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A GILBERT-VARSHAMOV TYPE BOUND FOR EUCLIDEAN PACKINGS

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Abstract. This paper develops a method to obtain a Gilbert-Varshamov type bound for dense packings in the Euclidean spaces using suitable lattices. For the Leech lattice the obtained bounds are quite reasonable for large dimensions, better than the Minkowski-Hlawka bound, but not as good as the lower bound given by Keith Ball in 1992.

1. Introduction

It is a classical problem to find dense sphere packings in Euclidean space; see for instance [2, 3, 4, 7] for an introduction to this topic.

For a real $b > 0$, let $\mathcal{A}_b^m$ be the ball of radius b in $\mathbb{R}^m$ defined by

$$\mathcal{A}_b^m := \{x := (x_1, \dots, x_m) \in \mathbb{R}^m : \ ||x||^2 := \sum_{i=1}^m x_i^2 \le b^2\}.$$

Then the volume of $\mathcal{A}_b^m$ is equal to $V_m \cdot b^m$, where $V_m = \frac{\pi^{m/2}}{\Gamma(m/2+1)}$ denotes the volume of a unit ball in $\mathbb{R}^m$. A packing in $\mathbb{R}^m$ is a set $\mathcal{P}$ of points in $\mathbb{R}^m$ such that the Euclidean distance of $\mathcal{P}$,

$$d(\mathcal{P}) := \inf_{\mathbf{u},\mathbf{v}\in\mathcal{P},\mathbf{u}\neq\mathbf{v}} d(\mathbf{u},\mathbf{v}),$$

is positive. It is clear that all balls with radius $d(\mathcal{P})/2$ and centers being points of $\mathcal{P}$ are not overlapping. Let $\mathcal{U}(\mathcal{P})$ be the union of all such balls, i.e.,

$$\mathcal{U}(\mathcal{P}) = \{\mathbf{x} \in \mathbb{R}^m : \ \exists \ \mathbf{u} \in \mathcal{P} \text{ such that } d(\mathbf{x},\mathbf{u}) \le d(\mathcal{P})/2\}.$$

Then the density of $\mathcal{P}$ is defined by

$$\Delta(\mathcal{P}) = \limsup_{b\to\infty} \frac{\mathrm{vol}\,(\mathcal{U}(\mathcal{P}) \cap \mathcal{A}_b^m)}{\mathrm{vol}\,(\mathcal{A}_b^m)},$$

where $\mathrm{vol}(\mathcal{T})$ denotes the volume of a subset $\mathcal{T}$ in $\mathbb{R}^m$. It is clear that $\Delta(\mathcal{P}) \le 1$. We are interested in dense packings, i.e., we want to find a packing with density close to the quantity

$$\Delta_m := \limsup_{\mathcal{P}} \Delta(\mathcal{P}),$$

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where $\mathcal{P}$ is extended over all packings in $\mathbb{R}^m$. Sometimes, it is more convenient to convert Δ_m into the center density which is defined by

$$\delta_m := \frac{\Delta_m}{V_m}.$$

The Gilbert-Varshamov (G-V, for short) bound is a benchmark for good codes in coding theory. The idea involved in the G-V bound can be employed for many other problems. In this paper, we use the idea to construct dense packings in $\mathbb{R}^m$ by choosing a subset of certain lattice points. It turns out that the obtained G-V type bound is quite reasonable. For the orthogonal sum of copies of the Leech lattice, it is better than the Minkowski-Hlawka bound, but not as good as the lower bound given in [1]. Up to now no sphere packings have been described that beat our bounds. A table for densities for large dimensions divisible by 720 is given in this paper.

2. Gilbert-Varshamov type bound

In this section we fix a lattice L of dimension m in Euclidean space $\mathbb{R}^m$ such that all norms in L are integers, i.e., $||\mathbf{c}||^2 \in \mathbb{Z}$ for all $\mathbf{c} \in L$. For a point $\mathbf{c} \in L$ and an integer $k > 0$, define the ball $\mathcal{B}_{L,k}(\mathbf{c})$ to be the set consisting of points in L with Euclidean distance from $\mathbf{c}$ at most $\sqrt{k}$, i.e.,

$$\mathcal{B}_{L,k}(\mathbf{c}) := \left\{\mathbf{b} \in L : \ ||\mathbf{b} - \mathbf{c}||^2 \le k\right\}.$$

It is clear that $\mathcal{B}_{L,k}(\mathbf{c})$ is a finite set and its cardinality, denoted by $B_{L,k}$, is independent of its center $\mathbf{c}$. In fact, we have

$$B_{L,k} = 1 + \sum_{i=1}^{k} S_{L,i}, \tag{2.1}$$

where $S_{L,i}$ is the number of lattice points of norm i in L.

One important invariant of L is its discriminant D, defined as the determinant of a Gram matrix of L, or equivalently as the square of the volume of a fundamental domain of L in $\mathbb{R}^m$ (see [3]). It measures the number of lattice points contained in a unit ball.

Lemma 2.1. *Let L be a lattice in $\mathbb{R}^m$ with discriminant D. Then*

$$\lim_{b\to\infty} \frac{B_{L,b}}{V_m b^{m/2}} = \frac{1}{\sqrt{D}},$$

where V_m is the volume of the unit ball in $\mathbb{R}^m$.

Proof. Clearly, $\sqrt{D}$ is the volume of the Dirichlet domain

$$\mathcal{D}_L = \{x \in \mathbb{R}^m : \ ||x|| < ||x - \ell|| \ \forall 0 \neq \ell \in L\}$$

around 0. For a fixed b, the $B_{L,b}$ translates $\ell + \mathcal{D}_L$ with $\ell \in \mathcal{B}_{L,b}(0)$ tile a subset of volume $B_{L,b}\sqrt{D}$ of $\mathbb{R}^m$, which tends to be the ball around 0 with radius $\sqrt{b}$ when b tends to infinity. □

Theorem 2.2. *Let $L \subseteq \mathbb{R}^m$ be a lattice of discriminant D, such that all norms in L are integral. Then for any integer $r \ge 1$, one has*

$$\Delta_m \ge \frac{1}{\sqrt{D}} \times \frac{1}{B_{L,r}} \times \left(\frac{\sqrt{r+1}}{2}\right)^m \times V_m, \tag{2.2}$$

hence

$$\delta_m \geq \frac{1}{2^m\sqrt{D}} \max_{r\geq 1}\left(\frac{(r+1)^{m/2}}{B_{L,r}}\right). \tag{2.3}$$

Proof. For a sufficiently large integer $b > 0$ choose an arbitrary point $\mathbf{c}_1$ from $\mathcal{B}_{L,b}(\mathbf{0})$. Pick up a point $\mathbf{c}_2$ from $\mathcal{B}_{L,b}(\mathbf{0}) \setminus \mathcal{B}_{L,r}(\mathbf{c}_1)$. Then pick up a point $\mathbf{c}_3$ from $\mathcal{B}_{L,b}(\mathbf{0}) \setminus \mathcal{B}_{L,r}(\mathbf{c}_1) \cup \mathcal{B}_{L,r}(\mathbf{c}_2)$.

This procedure constructs a subset $C := \{\mathbf{c}_1, \ldots, \mathbf{c}_{M+1}\}$ of $\mathcal{B}_{L,b}(\mathbf{0})$ with

$$M := \left\lfloor \frac{B_{L,b}-1}{B_{L,r}} \right\rfloor$$

such that

$$\mathbf{c}_i \in \mathcal{B}_{L,b}(\mathbf{0}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{L,r}(\mathbf{c}_j)$$

for any $1 \leq i \leq M+1$. This is because the set $\mathcal{B}_{L,b}(\mathbf{0}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{L,r}(\mathbf{c}_j)$ is not empty due to the fact that

$$\left|\mathcal{B}_{L,b}(\mathbf{0}) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_{L,r}(\mathbf{c}_j)\right| \geq |\mathcal{B}_{L,b}(\mathbf{0})| - \left|\bigcup_{j=1}^{M} \mathcal{B}_{L,r}(\mathbf{c}_j)\right| \geq B_{L,b} - M \cdot B_{L,r} > 0.$$

Since norms in L are integral and $||\mathbf{c}_i - \mathbf{c}_j||^2 > r$ for $i \neq j$ by construction, any two distinct points in the set C have Euclidean distance at least $\sqrt{r+1}$. Hence the balls around distinct $\mathbf{c}_j$ with radius $\sqrt{r+1}/2$ are disjoint. When b tends to ∞ this yields

$$\begin{aligned}
\Delta_m &\geq \lim_{b\to\infty} \frac{M}{V_m b^{m/2}} \times \left(\frac{\sqrt{r+1}}{2}\right)^m \times V_m \\
&= \lim_{b\to\infty} \frac{B_{L,b}}{V_m b^{m/2}} \times \frac{1}{B_{L,r}} \times \left(\frac{\sqrt{r+1}}{2}\right)^m \times V_m \\
&= \frac{1}{\sqrt{D}} \times \frac{1}{B_{L,r}} \times \left(\frac{\sqrt{r+1}}{2}\right)^m \times V_m.
\end{aligned}$$

The desired result follows. □

Remark 2.3. As r tends to infinity, the right hand side of the inequality (2.2) tends to $\frac{1}{2^m}$ yielding asymptotically the same bound as the Minkowski-Hlawka bound, $\frac{1}{m}\log_2(\Delta_m) \geq -1$. However, for smaller values of r the right hand side may be bigger, which yields a slight improvement of this bound.

3. Numerical results

In applications one usually chooses orthogonally decomposable lattices

$$L = L_1^{n_1} \perp L_2^{n_2} \perp \ldots \perp L_s^{n_s} \subseteq \mathbb{R}^m$$

of dimension $m = \sum_{i=1}^{s} n_i \dim(L_i)$ and discriminant $D = \prod_{i=1}^{s} D_i^{n_i}$ where D_i denotes the discriminant of L_i. We obtained good results by choosing

$$L = \frac{1}{\sqrt{2}}\Lambda_{24}^{n_1} \perp \frac{1}{\sqrt{2}}\mathbf{A}_2^{n_2} \perp \mathbb{Z}^{n_3}$$

with $m = 24n_1 + 2n_2 + n_3$, $0 \leq n_2 \leq 11$, $0 \leq n_3 \leq 1$ where $\Lambda_{24} \subseteq \mathbb{R}^{24}$ denotes the Leech lattice and $\mathbf{A}_2$ the 2-dimensional hexagonal lattice. Then $D_1 = 2^{-24}$,

$D_2 = \frac{3}{4}$ and $D_3 = 1$, and so $\log_2(\frac{1}{\sqrt{D}}) = 12n_1 + n_2\alpha$ with $\alpha = (1 - \log_2(\sqrt{3}))$ and hence $\log_2(\delta_m) \geq b_0(m)$ where

$$\begin{aligned}
&b_0(24n_1 + 2n_2 + n_3) \\
&\quad = 12n_1 + \alpha n_2 + \max_{r \geq 1} \big((12n_1 + n_2 + \frac{n_3}{2}) \log_2(\frac{r+1}{4}) - \log_2(B_{L,r})\big) \\
&\qquad \geq b_0(24n_1 + 2n_2 + n_3, r_0) \\
&\quad = 12n_1 + \alpha n_2 + (12n_1 + n_2 + \frac{n_3}{2}) \log_2(\frac{r_0+1}{4}) - \log_2(B_{L,r_0})
\end{aligned} \tag{3.1}$$

for any integer $r_0 \geq 1$. The following table lists a few examples (truncated to the second decimal place).

m	$b_0(m, r_0)$	r_0		m	$b_0(m, r_0)$	r_0		m	$b_0(m, r_0)$	r_0
1440	3176.35	161		1441	3178.99	162		1442	3181.81	163
1443	3184.46	165		1444	3187.29	167		1445	3189.95	169
1446	3192.78	171		1447	3195.46	174		1448	3198.31	178
1449	3201.01	184		1450	3203.88	192		1451	3206.63	209
1452	3209.54	218		1453	3212.34	235		1454	3215.27	240
1455	3218.10	253		1456	3221.03	257		1457	3223.88	268
1458	3226.82	270		1459	3229.68	281		1460	3232.62	282
1461	3235.50	291		1462	3238.44	292		1463	3241.32	301
1464	3246.59	164		1465	3249.24	165		1466	3252.07	166
1467	3254.73	168		1468	3257.57	170		1469	3260.24	172
1470	3263.09	174		1471	3265.78	177		1472	3268.64	181
1473	3271.36	188		1474	3274.24	195		1475	3277.00	212
1476	3279.92	221		1477	3282.74	238		1478	3285.67	244
1479	3288.52	258		1480	3291.46	261		1481	3294.32	273
1482	3297.27	275		1483	3300.15	285		1484	3303.10	287
1485	3305.99	296		1486	3308.94	297		1487	3311.83	306
1488	3317.11	167		1489	3319.77	168		1490	3322.61	169
1491	3325.29	171		1492	3328.14	173		1493	3330.82	175
1494	3333.68	177		1495	3336.38	180		1496	3339.26	184
1497	3341.98	191		1498	3344.88	198		1499	3347.66	216
1500	3350.59	225		1501	3353.41	242		1502	3356.36	248
1503	3359.21	262		1504	3362.17	265		1505	3365.05	277
1506	3368.01	279		1507	3370.89	290		1508	3373.86	291
1509	3376.75	301		1510	3379.72	302		1511	3382.62	311
1512	3387.90	170		1513	3390.58	171		1514	3393.43	172
1515	3396.12	174		1516	3398.98	176		1517	3401.68	178
1518	3404.55	180		1519	3407.27	184		1520	3410.15	187
1521	3412.89	194		1522	3415.80	201		1523	3418.58	219
1524	3421.53	229		1525	3424.36	246		1526	3427.32	252
1527	3430.19	266		1528	3433.16	269		1529	3436.04	282
1530	3439.02	284		1531	3441.91	295		1532	3444.89	296
1533	3447.80	306		1534	3450.77	307		1535	3453.69	316
1536	3458.97	173		1537	3461.66	174		1538	3464.53	175
1539	3467.23	177		1540	3470.10	178		1541	3472.81	181
1542	3475.70	183		1543	3478.42	187		1544	3481.32	190
1545	3484.07	197		1546	3486.99	205		1547	3489.79	223

m	$b_0(m,r_0)$	r_0		m	$b_0(m,r_0)$	r_0		m	$b_0(m,r_0)$	r_0
1548	3492.74	232		1549	3495.59	250		1550	3498.56	256
1551	3501.44	270		1552	3504.42	274		1553	3507.31	286
1554	3510.30	288		1555	3513.20	299		1556	3516.19	301
1557	3519.11	311		1558	3522.10	311		1559	3525.02	321
1560	3530.32	175		1561	3533.02	177		1562	3535.90	178
1563	3538.60	180		1564	3541.49	181		1565	3544.21	184
1566	3547.11	186		1567	3549.84	190		1568	3552.76	193
1569	3555.52	201		1570	3558.45	208		1571	3561.26	226
1572	3564.22	236		1573	3567.08	254		1574	3570.06	260
1575	3572.95	274		1576	3575.94	278		1577	3578.85	290
1578	3581.84	293		1579	3584.76	304		1580	3587.76	305

The Minkowski-Hlawka bound (see for instance [3, page 14]) yields $\Delta_m \geq \frac{\zeta(m)}{2^{m-1}}$. The paper [1] improves this bound and shows that

$$\Delta_m \geq (m-1)\frac{\zeta(m)}{2^{m-1}}$$

and hence

$$\log_2(\delta_m) \geq \mathrm{ball}(m) = \mathrm{mh}(m) + \log_2(m-1)$$

with

$$\mathrm{mh}(m) = \log_2(\zeta(m)) + \log_2((m/2)!) + 1 - m + (m/2)\log_2(\pi).$$

For dimensions $m = 720n$ we compare our bounds $b_1(m,r_1)$ and $b_2(m,r_2)$ obtained by taking $L = \frac{1}{\sqrt{2}}\Lambda_{24}^{30n}$, respectively $L = \frac{1}{\sqrt{2}}\Lambda_{72}^{10n}$ (for some putative extremal even unimodular lattice Λ_{72}) and choosing suitable integers $r_1, r_2 \geq 1$ to these two bounds. This shows that our lower bounds are better than the Minkowski-Hlawka bound, but not as good as the ones obtained by Keith Ball. Our construction is explicitly described, though it is not of polynomial-time, while the other two bounds, by Minkowski-Hlawka and Ball, are nonconstructive.

The results (rounded to 10 decimal places) were calculated with MAGMA [5] using the modular forms package.

m	$b_1(m,r_1)$	r_1	$b_2(m,r_2)$	r_2	$\mathrm{mh}(m)$	$\mathrm{ball}(m)$
1440	3176.358513	161	3177.780626	128	3173.388428	3183.879279
2160	5391.441687	249	5392.867890	196	5388.598648	5399.674795
2880	7782.965065	339	7784.398557	265	7780.204931	7791.696283
3600	10305.69411	430	10307.13442	334	10302.99474	10314.80812
4320	12932.98468	522	12934.43130	404	12930.33297	12942.40945
5040	15647.23054	615	15648.68263	474	15644.61786	15656.91678
5760	18435.91780	708	18437.37474	545	18433.33806	18445.82966
6480	21289.68966	803	21291.15088	615	21287.13832	21299.79988
7200	24201.28315	898	24202.74824	686	24198.75675	24211.57033
7920	27164.89608	993	27166.36461	757	27162.39186	27175.34296
8640	30175.78597	1089	30177.25765	829	30173.30171	30186.37836
9360	33230.00377	1185	33231.47830	900	33227.53761	33240.72975

m	$b_1(m, r_1)$	r_1	$b_2(m, r_2)$	r_2	$\mathrm{mh}(m)$	$\mathrm{ball}(m)$
10080	36324.20987	1282	36325.68702	972	36321.76029	36335.05935
10800	39455.54308	1379	39457.02261	1044	39453.10876	39466.50737
11520	42621.52459	1477	42623.00632	1115	42619.10439	42632.59612
12240	45819.98605	1574	45821.46984	1188	45817.57900	45831.15820
12960	49049.01471	1672	49050.50042	1260	49046.61995	49060.28161
13680	52306.91065	1771	52308.39816	1332	52304.52742	52318.26710
14400	55592.15318	1869	55593.64235	1404	55589.78081	55603.59449
15120	58903.37385	1968	58904.86459	1477	58901.01173	58914.89580

A MAGMA program producing this table is available from [6].

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