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THE RATIO MONOTONICITY OF THE BOROS-MOLL POLYNOMIALS

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Abstract. In their study of a quartic integral, Boros and Moll discovered a special class of Jacobi polynomials, which we call the Boros-Moll polynomials. Kauers and Paule proved the conjecture of Moll that these polynomials are log-concave. In this paper, we show that the Boros-Moll polynomials possess the ratio monotone property which implies the log-concavity and the spiral property. We conclude with a conjecture which is stronger than Moll's conjecture on the ∞-log-concavity.

1. Introduction

In this paper, we aim to show that the Boros-Moll polynomials satisfy the ratio monotone property which implies the log-concavity and the spiral property. Boros and Moll [3, 4, 5, 6, 7, 10] explored a special class of Jacobi polynomials in their study of a quartic integral. They have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}}P_m(a), \tag{1.1}$$

where

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(a+1)^j(a-1)^k}{2^{3(k+j)}}. \tag{1.2}$$

Using Ramanujan's Master Theorem, Boros and Moll [6, 10] derived the following formula

$$P_m(a) = 2^{-2m}\sum_k 2^k\binom{2m-2k}{m-k}\binom{m+k}{k}(a+1)^k, \tag{1.3}$$

which indicates that the coefficients of a^i in $P_m(a)$ are positive for $0 \le i \le m$. Let $d_i(m)$ be defined by

$$P_m(a) = \sum_{i=0}^m d_i(m)a^i. \tag{1.4}$$

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The polynomials $P_m(a)$ will be called the Boros-Moll polynomials, and the sequence $\{d_i(m)\}_{0\leq i\leq m}$ of the coefficients will be called a Boros-Moll sequence. From (1.4), it follows that

$$d_i(m) = 2^{-2m}\sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{1.5}$$

The readers can find in [2] many proofs of this formula. Recall that $P_m(a)$ can be expressed as a hypergeometric function

$$P_m(a) = 2^{-2m}\binom{2m}{m}{}_2F_1(-m, m+1; \frac{1}{2}-m; \frac{a+1}{2}),$$

from which one sees that $P_m(a)$ can be viewed as the Jacobi polynomial $P_m^{(\alpha,\beta)}(a)$ with $\alpha = m+\frac{1}{2}$ and $\beta = -(m+\frac{1}{2})$, where $P_m^{(\alpha,\beta)}(a)$ is given by

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m+\beta}{m-k}\binom{m+k+\alpha+\beta}{k}\left(\frac{1+a}{2}\right)^k.$$

Boros and Moll [4] proved that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is unimodal and the maximum element appears in the middle, namely,

$$d_0(m) < d_1(m) < \cdots < d_{\lfloor\frac{m}{2}\rfloor}(m) > d_{\lfloor\frac{m}{2}\rfloor+1}(m) > \cdots > d_m(m).$$

They also established the unimodality by taking a different approach [5]. Moll [10] conjectured that the sequence $\{d_i(m)\}_{0\leq i\leq m}$ is log-concave. Kauers and Paule [9] proved this conjecture based on four recurrence relations found using a computer algebra approach. Two of these four recurrences have been independently derived by Moll [11] using the WZ-method. Moreover, as will be seen, the two recurrences derived by Moll easily imply the other two given by Kauers and Paule. These recursions will be discussed in Section 2.

Recall that a sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{m-1}}{a_m}.$$

A polynomial is said to be log-concave if the sequence of its coefficients is log-concave. It is easy to see that if a sequence is log-concave, then it is unimodal. A sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be spiral if

$$a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{\lfloor\frac{m}{2}\rfloor}.$$

Similarly, a polynomial is said to be spiral if its sequence of coefficients is spiral. It is easily seen that a log-concave sequence is not necessarily spiral, and vice versa. For example, $(2, 10, 3, 1)$ is spiral but not log-concave, whereas $(3, 5, 4, 2, 1)$ is log-concave but not spiral. Chen and Xia [8] discovered that the q-derangement numbers are both spiral and log-concave, and introduced the *ratio monotone* property defined below, which implies both log-concavity and the spiral property. The purpose of this paper is to show that the Boros-Moll polynomials possess the ratio monotone property.

A sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be ratio monotone if

$$\frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \cdots \leq \frac{a_{i-1}}{a_{m-i}} \leq \frac{a_i}{a_{m-1-i}} \leq \cdots \leq \frac{a_{\lfloor\frac{m}{2}\rfloor-1}}{a_{m-\lfloor\frac{m}{2}\rfloor}} \leq 1 \tag{1.6}$$

and

$$\frac{a_m}{a_0} \le \frac{a_{m-1}}{a_1} \le \cdots \le \frac{a_{m-i}}{a_i} \le \frac{a_{m-1-i}}{a_{i+1}} \le \cdots \le \frac{a_{m-\left\lfloor \frac{m-1}{2} \right\rfloor}}{a_{\left\lfloor \frac{m-1}{2} \right\rfloor}} \le 1. \tag{1.7}$$

If every inequality relation in (1.6) and (1.7) becomes strict, we say that the sequence is strictly ratio monotone. It is easy to see that the ratio monotonicity implies log-concavity. Indeed, from (1.6) and (1.7), we deduce that

$$\frac{a_i}{a_{i-1}} \ge \frac{a_{m-1-i}}{a_{m-i}} \quad \text{and} \quad \frac{a_{i+1}}{a_i} \le \frac{a_{m-1-i}}{a_{m-i}}.$$

This gives

$$\frac{a_i}{a_{i-1}} \ge \frac{a_{i+1}}{a_i}.$$

The main result of this paper is stated as follows.

Theorem 1.1. *Let $m \ge 2$ be an integer. Then the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ satisfies the strictly ratio monotone property. To be precise, we have*

$$\frac{d_m(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_1(m)} < \cdots < \frac{d_{m-i}(m)}{d_i(m)} < \frac{d_{m-i-1}(m)}{d_{i+1}(m)} < \cdots < \frac{d_{m-\left\lfloor \frac{m-1}{2} \right\rfloor}(m)}{d_{\left\lfloor \frac{m-1}{2} \right\rfloor}(m)} < 1 \tag{1.8}$$

and

$$\frac{d_0(m)}{d_{m-1}(m)} < \frac{d_1(m)}{d_{m-2}(m)} < \cdots < \frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)} < \cdots < \frac{d_{\left\lfloor \frac{m}{2} \right\rfloor -1}(m)}{d_{m-\left\lfloor \frac{m}{2} \right\rfloor}(m)} < 1. \tag{1.9}$$

As a corollary of Theorem 1.1, we obtain the spiral property of the Boros-Moll sequences. It is not clear whether there is a simpler way to verify this property directly.

Corollary 1.2. *Let $m \ge 2$ be an integer. Then the Boros-Moll sequence $\{d_i(m)\}_{0\le i\le m}$ is spiral.*

The following example illustrates our main result. For $m = 8$, we have

$$\begin{aligned} P_8(a) =& \frac{4023459}{32768} + \frac{3283533}{4096}a + \frac{9804465}{4096}a^2 + \frac{8625375}{2048}a^3 + \frac{9695565}{2048}a^4 \\ &+ \frac{1772199}{512}a^5 + \frac{819819}{512}a^6 + \frac{109395}{256}a^7 + \frac{6435}{128}a^8. \end{aligned}$$

The strictly ratio monotone property is illustrated as follows:

$$\frac{\frac{6435}{128}}{\frac{4023459}{32768}} < \frac{\frac{109395}{256}}{\frac{3283533}{4096}} < \frac{\frac{819819}{512}}{\frac{9804465}{4096}} < \frac{\frac{1772199}{512}}{\frac{8625375}{2048}} < 1,$$

$$\frac{\frac{4023459}{32768}}{\frac{109395}{256}} < \frac{\frac{3283533}{4096}}{\frac{819819}{512}} < \frac{\frac{9804465}{4096}}{\frac{1772199}{512}} < \frac{\frac{8625375}{2048}}{\frac{9695565}{2048}} < 1.$$

The spiral property of $P_8(x)$ is reflected by the following order of the coefficients:

$$\begin{aligned} \frac{6435}{128} < \frac{4023459}{32768} < \frac{109395}{256} < \frac{3283533}{4096} < \frac{819819}{512} \\ < \frac{9804465}{4096} < \frac{1772199}{512} < \frac{8625375}{2048} < \frac{9695565}{2048}. \end{aligned}$$

Based on the Moll conjecture on the ∞-log-concavity of the sequences $\{d_i(m)\}_{0\leq i\leq m}$, we conclude this paper with a stronger conjecture that these polynomials are infinitely ratio monotone. Numerical evidence seems to be supportive of this conjecture.

2. Recurrence relations

We first give a brief review of Kauers and Paule's approach to proving the log-concavity of the Boros-Moll sequence [9]. Our work employs the four recurrences

$$d_i(m+1) = \frac{m+i}{m+1}d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}d_i(m), \quad 0 \leq i \leq m+1, \tag{2.1}$$

$$d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}d_i(m) - \frac{i(i+1)}{(m+1)(m+1-i)}d_{i+1}(m), \quad 0 \leq i \leq m, \tag{2.2}$$

$$d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)}d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)}d_i(m), \quad 0 \leq i \leq m+1, \tag{2.3}$$

and for $0 \leq i \leq m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{2.4}$$

These recurrences are derived by Kauers and Paule [9] with the RISC package MultiSum [12]. In fact, the recurrences (2.3) and (2.4) are also derived independently by Moll [11], and the other two relations (2.1) and (2.2) can be easily deduced from (2.3) and (2.4). Based on the four recurrence relations, Kauers and Paule [9] used a computer algebra system to derive the next theorem, from which the log-concavity of the Boros-Moll sequence is then derived.

Theorem 2.1. *For $0 < i < m$, we have*

$$d_i(m+1) \geq \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)}d_i(m). \tag{2.5}$$

The inequality (2.5) is also of vital importance for our proof of the ratio monotonicity of the Boros-Moll sequences. We note that the above inequality (2.5) is very tight. In other words, the ratio

$$\frac{(4m^2+7m+i+3)d_i(m)}{2(m+1-i)(m+1)d_i(m+1)},$$

seems to be very close to 1. For example, for $m = 100$, the smallest ratio is 0.998348.

In order to establish the strict ratio monotonicity, we need a slightly sharper version of (2.5). For example, we will show that the inequality in (2.5) is strict for $1 \leq i \leq m-1$.

Theorem 2.2. *Let $m \geq 2$. We have*

$$d_i(m+1) > \frac{4m^2+7m+i+3}{2(m+1-i)(m+1)} d_i(m), \qquad 1 \leq i \leq m-1, \tag{2.6}$$

and

$$d_0(m+1) = \frac{4m+3}{2(m+1)} d_0(m), \tag{2.7}$$

$$d_m(m+1) = \frac{(2m+3)(2m+1)}{2(m+1)} d_m(m) = \frac{(2m+3)(2m+1)}{2(m+1)} 2^{-m} \binom{2m}{m}. \tag{2.8}$$

To make this paper self-contained, we will present a detailed proof of the above improvement of Theorem 2.1. Before doing so, we remark that (2.3) and (2.4) can be also derived from (2.1) and (2.2). Equating the right-hand sides of (2.1) and (2.2) and replacing i by $i-1$, we get (2.4). Substituting i with $i+1$ and m with $m+1$ in (2.1) and (2.2), respectively, we obtain two expressions for $d_{i+1}(m+1)$. This yields

(2.9)

$$\begin{aligned} d_i(m+2) &= \frac{(4m-2i+7)(m+i+2)}{2(m+2)(m+2-i)} d_i(m+1) \\ &\quad - \frac{i(i+1)}{(m+2)(m+2-i)} \left(\frac{m+i+1}{m+1} d_i(m) + \frac{(4m+2i+5)}{2(m+1)} d_{i+1}(m) \right) \\ &= \frac{(4m-2i+7)(m+i+2)}{2(m+2)(m+2-i)} d_i(m+1) - \frac{i(i+1)(m+i+1)}{(m+2)(m+2-i)(m+1)} d_i(m) \\ &\quad - \frac{i(i+1)(4m+2i+5)}{(m+2)(m+2-i)(2m+2)} d_{i+1}(m). \end{aligned}$$

On the other hand, from (2.2), we have

(2.10)

$$d_{i+1}(m) = -\frac{(m+1)(m+1-i)}{i(i+1)} d_i(m+1) + \frac{(m+i+1)(4m-2i+3)}{2i(i+1)} d_i(m).$$

Substituting (2.10) into (2.9), we obtain (2.3).

We now present a proof of Theorem 2.2.

Proof. Clearly, (2.7) follows from (2.1) by setting $i=0$, and (2.8) can be obtained from (2.2) by setting $i=m$.

We proceed to prove (2.6) by induction on m. It is easy to verify that (2.6) holds for $m=2$. We assume that (2.6) holds for $n \geq 2$, namely,

$$d_i(n+1) > \frac{4n^2+7n+i+3}{2(n+1-i)(n+1)} d_i(n), \qquad 1 \leq i \leq n-1. \tag{2.11}$$

We aim to show that (2.6) holds for $n+1$, that is,

$$d_i(n+2) > \frac{4(n+1)^2+7(n+1)+i+3}{2(n+2)(n+2-i)} d_i(n+1), \quad 1 \leq i \leq n. \tag{2.12}$$

Observe that for $1 \leq i \leq n-1$,

$$2(n+i+1)(4n+3)(4n+5)(n+1-i)(n+1) - 2(4n^2+7n+i+3) \times (n+1)(n+1-i)(4n+4i+5) = -4i(1+2i)(n+1)(n+1-i) < 0.$$

Hence we have for $1 \leq i \leq n-1$,

$$\frac{4n^2+7n+i+3}{2(n+1-i)(n+1)} > \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)}. \tag{2.13}$$

From the inequalities (2.13) and (2.11), we find that for $1 \leq i \leq n-1$,

$$d_i(n+1) > \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)} d_i(n). \tag{2.14}$$

It is easy to check that

$$\frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)}} = \frac{(n+i+1)(4n+3)(4n+5)}{2(n+1)(n+1-i)(4n+4i+5)}.$$

Hence the inequality (2.14) can be rewritten as

$$d_i(n+1) > \frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)}} d_i(n).$$

It follows that

$$\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} d_i(n+1) - \frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)} d_i(n) > \frac{4(n+1)^2+7(n+1)+i+3}{2(n+2-i)(n+2)} d_i(n+1). \tag{2.15}$$

From the recurrence relation (2.3), the left-hand side of (2.15) equals $d_i(n+2)$. Thus we have verified the inequality (2.12) for $1 \leq i \leq n-1$. It is still necessary to show that (2.12) is true for $i = n$, that is,

$$d_n(n+2) > \frac{4(n+1)^2+8n+10}{4(n+2)} d_n(n+1). \tag{2.16}$$

Using the formula (1.5), we get

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1},$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}.$$

It is easily checked that for $n \geq 1$,

$$\frac{d_n(n+2)}{d_n(n+1)} = \frac{(n+1)(4n^2+18n+21)}{2(n+2)(2n+3)} > \frac{4(n+1)^2+8n+10}{4(n+2)}.$$

Hence the proof is complete by induction. □

3. Preliminary inequalities

To prove the ratio monotone property of the Boros-Moll polynomials, we will establish some inequalities based on the recurrence relations derived by Kauers and Paule [9] and Moll [11].

Lemma 3.1. *Let $m \geq 2$ be an integer. Then we have*

$$\frac{m-j}{j+1} > \frac{d_{j+1}(m)}{d_j(m)}, \qquad 1 \leq j \leq m-1. \tag{3.1}$$

Proof. From (2.2) and Theorem 2.2, we find that for $1 \leq j \leq m-1$,

$$\begin{aligned}(4m-2j+3)(m+j+1)d_j(m) &- 2j(j+1)d_{j+1}(m) \\ &= 2(m+1-j)(m+1)d_j(m+1) \\ &> (4m^2+7m+j+3)d_j(m),\end{aligned}$$

which implies (3.1). □

The following lemma gives an upper bound on the ratio $d_i(m+1)/d_i(m)$, which is crucial for the proof of the main result of this paper (Theorem 1.1).

Lemma 3.2. *Let $m \geq 2$ be a positive integer. We have for $0 \leq i \leq m$,*

$$d_i(m+1) \leq B(m,i)d_i(m), \tag{3.2}$$

where $B(m,i)$ is defined by

$$B(m,i) = \frac{A(m,i)}{2(i+2)(4m+2i+5)(m+1)(m-i+1)} \tag{3.3}$$

with

$$\begin{aligned}A(m,i) = &30 + 96m^2 + 94m + 37i + 72m^2i + 8m^2i^2 - i^3 \\ &+ 99mi + 5i^2 + 13mi^2 + 16m^3i + 32m^3.\end{aligned} \tag{3.4}$$

Proof. We proceed by induction on m. It is easily seen that the lemma holds for $m = 2$. We assume that the lemma is true for $n \geq 2$, i.e.,

$$d_i(n+1) \leq B(n,i)d_i(n), \qquad 0 \leq i \leq n, \tag{3.5}$$

where $B(n,i)$ is defined by (3.3). It will be shown that the lemma holds for $n+1$, that is,

$$d_i(n+2) \leq B(n+1,i)d_i(n+1), \quad 0 \leq i \leq n+1. \tag{3.6}$$

For $0 \leq i \leq n$, let

$$\begin{aligned}F(n,i) =&(4n+2i+9)(i+2)(4n+5)(4n+3)(n+i+1),\\ G(n,i) =&-2(-90-23i-202n+51i^3+60i^2-144n^2-32n^3\\ &-80n^2i-8n^2i^2-97ni+13ni^2-16n^3i+16ni^3+8i^4)(n+1).\end{aligned}$$

We claim that

$$\frac{F(n,i)}{G(n,i)} \geq B(n,i), \qquad 0 \leq i \leq n. \tag{3.7}$$

Keeping in mind that $A(n,i)$ is defined by (3.4), it is easy to check that

$$\begin{aligned}
&2(i+2)(4n+2i+5)(n+1)(n-i+1)F(n,i)-A(n,i)G(n,i)\\
&\quad=(128n^4i^4-32n^3i^5-80n^2i^6-16ni^7)+(618n^3i^4-222ni^6-16i^7-284n^2i^5)\\
&\qquad+(844ni^3-170i^4)+(1502n^2i^3-338i^5)+(984n^2i^4-142i^6)\\
&\qquad+(844n^3i^3-590ni^5)+256n^5i^2+720i+10i^3+788i^2+3984n^2i\\
&\qquad+2656ni+3568ni^2+3136n^3i+4600n^3i^2+256n^5i\\
&\qquad+1344n^4i+324ni^4+176n^4i^3+5908n^2i^2+1728n^4i^2.
\end{aligned}$$

We are now in a position to see that the above expression is always nonnegative since the expression in every parenthesis is nonnegative for $0\leq i\leq n$. For example,

$$128n^4i^4-32n^3i^5-80n^2i^6-16ni^7\geq 128n^4i^4-32n^4i^4-80n^4i^4-16n^4i^4=0.$$

Thus we have

$$2(i+2)(4n+2i+5)(n+1)(n-i+1)F(n,i)-A(n,i)G(n,i)\geq 0. \tag{3.8}$$

It is easy to see that $G(n,i)$ is positive for $0\leq i\leq n$, and hence (3.7) can be deduced from (3.8). From the inductive hypothesis (3.5) and (3.8), it follows that for $0\leq i\leq n$,

$$\frac{F(n,i)}{G(n,i)}d_i(n)\geq B(n,i)d_i(n)\geq d_i(n+1). \tag{3.9}$$

It is a routine to verify that

$$\frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)\left(\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)}-B(n+1,i)\right)}=\frac{F(n,i)}{G(n,i)}.$$

From the above identity and (3.9), it follows that for $0\leq i\leq n$,

$$\begin{aligned}
&\frac{(n+1+i)(4n+3)(4n+5)d_i(n)}{4(n+1)(n+2)(n+2-i)\left(\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)}-B(n+1,i)\right)}\\
&\qquad=\frac{F(n,i)}{G(n,i)}d_i(n)\geq d_i(n+1).
\end{aligned} \tag{3.10}$$

Since

$$\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)}-B(n+1,i)$$

is positive for $0\leq i\leq n$, (3.10) can be rewritten as

$$\begin{aligned}
&\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)}d_i(n+1)\\
&\qquad-\frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)}d_i(n)\leq B(n+1,i)d_i(n+1).
\end{aligned} \tag{3.11}$$

From the recurrence relation (2.3), we see that

$$\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)}d_i(n+1)-\frac{(n+1+i)(4n+3)(4n+5)}{4(n+1)(n+2)(n+2-i)}d_i(n)=d_i(n+2). \tag{3.12}$$

In view of (3.11) and (3.12), we find that the inequality (3.6) holds for $0 \leq i \leq n$. It remains to verify that (3.6) holds for $i = n+1$, that is,

$$d_{n+1}(n+2) \leq B(n+1,n+1)d_{n+1}(n+1). \tag{3.13}$$

By the definition (3.3) of $B(n,i)$, we have

$$B(n+1,n+1)=\frac{501+212n^3+692n^2+975n+24n^4}{2(n+3)(6n+11)(n+2)}.$$

From the formula (1.5) for $d_i(m)$, we get

$$d_{n+1}(n+1)=2^{-n-1}\binom{2n+2}{n+1}$$

and

$$d_{n+1}(n+2)=2^{-n-2}\binom{2n+3}{n+1}+2^{-n-2}(n+2)\binom{2n+4}{n+2}.$$

Therefore, for $n \geq 0$, we have

$$\frac{d_{n+1}(n+2)}{d_{n+1}(n+1)}=\frac{(2n+3)(2n+5)}{2(n+2)} \leq \frac{501+212n^3+692n^2+975n+24n^4}{2(n+3)(6n+11)(n+2)}.$$

This completes the proof of the lemma. □

Lemma 3.3. *Let $B(m,j)$ be defined by (3.3) and let $m \geq 2$ be an integer. Then we have for $1 \leq j \leq m$,*

$$d_{j-1}(m) \leq \frac{2(m+1)B(m,j)-(4m+2j+3)}{2(m+j)}d_j(m). \tag{3.14}$$

Proof. From the recurrence relation (2.1) and Lemma 3.2, we find that for $0 \leq j \leq m$,

$$\begin{aligned} 2(m+1)d_j(m+1) &= 2(m+j)d_{j-1}(m)+(4m+2j+3)d_j(m) \\ &\leq 2(m+1)B(m,j)d_j(m), \end{aligned} \tag{3.15}$$

where $B(m,j)$ is defined by (3.3). Then (3.15) implies (3.14). □

Lemma 3.4. *Let m be a positive integer. For $0 \leq i \leq \frac{m}{2}$, we have*

$$\frac{2(2m-i)}{2(m+1)B(m,m-i)-(6m-2i+3)} > \frac{2(m+1)B(m,i)-(4m+2i+3)}{2(m+i)}, \tag{3.16}$$

where $B(m,i)$ is defined by (3.3).

Proof. For $0 \leq i \leq m$, let

$$N(m,i) = 2(2m-i)(m-i+2)(6m-2i+5)(i+1), \tag{3.17}$$

$$M(m,i) = 4(3m-i)(2m-i)(m-i)^2 + (80m^3 - 155m^2 i) + (80m^2 - 108mi) + (20m - 20i) + (94mi^2 - 19i^3) + 28i^2, \tag{3.18}$$

$$C(m,i) = i(24m^2+52m+8m^2 i+37mi+4i^3+12mi^2+20+19i^2+28i), \tag{3.19}$$

$$D(m,i) = 2(i+2)(4m+2i+5)(m-i+1)(i+m). \tag{3.20}$$

Note that $N(m,i), M(m,i), C(m,i)$ and $D(m,i)$ are all nonnegative for $0 \leq i \leq \frac{m}{2}$, since the sum in every parenthesis in (3.17), (3.18), (3.19) and (3.20) is nonnegative for $0 \leq i \leq \frac{m}{2}$. It is easy to check that

$$\begin{aligned} &N(m,i)D(m,i) - C(m,i)M(m,i) \\ &= (312m^5 i^2 + 36m^2 i^5 + 276m^3 i^4 - 612m^4 i^3 - 12mi^6) + (2040m^4 i^2 - 2533m^3 i^3) \\ &\quad + (129mi^5 - 43i^6) + (384m^6 - 752m^5 i) + (3568m^4 - 3328m^3 i) \\ &\quad + (1952m^5 - 2792m^4 i) + (4280m^3 i^2 - 2976m^2 i^3) + (2800m^3 - 1240m^2 i) \\ &\quad + (3868m^2 i^2 - 1080mi^3) + 1240mi^2 + 1488mi^4 + 540i^4 + 800m^2 + 1159m^2 i^4. \end{aligned}$$

Observe that the expression in every parenthesis in the above sum is nonnegative for $0 \leq i \leq \frac{m}{2}$. Moreover, one sees that the term $800m^2$ is certainly positive. It follows that

$$N(m,i)D(m,i) - C(m,i)M(m,i) > 0, \qquad 0 \leq i \leq \frac{m}{2}. \tag{3.21}$$

Recall that $B(n,i)$ is defined by (3.3). It is easy to check that

$$\frac{2(m+1)B(m,i) - (4m+2i+3)}{2(m+i)} = \frac{C(m,i)}{D(m,i)},$$

$$\frac{2(2m-i)}{2(m+1)B(m,m-i) - (6m-2i+3)} = \frac{N(m,i)}{M(m,i)}.$$

Thus the inequality (3.21) is equivalent to (3.16). This completes the proof of the lemma. □

4. Proof of the Main Theorem

Using the preliminary inequalities presented in the previous section, we are ready to give a proof of Theorem 1.1.

Proof. It is clear that Theorem 1.1 holds for $m = 2, 3, 4$. We now assume that $m \geq 5$. First we consider (1.8). In order to verify

$$\frac{d_m(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_1(m)}, \tag{4.1}$$

we invoke the formula (1.5) to get

$$\frac{d_1(m)}{d_0(m)} = \frac{2^{-2m}\sum_{k=1}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m} k}{2^{-2m}\sum_{k=0}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}} < \frac{\sum_{k=1}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m} m}{\sum_{k=1}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}} = m \tag{4.2}$$

and

$$\frac{d_{m-1}(m)}{d_m(m)} = \frac{2^{-m}\binom{2m-1}{m} + 2^{-m}\binom{2m}{m} m}{2^{-m}\binom{2m}{m}} > m. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\frac{d_1(m)}{d_0(m)} < \frac{d_{m-1}(m)}{d_m(m)},$$

which yields (4.1).

The next step is to show that

$$\frac{d_{m-i}(m)}{d_i(m)} < \frac{d_{m-i-1}(m)}{d_{i+1}(m)}, \qquad 1 \le i \le \left[\frac{m-1}{2}\right] - 1. \tag{4.4}$$

By the assumption $m \ge 5$, we have $\left[\frac{m-1}{2}\right] - 1 \ge 1$. Substituting j with i in (3.1), we have for $1 \le i \le \left[\frac{m-1}{2}\right] - 1$,

$$\frac{d_{i+1}(m)}{d_i(m)} < \frac{m-i}{i+1}. \tag{4.5}$$

On the other hand, since $1 \le i \le \left[\frac{m-1}{2}\right] - 1$, we have $m - \left[\frac{m-1}{2}\right] \le m-i-1 \le m-2$. Hence we may substitute j with $m-i-1$ in (3.1) to deduce that

$$\frac{d_{m-i-1}(m)}{d_{m-i}(m)} > \frac{m-i}{i+1}. \tag{4.6}$$

From (4.5) and (4.6), it follows that for $1 \le i \le \left[\frac{m-1}{2}\right] - 1$,

$$\frac{d_{i+1}(m)}{d_i(m)} < \frac{m-i}{i+1} < \frac{d_{m-i-1}(m)}{d_{m-i}(m)}.$$

Hence we have verified (4.4).

It remains to show that the last ratio in (1.8) is smaller than 1. Since $\left[\frac{m-1}{2}\right] < m - \left[\frac{m-1}{2}\right]$, it is easily seen that for $m - \left[\frac{m-1}{2}\right] \le k \le m$, we have

$$\binom{k}{\left[\frac{m-1}{2}\right]} \ge \binom{k}{m - \left[\frac{m-1}{2}\right]}.$$

Based on the formula (1.5) and the above relation, we obtain that

$$\begin{aligned} d_{\lfloor\frac{m-1}{2}\rfloor}(m) &= 2^{-2m} \sum_{k=\lfloor\frac{m-1}{2}\rfloor}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\lfloor\frac{m-1}{2}\rfloor} \\ &> 2^{-2m} \sum_{k=m-\lfloor\frac{m-1}{2}\rfloor}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{\lfloor\frac{m-1}{2}\rfloor} \\ &\geq 2^{-2m} \sum_{k=m-\lfloor\frac{m-1}{2}\rfloor}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{m-\lfloor\frac{m-1}{2}\rfloor} \\ &= d_{m-\lfloor\frac{m-1}{2}\rfloor}(m), \end{aligned}$$

leading to the relation

$$\frac{d_{m-\lfloor\frac{m-1}{2}\rfloor}(m)}{d_{\lfloor\frac{m-1}{2}\rfloor}(m)} < 1.$$

This completes the proof of (1.8).

We now turn our attention to the proof (1.9), which will rely on the bound $B(n,i)$ and Lemmas 3.3 and 3.4. First, rewrite (3.14) as

$$\frac{d_{i-1}(m)}{d_i(m)} \leq \frac{2(m+1)B(m,i)-(4m+2i+3)}{2(m+i)}, \quad 1 \leq i \leq m. \tag{4.7}$$

For $1 \leq i \leq \lfloor\frac{m}{2}\rfloor$, we have $m-\lfloor\frac{m}{2}\rfloor \leq m-i \leq m-1$. It follows that

$$\begin{aligned} &2(m+1)B(m,j)-(4m+2j+3) \\ &\quad = \frac{j(24m^2+8m^2j+52m+37mj+19j^2+28j+20+12mj^2+4j^3)}{(j+2)(4m+2j+5)(m-j+1)}, \end{aligned} \tag{4.8}$$

which is positive for $1 \leq j \leq m$. Substituting j with $m-i$ in (4.8), we obtain that

$$2(m+1)B(m,m-i)-(6m-2i+3) > 0, \qquad 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Hence we can substitute j with $m-i$ in (3.14) to deduce that for $1 \leq i \leq \lfloor\frac{m}{2}\rfloor$,

$$\frac{d_{m-i}(m)}{d_{m-i-1}(m)} \geq \frac{2(2m-i)}{2(m+1)B(m,m-i)-(6m-2i+3)}. \tag{4.9}$$

Combining (4.7), (4.9) and Lemma 3.4, we obtain that for $1 \leq i \leq \lfloor\frac{m}{2}\rfloor$,

$$\frac{d_{i-1}(m)}{d_i(m)} < \frac{d_{m-i}(m)}{d_{m-i-1}(m)},$$

which can be restated as

$$\frac{d_{i-1}(m)}{d_{m-i}(m)} < \frac{d_i(m)}{d_{m-i-1}(m)}, \qquad 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor. \tag{4.10}$$

At this point, it is necessary to show that

$$\frac{d_{\lfloor\frac{m}{2}\rfloor-1}(m)}{d_{m-\lfloor\frac{m}{2}\rfloor}(m)} < 1. \tag{4.11}$$

For $i = \left\lceil \frac{m}{2} \right\rceil$, (4.10) becomes

$$\frac{d_{\left\lceil \frac{m}{2} \right\rceil - 1}(m)}{d_{m-\left\lceil \frac{m}{2} \right\rceil}(m)} < \frac{d_{\left\lceil \frac{m}{2} \right\rceil}(m)}{d_{m-\left\lceil \frac{m}{2} \right\rceil - 1}(m)}. \tag{4.12}$$

When m is even, we have $\left\lceil \frac{m}{2} \right\rceil = m - \left\lceil \frac{m}{2} \right\rceil$. From (4.12) it follows that

$$\frac{d_{\left\lceil \frac{m}{2} \right\rceil - 1}(m)}{d_{m-\left\lceil \frac{m}{2} \right\rceil}(m)} < \frac{d_{m-\left\lceil \frac{m}{2} \right\rceil}(m)}{d_{\left\lceil \frac{m}{2} \right\rceil - 1}(m)},$$

which implies (4.11). When m is odd, we have $\left\lceil \frac{m}{2} \right\rceil = m - \left\lceil \frac{m}{2} \right\rceil - 1$. Then (4.11) immediately follows from (4.12). This completes the proof of Theorem 1.1. □

We remark that the proof of Theorem 1.1 does not explain how the expression (3.3) for $B(m, i)$ is derived. In fact, it has been found by a heuristic approach by considering an approximate equation. It would be interesting to find a proof without guessing a formula for $B(m, i)$.

5. A conjecture

Moll made a conjecture on a property of the Boros-Moll sequences which is stronger than the log-concavity. Given a sequence $A = \{a_i\}_{0 \le i \le n}$, define the operator $\mathcal{L}$ by $\mathcal{L}(A) = S = \{b_i\}_{0 \le i \le n}$, where

$$b_i = a_i^2 - a_{i-1}a_{i+1}, \qquad 0 \le i \le n,$$

with the convention that $a_{-1} = a_{n+1} = 0$. We say that $\{a_i\}_{0 \le i \le n}$ is k-log-concave if $\mathcal{L}^j(\{a_i\}_{0 \le i \le n})$ is log-concave for every $0 \le j \le k-1$, and that $\{a_i\}_{0 \le i \le n}$ is ∞-log-concave if $\mathcal{L}^k(\{a_i\}_{0 \le i \le n})$ is log-concave for every $k \ge 0$. Similarly, we say that $\{a_i\}_{0 \le i \le n}$ is j-ratio-monotone (resp. j-strictly-ratio-monotone) if $\mathcal{L}^k(\{a_i\}_{0 \le i \le n})$ is ratio monotone (resp. strictly ratio monotone) for every $0 \le k \le j-1$, and that $\{a_i\}_{0 \le i \le n}$ is ∞-ratio-monotone (resp. ∞-strictly-ratio-monotone) if $\mathcal{L}^k(\{a_i\}_{0 \le i \le n})$ is ratio monotone (resp. strictly ratio monotone) for every $k \ge 0$.

Moll [10] has conjectured that the Boros-Moll sequence $\{d_i(m)\}_{0 \le i \le m}$ is ∞-log-concave. We propose a stronger conjecture.

Conjecture 5.1. *Suppose that $m \ge 2$ is a positive integer, then the Boros-Moll sequence $\{d_i(m)\}_{0 \le i \le m}$ is ∞-strictly-ratio-monotone.*

We have verified that the Boros-Moll sequence $\{d_i(m)\}_{0 \le i \le m}$ is 2-strictly-ratio-monotone for $2 \le m \le 100$. For example, $\mathcal{L}(\{d_i(8)\}_{0 \le i \le 8})$ is given by

$$b_0 = \frac{16188222324681}{1073741824}, \quad b_1 = \frac{46804848752277}{134217728}, \quad b_2 = \frac{39484127036475}{16777216},$$

$$b_3 = \frac{53734360083525}{8388608}, \quad b_4 = \frac{32860456870725}{4194304}, \quad b_5 = \frac{4614148779669}{1048576},$$

$$b_6 = \frac{284363773551}{262144}, \quad b_7 = \frac{836466345}{8192}, \quad b_8 = \frac{41409225}{16384}.$$

We see that

$$\frac{b_8}{b_0} < \frac{b_7}{b_1} < \frac{b_6}{b_2} < \frac{b_5}{b_3} < 1, \qquad \frac{b_0}{b_7} < \frac{b_1}{b_6} < \frac{b_2}{b_5} < \frac{b_3}{b_4} < 1.$$

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