, a\right)$ via succesive integral averaging. In fact, their methods amount to weighting the primes with a smooth function. Our approach will be to introduce directly a smooth positive weight into the difference $\psi\left(e^{x+\epsilon}; q, a\right) - \psi\left(e^x; q, a\right)$:

$$\sum_{\substack{e^x < n \leq e^{x+\epsilon} \\ n \equiv a[q]}} \frac{\Lambda(n)}{n} f(\log n). \tag{1.2}$$

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We choose the function f so that it has compact support contained in $[x, x+\epsilon]$ and so that the peak of the function is near the prime we want to locate. We have an explicit formula for the sum (1.2):

$$(1.3) \qquad (1+o(1))\,\frac{F(0)}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{\varrho \in Z(\chi)} F(1-\varrho),$$

where F is the Laplace transform of f, and $Z(\chi)$ is the set of non-trivial zeros of $L(s,\chi)$. Note that this formula generalizes the classical formula for ψ (see pp.121–122 of [1]):

$$\psi(x;q,a) = \frac{x}{\phi(q)} - \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi}(a) \sum_{|\gamma|<T} \frac{x^{\varrho}}{\varrho} + \mathcal{O}\left(\frac{x \log^2(qx)}{\phi(q)T} + \frac{x e^{-c_1\sqrt{\log x}}}{\phi(q)} \right).$$

The second argument relies on finding the largest real part for the zeros of the L-functions modulo q, in particular in the case of the low lying zeros. The key result is due to Liu and Wang [6]. It asserts that the zeros $\varrho = \beta + i\gamma$ with $|\gamma| \le H$, except for at most four of them, satisfy:

$$\beta \le 1 - \frac{1}{R_1 \log(qH)}, \quad \text{where } R_1 = 3.82.$$

For the zeros of larger imaginary part, we use the latest effective result on the classical zero-free region (see [4]):

$$\beta \le 1 - \frac{1}{R \log(q|\gamma|)}, \quad \text{where } R = 6.50.$$

We shall study the expression in (1.3) with $x = \alpha \log^2 q$. We deduce a lower bound for non-exceptional moduli q:

$$\sum_{\substack{e^x < n \le e^{x+\epsilon} \\ n \equiv a[q]}} \frac{\Lambda(n)}{n} \frac{f(\log n)}{\|f\|_1} \ge \frac{1}{q} - (1+o(1)) \frac{(\log H)\log(q^2 H)}{2\pi\epsilon}\; q^{-\frac{\alpha}{R_1}\frac{\log q}{\log(qH)}},$$

where H depends essentially on ϵ, i.e. $H \asymp_q \epsilon^{-1}$. From this we shall deduce that the sum on the primes is positive when:

$$\alpha \ge R_1 \frac{\log(qH)}{\log^2 q} \log\left(\frac{q(\log H)\log(q^2 H)}{2\pi\epsilon} \right) (1+o(1)),$$

which gives values for α approaching R_1 as ϵ decreases. Our main result is the following:

Theorem 1.1. *Let $q \ge 3$ be a non-exceptional modulus and let $(a,q)=1$. For any $\epsilon > 0$, there exists an $\alpha > 0$ such that, if $x \ge \alpha \log^2 q$, then the interval $[e^x, e^{x+\epsilon}]$ contains a prime $p \equiv a \pmod q$. Table 1 gives the values of α for various choices of ϵ and $q \ge q_0$.*

In section 4, we describe the general algorithm to compute α as a function of q and ϵ. In comparison, for $q \ge 10^{30}$ and $\epsilon = \ln 3$, McCurley's bounds on $\psi(x;q,a)$ would give $\alpha = 10.690$ (see Theorem 1.2 of [7]). With our new smoothing function, this result may first be improved to $\alpha = 10.562$, and with the new zero-free region ($R = 6.50$ instead of $R = 9.65$) to $\alpha = 7.281$. Using the region with a finite number of zeros ($R_1 = 3.82$), we finally obtain $\alpha = 4.401$.

Table 1

q_0 \ ϵ	0.0001	0.001	0.01	0.1	1	10
$5 \cdot 10^4$	19.228	15.550	12.245	9.4357	6.9684	4.8430
10^{10}	9.8356	8.5912	7.4255	6.3398	5.3418	4.4761
10^{15}	7.6121	6.8799	6.1816	5.5174	4.8905	4.3256
10^{20}	6.5919	6.0799	5.5864	5.1114	4.6565	4.2373
10^{25}	6.0079	5.6164	5.2364	4.8678	4.5116	4.1783
10^{30}	5.6298	5.3137	5.0053	4.7047	4.4123	4.1357
10^{35}	5.3649	5.0102	4.8411	4.5875	4.3396	4.1032
10^{40}	5.1688	4.9414	4.7181	4.4989	4.2839	4.0776
10^{45}	5.0178	4.8185	4.6225	4.4295	4.2398	4.0567
10^{50}	4.8979	4.7205	4.5459	4.3737	4.2039	4.0394
10^{55}	4.8003	4.6407	4.4832	4.3276	4.1740	4.0247
10^{60}	4.7192	4.5742	4.4308	4.2890	4.1488	4.0121
10^{65}	4.6509	4.5179	4.3864	4.2562	4.1272	4.0011
10^{70}	4.5924	4.4697	4.3482	4.2278	4.1084	3.9915
10^{75}	4.5418	4.4280	4.3151	4.2031	4.0920	3.9829
10^{80}	4.4976	4.3914	4.2860	4.1814	4.0774	3.9753
10^{85}	4.4587	4.3591	4.2603	4.1621	4.0645	3.9684
10^{90}	4.4240	4.3304	4.2373	4.1448	4.0528	3.9622
10^{95}	4.3931	4.3046	4.2168	4.1293	4.0423	3.9565
10^{100}	4.3652	4.2815	4.1982	4.1153	4.0328	3.9513

Note that an explicit bound for the size of the least prime $p \equiv a \pmod q$, namely $P(a,q)$, follows immediately:

$$P(a,q) \leq e^{\alpha \log^2 q}. \tag{1.4}$$

In [14], Wagstaff computes the size of $P(a,q)$ for all possible arithmetic progressions up to modulus $5 \cdot 10^4$. For this reason, the data presented in Table 1 begins with moduli q_0 greater than $5 \cdot 10^4$. There exists a stronger result than (1.4), and we refer the reader to the work of Heath-Brown on the subject. In [2], he proved:

$$P(a,q) \ll q^{5.5}.$$

Unfortunately, this is only valid for asymptotically large q. Moreover, if the proof is made effective, it is likely that this result would be weaker than (1.4) in the range we are considering. Also it can be applied to solve some effective problems. We give an example in the second part of the article for which we will apply Theorem 1.1 for $q \geq 10^{32}$.

It concerns Waring's problem for sums of seven cubes. Landau proved in 1909 that every sufficiently large integer may be represented as a sum of eight non-negative cubes. His proof used results on the representation of integers as a sum of three squares. In 1943, Linnik used a theorem on the representation of integers by ternary quadratic forms and proved in [5] that it was also true with seven cubes. In 1939, Dickson completed Landau's statement by showing that all integers, except 23 and 239, are a sum of 8 cubes. It is widely expected that every integer ≥ 455 is a sum of seven cubes.

In 1951, Watson simplified Linnik's proof in [15] by using a lemma establishing some conditions on n to be represented as a sum of seven cubes. This lemma has recently been improved by Ramaré in [11]. The main condition consists of finding prime integers in an arithmetic progression as small as possible. For example, McCurley found $n \geq \exp(1\,077\,334)$ in [8] and Ramaré $n \geq \exp(205\,000)$ in [11]. These authors use Chebyshev's estimates for $\theta(x; q, a)$ that McCurley previously established in [7]. We replace this argument with our result concerning small intervals containing a prime mod q. Since this assertion is only proven for non-exceptional moduli, we give an explicit description of the scarcity of exceptional moduli. We prove the following in section 5:

Theorem 1.2. *Every integer n larger than $N_0 = \exp(71\,000)$ is a sum of seven cubes.*

2. Preliminary lemmas

2.1. **Zeros of the Dirichlet L-functions.** The proof depends essentially on the distribution of the zeros of Dirichlet L-functions. The first theorem states an explicit zero free region for all moduli q, even for those of not too large a size.

Theorem 2.1 (Theorem 1.1 of [4])**.** *Let q be an integer, $q \geq 3$, and let $\mathcal{L}_q(s)$ be the product of Dirichlet L-functions modulo q. Then $\mathcal{L}_q(s)$ has at most one zero in the region*

$$\sigma \geq 1 - \frac{1}{R \log\left(q \max(1, |t|)\right)}, \text{ where } R = 6.50.$$

Such a zero, if it exists, is real, simple and corresponds to a real non-principal character modulo q. We shall refer to it as an exceptional zero and q as an exceptional modulus.

The next theorem illustrates the fact that the zeros do not cluster near the one-line. In fact, there are few of them:

Theorem 2.2 (Theorem 1 of [6])**.** *Suppose q is an integer satisfying $1 \leq q \leq x$, and x is a real number, $x \geq 8 \cdot 10^9$. Then the function $\mathcal{L}_q(s)$ has at most four zeros in the region*

$$|\Im s| \leq x/q, \quad \sigma \geq 1 - \frac{1}{R_1 \log x}, \text{ where } R_1 = 3.82.$$

We will apply this theorem for the case $x = q$ and $x = qH$. We describe explicitly the following phenomenon: the exceptional zero tends to repel the zeros of close conductor.

Theorem 2.3 (Theorem 1.3 of [4])**.** *If χ_1 and χ_2 are two distinct real primitive characters modulo q_1 and q_2 respectively and if β_1 and β_2 are real zeros of $L(s, \chi_1)$ and $L(s, \chi_2)$ respectively, then:*

$$\min\left(\beta_1, \beta_2\right) \leq 1 - \frac{1}{R_2 \log(q_1 q_2)}, \text{ where } R_2 = 2.05.$$

When $q_1 < q_2$, then both q_1 and q_2 cannot be exceptional, unless $q_2 \geq q_1^{2.12}$. The next theorems gives explicit density for the zeros associated to each character χ modulo q (see p. 267 of [7]).

Lemma 2.4. *Let $T \geq 1$. We denote by $N(T, \chi)$ the number of zeros of the Dirichlet L-function $L(s, \chi)$ in the rectangle $\{s \in \mathbb{C} : 0 \leq \Re s \leq 1, |\Im s| \leq T\}$. Then $N(T, \chi) = P(T) + r(T)$, with*

$$P(T) := \frac{T}{\pi} \log \frac{qT}{2\pi e}, \; |r(T)| \leq R(T) := a_1 \log(qT) + a_2, \; a_1 = 0.92, \; a_2 = 5.37.$$

The next lemma establishes a bound for

$$\mathcal{S}(H,\chi) := \sum_{\substack{1<|\gamma|<H \\ L(\beta+i\gamma,\chi)=0}} \frac{1}{|\gamma|}.$$

Lemma 2.5. *Let q be the conductor of χ and $H \geq 1$. Then $\mathcal{S}(H,\chi) \leq \tilde{E}(H)$, where*

$$\tilde{E}(H) := \frac{1}{\pi}(\log q)(\log H) + \frac{1}{2\pi}\log^2 H + \left(\frac{1}{\pi} + a_1\right)\log q - \frac{\log(2\pi)}{\pi}\log H - \frac{1}{\pi}\log(2\pi e) + a_2 + a_1 - \frac{a_1}{H}.$$

Proof. We have:

$$\mathcal{S}(H,\chi) = \frac{N(H,\chi)}{H} - N(1,\chi) + \int_1^H \frac{N(t,\chi)}{t^2}dt.$$

We use Lemma 2.4 to bound $N(t,\chi)$ in the integral and we integrate by parts to obtain:

$$\mathcal{S}(H,\chi) \leq P(1) + R(1) + \int_1^H \frac{P'(t)+R'(t)}{t}dt.$$

We conclude by computing the last integral:

$$\int_1^H \frac{P'(t)+R'(t)}{t}dt = \frac{1}{\pi}(\log q)(\log H) + \frac{1}{2\pi}\log^2 H - \frac{\log(2\pi)}{\pi}\log H + a_1 - \frac{a_1}{H}.$$

□

2.2. **Bounds for $\Gamma'/\Gamma(s)$.**

Lemma 2.6. *If $T \geq 0$, then*

$$\left|\Re\frac{\Gamma'}{\Gamma}\left(\frac{2-\chi(-1)}{4} + i\frac{T}{2}\right)\right| \leq U(T) := \log\left(6(T+12)\right).$$

Proof. See [4]. □

2.3. **Properties of the weight function.** Let m be a positive integer, L and ϵ some positive constants. Our choice for f is inspired by the function Ramaré and Saouter used on p. 17 of [10]. They call f m-admissible when it satisfies:

- f is an m-times differentiable function,
- $f^{(k)}(0) = f^{(k)}(1) = 0$ if $0 \leq k \leq m-1$,
- $f \geq 0$,
- f is non-identically zero.

The specific function we use is

$$f(t) = (t-L)^m\,(L+\epsilon-t)^m, \text{ if } L \leq t \leq L+\epsilon, \tag{2.1}$$

and $f(t) = 0$ otherwise. Furthermore, we notice that f and its derivative are symmetric with respect to $L + \epsilon/2$.

Lemma 2.7.

$$\frac{\|f^{(m)}\|_2}{\|f\|_1} = \frac{\mu_m}{\epsilon^{m+1/2}}, \text{ with } \mu_m = \frac{(2m+1)!}{m!\sqrt{2m+1}}. \tag{2.2}$$

$$\frac{\|f\|_\infty}{\|f\|_1} = \frac{\nu_m}{\epsilon}, \text{ with } \nu_m = \frac{(2m+1)!}{4^m(m!)^2}. \tag{2.3}$$

Proof. This is an exercise, or else, see p. 17 of [10] for the derivation. □

Let F be the Laplace transform of the function f as defined in (2.1):

$$F(s) = \int_L^{L+\epsilon} f(t)e^{-st}dt.$$

Lemma 2.8. *If $\sigma \geq 0$, then*

$$F(\sigma) \geq e^{-\sigma(L+\epsilon)}\|f\|_1, \tag{2.4}$$

$$|F(s)| \leq e^{-\sigma L}\|f\|_1, \tag{2.5}$$

$$|F(s)| \leq \frac{e^{-\sigma L}}{|s|}\frac{2(2m+1)}{\epsilon m}\|f\|_1, \tag{2.6}$$

$$|F(s)| \leq \sqrt{\epsilon}\, e^{-\sigma L}\frac{\|f^{(m)}\|_2}{|s|^m}. \tag{2.7}$$

Proof. The proof makes use of the symmetry of f and $f^{(m)}$ and, in the last case, the Cauchy-Schwarz inequality. □

2.4. **An explicit formula.** Let q be an integer, $q \geq 3$. For each character χ modulo q, we denote by χ_1 the primitive character associated with χ.

Lemma 2.9. *Let f be the function given by* (2.1)*. Then*

$$\begin{aligned}\sum_{\chi \bmod q} \overline{\chi(a)} \sum_{n\geq 1} \frac{\Lambda(n)\chi_1(n)}{n} f(\log n) =& F(0) + c(a,q)F(1) \\ & - \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{\varrho \in Z(\chi_1)} F(1-\varrho) + I(a,q),\end{aligned}$$

where $c(a,q) \geq \frac{1}{2}$, $Z(\chi_1)$ is the set of non-trivial zeros of $L(s,\chi_1)$ and

$$I(a,q) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{\chi \bmod q} \overline{\chi(a)}\Re\frac{\Gamma'}{\Gamma}\left(\frac{2-\chi_1(-1)}{4} + i\frac{T}{2}\right) F\left(1/2 - iT\right) dT.$$

Proof. This is a special case of the explicit formula of Theorem 3.1, p. 314 of [3], applied to the smooth function $\phi(x) = f(x)e^{-x}$ if $x \geq 0$ and $\phi(x) = 0$ otherwise. The constant $c(a,q)$ is given by

$$\frac{1}{4}\sum_{\chi \bmod q} \overline{\chi(a)} + \frac{1}{4}\sum_{\chi \bmod q} \overline{\chi(a)}\chi_1(-1) + \frac{1}{2} \geq \frac{1}{2}$$

(see p. 414 of [9] for the details). □

3. Main lemma

We place our study in the case of modulus q not studied by Wagstaff, that is to say, for those q larger than $5 \cdot 10^4$. Let $\epsilon > 0$, $H \geq 1$ and α be positive reals such that

$$\alpha < R\left(\frac{\log(qH)}{\log q}\right)^2. \tag{3.1}$$

We define L as a parameter depending only on q:

$$L := \alpha \log^2 q.$$

Throughout the paper, $\varrho = \beta + i\gamma$ always stands for a non-trivial zero of a Dirichlet L-function. We prove in this section that

Lemma 3.1. *If $m \geq 3$, $\epsilon > 0$ and if α satisfies the condition* (3.1)*, then the sum over the primes* $\Sigma(a,q) := \sum_{p \equiv a \bmod q} \frac{\log p}{p} f(\log p)$ *satisfies*

$$\frac{\Sigma(a,q)}{\|f\|_1} \geq \frac{1}{q} - r(\alpha, \epsilon, H, m, q),$$

where $r := \sum_{i=1}^{5} r_i$ *and the r_i's are given by* (3.4) , (3.7) , (3.9) , (3.16) *and* (3.18).

Note that $\Sigma(a,q)$ is actually close to the sum appearing in Lemma 2.9:

$$\Sigma = \Sigma_{11} + \Sigma_{12} - \Sigma_2$$

with

$$\Sigma_{11}(a,q) := \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{n \geq 1} \frac{\Lambda(n)\chi_1(n)}{n} f(\log n),$$

$$\Sigma_{12}(a,q) := \frac{1}{\phi(q)} \sum_{\chi \bmod q} \overline{\chi(a)} \sum_{n \geq 1} \frac{\Lambda(n)(\chi(n) - \chi_1(n))}{n} f(\log n),$$

where χ_1 is the primitive character associated to χ, and

$$\Sigma_2(a,q) := \sum_{\substack{k \geq 2 \\ p^k \equiv a \bmod q}} \frac{\log p}{p^k} f(k \log p).$$

We will prove in sections 3.5 and 3.6 that the last two sums are small in comparison to Σ_{11}. We use Lemma 2.9 to bound Σ_{11}:

$$\Sigma(a,q) \geq A_1(q) - A_2^-(q) - A_2^+(q) - A_3(q) - |\Sigma_{12}(a,q)| - |\Sigma_2(a,q)|,$$

with

$$A_1(q) := \frac{1}{\phi(q)} \left(F(0) + c(a,q)F(1)\right),$$

$$A_2^-(q) := \frac{1}{\phi(q)} \sum_{\chi \bmod q} \sideset{}{'}\sum_{\substack{\varrho \in Z(\chi) \\ |\gamma| \leq H}} |F(1-\varrho) + F(\overline{\varrho})|,$$

$$A_2^+(q) := \frac{1}{\phi(q)} \sum_{\chi \bmod q} \sideset{}{'}\sum_{\substack{\varrho \in Z(\chi) \\ |\gamma| > H}} |F(1-\varrho) + F(\overline{\varrho})|,$$

$$A_3(q) := \frac{1}{\phi(q)} |I(a,q)|,$$

where

$$\sideset{}{'}\sum_{\beta} = \sum_{1/2 < \beta < 1} + \frac{1}{2} \sum_{\beta = 1/2}$$

(we use the symmetry property of the zeros). We extend the sum over the zeros of $L(s,\chi_1)$ to the zeros of $L(s,\chi)$ to simplify our argument.

The sections 3.1 to 3.4 study these A_i's.

3.1. **Study of A_1.** It is immediate that

$$\frac{A_1(q)}{\|f\|_1} \geq \frac{1}{q}. \tag{3.2}$$

3.2. **Study of A_2^-.** Since we are in the case where q is non-exceptional, we do not worry about the existence of a Siegel zero. Thanks to Theorems 2.1 and 2.2 we can split the sum A_2^- as follows:

$$A_2^-(q) = \frac{1}{\phi(q)}\sum_{k=1}^{8}|F(1-\varrho_k)+F(\overline{\varrho}_k)| + \frac{1}{\phi(q)}\sum_{\chi \bmod q}\ \sideset{}{'}\sum_{\substack{|\gamma|\leq 1\\ \beta\leq 1-\frac{1}{R_1\log q}}}|F(1-\varrho)+F(\overline{\varrho})|$$
$$+\frac{1}{\phi(q)}\sum_{\chi \bmod q}\ \sideset{}{'}\sum_{\substack{1<|\gamma|\leq H\\ \beta\leq 1-\frac{1}{R_1\log(qH)}}}|F(1-\varrho)+F(\overline{\varrho})|,$$

where the zeros $\varrho_k=\beta_k+i\gamma_k$ satisfy:

$$|\gamma_k|\leq 1 \text{ and } 1-\frac{1}{R_1\log q}\leq \beta_k\leq 1-\frac{1}{R\log q}\ , \text{ for } k=1,2,3,4,$$
$$|\gamma_k|\leq H \text{ and } 1-\frac{1}{R_1\log(qH)}\leq \beta_k\leq 1-\frac{1}{R\log(qH)}\ , \text{ for } k=5,6,7,8.$$

We use the inequalities (2.5) and (2.6) for the first and second lines respectively:

$$|F(1-\varrho)+F(\overline{\varrho})|\leq\left(e^{-\beta L}+e^{-(1-\beta)L}\right)\|f\|_1$$

and

$$|F(1-\varrho)+F(\overline{\varrho})|\leq\frac{1}{|\gamma|}\left(e^{-\beta L}+e^{-(1-\beta)L}\right)\frac{2(2m+1)}{\epsilon m}\|f\|_1.$$

Then

$$A_2^-(q)\leq\frac{4}{\phi(q)}b_2(\alpha,R,q)\left(q^{-\frac{\alpha}{R}}+q^{-\frac{\alpha}{R}\frac{\log q}{\log(qH)}}\right)\|f\|_1$$
$$+\frac{1}{2}b_2(\alpha,R_1,q)q^{-\frac{\alpha}{R_1}}\|f\|_1\max_{\chi \bmod q}N(1,\chi)$$
$$+b_2(\alpha,R_1,q)\frac{2m+1}{\epsilon m}\|f\|_1\,q^{-\frac{\alpha}{R_1}\frac{\log q}{\log(qH)}}\max_{\chi \bmod q}\left(\sideset{}{'}\sum_{1<|\gamma|\leq H}\frac{1}{|\gamma|}\right),$$

where $b_2(\alpha,r,q):=1+q^{-\alpha\log q+\frac{2\alpha}{r}}$. We conclude by bounding the sum over the zeros as in Lemma 2.5, $N(1,\chi)$ as in Lemma 2.4 and $\phi(q)$ as on page 72 of [13]:

$$\frac{q}{\phi(q)}<e^C\log\log q+\frac{2.51}{\log\log q}\text{ for } q\geq 3,$$

where C stands for the Euler constant. Then

$$\frac{A_2^-(q)}{\|f\|_1}\leq 4b_2(\alpha,R,q)\left(q^{-1-\frac{\alpha}{R}}+q^{-1-\frac{\alpha}{R}\frac{\log q}{\log(qH)}}\right)\left(e^C\log\log q+\frac{2.51}{\log\log q}\right)$$
$$+\left(\frac{1+a_1\pi}{2\pi}\log q-\frac{\log(2\pi e)+a_2\pi}{2\pi}\right)b_2(\alpha,R_1,q)q^{-\frac{\alpha}{R_1}}$$
$$+\frac{(2m+1)\tilde{E}(H)}{2\epsilon m}b_2(\alpha,R_1,q)q^{-\frac{\alpha}{R_1}\frac{\log q}{\log(qH)}}.$$

We obtain

$$\frac{A_2^-(q)}{\|f\|_1} \leq r_1(\alpha,\epsilon,H,m,q), \tag{3.3}$$

$$\begin{aligned} r_1(\alpha,\epsilon,H,m,q) &:= b_2(\alpha,R_1,q)\frac{(2m+1)}{2\epsilon m}\Bigg(\frac{(\log H)(\log(q^2H))}{2\pi} \\ &+ \Big(\frac{1}{\pi}+a_1\Big)\log q - \frac{\log(2\pi)}{\pi}\log H - \frac{\log(2\pi e)}{\pi} + a_2 + a_1 - \frac{a_1}{H}\Bigg) q^{-\frac{\alpha}{R_1}\frac{\log q}{\log(qH)}} \\ &+ b_2(\alpha,R_1,q)\Big(\frac{1+a_1\pi}{2\pi}\log q - \frac{\log(2\pi e)+a_2\pi}{2\pi}\Big)q^{-\frac{\alpha}{R_1}} \\ &+ 4b_2(\alpha,R,q)\left(e^C\log\log q + \frac{2.51}{\log\log q}\right)\left(q^{-1-\frac{\alpha}{R}} + q^{-1-\frac{\alpha}{R}\frac{\log q}{\log(qH)}}\right). \end{aligned} \tag{3.4}$$

3.3. **Study of A_2^+.** Theorem 2.1 allows us to restrict $\sum'$ to the zeros in the region:

$$|\gamma| > H,\ 1/2 \leq \beta \leq 1 - \frac{1}{R\log(q|\gamma|)}.$$

We use (2.7) to bound $|F(1-\varrho)|$ and$|F(\overline{\varrho})|$,

$$\begin{aligned} &|F(1-\varrho)+F(\overline{\varrho})| \\ &\quad\leq \sqrt{\epsilon}\|f^{(m)}\|_2\Big[\exp\Big(\frac{-L}{R\log(q|\gamma|)}\Big) + \exp\Big(-L\Big(1-\frac{1}{R\log(qH)}\Big)\Big)\Big]\frac{1}{|\gamma|^m}. \end{aligned}$$

We follow Lemma 4.1.3 and Lemma 4.2.1 of [9] and obtain that if $L \leq R\log^2(qH)$, then:

$$\sum_{\substack{\varrho\in Z(\chi_1)\\|\gamma|>H}}{}' \frac{\exp\left(\frac{-L}{R\log(q|\gamma|)}\right)}{|\gamma|^m} \leq \frac{\tilde{A}+\tilde{B}}{2} \quad\text{and}\quad \sum_{\substack{\varrho\in Z(\chi_1)\\|\gamma|>H}}{}' \frac{1}{|\gamma|^m} \leq \frac{\tilde{C}+\tilde{D}}{2},$$

with

$$\begin{aligned} \tilde{A} &:= \frac{1}{\pi(m-2)H^{m-1}}\exp\left(-\frac{L}{R\log(qH)}\right)\left(\log\frac{qH}{2\pi} + \frac{1}{m-2} + \frac{a_1}{(m-1)H}\right), \\ \tilde{B} &:= \frac{2(a_1\log(qH)+a_2)}{H^m}\exp\left(-\frac{L}{R\log(qH)}\right), \\ \tilde{C} &:= \frac{1}{\pi(m-1)H^{m-1}}\left(\log\frac{qH}{2\pi} + \frac{1}{m-1}\right), \\ \tilde{D} &:= \frac{2a_1\log(qH)+2a_2+\frac{a_1}{m}}{H^m}. \end{aligned}$$

We deduce the bound:

$$\frac{A_2^+(q)}{\sqrt{\epsilon}\|f^{(m)}\|_2} \leq \frac{\tilde{A}+\tilde{B}}{2} + \frac{\tilde{C}+\tilde{D}}{2}\exp\Big(-L\Big(1-\frac{1}{R\log(qH)}\Big)\Big) \tag{3.5}$$

and together with (2.2):

$$\frac{A_2^+(q)}{\|f\|_1} \le r_2(\alpha,\epsilon,H,m,q), \tag{3.6}$$

$$\begin{aligned} r_2(\alpha,\epsilon,H,m,q) &:= q^{-\frac{\alpha}{R}\frac{\log q}{\log(qH)}}\frac{\mu_m}{(H\epsilon)^m}\Big[\frac{H\log\frac{qH}{2\pi}}{2\pi(m-2)}+\frac{H}{2\pi(m-2)^2} \\ &+\frac{a_1}{2\pi(m-2)(m-1)}+a_1\log(qH)+a_2\Big]+q^{-\alpha\log q+\frac{\alpha}{R}\frac{\log q}{\log(qH)}}\frac{\mu_m}{(H\epsilon)^m} \\ &\times\Big[\frac{H}{2\pi(m-1)}\left(\log\frac{qH}{2\pi}+\frac{1}{m-1}\right)+a_1\log(qH)+a_2+\frac{a_1}{2m}\Big]. \end{aligned} \tag{3.7}$$

3.4. **Study of A_3.**

$$A_3(q) \le \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|\Re\frac{\Gamma'}{\Gamma}\left(\frac{2-\chi_1(-1)}{4}+i\frac{T}{2}\right)\right||F(1/2-iT)|\,dT.$$

We use Lemma 2.6 to bound Γ'/Γ, (2.5) to bound F when $T\le 1$ and (2.7) otherwise:

$$\frac{A_3(q)}{\|f\|_1} \le r_3(\alpha,\epsilon,m,q), \tag{3.8}$$

$$r_3(\alpha,\epsilon,m,q) := \frac{1}{2\pi}\left(J_0+\frac{\mu_m J(m)}{\epsilon^m}\right)q^{-\frac{\alpha}{2}\log q} \tag{3.9}$$

with $J_0 := \int_{|T|\le 1}\log(6(|T|+12))dT$ and $J(m) := \int_{|T|>1}\frac{\log(6(|T|+12))}{|T|^m}dT$.

3.5. **Study of $\Sigma_{12}(a,q)$.** For n fixed, we denote by Q_n the largest divisor of q coprime with n. Then

$$\frac{1}{\phi(q)}\sum_{\chi \bmod q}\chi_1(n)\overline{\chi(a)} = \begin{cases}\frac{\phi(Q_n)}{\phi(q)} & \text{if } n\equiv a \pmod{Q_n},\\ 0 & \text{else.}\end{cases}$$

For a proof, see p. 414 of [9]. It implies that

$$\Sigma_{12}(a,q) = \sum_{\substack{n\equiv a \bmod Q_n\\ Q_n<q}}\frac{\phi(Q_n)}{\phi(q)}\frac{\Lambda(n)f(\log n)}{n}.$$

In this sum, we have

$$\frac{\phi(Q_n)}{\phi(q)} = \frac{1}{p^{\nu_p(q)-1}(p-1)}$$

since n is a prime power, $n=p^k$, coprime with Q_n but not with q. Therefore

$$\Sigma_{12}(a,q) \le \|f\|_\infty \sum_{p^{\nu_p(q)}|q}\frac{\log p}{p^{\nu_p(q)-1}(p-1)}\sum_{e^L<p^k<e^{L+\epsilon}}\frac{1}{p^k}. \tag{3.10}$$

We compute the geometric sum

$$\sum_{e^L<p^k<e^{L+\epsilon}}\frac{1}{p^k} \le \sum_{k\ge[\frac{L}{\log p}]+1}\frac{1}{p^k} = \frac{e^{-L}}{p-1}. \tag{3.11}$$

We reinsert the last bound in the summand and split the obtained sum:

$$\sum_{p^{\nu_p(q)}|q}\frac{\log p}{p^{\nu_p(q)-1}(p-1)^2} \le \sum_{p|q}\frac{\log p}{(p-1)^2}+\sum_{p^j|q,j\ge 2}\frac{\log p}{p^{j-1}(p-1)^2}, \tag{3.12}$$

where

$$\sum_{p^j|q, j\geq 2} \frac{\log p}{p^{j-1}(p-1)^2} \leq \sum_{p|q} \frac{\log p}{(p-1)^2} \sum_{j\geq 2} \frac{1}{p^{j-1}} = \sum_{p|q} \frac{\log p}{(p-1)^3} \tag{3.13}$$

and

$$\sum_{p\geq 2} \log p \left(\frac{1}{(p-1)^2} + \frac{1}{(p-1)^3} \right) \leq 2.10. \tag{3.14}$$

Together with (2.3) and (3.10) to (3.14), we conclude that

$$\frac{|\Sigma_{12}(a,q)|}{\|f\|_1} \leq 2.10 \frac{\|f\|_\infty}{\|f\|_1} e^{-L} \leq r_4(\alpha, \epsilon, m, q), \tag{3.15}$$

$$r_4(\alpha, \epsilon, m, q) := 2.10 \, \frac{\nu_m}{\epsilon} \, q^{-\alpha \log q}. \tag{3.16}$$

3.6. **Study of** $\Sigma_2(a,q)$**.** We have

$$|\Sigma_2(a,q)| = \sum_{\substack{k\geq 2 \\ p^k \equiv a \bmod q}} \frac{\log p}{p^k} f(k \log p) \leq \|f\|_\infty \sum_{2\leq p\leq e^{\frac{L+\epsilon}{2}}} \log p \sum_{e^L < p^k < e^{L+\epsilon}} \frac{1}{p^k}.$$

From (3.11) and

$$\sum_{2\leq p\leq e^{\frac{L+\epsilon}{2}}} \frac{\log p}{p-1} \leq 2 \log \left(e^{\frac{L+\epsilon}{2}} \right) = L + \epsilon \text{ (see equation (3.24) of [13])},$$

it follows that

$$\frac{|\Sigma_2(a,q)|}{\|f\|_1} \leq \frac{\|f\|_\infty}{\|f\|_1} (L+\epsilon) e^{-L} \leq r_5(\alpha, \epsilon, m, q), \tag{3.17}$$

$$r_5(\alpha, \epsilon, m, q) := \frac{\nu_m}{\epsilon} (\alpha \log^2 q + \epsilon) q^{-\alpha \log q}. \tag{3.18}$$

4. Proof of Theorem 1.1

We gather the inequalities (3.2), (3.3), (3.6), (3.8), (3.15) and (3.17) and obtain:

$$\frac{\Sigma(a,q)}{\|f\|_1} \geq q^{-1} - r(\alpha, \epsilon, H, m, q) \geq q^{-1} \left(1 - q_0 r(\alpha, \epsilon, H, m, q_0) \right).$$

Let $u \in [0.001, 0.2]$, $q \geq q_0$ with $q_0 = 5\cdot 10^4, 10^{10}, ..., 10^{100}$ and $\epsilon = 10^{-3}, 10^{-2}, ..., 10$ be fixed. We will choose H and m such that α is as small as possible and satisfies

$$1 - q_0 r(\alpha, \epsilon, H, m, q_0) = 10^{-6} \tag{4.1}$$

and r_1 and r_2 are of comparable size:

$$r_2(\alpha, \epsilon, H, m, q_0) = u r_1(\alpha, \epsilon, H, m, q_0). \tag{4.2}$$

We approximate r_1, r_2 and r with $\tilde{r}_1$, $\tilde{r}_2$ and $\tilde{r}_1 + \tilde{r}_2 = (1+u)\tilde{r}_1$ respectively, where

$$\tilde{r}_1(\alpha, H, m) := \frac{(2m+1)(\log H)(\log(q_0^2 H))}{4\pi\epsilon m} q_0^{-\frac{\alpha}{R_1} \frac{\log q_0}{\log(q_0 H)}},$$

$$\tilde{r}_2(\alpha, H, m) := \frac{H \log(q_0 H)}{\pi\sqrt{m}} q_0^{-\frac{\alpha}{R} \frac{\log q_0}{\log(q_0 H)}} \left(\frac{4m}{eH\epsilon} \right)^m.$$

We approximate (4.1) by the equation

$$1 - q_0(1+u)\tilde{r}_1(\alpha, H, m) = 10^{-6}.$$

Its solution is close to

$$\tilde{\alpha}(H,m) := R_1 \frac{\log(q_0 H)}{\log^2 q_0} \log\left(\frac{q_0(\log H)(\log(q_0^2 H))}{2\pi\epsilon}\right). \tag{4.3}$$

It remains to find appropriate values of H which will satisfy (4.2). The solution of the equation

$$\tilde{r}_2(\tilde{\alpha}(H,m),H,m) = u\tilde{r}_1(\tilde{\alpha}(H,m),H,m)$$

is close to

$$\tilde{H}(m) = \frac{1}{\epsilon}\left(\frac{4}{u\sqrt{m}}\left(\frac{4m}{e}\right)^m\left(\frac{q_0\log q_0}{4\pi\epsilon}\right)^{1-\frac{R_1}{R}}\right)^{\frac{1}{m-1}}. \tag{4.4}$$

We minimize the value of $\tilde{\alpha}(\tilde{H}(m),m)$ and find that m is close to

$$\tilde{m} := \frac{1}{2} + \log\left(\frac{16}{u}\left(\frac{q_0\log q_0}{4\pi\epsilon}\right)^{1-\frac{R_1}{R}}\right).$$

We now describe the algorithm to compute α. For u and m fixed (the value of m is chosen close to $\tilde{m}$):

- We compute $\tilde{H}(m)$ and $\tilde{\alpha}(\tilde{H}(m),m)$ as given in (4.4) and (4.3) respectively.
- We choose for H the value of the solution of the following approximation of equation (4.2):

  $$r_2(\tilde{\alpha}(\tilde{H}(m),m),H,m) = ur_1(\tilde{\alpha}(\tilde{H}(m),m),H,m).$$

  With this value for H, we solve (4.1) with respect to α. It is not difficult to see that the function $r(\alpha,\epsilon,H,m,q)$ decreases when α increases. Therefore we are insured of the uniqueness of the solution of the equation.
- We choose u and m so that the value of α is as small as possible.

Table 2 records the values of the parameters m, H and u. They have been rounded up in the last decimal place.

For the next section, we will use the following result: when $q \geq 10^{32}$, $\epsilon = 1.9$, then $u = 0.022$, $H = 80.8$, $m = 38$ and $\alpha = 4.3060$.

5. A seven cubes problem

Watson's proof in [15] relies on the fact that, for $X > \exp(q^{1/100})$, the existence of a prime $p \equiv a (\bmod\, q)$ in the interval $[X, 1.01\,X]$ makes it possible to write a sufficiently large integer n as a sum of seven cubes, and the size of the smallest of these n's depends on the size of X. We will follow the latest version of this algorithm, due to Ramaré ([11]).

5.1. **A modified form of Watson's lemma (Lemma 5 of [8]).** The next lemma provides conditions for an integer to be a sum of seven cubes.

Lemma 5.1 (Lemma 2.1 of [11]). *Let n, a, u, v and w be positive integers and t a non-negative integer. We assume that*

$$1 \le u \le v \le w \le (3/4)^{1/3} uv/24, \tag{5.1}$$

$$\gcd(uvw, 6n) = 1 \text{ and } a \text{ is odd}, \tag{5.2}$$

$$u, v, w \text{ and } a \text{ are pairwise co-prime}, \tag{5.3}$$

$$n - t^3 \equiv 1\ [2], \tag{5.4}$$

$$n - t^3 \equiv 0\ [3a], \tag{5.5}$$

$$\begin{cases} 4(n-t^3) \equiv v^6 w^6 a^3\ [u^2], \\ 4(n-t^3) \equiv u^6 w^6 a^3\ [v^2], \\ 4(n-t^3) \equiv u^6 v^6 a^3\ [w^2]. \end{cases} \tag{5.6}$$

Set $\delta = \left(1 + (w/u)^6 + (w/v)^6\right)/4$. *If*

$$0 \le \frac{uv}{6w}\left(\frac{n}{u^6v^6a^3} - \delta - \frac{3}{4}\right)^{1/3} \le \frac{t}{6uvwa} \le \frac{uv}{6w}\left(\frac{n}{u^6v^6a^3} - \delta\right)^{1/3}, \tag{5.7}$$

then n is a sum of seven non-negative cubes.

5.2. **Reducing to finding a prime in an arithmetic progession.** Suppose the integer n is given. We need to find u, v, w, a and t such that the conditions of our lemma are fulfilled. Let u, v, w be prime numbers $\equiv 5\ [6]$ that satisfy (5.1) and are coprime with n. Then $(4n)/(v^6w^6)$ is a cube, modulo u^2. We have the same for $(4n)/(u^6w^6)$ modulo v^2 and $(4n)/(u^6v^6)$ modulo w^2. This is easy to prove, knowing that, if p is a prime $\equiv 5\ [6]$, then every invertible residue class modulo p is a cube modulo p^2. Moreover u^2, v^2 and w^2 are pairwise coprime and, by the Chinese remainder theorem, there exists an integer a' such that

$$\begin{cases} 4n \equiv (a'v^2w^2)^3\ [u^2], \\ 4n \equiv (a'u^2w^2)^3\ [v^2], \\ 4n \equiv (a'u^2v^2)^3\ [w^2]. \end{cases} \tag{5.8}$$

We choose a to be $a \equiv a'\ [u^2v^2w^2]$, so that we can replace a' by a in the system (5.8). Also we can choose a to be prime and $a \equiv 5\ [6]$, so that we are insured that there exists an integer n cubic modulo $3a$. We deduce that

Condition 1. There exists a prime a such that $a \equiv \ell\ [6u^2v^2w^2]$.

Since the integers u, v, w and $6a$ are coprime, there exist integers t satisfying:

$$t^3 \equiv n\ [3a],\ t^3 \equiv n-1\ [2],\ t \equiv 0\ [uvw].$$

Up to now, the conditions (5.1) to (5.6) are satisfied. In order to find $\frac{t}{6auvw}$ bounded as in (5.7), we need to add some conditions on a, namely that

Condition 2. $\dfrac{Y}{\kappa} \le a \le Y$,

where $Y := \dfrac{n^{1/3}}{u^2v^2(3/4+\delta)^{1/3}}$, $\kappa^3 := \dfrac{1}{3/4+\delta}\left[\left(\dfrac{uv}{24w(\rho+1)}\right)^{3/2} + \delta\right]$, $\rho := \dfrac{1}{6uvwa}$.

Somemore explanation is provided on pp. 377–378 of [11]. We will see that Theorem 1.1 insures us of the existence of a prime a satisfying conditions 1 and 2. However, this theorem is established for non-exceptional moduli. We explain how to avoid the case of exceptional zeros in the next section.

5.3. **Creating a non-exceptional modulus.**

Theorem 5.2 (Theorem 2 of [9]). *For all $q \le 72$, and for all a prime to q, uniformly for $1 \le x \le 10^{10}$,*

$$\max_{1\le y\le x}\left|\theta(y;q,a)-\frac{y}{\phi(q)}\right| \le 2.072\sqrt{x}.$$

Lemma 5.3. *There are more than 12 prime numbers coprime to n and congruent to 5 modulo 6 lying in the interval $[0.521\log n, 2.562\log n]$ if $\log n$ is larger than 68 509.*

Proof. See the proof of Lemma 4.5 of [11]. The constants c_1 and c_2 are chosen to optimize the lower bound of $\log n$ given in the equation (5.9) below under the conditions that $c_2 - c_1 > \phi(6)$ and

$$\left(\frac{c_2-c_1}{2}-1\right)\log n - 2.072\left(\sqrt{c_1}+\sqrt{c_2}\right)\sqrt{\log n} \ge 12\log\left(c_2\log n\right). \qquad \square$$

We note $c_3 = \left(\frac{c_2}{c_1}\right)^{\frac12}$. Then we deduce by the pigeon hole principle that there exists an interval $[A, c_3A]$ with A in $[c_1\log n, c_2/c_3\log n]$ that contains more than 6 primes coprime to n and congruent to 5 modulo 6. We denote by $u_1 < v_1 < w_1 < u_2 < v_2 < w_2$, 6 of these primes, $k_1 = 3(u_1v_1w_1)^2$ and $k_2 = 3(u_2v_2w_2)^2$. To prove that one of the coprime integers k_1 and k_2 has to be non-exceptional, we use Theorem 2.3 and the inequalities

$$k_1^{2.12} \ge \left(3\left(c_1\log n\right)^6\right)^{2.12} > 3\left(c_2\log n\right)^6 \ge k_2$$

for $n \ge 150$. We denote simply by k the non-exceptional modulus and by u, v, w the associated integers in $[A, c_3A]$. It remains to find for which n, the interval $[Y/\kappa, Y]$ satisfies the hypotheses of Theorem 1.1.

5.4. **Finding a prime in a progression with a large modulus.**

Lemma 5.4. *Assume $\log n \ge 71\,000$. For any invertible residue class l modulo k, there is a prime a, congruent to l modulo k contained in $[Y/\kappa, Y]$.*

Proof. We use the bounds:

$$c_1\log n \le A \le u,v,w \le c_3A \le c_2\log n,\ \frac14+\frac{1}{2c_3^6} \le \delta \le \frac14+\frac{c_3^6}{2},\ \rho \le \frac{1}{6(c_1\log n)^3}.$$

We deduce that $\kappa \ge \kappa_0(n)$ and $Y \ge Y_0(n)$, with

$$\kappa_0(n)^3 := \frac{1}{1+\frac{c_3^6}{2}}\left(\left(\frac{c_1\log n}{24c_3\left(\frac{1}{6(c_1\log n)^3}+1\right)}\right)^{3/2}+\frac14+\frac{1}{2c_3^6}\right),$$

$$Y_0(n) := \frac{n^{1/3}}{(c_2\log n)^4\left(1+\frac{c_3^6}{2}\right)^{1/3}}.$$

For the values $k \ge 10^{32}$, $\alpha = 4.3060$ and $\epsilon = 1.9$, the inequality $Y \ge e^{\alpha\log^2 k+\epsilon}$ is satisfied when

$$\frac{\log n}{3} - 4\log(c_2\log n) - \frac13\log\left(1+\frac{c_3^6}{2}\right) \ge \alpha\log^2\left(3(c_2\log n)^6\right)+\epsilon, \tag{5.9}$$

that is to say, for $\log n \ge 70\,341$. This also warrants $\kappa_0(n) \ge e^{\epsilon}$. $\square$

Table 2

q_0	ϵ	u	m	H	α	q_0	ϵ	u	m	H	α
5.10^4	0.0001	0.086	14	514 998	19.228	10^{55}	0.0001	0.016	64	2 538 632	4.8003
	0.001	0.092	13	47 292	15.550		0.001	0.016	64	250 167	4.6407
	0.01	0.098	12	4 311	12.245		0.01	0.016	63	24 661	4.4832
	0.1	0.004	14	528	9.4357		0.1	0.015	62	2 338	4.3276
	1	0.01	15	57.8	6.9684		1	0.015	61	244	4.1740
	10	0.037	11	4.4219	4.8430		10	0.012	62	24.8	4.0247
10^{10}	0.0001	0.056	20	741 876	9.8356	10^{60}	0.0001	0.015	69	2 731 576	4.7192
	0.001	0.058	19	70 330	8.5912		0.001	0.015	68	269 639	4.5742
	0.01	0.060	18	6 632	7.4254		0.01	0.014	68	26 639	4.4308
	0.1	0.061	17	630	6.3398		0.1	0.014	67	2 633	4.2890
	1	0.057	16	62.5	5.3418		1	0.014	66	263	4.1488
	10	0.028	17	6.75	4.4761		10	0.011	67	26.8	4.0121
10^{15}	0.0001	0.043	25	948 594	7.6121	10^{65}	0.0001	0.014	74	2 927 544	4.6509
	0.001	0.045	24	90 920	6.8799		0.001	0.014	73	289 140	4.5179
	0.01	0.046	23	8 713	6.1816		0.01	0.014	72	28 660	4.3864
	0.1	0.046	22	839	5.5174		0.1	0.013	72	2 829	4.2562
	1	0.043	21	83.6	4.8905		1	0.013	71	283	4.1272
	10	0.024	22	8.84	4.3256		10	0.010	72	28.7	4.0011
10^{20}	0.0001	0.035	30	1 152 223	6.5919	10^{70}	0.0001	0.013	79	3 122 581	4.5924
	0.001	0.036	29	111 390	6.0799		0.001	0.013	78	308 647	4.4697
	0.01	0.037	28	10 762	5.5864		0.01	0.013	77	30 511	4.3482
	0.1	0.037	27	1 045	5.1114		0.1	0.013	76	3 021	4.2278
	1	0.035	26	105	4.6565		1	0.012	76	302	4.1084
	10	0.021	27	10.9	4.2373		10	0.010	77	30.7	3.9915
10^{25}	0.0001	0.030	35	1 353 117	6.0079	10^{75}	0.0001	0.012	84	3 317 736	4.5418
	0.001	0.030	34	131 618	5.6164		0.001	0.012	83	328 165	4.4280
	0.01	0.031	33	12 786	5.2364		0.01	0.012	82	32 465	4.3151
	0.1	0.031	32	1 247	4.8678		0.1	0.012	81	3 216	4.2031
	1	0.029	31	125	4.5116		1	0.011	81	322	4.0920
	10	0.019	32	12.9	4.1783		10	0.009	81	32.6	3.9829
10^{30}	0.0001	0.026	40	1 553 007	5.6298	10^{80}	0.0001	0.011	89	3 513 060	4.4976
	0.001	0.026	39	151 626	5.3137		0.001	0.011	88	347 700	4.3914
	0.01	0.026	38	14 802	5.0053		0.01	0.011	87	34 417	4.2860
	0.1	0.026	37	1 449	4.7046		0.1	0.011	86	3 311	4.1814
	1	0.025	36	145	4.4123		1	0.011	86	341	4.0774
	10	0.017	37	15	4.1357		10	0.009	86	34.5	3.9753
10^{35}	0.0001	0.023	45	1 751 630	5.3649	10^{85}	0.0001	0.011	94	3 704 920	4.4587
	0.001	0.023	44	171 503	5.1002		0.001	0.011	93	366 878	4.3591
	0.01	0.023	43	16 791	4.8411		0.01	0.011	92	36 335	4.2603
	0.1	0.023	42	1 648	4.5875		0.1	0.010	91	3 607	4.1621
	1	0.022	41	165	4.3396		1	0.010	90	361	4.0645
	10	0.016	42	16.9	4.1032		10	0.008	91	35.5	3.9684
10^{40}	0.0001	0.020	50	1 950 568	5.1688	10^{90}	0.0001	0.010	98	3 900 103	4.4240
	0.001	0.021	49	191 213	4.9414		0.001	0.010	98	386 427	4.3331
	0.01	0.021	48	18 763	4.7181		0.01	0.010	97	38 290	4.2373
	0.1	0.021	47	1 846	4.4989		0.1	0.010	96	3 799	4.1448
	1	0.020	46	185	4.2839		1	0.010	95	380	4.0528
	10	0.014	47	18.9	4.0776		10	0.008	96	37.4	3.9623
10^{45}	0.0001	0.018	55	2 114 784	5.0178	10^{95}	0.0001	0.010	103	4 091 636	4.3931
	0.001	0.019	54	210 928	4.8185		0.001	0.010	102	405 566	4.3046
	0.01	0.019	53	20 736	4.6225		0.01	0.010	101	40 204	4.2168
	0.1	0.019	52	2 043	4.4295		0.1	0.009	101	3 995	4.1293
	1	0.018	51	204	4.2398		1	0.009	100	399	4.0423
	10	0.013	52	20.9	4.0567		10	0.008	101	40.3	3.9565
10^{50}	0.0001	0.017	60	2 342 931	4.8979	10^{100}	0.0001	0.009	108	4 287 331	4.3652
	0.001	0.017	59	230 659	4.7206		0.001	0.009	107	425 137	4.2815
	0.01	0.017	58	22 710	4.5459		0.01	0.009	106	41 161	4.1982
	0.1	0.017	57	2 240	4.3737		0.1	0.009	105	4 186	4.1153
	1	0.016	56	224	4.2039		1	0.009	105	419	4.0328
	10	0.012	57	22.9	4.0394		10	0.007	106	42.3	3.9513

References

1. H. Davenport, *Multiplicative Number Theory*, Graduate Texts in Mathematics, third edition (2000). MR1790423 (2001f:11001)
2. D.R. Heath-Brown, *Zero-free regions for Dirichlet L-functions and the least prime in an arithmetic progression.* Proc. London Math. Soc. **64** (1992) 265–338. MR1143227 (93a:11075)
3. H. Kadiri, *Une région explicite sans zéro pour la fonction zeta de Riemann.* Acta Arith. **117.4** (2005) 303–339. MR2140161 (2005m:11159)
4. H. Kadiri, *An explicit zero-free region for Dirichlet L-functions.* submitted, can be found at http://arxiv.org/pdf/math.NT/0510570.
5. U.V. Linnik, *On the representation of large numbers as sums of seven cubes.* Rec. Math. [Mat. Sbornik] N. S. 12**54** (1943) 218–224 MR0009388 (5:142c)

6. M-C. Liu and T. Wang, *Distribution of zeros of Dirichlet L-functions and an explicit formula for $\psi(t,\chi)$.* Acta Arith. **102** (2002), no. 3, 261–293. MR1884719 (2003f:11125)
7. K.S. McCurley, *Explicit estimates for the error term in the prime number theorem for arithmetic progressions.* Math. Comp. **42** (1984), no. 165, 265–285. MR726004 (85e:11065)
8. K.S. McCurley, *An effective seven cube theorem.* J. Number Theory **19** (1984), no. 2, 176–183. MR762766 (86c:11078)
9. O. Ramaré and R. Rumely, *Primes in arithmetic progressions.* Math. Comp. **65** (1996) 397–425. MR1320898 (97a:11144)
10. O. Ramaré and Y. Saouter, *Short effective intervals containing primes.* J. Number Theory **98** (2003) 10–33. MR1950435 (2004a:11095)
11. O. Ramaré, *An explicit seven cube theorem.* Acta Arith. **118** (2005) no. 4, 375–382. MR2165551 (2006g:11202)
12. J.B. Rosser, *Explicit bounds for some functions of prime numbers.* American Journal of Math. **63** (1941) 211–232. MR0003018 (2:150e)
13. J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers.* Illinois J. Math. **6** (1962) 64–94. MR0137689 (25:1139)
14. S. Wagstaff, *Greatest of the least primes in arithmetic progressions having a given modulus.* Math. of Comp. **33** (1979) 1073–1080. MR528061 (81e:10038)
15. G.L. Watson, *A proof of the seven cube theorem.* J. London Math. Soc. **26** (1951), 153–156. MR0047691 (13:915a)

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