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FAST INTEGRATION OF HIGHLY OSCILLATORY INTEGRALS WITH EXOTIC OSCILLATORS

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ABSTRACT. In this paper, we present an efficient Filon-type method for the integration of systems containing Bessel functions with exotic oscillators based on a diffeomorphism transformation and give applications to Airy transforms. Preliminary numerical results show the effectiveness and accuracy of the quadrature for large arguments of integral systems.

1. INTRODUCTION

In many areas of applied mathematics one encounters the problem of computing rapidly oscillatory integrals of the type

$$I[f] = \int_a^b f(x)S(\omega g(x))dx, \tag{1.1}$$

where S is an oscillatory function, $f(x)$ and $g(x)$ are sufficiently smooth functions, ω is a large parameter and a and b are real and finite. In most of the cases, such transforms cannot be calculated analytically and one has to resort to numerical methods. When the integrand becomes highly oscillatory, it presents serious difficulties in obtaining numerical convergence of the integration.

Many efficient methods for computing (1.1) when $S(\omega g(x)) = e^{i\omega g(x)}$, where $g'(x) \neq 0$ for all $x \in [a, b]$, have been devised. See, for example, the Filon method [2, 5, 6, 16], the Levin method [14], generalized quadrature rules [3, 4], the asymptotic method [11], the Filon-type method [10, 11, 25], the Levin-type method [17] and the steepest descent method [8]. When S is a Bessel function of the first kind $J_v(\omega x)$, there are also a few methods available. For example, the modified Clenshaw-Curtis method [20, 21] is efficient for computing $\int_0^1 f(x)J_\nu(\omega x)dx$ for nonnegative orders ν; the Levin-type method [15, 18, 26, 27] and generalized quadrature rules [3, 4, 28] are efficient for approximating $\int_a^b f(x)J_\nu(\omega x)dx$ for $Re(\nu) > -1$ and $0 \notin [a, b]$. Moreover, the Levin-type method and generalized quadrature rules can be extended to approximate the integral $\int_a^b f(x)J_v(\omega g(x))dx$ if $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in [a, b]$.

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Once g' vanishes at one or more points in $[a,b]$, which are known as critical points, the above numerical evaluation may be difficult. For treatment of critical points for $\int_a^b f(x)e^{i\omega g(x)}dx$, without loss of generality, assume $\xi_0 \in [a,b]$ is the unique turning point of g and for an integer $r \geq 1$,

$$g'(\xi_0) = g''(\xi_0) = \cdots = g^{(r)}(\xi_0) = 0,\ g^{(r+1)}(\xi_0) \neq 0,$$
$$\text{and } g'(x) \neq 0 \text{ for } x \neq \xi_0,\ x \in [a,b].$$

From Stein ([22], p. 334), it is known from the classical method of the stationary phase that for every smooth function f with compact support near ξ_0, there exist linear differential functionals A_k, $k = 0, 1, \ldots$, such that

$$I[f] \sim \omega^{-1/(r+1)} \sum_{k=0}^{\infty} A_k[f]\omega^{-k/(r+1)}, \qquad \omega \gg 1. \tag{1.2}$$

The following asymptotic expansion developed by Iserles and Nørsett [11] provides an invaluable tool for computing highly oscillatory integrals:

$$\begin{aligned} I[f] \sim & \sum_{j=0}^{r-1} \frac{1}{j!}\mu_j(\omega,\xi_0) \sum_{k=0}^{\infty} \frac{1}{(-i\omega)^k} \rho_k^{(j)}[f](\xi_0) \\ & - \sum_{k=1}^{\infty} \frac{1}{(-i\omega)^k} \left(\frac{e^{i\omega g(b)}}{g'(b)} \left\{ \rho_{k-1}[f](b) - \sum_{j=0}^{r-1} \frac{\rho_{k-1}[f]^{(j)}(\xi_0)(b-\xi)^j}{j!} \right\} \right. \\ & \left. - \frac{e^{i\omega g(a)}}{g'(a)} \left\{ \rho_{k-1}[f](a) - \sum_{j=0}^{r-1} \frac{\rho_{k-1}[f]^{(j)}(\xi_0)(a-\xi)^j}{j!} \right\} \right), \quad \omega \gg 1, \end{aligned} \tag{1.3}$$

where $\mu_j(\omega,\xi_0) = \int_a^b (x-\xi_0)^j e^{i\omega g(x)}dx$, $j = 0, 1, \ldots, r-1$, and

$$\rho_0[f](x) = f(x), \quad \rho_{k+1}[f](x) = \frac{d}{dx} \frac{\rho_k[f](x) - \sum_{j=0}^{r-1} \frac{1}{j!}\rho_k[f]^{(j)}(\xi_0)(x-\xi_0)^j}{g'(x)}, k \geq 0. \tag{1.4}$$

The generalized asymptotic method (the finite sums of the infinite series (1.3)) and the generalized Filon-type method presented in [11] are efficient in dealing with highly oscillatory integrals involving critical points under the condition that the first few moments

$$\mu_k(\omega,\xi_0) = \int_a^b (x-\xi_0)^k e^{i\omega g(x)}dx, \quad k = 0, 1, \ldots, r-1,$$

are in explicit forms. Unfortunately, the moments are often unknown. This situation is readily remedied by a Filon-type method recently developed by Olver [19], where the generalized moments for the basis functions can be computed explicitly by the solutions of the collocation differential equations

$$u' + i\omega g'(x)u = x^k, \quad k = 0, 1, \ldots.$$

However, it is difficult to extend this method to approximate Bessel or Airy transforms since in these cases the corresponding collocation differential equations cannot be solved exactly.

The critical case for $S(\omega g(x)) = e^{i\omega g(x)}$ was also handled in Huybrechs and Vandewalle [8] by going to the complex plane, and integrating along a path that approximated the path of steepest descent. Numerical steepest descent methods for evaluating the integrals involving Hankel functions of the form

$$\int_a^b f(x)H_\nu^{(1)}(\omega g_1(x))e^{i\omega g_2(x)}dx \tag{1.5}$$

were proposed in [9]. Since the Bessel function is the real part of the Hankel function and decays exponentially in the complex plane, then numerical steepest descent methods can also be used to evaluate the integrals involving the Bessel function. However, these methods require the functions f, g_1 and g_2 should be analytic in the complex plane containing $[a, b]$.

The purpose of this paper is to solve the open problems discussed in Iserles et al. [12, 13] and Olver [18] regarding the efficient computation of the integrals

$$\int_0^1 f(x)J_m(\omega g(x))dx, \qquad \int_0^1 f(x)Ai(-\omega g(x))dx, \tag{1.6}$$

with the exotic oscillator $g(x)$ that satisfies for $r \geq 0$,

$$g(0) = g'(0) = \cdots = g^{(r)}(0) = 0, \;\; g^{(r+1)}(0) \neq 0, \quad g'(x) \neq 0 \text{ for } x \in (0, 1], \tag{1.7}$$

where $Re(m) > -\frac{1}{r+1}$. (For $Re(m) \leq -\frac{1}{r+1}$, in general, the integral (1.7) is not defined. Consider $\int_0^1 \sin(x)J_{-0.5}(rx^2)dx$ as an example.) Without loss of generality, we assume $\omega > 0$ and $g^{(r+1)}(0) > 0$ (replacing g by $-g$ if necessary). From (1.7) we see that $g(x)$ is strictly monotonic on $[0, 1]$ and $g(x) > 0$ for $x \in (0, 1]$. Following [25], we apply a diffeomorphism transformation $t^{r+1} = g(x)$ for the exotic oscillator of the integrand $\int_0^1 f(x)S(\omega g(x))dx$ such that the moments $I[t^k] = \int_0^{\sqrt[r+1]{g(1)}} t^k S(\omega t^{r+1})dt$ can be calculated explicitly. Then we can compute the resulting integrals efficiently.

The rest of the paper is organized as follows. In Section 2 we introduce a Filon-type method for (1.6) and present explicit expressions for the moments of the Bessel and Airy transforms. The details of the computation of Lommel functions are also discussed there. In Section 3 we first present the asymptotic expansion for the Bessel transform $\int_0^1 f(x)J_m(\omega x)dx$, and then we give the error analysis of the Filon-type method for $\int_a^b f(x)J_m(\omega g(x))dx$. Preliminary numerical examples show that the Filon-type method is efficient and accurate for approximating the integral considered. Finally, we consider applications to Airy transforms in Section 4.

A word about our notation. (i) If $f(x)/\phi(x)$ tends to unity as $x \to x_0$, we write

$$f(x) \sim \phi(x) \qquad (x \to x_0).$$

In words, f is asymptotic to ϕ.

(ii) If $f(x)/\phi(x) \to 0$ as $x \to x_0$, we write

$$f(x) = o(\phi(x)) \qquad (x \to x_0).$$

(iii) If $f(x)/\phi(x)$ is bounded as $x \to x_0$, we write

$$f(x) = O(\phi(x)) \qquad (x \to x_0).$$

2. Filon-type method and efficient evaluation of the moments

We investigate the following highly oscillatory integral:

$$I[f]=\int_0^1 f(x)S(\omega g(x))dx, \tag{2.1}$$

where $f(x)$ and $g(x)$ are smooth functions and $g(x)$ satisfies (1.7) with $g^{(r+1)}(0)>0$.

Let $t^{r+1}=g(x)$. Then from Stein ([22], pp. 336-337), t is well defined in $[0,1]$ and is a diffeomorphism in a small neighborhood of 0. Thus t is a diffeomorphism in $[0,1]$ since $g'(x)\neq 0$ for $x\in(0,1]$. Therefore, $x(t)$ is well defined and $I[f]$ can be rewritten as

$$\begin{aligned} I[f]=\int_0^1 f(x)S(\omega g(x))dx &= (r+1)\int_0^{y_0}\frac{f(x)g(x)^{r/(r+1)}}{g'(x)}S(\omega t^{r+1})dt \\ &= (r+1)\int_0^{y_0}\widetilde{f}(t)S(\omega t^{r+1})dt, \end{aligned} \tag{2.2}$$

where $\widetilde{f}(t)=\frac{f(x)g(x)^{r/(r+1)}}{g'(x)}$ and $y_0=\sqrt[r+1]{g(1)}$.

Filon-type method. Let $\{c_j\}_{j=1}^{v}$ be a set of node points such that $0=c_1<c_2<\cdots<c_v=1$ and denote

$$d_j=\sqrt[r+1]{g(c_j)},\quad j=1,2,\ldots,v.$$

Assume that s is a nonnegative integer and $\{m_k\}_1^v$ is a set of multiplicities associated with the node points $0=d_1<d_2<\cdots<d_v=y_0$ such that $m_1=s(r+1)+k_0$ $(0\leq k_0\leq r, m_1\geq 1)$ and $m_v\geq s$. Suppose that $p(t)=\sum_{k=0}^{n}a_kt^k$, where $n=\sum_{k=1}^{v}m_k-1$ is the solution of the system of equations

$$p(d_k)=\widetilde{f}(d_k), p'(d_k)=\widetilde{f}'(d_k),\ldots,p^{(m_k-1)}(d_k)=\widetilde{f}^{(m_k-1)}(d_k), \tag{2.3}$$

for every integer $1\leq k\leq v$. Then the Filon-type method is defined as follows:

$$Q_s^F[f]=(r+1)\int_0^{y_0}p(t)S(\omega t^{r+1})dt=(r+1)\sum_{k=0}^{n}a_kI[t^k], \tag{2.4}$$

where $I[t^k]=\int_0^{y_0}t^kS(\omega t^{r+1})dt$. For $d_k\in[0,y_0]$, $k=1,2,\ldots,v$, we obtain $\widetilde{f}(d_k)=\dfrac{f(c_k)g(c_k)^{r/(r+1)}}{g'(c_k)}$. The derivatives of $\widetilde{f}^{(\ell)}(t)$ can be computed by

$$\frac{d^\ell\left[\frac{f(x)t^r}{g'(x)}\right]}{dt^\ell},$$

where x is considered as a function of t and $x'(t)=\dfrac{(r+1)t^r}{g'(x)}$. In particular, for $d_0=0$, $\widetilde{f}(0)$ and $\widetilde{f}^{(k)}(0)$ are computed by their limits using the Taylor expansion of $g(x)$. In this paper we choose the quadrature points as the shifted Chebyshev points $c_k=\dfrac{1+\cos(\frac{(v-k)\pi}{v-1})}{2}$, $k=1,\ldots,v$.

- For $S(\omega t^{r+1}) = J_m(\omega t^{r+1})$: If $\omega > 0$ and $Re(\mu+\nu) > -1$, then it follows from Gradshteyn ([7], p. 707) that

$$\int_0^1 x^\mu J_\nu(\omega x)dx = \frac{2^\mu \Gamma(\frac{\mu+\nu+1}{2})}{\omega^{\mu+1}\Gamma(\frac{\nu-\mu+1}{2})} + \omega^{-\mu}\{(\mu+\nu-1)J_\nu(\omega)s^{(2)}_{\mu-1,\nu-1}(\omega) - J_{\nu-1}(\omega)s^{(2)}_{\mu,\nu}(\omega)\}, \tag{2.5}$$

where $s^{(2)}_{\mu,\nu}(z)$ denotes the second kind of Lommel function. According to the preceding conditions $Re(m) > -\frac{1}{r+1}$ and $y_0 > 0$, we can deduce that $Re(m) + \frac{k+1}{r+1} - 1 > -1$ and $\omega y_0^{r+1} > 0$. Therefore, we can obtain the moments for $J_m(\omega t^{r+1})$ immediately from (2.5):

$$\begin{aligned} I[t^k] &= \int_0^{y_0} t^k J_m(\omega t^{r+1})dt \\ &= \frac{y_0^{k+1}}{r+1}\int_0^1 x^{\frac{k+1}{r+1}-1} J_m(\omega y_0^{r+1}x)dx \\ &= \frac{y_0^{k+1}}{r+1}\Big[\frac{2^{\frac{k+1}{r+1}-1}\Gamma(\frac{m+\frac{k+1}{r+1}}{2})}{(\omega y_0^{r+1})^{\frac{k+1}{r+1}}\Gamma(\frac{m-\frac{k+1}{r+1}+2}{2})} + (\omega y_0^{r+1})^{-\frac{k+1}{r+1}+1}\cdot \\ &\quad \big\{(\frac{k+1}{r+1}+m-2)J_m(\omega y_0^{r+1})s^{(2)}_{\frac{k+1}{r+1}-2,m-1}(\omega y_0^{r+1}) \\ &\quad -J_{m-1}(\omega y_0^{r+1})s^{(2)}_{\frac{k+1}{r+1}-1,m}(\omega y_0^{r+1})\big\}\Big]. \end{aligned} \tag{2.6}$$

For large values of z, $s^{(2)}_{\mu,\nu}(z)$ admits the following asymptotic expansion ([24], p. 351):

$$\begin{aligned} & s^{(2)}_{\mu,\nu}(z) \\ =\ & z^{\mu-1}\left[1 - \frac{(\mu-1)^2-\nu^2}{z^2} + \frac{\{(\mu-1)^2-\nu^2\}\{(\mu-3)^2-\nu^2\}}{z^4} - \cdots\right] \\ =\ & z^{\mu-1}\left[1 - \frac{(\mu-1)^2-\nu^2}{z^2} + \cdots \right. \\ & \left. +(-1)^p\frac{\{(\mu-1)^2-\nu^2\}\cdots\{(\mu-2p+1)^2-\nu^2\}}{z^{2p}}\right] + O(z^{\mu-2p-2}). \end{aligned} \tag{2.7}$$

Therefore, $s^{(2)}_{\mu,\nu}(z)$ can be efficiently approximated by truncating (2.7) once ω is large. Furthermore, if either of the numbers $\mu \pm \nu$ is an odd positive integer, then the Lommel function $s^{(2)}_{\mu,\nu}(z)$ has a finite representation of (2.7) ([24], p. 347).

- For $S(\omega t^{r+1}) = Ai(-\omega t^{r+1})$: By

$$Ai(-x) = \frac{1}{3}\sqrt{x}\left[J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right] \quad \text{([1], p. 447)},$$

the moment $I[t^k] = \int_0^1 t^k Ai(-\omega t^{r+1})dt$ can be represented by

(2.8)
$$
\begin{aligned}
I[t^k] &= \int_0^{y_0} t^k Ai(-\omega t^{r+1})dt = \frac{y_0^{r+1}}{r+1}\int_0^1 t^{\frac{k-r}{r+1}} Ai(-\omega y_0^{r+1} t)dt \\
&= \frac{\sqrt{\omega} y_0^{k+1+\frac{r+1}{2}}}{9(r+1)}\Bigg[\frac{2^{\frac{2k-2r}{3r+3}}\Gamma(\frac{k+1}{3r+3})}{(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})^{\frac{2k+r+3}{3r+3}}\Gamma(\frac{2r-k+1}{3r+3})} + \frac{2^{\frac{2k-2r}{3r+3}}\Gamma(\frac{k+r+2}{3r+3})}{(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})^{\frac{2k+r+3}{3r+3}}\Gamma(\frac{3r-k+2}{3r+3})} \\
&\quad + (\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})^{-\frac{2k-2r}{3r+3}}\Bigg\{(\frac{2k-6r-4}{3r+3})J_{-\frac{1}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})s^{(2)}_{\frac{2k-5r-3}{3r+3},-\frac{4}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}}) \\
&\qquad - J_{-\frac{4}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})s^{(2)}_{\frac{2k-2r}{3r+3},-\frac{1}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}}) \\
&\qquad + (\frac{2k-4r-2}{3r+3})J_{\frac{1}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})s^{(2)}_{\frac{2k-5r-3}{3r+3},-\frac{2}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}}) \\
&\qquad - J_{-\frac{2}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})s^{(2)}_{\frac{2k-2r}{3r+3},\frac{1}{3}}(\frac{2}{3}(\omega y_0^{r+1})^{\frac{3}{2}})\Bigg\}\Bigg].
\end{aligned}
$$

3. Numerical analysis for the Filon-type method for the Bessel transform

In this section, we consider the Filon-type method (2.4) for

$$I[f] = \int_0^1 f(x)J_m(\omega g(x))dx, \tag{3.1}$$

where $f(x)$ and $g(x)$ are smooth functions and $g(x)$ satisfies (1.7) with $g^{(r+1)}(0) > 0$. In the case of $g^{(r+1)}(0) < 0$, by using $J_m(x) = e^{-im\pi}J_m(-x)$ ([1], p. 361]), the integral (3.1) can be transformed into $e^{-im\pi}\int_0^1 f(x)J_m(-\omega g(x))dx$ with $-g^{(r+1)}(0) > 0$.

Lemma 3.1. *Suppose that α is a real number with $0 \le \alpha \le r$. Then for every function $w(t) \in C^1[0, y_0]$, we have*

$$|\int_0^{y_0} w(t)t^\alpha J_m(\omega t^{r+1})dt| \le \frac{C}{\omega^{(\alpha+1)/(r+1)}}\left(|w(y_0)| + \int_0^{y_0}|w'(t)|dt\right) \tag{3.2}$$

for some constants C independent of $w(t)$ and ω.

Proof. We first show that for all $x \in [0, y_0]$,

$$|\int_0^x t^\alpha J_m(\omega t^{r+1})dt| \le C\omega^{-(\alpha+1)/(r+1)} \tag{3.3}$$

for some constant C independent of x, ω and $w(t)$. Let $u = \omega t^{r+1}$. Then, the integral in (3.3) is expressed by

$$\int_0^x t^\alpha J_m(\omega t^{r+1})dt = \frac{\omega^{-(\alpha+1)/(r+1)}}{r+1}\int_0^{\omega x^{r+1}} u^{\frac{\alpha-r}{r+1}} J_m(u)du. \tag{3.4}$$

Therefore, it is trivial to prove (3.3) if the integral in the right-hand side of (3.4) is uniformly bounded on x and ω. Recalling from the asymptotic expansion of $s^{(2)}_{\mu,\nu}(u)$ in (2.7), for large u one can obtain

$$s^{(2)}_{\frac{\alpha-r}{r+1}-1,m-1}(u) = O(u^{\frac{\alpha-r}{r+1}-2}), \quad s^{(2)}_{\frac{\alpha-r}{r+1},m}(u) = O(u^{\frac{\alpha-r}{r+1}-1}). \tag{3.5}$$

This, together with the following asymptotic expansion ([1], p. 364; [22], p. 338)

$$J_\nu(u) = (\frac{2}{\pi})^{\frac{1}{2}} u^{-\frac{1}{2}} \cos(u - \frac{\nu\pi}{2} - \frac{\pi}{4}) + O(u^{-\frac{3}{2}}), \quad u \gg 1, \tag{3.6}$$

and with (2.6), deduces that $\int_0^{+\infty} u^{\frac{\alpha-r}{r+1}} J_m(u)du$ is convergent, and then the function $G(z) = \int_0^z u^{\frac{\alpha-r}{r+1}} J_m(u)du$ is continuous and uniformly bounded in $[0, +\infty)$, which implies that $\int_0^{\omega x^{r+1}} u^{\frac{\alpha-r}{r+1}} J_m(u)du$ is uniformly bounded on $[0, y_0]$.

Expression (3.2) is proved by writing $\int_0^{y_0} w(t)t^\alpha J_m(\omega t^{r+1})dt$ as $\int_0^{y_0} w(t)F'(t)dt$, with

$$F(t) = \int_0^t u^\alpha J_m(\omega u^{r+1})du.$$

Integrating by parts one derives

$$\int_0^{y_0} w(t)t^\alpha J_m(\omega t^{r+1})dt = w(y_0)F(y_0) + \int_0^{y_0} w'(t)F(t)dt,$$

which together with (3.3) gives the desired result. □

Example 3.1. Let us consider the asymptotic behaviour of $I[t^k] = \int_0^1 t^k J_0(\omega t^3)dt$ for large ω (see Figure 1).

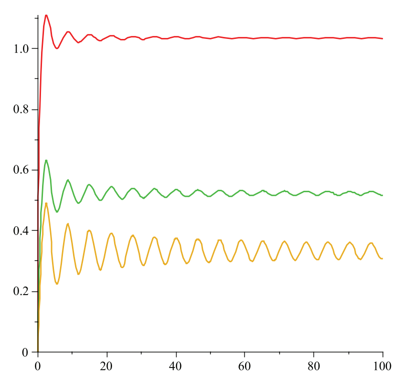

FIGURE 1. The absolute values of the moment $I[t^k] = \int_0^1 t^k J_0(\omega t^3)dt$ scaled by $\omega^{(k+1)/3}$ for $k = 0, 1, 2$ (the top, middle and bottom, respectively) and $0 \le \omega \le 100$.

The asymptotic expansion (1.3) of the highly oscillatory integral $\int_0^1 f(x)e^{i\omega g(x)}dx$ provides an invaluable tool in numerical analysis [11]. For the integral $\int_0^{y_0} f(x)J_m(\omega x)dx$, the asymptotic expansion can be obtained from the following lemma.

Lemma 3.2. *For every smooth f it is true that*

$$\begin{aligned}\int_0^{y_0} f(x)J_m(\omega x)dx \sim& \sum_{k=0}^{+\infty}\frac{(-1)^k}{\omega^k}\sigma_k[f](0)\int_0^{y_0} J_{m+k}(\omega x)dx\\ &-\sum_{k=1}^{+\infty}\frac{(-1)^k}{\omega^k}\left\{\sigma_{k-1}[f](y_0)-\sigma_{k-1}[f](0)\right\}J_{m+k}(\omega y_0),\end{aligned} \tag{3.7}$$

where $\int_0^{y_0} J_{m+k}(\omega x)dx$ can be computed explicitly by (2.6) and

$$\sigma_0[f](x) = f(x),$$
$$\sigma_{k+1}[f](x) = \frac{x\sigma_k[f]'(x) - (m+k+1)(\sigma_k[f](x) - \sigma_k[f](0))}{x}, \quad k \geq 0.$$

Proof. Following Iserles and Nørsett [11], we prove by induction on s the identity

$$\begin{aligned}&\int_0^{y_0} f(x)J_m(\omega x)dx\\ &\quad= \sum_{k=0}^{s-1}\frac{(-1)^k}{\omega^k}\sigma_k[f](0)\int_0^{y_0} J_{m+k}(\omega x)dx\\ &\qquad-\sum_{k=1}^{s}\frac{(-1)^k}{\omega^k}\left\{\sigma_{k-1}[f](y_0)-\sigma_{k-1}[f](0)\right\}J_{m+k}(\omega y_0)\\ &\qquad+\frac{(-1)^s}{\omega^s}\int_0^{y_0}\sigma_s[f](x)J_{m+s}(\omega x)dx.\end{aligned} \tag{3.8}$$

Suppose that $s = 1$. From $\frac{d}{dx}\left[x^{m+1}J_{m+1}(x)\right] = x^{m+1}J_m(x)$ ([1], p. 361]), we have

$$\begin{aligned}&\int_0^{y_0} f(x)J_m(\omega x)dx\\ &\quad= \sigma_0[f](0)\int_0^{y_0} J_m(\omega x)dx + \frac{1}{\omega}\int_0^{y_0}\frac{\sigma_0[f](x)-\sigma_0[f](0)}{x^{m+1}}d[x^{m+1}J_{m+1}(\omega x)]\\ &\quad= \sigma_0[f](0)\int_0^{y_0} J_m(\omega x)dx + \frac{1}{\omega}\left(\sigma_0[f](y_0)-\sigma_0[f](0)\right)J_{m+1}(\omega y_0)\\ &\qquad-\frac{1}{\omega}\int_0^{y_0}\frac{x\sigma_0[f]'(x)-(m+1)(\sigma_0[f](x)-\sigma_0[f](0))}{x}J_{m+1}(\omega x)dx\\ &\quad= \sigma_0[f](0)\int_0^{y_0} J_m(\omega x)dx + \frac{1}{\omega}\left(\sigma_0[f](y_0)-\sigma_0[f](0)\right)J_{m+1}(\omega y_0)\\ &\qquad-\frac{1}{\omega}\int_0^{y_0}\sigma_1[f](x)J_{m+1}(\omega x)dx.\end{aligned}$$

So it is true for $s = 1$. For $s \geq 2$, integration by parts on the right yields

$$\begin{aligned}
&\frac{(-1)^s}{\omega^s}\int_0^{y_0}\sigma_s[f](x)J_{m+s}(\omega x)dx\\
&\quad= \frac{(-1)^s\sigma_s[f](0)}{\omega^s}\int_0^{y_0}J_{m+s}(\omega x)dx\\
&\qquad+\frac{(-1)^s}{\omega^{s+1}}\int_0^{y_0}\frac{\sigma_s[f](x)-\sigma_s[f](0)}{x^{m+s+1}}d[x^{m+s+1}J_{m+s+1}(\omega x)]\\
&\quad= \frac{(-1)^s\sigma_s[f](0)}{\omega^s}\int_0^{y_0}J_{m+s}(\omega x)dx\\
&\qquad+\frac{(-1)^s(\sigma_s[f](x)-\sigma_s[f](0))}{\omega^{s+1}}J_{m+s+1}(\omega x)|_0^{y_0}\\
&\qquad+\frac{(-1)^{s+1}}{\omega^{s+1}}\int_0^{y_0}\sigma_{s+1}[f](x)J_{m+s+1}(\omega x)dx.
\end{aligned}$$

This proves (3.8). Letting $s \to \infty$ yields the asymptotic expansion (3.7). □

The following asymptotic quadrature for $\int_0^{y_0} f(x)J_m(\omega x)dx$, defined by

$$\begin{aligned}
Q_s^A[f] &= \sum_{k=0}^{s-1}\frac{(-1)^k}{\omega^k}\sigma_k[f](0)\int_0^{y_0}J_{m+k}(\omega x)dx\\
&\quad-\sum_{k=1}^{s}\frac{(-1)^k}{\omega^k}\left\{\sigma_{k-1}[f](y_0)-\sigma_{k-1}[f](0)\right\}J_{m+k}(\omega y_0),
\end{aligned} \tag{3.9}$$

a truncation of the asymptotic expansion (3.7), is efficient for large values of ω.

Theorem 3.1. *For every smooth f it is true that*

$$Q_s^A[f]-\int_0^{y_0}f(x)J_m(\omega x)dx = O\left(\frac{1}{\omega^{s+1}}\right), \quad \omega \gg 1.$$

Proof. This follows directly from Lemma 3.1 and (3.8). □

Throughout the rest of the paper, we evaluate $s_{\mu,\nu}^{(2)}(z)$ by truncating after the first 10 terms of (2.7). The exact values of the integrals in the numerical examples are computed by Maple 11 with 100-digit arithmetic; they are also tested by using the code of Clenshaw-Curtis methods presented by Trefethen in [23] with 32-digit or 64-digit arithmetic, respectively.

Example 3.2. Let us consider the asymptotic quadrature (3.9) for approximating

$$\int_0^1 \cos(x)J_0(\omega x)dx \text{ (see Figure 2).}$$

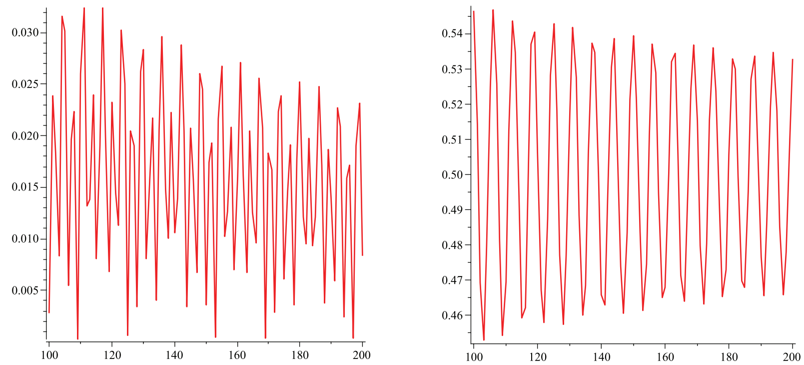

FIGURE 2. The absolute error for $Q_1^A[\cos(x)]$ (the left) and $Q_2^A[\cos(x)]$ (the right) scaled by ω^2 and ω^3, respectively, for $\int_0^1 \cos(x) J_0(\omega x)dx$ and $100 \le \omega \le 200$.

Applying the asymptotic expansion (3.7), we give the error analysis of the Filon-type method (2.4) for the Bessel transform in (1.6).

Theorem 3.2. *The Filon-type method defined by* (2.4) *for*

$$I[f] = \int_0^1 f(x) J_m(\omega g(x))dx$$

satisfies

$$I[f] - Q_s^F[f] = O\left(\frac{1}{\omega^{s+(k_0+1)/(r+1)}}\right). \tag{3.10}$$

Proof. Let $h(t) = \widetilde{f}(t) - p(t)$ and $x = t^{r+1}$. Then from (2.2) and (2.4) the error can be written as

$$\begin{aligned} I[f] - Q_s^F[f] &= (r+1)\int_0^{y_0} h(t) J_m(\omega t^{r+1})dt \\ &= \int_0^{y_0^{r+1}} \frac{h(x^{1/(r+1)})}{x^{r/(r+1)}} J_m(\omega x)dx \\ &= \int_0^{y_0^{r+1}} \psi(x) J_m(\omega x)dx, \end{aligned} \tag{3.11}$$

where $\psi(x) = \frac{h(x^{1/(r+1)})}{x^{r/(r+1)}}$. From Maclaurin's expansion of

$$h(t) = \frac{\widetilde{f}^{(s(r+1)+k_0)}(\xi) - p^{(s(r+1)+k_0)}(\xi)}{(s(r+1)+k_0)!} t^{s(r+1)+k_0} \quad \text{for some } \xi \in [0,t],$$

we have

$$\begin{cases} \psi(x) = \dfrac{\widetilde{f}^{(s(r+1)+k_0)}(\xi) - p^{(s(r+1)+k_0)}(\xi)}{(s(r+1)+k_0)!} x^{s+\frac{k_0-r}{r+1}} \\ \psi(0) = \psi'(0) = \cdots = \psi^{(s-1)}(0) = 0, \psi(y_0^{r+1}) \\ \qquad\qquad = \psi'(y_0^{r+1}) = \cdots = \psi^{(s-1)}(y_0^{r+1}) = 0. \end{cases} \tag{3.12}$$

It is not difficult from (3.12) and the definitions of $\sigma_k[\psi](x)$ to verify that

$$\sigma_1[\psi](x) = O\left(x^{s-1+\frac{k_0-r}{r+1}}\right), \ldots, \sigma_s[\psi](x) = O\left(x^{\frac{k_0-r}{r+1}}\right)$$

and

$$\sigma_k[\psi](0) = \sigma_k[\psi](y_0^{r+1}) = 0, \quad k = 0, \ldots, s-1.$$

Therefore

$$\sigma_1[\psi](t^{r+1}) = O\left(t^{(s-1)(r+1)+k_0-r}\right), \ldots, \sigma_s[\psi](t^{r+1}) = O\left(t^{k_0-r}\right),$$
$$\sigma_s[\psi](t^{r+1})t^r = O\left(t^{k_0}\right).$$

This, together with (3.8) and Lemma 3.1, implies

$$\begin{aligned} I[f] - Q_s^F[f] &= \frac{(-1)^s}{\omega^s} \int_0^{y_0^{r+1}} \sigma_s[\psi](x) J_{m+s}(\omega x) dx \\ &= \frac{(-1)^s(r+1)}{\omega^s} \int_0^{y_0} \sigma_s[\psi](t^{r+1}) t^r J_{m+s}(\omega t^{r+1}) dt \\ &= O\left(\frac{1}{\omega^{s+(k_0+1)/(r+1)}}\right). \end{aligned}$$ □

Example 3.3. We now demonstrate the application of the Filon-type method to several numerical examples. The first is the computation of $\int_0^1 e^x J_0(\omega(1 - \cos(x)))dx$. It is evident that the Filon-type method is applicable. We compare two

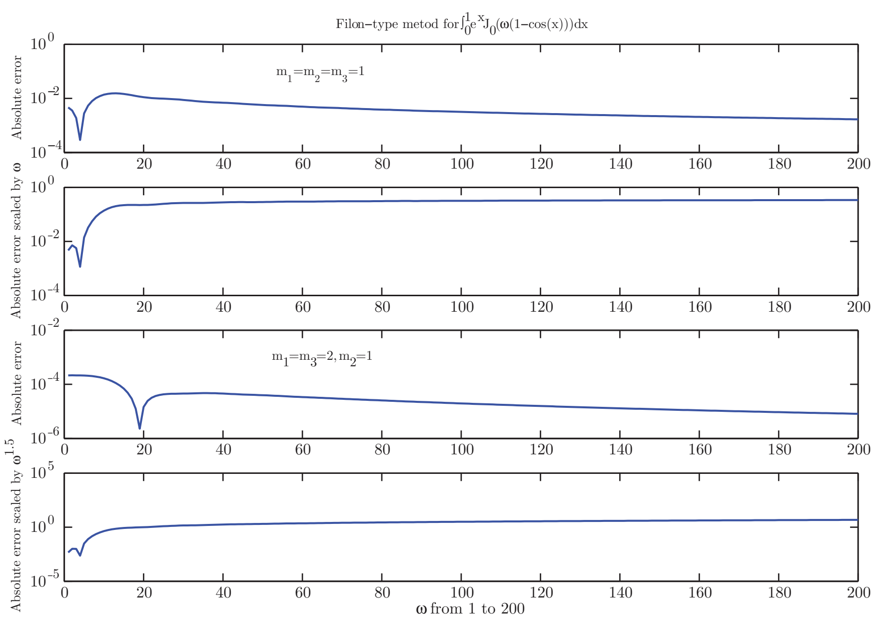

FIGURE 3. The error (the top), error scaled by ω (the second), the error (the third), and error scaled by $\omega^{1.5}$ (bottom) with the same nodes $\{x_j = \frac{(\cos(\frac{(n-j)\pi}{2})+1)}{2}, j = 1, 2, 3\}$, and multiplicities all one and $\{2, 1, 2\}$, respectively, for $\int_0^1 e^x J_0(\omega(1 - \cos(x)))dx$.

Filon-type methods with the same three nodes and multiplicities all 1 and $\{2,1,2\}$, respectively. As seen in Figure 3, the first two figures illustrate the errors (the top) and errors scaled by ω (the second) with nodes $\{x_j = \frac{(\cos(\frac{(3-j)\pi}{2})+1)}{2}, j = 1,2,3\}$ and multiplicities all 1. The bottom two figures illustrate the errors and errors scaled by $\omega^{1.5}$ with the same nodes and multiplicities $\{2,1,2\}$, respectively.

The other two examples are $I[\frac{1}{1+x}] = \int_0^1 \frac{1}{1+x} J_1(\omega(e^x - x - 1))dx$ and $I[e^x] = \int_0^1 e^x J_2(\omega(x - \sin(x)))dx$, which seem more complicated to handle. Tables 1 and 2 illustrate the relative errors in v-points approximation with multiplicities all 1. From these two tables, we can see that the Filon-type method exhibits the fast convergence as v increases. Tables 3 and 4 illustrate that as the multiplicities increase at the end points, the approximate accuracy can be enhanced.

TABLE 1. Relative errors in v-points approximation to $I[\frac{1}{1+x}] = \int_0^1 \frac{1}{1+x} J_1(\omega(e^x - x - 1))dx$ with multiplicities all one.

ω	$v = 4$	$v = 8$	$v = 12$
200	0.0046263225	0.0000085084	$1.427796502E-8$
500	0.0039959151	0.0000036176	$2.983185440E-9$
1000	0.0032365958	$2.200974762E-7$	$8.707204300E-9$
2000	0.0025035221	0.0000023876	$6.252544276E-9$

TABLE 2. Relative errors in v-points approximation to $I[e^x] = \int_0^1 e^x J_2(\omega(x - \sin(x)))dx$ with multiplicities all one.

ω	$v = 4$	$v = 8$	$v = 12$
200	0.001656537	$4.586916397E-8$	$7.026658595E-14$
500	0.0005901539	$1.574877810E-7$	$3.537131115E-12$
1000	0.0002677535	$1.160995982E-7$	$9.436098047E-12$
2000	0.0009114874	$7.731021307E-9$	$4.422931661E-12$

TABLE 3. Relative errors in v-points approximation to $I[\frac{1}{1+x}] = \int_0^1 \frac{1}{1+x} J_1(\omega(e^x - x - 1))dx$ with $m_1, m_v = 2$ and the others all one.

ω	$v = 4$	$v = 8$	$v = 12$
200	0.0003147118	$6.597628622E-7$	$1.309191824E-9$
500	0.0001758496	$1.734538903E-7$	$1.591356020E-10$
1000	0.0001015467	$9.101387022E-9$	$3.248618020E-10$
2000	0.0000558027	$5.724619643E-8$	$1.581881723E-10$

TABLE 4. Relative errors in v-points approximation to $I[e^x] = \int_0^1 e^x J_2(\omega(x-\sin(x)))dx$ with $m_1, m_v = 2$ and the others all one.

ω	$v=4$	$v=8$	$v=12$
200	0.0000618306	$1.673703661E-9$	$2.112253035E-14$
500	0.0000196664	$4.445163366E-9$	$1.098853818E-13$
1000	0.0000067988	$2.896855435E-9$	$2.289120828E-13$
2000	0.0000208972	$9.401439940E-11$	$9.561286297E-14$

4. An application to the Airy transform

In this section, we consider the Filon-type method for the integral

$$I[f] = \int_0^1 f(x)Ai(-\omega g(x))dx, \tag{4.1}$$

where $f(x)$ and $g(x)$ are smooth functions and $g(x)$ satisfies (1.7) with $g^{(r+1)}(0) > 0$. Using the transformation $t^{r+1} = g(x)$, (4.1) can be transformed into

$$I[f] = (r+1)\int_0^{y_0} \widetilde{f}(t)Ai(-\omega t^{r+1})dt, \tag{4.2}$$

where $\widetilde{f}(t)$ is a smooth function.

Assume that s is a positive integer and that $\{m_k\}_1^v$ is a set of multiplicities associated with the node points $0 = d_1 < d_2 < \cdots < d_v = y_0$ such that $m_1 \geq floor\left(\frac{(3s-1)(r+1)+k_0}{2}\right)$ $(0 \leq k_0 < 3r+3)$ and $m_v \geq s$, where $floor(x)$ rounds the elements of x to the nearest integers towards minus infinity. Suppose that $p(t)$ is the Hermite interpolating polynomial defined by (2.3) and $h(t) = \widetilde{f}(t) - p(t)$.

Based on

$$Ai(-x) = \frac{1}{3}\sqrt{x}\left[J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right],$$

the error can be rewritten by $t = u^2$ as

$$\begin{aligned} I[f] - Q_s^F[f] = \frac{2(r+1)\sqrt{\omega}}{3}\int_0^{\sqrt{y_0}} h(u^2)u^{r+2}\Bigg[&J_{-\frac{1}{3}}\left(\frac{2}{3}\omega^{\frac{3}{2}}u^{3r+3}\right) \\ &+J_{\frac{1}{3}}\left(\frac{2}{3}\omega^{\frac{3}{2}}u^{3r+3}\right)\Bigg]du. \end{aligned} \tag{4.3}$$

Theorem 4.1. *The Filon-type method defined by (2.4) with*

$$m_1 \geq floor\left(\frac{(3s-1)(r+1)+k_0}{2}\right), \quad m_v \geq s,$$

for $I[f] = \int_0^1 f(x)Ai(-\omega g(x))dx$ satisfies

$$I[f] - Q_s^F[f] = O\left(\frac{1}{\omega^{\frac{3s-1}{2}+\frac{k_0+1}{2(r+1)}}}\right). \tag{4.4}$$

Proof. From (4.3), we see that $u = 0$ and $u = \sqrt{y_0}$ are two roots of $h(u^2)u^{r+2} = 0$ with multiplicities $\widetilde{m_1} \geq 3s(r+1) + k_0$ and $\widetilde{m_v} \geq s$, respectively. Therefore, from

Theorem 3.2, we have

$$I[f] - Q_s^F[f] = O\left(\frac{\omega^{\frac{1}{2}}}{\left(\omega^{\frac{3}{2}}\right)^{\frac{3s(r+1)+k_0+1}{3(r+1)}}}\right) = O\left(\frac{1}{\omega^{\frac{3s-1}{2}+\frac{k_0+1}{2(r+1)}}}\right). \qquad \square$$

Example 4.1. As an illustration of the effectiveness of the Filon-type method, we consider the Filon-type method $Q_s^F[\frac{1}{1+x}]$ $(s = 1, 2)$ in the case $r = 0$ for approximating

$$I[\frac{1}{1+x}] = \int_0^1 \frac{1}{1+x} Ai(-\omega x)dx \text{ (see Figure 4).}$$

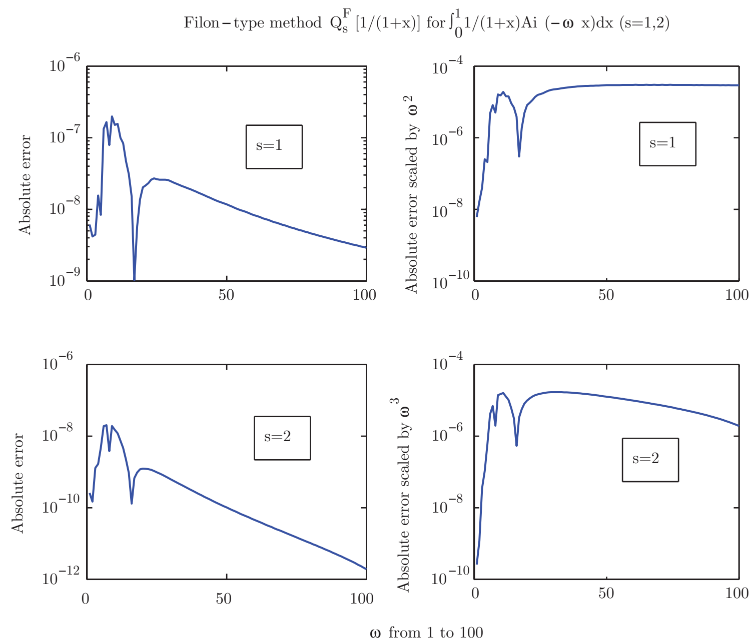

FIGURE 4. The absolute error and the absolute error scaled by ω^2 and ω^3, respectively, for the Filon-type method with $m_1 = \cdots = m_8 = 1$ (the first row), and $m_1 = m_8 = 2$, $m_2 = \cdots = m_7 = 1$ (the second row) at shifted Chebyshev points $\{x_j = \frac{(\cos(\frac{(8-j)\pi}{7})+1)}{2}, j = 1, 2, \ldots, 8\}$, for $I[f] = \int_0^1 \frac{1}{1+x} Ai(-\omega x)dx$.

Remark 1. The asymptotic order on ω of the error bound (4.4) of the Filon-type method (2.4) for the Airy transform is sharp. For example, let us consider

$$\int_0^1 (x^3 + 2x + 1 + x^2(1 + \cos(\pi x)))Ai(-\omega x)dx.$$

The polynomial x^3+2x+1 is the Hermite interpolating polynomial of $x^3+2x+1+x^2(1+\cos(\pi x))$ at 0 and 1 with multiplicities all 2. So the error for the Filon-type method can be written as

$$I[f] - Q_2^F[f] = \int_0^1 x^2(1+\cos(\pi x))Ai(-\omega x)dx.$$

Since $[Ai(x)]'' = xAi(x)$ ([1], p. 446), it follows that $\omega^3 xAi(-\omega x) = -[Ai(-\omega x)]''$. Integrating by parts twice we get for $\omega \gg 1$,

$$\begin{aligned} I[f] - Q_2^F[f] &= -\frac{1}{\omega^3}\int_0^1 x(1+\cos(\pi x))[Ai(-\omega x)]''dx \\ &= -\frac{2}{\omega^3}Ai(0) + \frac{\pi}{\omega^3}\int_0^1 (2\sin(\pi x) + \pi x\cos(\pi x))Ai(-\omega x)dx \\ &= O\left(\frac{1}{\omega^3}\right) \end{aligned}$$

since $Ai(0) = 3^{-2/3}/\Gamma\left(\frac{2}{3}\right) \neq 0$ and

$$\begin{aligned} & \frac{\pi}{\omega^3}\int_0^1 (2\sin(\pi x) + \pi x\cos(\pi x))Ai(-\omega x)dx \\ = & \frac{\pi}{3\omega^3}\int_0^1 (2\sin(\pi x) + \pi x\cos(\pi x))\sqrt{x}\left[J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right] \\ = & O\left(\frac{1}{\omega^4}\right) \text{(Lemma 3.1).} \end{aligned}$$

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