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COMPUTING THE HILBERT TRANSFORM AND ITS INVERSE

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Abstract. We construct a new method for approximating Hilbert transforms and their inverse throughout the complex plane. Both problems can be formulated as Riemann–Hilbert problems via Plemelj's lemma. Using this framework, we rederive existing approaches for computing Hilbert transforms over the real line and unit interval, with the added benefit that we can compute the Hilbert transform in the complex plane. We then demonstrate the power of this approach by generalizing to the half line. Combining two half lines, we can compute the Hilbert transform of a more general class of functions on the real line than is possible with existing methods.

1. Introduction

We consider the computation of the Hilbert transform

$$\mathcal{H}_\Gamma f(z) = \frac{1}{\pi} \fint_\Gamma \frac{f(t)}{t-z}\, \mathrm{d}t, \tag{1.1}$$

where Γ is an oriented curve in the extended complex plane $\mathbb{C}$, $f : \Gamma \to \mathbb{C}$ satisfies a Hölder condition and $z \in \mathbb{C}$, including the possibility of z lying on Γ itself. (Note that the Hilbert transform is often defined with the opposite sign as (1.1).) We also consider the inverse problem, i.e., finding a continuous function $u : \Gamma \to \mathbb{C}$ which satisfies

$$\mathcal{H}_\Gamma u(z) = f(z) \quad \text{for } z \in \Gamma, \tag{1.2}$$

or in other words, computing $\mathcal{H}_\Gamma^{-1} f$. In particular, we consider the case where Γ is one of the following domains (using $\mathbb{T} = [-\pi, \pi)$ to denote the periodic interval):

$$\begin{aligned}
&\text{unit circle} && \mathbb{U} = \mathrm{e}^{\mathrm{i}\mathbb{T}} = \{z : |z| = 1\}, \\
&\text{real line} && \mathbb{R} = (-\infty, \infty), \\
&\text{unit interval} && \mathbb{I} = [-1, 1], \\
&\text{half line} && \mathbb{R}^+ = [0, \infty).
\end{aligned}$$

We use the notation $\mathcal{H}$ when Γ is clear from the context.

There are many applications for the computation of (1.1) and (1.2), including computing the analytic signal [12] and the Benjamin–Ono equation [5, 24]. Our interest stems from the numerical solution of gravity waves and the computation

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of solutions to matrix-valued Riemann–Hilbert problems. Gravity wave flow over a step satisfies the equation [7]

$$\log q(t) = \frac{1}{2}\log\frac{t+b}{t+a} - \mathcal{H}_{\mathbb{R}^+}\theta,$$
$$\epsilon q(t)^2 t q'(t) = \sin\theta(t).$$

The principal value integral in the Hilbert transform means that the solution to this equation is global; we cannot time-step as in an ODE. If we attempt to solve this equation using an iterative scheme, we invariably need to either compute $\mathcal{H}\theta$ to determine q from θ, or compute $\mathcal{H}^{-1}[\log q - \frac{1}{2}\log\frac{t+b}{t+a}]$ to determine θ from q.

Matrix-valued Riemann–Hilbert problems can be used to solve nonlinear ordinary and partial differential equations such as the nonlinear Schrödinger equation, the KdV equation and Painlevé equations [10, 11]. Such formulations have been used with great success to determine the asymptotics of solutions, but, to the best of this author's knowledge, have not been used to compute solutions numerically. In Section 7 we describe a possible approach in which the results of this paper can be utilized for computing the solutions to matrix-valued Riemann–Hilbert problems, and hence to the solution of the associated nonlinear differential equations. To accomplish this, we need to compute not the Hilbert transform itself, but its limit as z approaches Γ from the left or right.

There are several existing methods for computing $\mathcal{H}f$, with a recent review found in [15]. The simplest method is to subtract out the singularity:

$$\fint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t = \int_\Gamma \frac{f(t)-f(z)}{t-z}\,\mathrm{d}t + f(z)\fint_\Gamma \frac{1}{t-z}\,\mathrm{d}t. \tag{1.3}$$

The singularity in the first integral is now removable, hence — ignoring round-off error caused by the removable singularity — it can be computed effectively using a standard quadrature method. The latter integral, on the other hand, is typically known in closed form. In particular [19]:

$$\fint_{-1}^{1} \frac{1}{t-x}\,\mathrm{d}t = \log\frac{1-x}{1+x} \quad \text{for } x \in \mathbb{I}, \tag{1.4}$$

$$\fint_{-\infty}^{\infty} \frac{1}{t-y}\,\mathrm{d}t = 0 \quad \text{for } y \in \mathbb{R}, \tag{1.5}$$

$$\fint_{\mathbb{U}} \frac{1}{t-z}\,\mathrm{d}t = \mathrm{i}\pi \quad \text{for } z \in \mathbb{U}.$$

(We use the convention that $x \in \mathbb{I}$, $y \in \mathbb{R}$ and $z \in \mathbb{U}$, and for functions, $f : \mathbb{I} \to \mathbb{C}$, $r : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{U} \to \mathbb{C}$. When Γ is a general curve, we use z as the variable and f as the function.)

If Gaussian quadrature is used, each value of z for which we wish to evaluate the Hilbert transform costs $\mathcal{O}(n^2)$ operations [25]. Thus if we wish to compute the Hilbert transform for n points in Γ, the total cost is $\mathcal{O}(n^3)$. On the other hand, for Γ equal to $\mathbb{R}$ or $\mathbb{U}$, the method we develop takes only $\mathcal{O}(n\log n)$ operations to compute the solution at n points, which is a considerable improvement. At each additional point, including z throughout the complex plane, only an additional $\mathcal{O}(n)$ operation is required. On the unit interval we could replace Gaussian quadrature with Clenshaw–Curtis quadrature [8] in (1.3), for a total cost of $\mathcal{O}(n^2\log n)$ operations when evaluated at n points. However, this still suffers from issues with removable singularities, as well as issues when x is near the endpoints of $\mathbb{I}$, where the Hilbert transform blows up.

Over the real line, an approach developed by Weideman [27] is to use the FFT to expand f in terms of the eigenfunctions of $\mathcal{H}_{\mathbb{R}}$, i.e., writing

$$r(y) = \sum_{k=-\infty}^{\infty} c_k \frac{(1+\mathrm{i}y)^k}{(1-\mathrm{i}y)^{k+1}},$$

and applying the formulæ

$$\mathcal{H}_{\mathbb{R}} \frac{(1+\mathrm{i}y)^k}{(1-\mathrm{i}y)^{k+1}} = -\mathrm{i}\,\mathrm{sgn}\,k \frac{(1+\mathrm{i}y)^k}{(1-\mathrm{i}y)^{k+1}} \quad \text{for } k \neq 0, \tag{1.6}$$
$$\mathcal{H}_{\mathbb{R}} \frac{1}{1-\mathrm{i}y} = -\mathrm{i}\ \mathrm{sgn}\ \frac{1}{1-\mathrm{i}y} \qquad [18].$$

Similarly, on the interval $\mathbb{I}$ the formula

$$\mathcal{H}_{\mathbb{I}} \frac{T_k(x)}{\sqrt{1-x^2}} = -U_{k-1} \qquad [15] \tag{1.7}$$

can be utilized to compute $\mathcal{H}f$ (by expanding $f(x)\sqrt{1-x^2}$ into Chebyshev T_k series) or $\mathcal{H}^{-1}f$ (by expanding f into Chebyshev U_k series). To compute $\mathcal{H}_{\mathbb{I}}f$ efficiently for smooth f, the formula

$$\mathcal{H}_{\mathbb{I}} T_k(x) = \frac{1}{\pi} T_k(x) \log\left(\frac{1+x}{1-x}\right) - \frac{1}{\pi} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} x^{k-2j-1} \sum_{v=0}^{j} \frac{c_{k,v}}{2j+1-2v}, \tag{1.8}$$

can be used, where $c_{n,v}$ are defined so that

$$T_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} c_{k,j} x^{k-2j},$$

or, equivalently,

$$c_{0,j} = 1,$$
$$c_{k,j} = \frac{2^{k-2j-1}(-1)^j k(k-j-1)!}{j!(k-2j)!}, \qquad k = 1, 2, \ldots \qquad [15].$$

Equations (1.6) and (1.7) can also be applied to compute the Hilbert transform globally in $\mathcal{O}(n \log n)$ time and (1.8) in $\mathcal{O}(n^3)$ time. Our approach is in some sense equivalent to (1.6) throughout the complex plane, and (1.7) and (1.8) on $\mathbb{I}$ itself, though our version of (1.8) requires only $\mathcal{O}(n \log n)$ operations. Furthermore, the known expansion in terms of Chebyshev polynomials cannot be used for x off the interval, whereas our approach can be used throughout the complex plane in a numerically stable manner. Moreover, the additional terms in our version of (1.8) can be written in terms of Chebyshev series, not power series, making the method numerically stable.

In the following section, we setup the computation of $\mathcal{H}$ in terms of the solution of a Riemann–Hilbert problem. In Section 3 we construct our method for the circle, and describe the rate of convergence, which is based on the standard FFT convergence theory. In Section 4 we construct the method for the real line by mapping it to the circle. In Section 5 we use the Joukowsky map to solve the Riemann–Hilbert problem on the interval. We can then compute the semi-infinite

Hilbert transform by mapping the half line to the interval. We can combine the computation of two half lines to compute the Hilbert transform over the real line efficiently, even when the behaviour at $\pm\infty$ differ. This is unlike the method of Section 4 and [27], which requires that the function has the same asymptotic series at both $+\infty$ and $-\infty$.

2. Riemann–Hilbert problems and the Plemelj lemma

To construct our method, we rewrite it as a Riemann–Hilbert problem.

Definition 2.1. Given a piecewise smooth oriented curve in the complex plane Γ and $z \in \Gamma$ not at any endpoint or discontinuity of Γ, $\Phi^+(t)$ is the limit of $\Phi(p)$ as $p \to z$ with p lying on the left of Γ. Likewise, $\Phi^-(z)$ is the limit of $y(p)$ as $p \to z$ with p lying on the right of Γ. See [19] for a more detailed definition.

Problem 2.2. Suppose we are given a piecewise smooth oriented curve in the complex plane Γ and $b, f : \Gamma \to \mathbb{C}$ which satisfy a Hölder condition. Find a function Φ which is analytic in $\mathbb{C}\backslash\Gamma$ such that

$$\Phi^+(z) + b(z)\Phi^-(z) = f(z) \quad \text{for } z \in \Gamma \quad \text{and } \Phi(\infty) = 0.$$

We use $\Phi(\infty)$ to denote the limit $\Phi(z)$ as $z \to \infty$ from any direction, assuming it exists. Likewise, when f is defined only on Γ containing ∞, we use $f(\infty)$ to denote the limit as $z \to \infty$ from any direction along Γ.

In our case, $b(x)$ will be either 1 or -1. The following theorem follows from Plemelj's lemma:

Theorem 2.3 ([19])**.** *Let Γ be a piecewise smooth oriented curve in the complex plane and $f : \Gamma \to \mathbb{C}$ a function which satisfies a Hölder condition. The function*

$$\Phi(z) = \frac{1}{2\mathrm{i}}\mathcal{H}f(z) = \frac{1}{2\mathrm{i}\pi}\int_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t$$

is analytic in $\mathbb{C}\backslash\Gamma$ and satisfies $\Phi(\infty) = 0$. Let $z \in \Gamma$ such that z is not an endpoint or discontinuity of Γ. Then

$$\Phi^+(z) - \Phi^-(z) = f(z)$$

and

$$\Phi^+(z) + \Phi^-(z) = -\mathrm{i}\mathcal{H}f(z).$$

The solution to the Riemann–Hilbert problem

$$\Phi^+ - \Phi^- = f \qquad \text{and} \qquad \Phi(\infty) = 0 \tag{2.1}$$

is unique [19], hence solving this problem allows us to compute the Hilbert transform of f. The solution to

$$\Phi^+ + \Phi^- = f \quad \text{and} \quad \Phi(\infty) = 0 \tag{2.2}$$

is unique on closed curves (such as $\mathbb{U}$ and $\mathbb{R}$), but not necessarily on open curves (such as $\mathbb{I}$ and $\mathbb{R}^+$) without additional conditions imposed. However, if Φ is a solution to (2.2), then it has the property that the Hilbert transform of $\Phi^+ - \Phi^-$ is equal to f [19]. If $\Gamma = \mathbb{I}$, then we can determine Φ uniquely if we impose that it must be bounded at either ± 1. If the zeroth Chebyshev coefficient of f is zero, then Φ can be uniquely determined by imposing that it must be bounded at both ± 1.

Definition 2.4. We define $\mathcal{P}_\Gamma f$ as the equivalency class of solutions to (2.2) and $\mathcal{M}_\Gamma f$ as the solution to equation (2.1), namely

$$\mathcal{M}_\Gamma f = \frac{1}{2\mathrm{i}}\mathcal{H}_\Gamma f,$$

which is the Cauchy transform (cf. [10], where it is denoted $\mathcal{C}$). When Γ is clear from context, we use $\mathcal{P}$ and $\mathcal{M}$. We also use the notation

$$\mathcal{P}^\pm f = (\mathcal{P}f)^\pm \qquad \text{and} \qquad \mathcal{M}^\pm f = (\mathcal{M}f)^\pm.$$

In other words, computing $\mathcal{M}f$ allows us to compute the Hilbert transform of f throughout the complex plane off Γ, and computing $\mathcal{M}^\pm f$ allows us to compute the Hilbert transform on Γ itself. Likewise, computing $\mathcal{P}^\pm f$ allows us to compute $\mathcal{H}^{-1}f$ on Γ. Thus our primary goal is the computation of $\mathcal{P}f$, $\mathcal{P}^\pm f$, $\mathcal{M}f$ and $\mathcal{M}^\pm f$.

When Γ is a simple closed curve — such as the unit circle or real line — we can regard Φ^+ and Φ^- as independent analytic functions in the interior and exterior of the curve; see Figure 1. Thus if $\Phi^+ - \Phi^- = f$, then

$$\mathrm{i}\mathcal{H}^{-1}f = \Phi^+ - (-\Phi^-) = \Phi^+ + \Phi^- = -\mathrm{i}\mathcal{H}f,$$

which is equivalent to the well-known identity

$$\mathcal{H}^{-1} = -\mathcal{H}.$$

This is no longer the case when Γ is either not closed or not simple.

There are existing methods for solving Riemann–Hilbert problems. One approach is based on rewriting the Riemann–Hilbert problem as a principal value integral or singular integral equation [20]. In our case, such an approach would return us to our original problem, hence it is not useful. Another approach is the conjugation method [26], for solving a Riemann–Hilbert problem of the form $a(z)\Phi^+(z) + b(z)\Phi^-(z) = f(z)$ on closed curves. Our approach is related to the conjugation method, however, since our Riemann–Hilbert problem has constant a and b, it is significantly simpler. Furthermore, unlike our approach, the conjugation method has not been generalized to open curves such as the unit interval $\mathbb{I}$.

3. The unit circle

We use the standard, counterclockwise orientation for the unit circle; cf. Figure 1. Consider a function $g : \mathbb{U} \to \mathbb{C}$ such that it is $\mathcal{C}^1[\mathbb{U}]$ and its first derivative has bounded variation. Then its Fourier coefficients converge absolutely, and we can express g in terms of its Fourier series:

$$g\big(\mathrm{e}^{\mathrm{i}\theta}\big) = \sum_{k=-\infty}^{\infty} \hat{g}_k \mathrm{e}^{\mathrm{i}k\theta} \quad \text{for } \theta \in \mathbb{T}.$$

Alternatively, we can express g in terms of its Laurent series:

$$g(z) = \sum_{k=-\infty}^{\infty} \hat{g}_k z^k.$$

If g is analytic in an annulus

$$A_\rho = \left\{ z : \frac{1}{\rho} < |z| < \rho \right\},$$

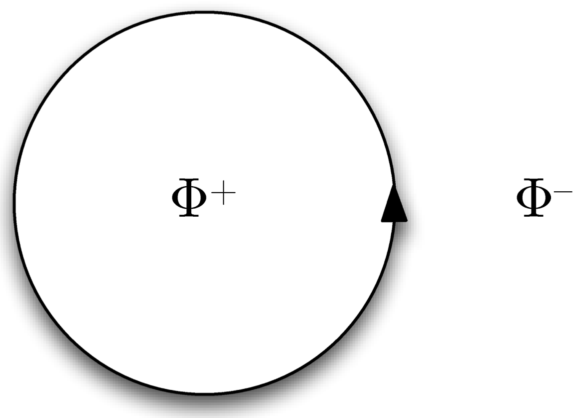

FIGURE 1. Riemann–Hilbert problem on the circle

then this series is guaranteed to converge in A_ρ. Otherwise, it will only converge on $\mathbb{U}$. On the other hand,

$$g_+ = \sum_{k=0}^{\infty} \hat{g}_k z^k \quad \text{and} \quad g_- = \sum_{k=-\infty}^{-1} \hat{g}_k z^k$$

are analytic in the interior and exterior of the unit disk, respectively, with $g = g_+ + g_-$ on $\mathbb{U}$. Therefore, we obtain:

Theorem 3.1. *Suppose* $g : \mathbb{U} \to \mathbb{C}$ *is* $\mathcal{C}^1[\mathbb{U}]$ *and its first derivative has bounded variation. Then*

$$\mathcal{P}g(z) = \begin{cases} g_+(z) & \text{for } |z| < 1, \\ g_-(z) & \text{for } |z| > 1 \end{cases} \quad \textit{and} \quad \mathcal{M}g(z) = \mathcal{P}g(z) \begin{cases} 1 & \text{for } |z| < 1, \\ -1 & \text{for } |z| > 1. \end{cases}$$

In other words, computing the Fourier series allows us to compute the solution to either Riemann–Hilbert problem on the unit circle.

Computation of the Fourier series can be accomplished efficiently using the FFT. Denote n evenly spaced points in $\mathbb{T}$ as

$$\theta_n = \left(-\pi, -\pi + \frac{2}{n}\pi, \ldots, \pi - \frac{2}{n}\pi \right)^\top$$

and n evenly spaced points in $\mathbb{U}$ as

$$\boldsymbol{z}_n = (z_1, \ldots, z_n)^\top = \mathrm{e}^{\mathrm{i}\theta_n} = \left(-1, \mathrm{e}^{\mathrm{i}\pi\left(\frac{2}{n}-1\right)}, \ldots, \mathrm{e}^{\mathrm{i}\pi\left(1-\frac{2}{n}\right)} \right)^\top.$$

The sample vector of g at the points $\boldsymbol{z}_n$ is $\boldsymbol{g} = g(\boldsymbol{z}_n)$. Let

$$\hat{\boldsymbol{g}} = \left(\hat{g}^n_{-\lfloor n/2 \rfloor}, \ldots, \hat{g}^n_0, \ldots, \hat{g}^n_{\lfloor (n+1)/2 \rfloor - 1} \right)$$

so that

$$\sum_{k=-\lfloor n/2 \rfloor}^{\lfloor (n+1)/2 \rfloor - 1} \hat{g}^n_k \mathrm{e}^{\mathrm{i}k\theta}$$

takes the values $\boldsymbol{g}$ at $\boldsymbol{t}_n$. Then

$$g_n(z) = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor (n+1)/2 \rfloor - 1} \hat{g}_k^n z^k$$

interpolates g at $\boldsymbol{z}_n$: $g_n(\boldsymbol{z}_n) = \boldsymbol{g}$. The vector $\hat{\boldsymbol{g}}$ can be written in terms of the trapezium rule, or computed with $\mathcal{O}(n \log n)$ operations using the FFT.

We will express this transformation as an operator applied to $\boldsymbol{g}$:

Definition 3.2. We denote the Laurent polynomial which takes the data $\boldsymbol{g}$ at the points $\boldsymbol{z}_n$ as

$$\boldsymbol{e}(z)^\top \boldsymbol{g} = g_n(z) = \left(z^{-\lfloor n/2 \rfloor}, \ldots, z^{\lfloor (n+1)/2 \rfloor - 1}\right) \hat{\boldsymbol{g}} = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor (n+1)/2 \rfloor - 1} \hat{g}_k^n z^k.$$

The nonnegative and negative components are denoted

$$\boldsymbol{e}_+(z)^\top \boldsymbol{g} = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor - 1} \hat{g}_k^n z^k \qquad \text{and} \qquad \boldsymbol{e}_-(z)^\top \boldsymbol{g} = \sum_{k=-\lfloor n/2 \rfloor}^{-1} \hat{g}_k^n z^k.$$

We also use $\boldsymbol{g}_+ = \boldsymbol{e}_+(\boldsymbol{z}_n)^\top \boldsymbol{g}$ and $\boldsymbol{g}_- = \boldsymbol{e}_-(\boldsymbol{z}_n)^\top \boldsymbol{g}$ for the values these functions take at $\boldsymbol{z}_n$. Both of these can be computed with $\mathcal{O}(n \log n)$ operations by applying the FFT to compute $\hat{\boldsymbol{g}}$, dropping the nonnegative/positive entries and applying the inverse FFT.

The notation $\boldsymbol{e}(z)$ is chosen to emphasize that $\boldsymbol{e}(z_k)^\top \boldsymbol{g} = \boldsymbol{e}_k^\top \boldsymbol{g}$, $k = 1, \ldots, n$. In practice, $\boldsymbol{e}_\pm(z)$ can be evaluated efficiently and in a stable manner [13] using the barycentric formula [6]:

$$\boldsymbol{e}_+(z)^\top \boldsymbol{g} = \frac{\sum_{k=1}^n \frac{z_k}{z - z_k} \boldsymbol{e}_k^\top \boldsymbol{g}_+}{\sum_{k=1}^n \frac{z_k}{z - z_k}} \quad \text{for } z \notin \boldsymbol{z}_n,$$

where $\boldsymbol{e}_k$ is the kth basis vector of $\mathbb{C}^n$. The function $\frac{1}{z}$ maps a series in inverse polynomials to a series in polynomials, and the values of the series at $\boldsymbol{z}_n$ to the values at $\bar{\boldsymbol{z}}_n = (z_1, z_n, \ldots, z_2)$. Applying this map and then the barycentric formula, we obtain

$$\boldsymbol{e}_-(z)^\top \boldsymbol{g} = \frac{\frac{z_1}{z^{-1} - z_1} \boldsymbol{e}_1^\top \boldsymbol{g}_- + \sum_{k=2}^n \frac{z_k}{z^{-1} - z_k} \boldsymbol{e}_{n-k+2}^\top \boldsymbol{g}_-}{\sum_{k=1}^n \frac{z_k}{z^{-1} - z_k}} \quad \text{for } z \notin \boldsymbol{z}_n.$$

Replacing g by its approximation $\boldsymbol{e}(z)^\top \boldsymbol{g}$, we can solve the Riemann–Hilbert problem exactly:

Definition 3.3. We define approximate solutions to $\mathcal{P}g$ and $\mathcal{M}g$ as

$$\mathcal{P}_n g(z) = \mathcal{P}\boldsymbol{e}(z)^\top \boldsymbol{g} = \begin{cases} \boldsymbol{e}_+(z)^\top \boldsymbol{g} & \text{for } |z| < 1, \\ \boldsymbol{e}_-(z)^\top \boldsymbol{g} & \text{for } |z| > 1, \end{cases} \quad \text{and}$$

$$\mathcal{M}_n g(z) = \mathcal{P}_n g(z) \begin{cases} 1 & \text{for } |z| < 1, \\ -1 & \text{for } |z| > 1. \end{cases}$$

For $z \in \mathbb{U}$, we obtain

$$\mathcal{P}_n^\pm g(z) = \boldsymbol{e}_\pm(z)^\top \boldsymbol{g} = \boldsymbol{e}(z)^\top \boldsymbol{g}_\pm \quad \text{and} \quad \mathcal{M}_n^\pm g(z) = \pm \boldsymbol{e}_\pm(z)^\top \boldsymbol{g} = \pm \boldsymbol{e}(z)^\top \boldsymbol{g}_\pm.$$

Let $u = \lim_{n\to\infty} \mathcal{P}_n g$. From classical Fourier analysis, we know that

$$u^+ + u^- = \lim_{n\to\infty} \left[\mathcal{P}_n^+ g + \mathcal{P}_n^- g\right] = \lim_{n\to\infty} g_n = g.$$

Furthermore,

$$\mathcal{P}_n g = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor -1} \hat{g}_k^n z^k$$

must converge to an analytic function inside the unit circle. Similarly, $\mathcal{P}_n g$ converges to an analytic function outside the unit circle. By uniqueness, we thus obtain $\mathcal{P}g = u$. For $z \in \mathbb{U}$, we thus obtain the following approximations:

$$-\mathrm{i}\mathcal{H}g = \mathrm{i}\mathcal{H}^{-1} g \approx \boldsymbol{e}(z)^\top (\boldsymbol{g}_+ - \boldsymbol{g}_-).$$

By utilizing results for the convergence rate of Fourier series, we find that the convergence rate of the approximation depends on the smoothness of g:

- If $g \in \mathcal{C}^\rho[\mathbb{U}]$ such that the ρth derivative of g has bounded variation, then the rate of convergence is algebraic, on the order of $\mathcal{O}\big(n^{\rho+1}\big)$.
- If $g \in \mathcal{C}^\infty[\mathbb{U}]$, then the rate of convergence is superalgebraic.
- If g is analytic in an annulus A_ρ, then the rate of convergence is geometric, on the order of $\mathcal{O}(\mathrm{e}^{-\rho n})$.
- If g is analytic everywhere, except the possible exceptions of zero and ∞, then the rate of convergence is supergeometric.

4. The real line

We now consider the computation of $\mathcal{M}_{\mathbb{R}} r$, where $r(\pm\infty) = 0$. The Möbius transformation

$$R(z) = \mathrm{i}\frac{1-z}{1+z}$$

conformally maps the unit circle onto the real line with the interior of the circle mapped to the upper half plane and the exterior mapped to the lower half plane. We can then project r onto the unit circle as $g(z) = r(R(z))$. Computing $\mathcal{M}_{\mathbb{U}} g$ and $\mathcal{P}_{\mathbb{U}} g$ allows us to compute $\mathcal{M}_{\mathbb{R}} r$ and $\mathcal{P}_{\mathbb{R}} r$:

Theorem 4.1. *Suppose that r is $\mathcal{C}^1[\mathbb{R}]$, its first derivative has bounded variation and $r(y) \sim \frac{\alpha_1}{y} + O(\frac{1}{y^2})$ as $y \to \pm\infty$. Let $\Phi = \mathcal{M}_{\mathbb{U}} g$, for $g(z) = r(R(z))$. Then*

$$\mathcal{M}_{\mathbb{R}} r(y) = \Phi(R^{-1}(y)) - \Phi^+(-1) \quad \text{and} \quad \mathcal{P}_{\mathbb{R}} r(y) = \mathcal{M}_{\mathbb{R}} r(y) \begin{cases} 1 & \text{for } \Im y > 1, \\ -1 & \text{for } \Im y < 1. \end{cases}$$

Proof. The first hypothesis ensures that g is $\mathcal{C}^1[\mathbb{U}]$ and its first derivative has bounded variation. Let $\varphi(y) = \Phi(R^{-1}(y)) - \Phi^+(-1)$. Note that, if $\Phi^+ - \Phi^- = r$, then $(\Phi + c)^+ - (\Phi + c)^- = r$, and adding a constant to a function does not alter its analyticity. Thus,

$$\varphi^+(y) - \varphi^-(y) = \Phi^+(R^{-1}(y)) - \Phi^+(R^{-1}(y)) = g(R^{-1}(y)) = r(y).$$

Furthermore, $\Phi^+(-1) - \Phi^-(-1) = g(-1) = r(\infty) = 0$, hence, for y lying off $\mathbb{R}$,

$$\lim_{y\to\infty} \varphi(y) = \lim_{z\to -1} \Phi(z) - \Phi^+(-1) = 0.$$

Thus we know that $\mathcal{M}_{\mathbb{R}} r = \varphi$. ∎

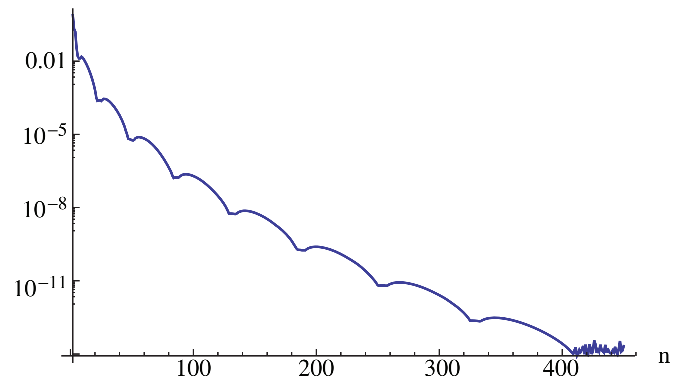

FIGURE 2. The maximum error in approximating the Hilbert transform of $\frac{1-\operatorname{sech} y}{y}$ by (4.2) at the points $R(\boldsymbol{z}_n)$

Definition 4.2. Let $\boldsymbol{r} = r(R(\boldsymbol{z}_n)) = g(\boldsymbol{z}_n)$. We can define

$$\mathcal{M}_{\mathbb{R},n} r(y) = \mathcal{M}_{\mathbb{R}} \boldsymbol{e}(R^{-1}(y))^\top \boldsymbol{r} = \begin{cases} \boldsymbol{e}_+(R^{-1}(y))^\top \boldsymbol{r} & \text{for } |z| < 1, \\ -\boldsymbol{e}_-(R^{-1}(y))^\top \boldsymbol{r} & \text{for } |z| > 1 \end{cases} - \boldsymbol{e}_+(-1)^\top \boldsymbol{r},$$

$$\mathcal{P}_{\mathbb{R},n} r(y) = \mathcal{M}_{\mathbb{R},n} r(y) \begin{cases} 1 & \text{for } \Im y > 1, \\ -1 & \text{for } \Im y < 1. \end{cases}$$

For $y \in \mathbb{R}$, we define

$$\mathcal{M}^{\pm}_{\mathbb{R},n} r(y) = \left[\pm \boldsymbol{e}_\pm(R^{-1}(y))^\top - \boldsymbol{e}_+(-1)^\top \right] \boldsymbol{r} = \boldsymbol{e}(R^{-1}(y))^\top (\pm \boldsymbol{r}_\pm - \boldsymbol{e}_1^\top \boldsymbol{r}_+),$$
$$\mathcal{P}^{\pm}_{\mathbb{R},n} r(y) = \left[\boldsymbol{e}_\pm(R^{-1}(y))^\top \mp \boldsymbol{e}_+(-1)^\top \right] \boldsymbol{r} = \boldsymbol{e}(R^{-1}(y))^\top (\boldsymbol{r}_\pm \mp \boldsymbol{e}_1^\top \boldsymbol{r}_+).$$

If $r(+\infty) = r(-\infty) \neq 0$, we can use the fact that $\mathcal{H}1 = 0$ (cf. (1.5)) to compute

$$\mathcal{H}r = \mathcal{H}[r - r(\infty)].$$

Suppose that

$$r(y) \sim \sum_{k=1}^{\rho+1} \frac{\alpha_k}{y^k} \tag{4.1}$$

as $y \to \pm\infty$. Then, if $r \in \mathcal{C}^\rho[\mathbb{R}]$, $g \in \mathcal{C}^\rho[\mathbb{U}]$. Thus, if $r \in \mathcal{C}^\infty[\mathbb{R}]$ and has the same asymptotic series at both $\pm\infty$, then $\mathcal{M}_{\mathbb{R},n}$ and $\mathcal{P}_{\mathbb{R},n}$ converge superalgebraically.

As a numerical example, we consider the computation of the Hilbert transform of

$$r(y) = \frac{1 - \operatorname{sech} y}{y},$$

whose Hilbert transform is

$$\mathcal{H}r(y) = \frac{\mathrm{i}}{\pi y} \left[\psi\left(\frac{1}{4} - \frac{\mathrm{i}y}{2\pi}\right) - \psi\left(\frac{1}{4} + \frac{\mathrm{i}y}{2\pi}\right) \right] - \frac{\tanh y}{y} \qquad [16],$$

where ψ is the polygamma function [2] and both r anf $\mathcal{H}r$ are defined at zero by taking their limits. In Figure 2, we compare

$$\mathrm{i}\left(\mathcal{M}^+_{\mathbb{R},n} r + \mathcal{M}^-_{\mathbb{R},n} r\right) \tag{4.2}$$

with the Hilbert transform computed by the exact formula using MATHEMATICA's built-in `PolyGamma` routine. Note that (4.2) is a rederivation of Weideman's method [27].

5. The unit interval

We now consider the case of computing $\mathcal{P}_{\mathbb{I}}$ and $\mathcal{M}_{\mathbb{I}}$. Unlike the previous two cases, the unit interval $\mathbb{I} = [-1, 1]$ is not a simple closed curve, hence we cannot compute one from the other. We will, however, compute both of these functions by mapping the associated Riemann–Hilbert problems to the unit circle.

The Joukowsky map and Chebyshev series. The Joukowsky map

$$T(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps both the upper half-circle $\mathbb{U}^{\uparrow} = \{z \in \mathbb{U} : \operatorname{Im} z \geq 0\}$ and the lower half-circle $\mathbb{U}^{\downarrow} = \{z \in \mathbb{U} : \operatorname{Im} z \leq 0\}$ onto the unit interval, with the interior and exterior of the circle both being mapped to $\mathbb{C}\backslash\mathbb{I}$ (z and $\frac{1}{z}$ are mapped to the same point with zero and ∞ mapped to ∞). Let $g(z) = f(T(z))$, which is equivalent to projecting f to both the upper and lower half circles simultaneously. Now on the unit circle $g(\mathrm{e}^{\mathrm{i}\theta}) = f(T(\mathrm{e}^{\mathrm{i}\theta})) = f(\cos\theta)$ is 2π periodic, and if $f \in \mathcal{C}^{\rho}[-1, 1]$, then $f(\cos\theta) \in \mathcal{C}^{\rho}[\mathbb{T}]$. Furthermore, if f is analytic in the Chebyshev ellipse

$$E_{\rho} = T(A_{\rho}),$$

then g is analytic in the annulus A_{ρ} [25].

Two one-sided inverses for $T(z)$ are

$$T_{\pm}^{-1}(x) = x \mp \sqrt{x-1}\sqrt{x+1},$$

which map points in $\mathbb{C}\backslash\mathbb{I}$ to the interior and exterior of the circle, respectively. These satisfy $T(T_{\pm}^{-1}(x)) = x$. Furthermore, $T_{+}^{-1}(x) = \frac{1}{T_{-}^{-1}(x)}$. Taking the standard branch cuts for the square root, these inverses have a branch cut along $\mathbb{I}$ and are analytic elsewhere.

On the interval itself, with the standard choice of branch, T_{+} maps $\mathbb{I}$ onto $\mathbb{U}^{\downarrow}$ and T_{-} maps $\mathbb{I}$ onto $\mathbb{U}^{\uparrow}$. However, since $\mathbb{I}$ lies on the branch cut of each of these functions, round-off error can produce unreliable results if it introduces a nonzero imaginary part. Thus we introduce two other choices for the inverse of T:

$$T_{\uparrow}^{-1}(x) = x + \mathrm{i}\sqrt{1-x}\sqrt{1+x} \qquad \text{and} \qquad T_{\downarrow}^{-1}(x) = x - \mathrm{i}\sqrt{1-x}\sqrt{1+x}.$$

These functions have branch cuts along $(-\infty, -1)$ and $(1, \infty)$, thus are analytic for $x \in \mathbb{I}$. $T_{\uparrow}$ maps the unit interval to the upper half circle, with any perturbation above or below $\mathbb{I}$ mapping to a perturbation above or below $\mathbb{U}^{\uparrow}$. Likewise, $T_{\downarrow}$ reliably maps $\mathbb{I}$ to the lower half circle. We also use the notation $\updownarrow$ in the same way as $\pm$ is used (we do not define any equivalent to $\mp$).

We denote the Chebyshev series of f as

$$f = \sum_{k=0}^{\infty} \check{f}_k T_k.$$

From classical Chebyshev polynomial theory, we know that

$$\hat{g}_0 = \check{f}_0, \qquad \hat{g}_k = \hat{g}_{-k} = \frac{\check{f}_k}{2}.$$

Let $\boldsymbol{f} = f(\boldsymbol{x}_n) = (f(x_1), \ldots, f(x_{n-1}), f(x_n))^\top$, where

$$\boldsymbol{x}_n = (x_1, \ldots, x_n)^\top = \left(-1, \cos\pi\left(-1 + \frac{1}{n-1}\right), \ldots, \cos\pi\left(1 - \frac{1}{n-1}\right), 1\right)^\top$$

are the n Chebyshev–Lobatto points. Then the sample of g at $2n-2$ evenly spaced points on the circle is

$$\boldsymbol{g} = g(\boldsymbol{z}_{2n-2}) = f(\cos\theta_{2n-2}) = (f(x_1), \ldots, f(x_{n-1}), f(x_n), f(x_{n-1}), \ldots, f(x_2))^\top.$$

Because $\boldsymbol{g}$ is symmetric, we can use the fast cosine transform (FCT) in place of the FFT to compute

$$\hat{\boldsymbol{g}} = \left(\hat{g}^n_{1-n}, \ldots, \hat{g}^n_0, \ldots \hat{g}^n_{n-1}\right)^\top = \left(\hat{g}^n_{n-1}, \ldots, \hat{g}^n_1, \hat{g}^n_0, \hat{g}^n_1, \ldots, \hat{g}^n_{n-1}\right)^\top,$$

which gives us the approximate Chebyshev series

$$\check{\boldsymbol{f}} = \left(\check{f}^n_0, \check{f}^n_1, \ldots, \check{f}^n_{n-1}\right)^\top = \left(\hat{g}^n_0, 2\hat{g}^n_1, \ldots, 2\hat{g}^n_{n-1}\right)^\top.$$

The barycentric formula at Chebyshev–Lobatto points, i.e., the polynomial which takes the given data at Chebyshev–Lobatto points, can be expressed in the following form:

$$\boldsymbol{e}_T(x)^\top \boldsymbol{f} = \frac{\sum\nolimits'^{n}_{k=1} \frac{(-1)^k}{x - x_k} \boldsymbol{e}_k^\top \boldsymbol{f}}{\sum\nolimits'^{n}_{k=1} \frac{(-1)^k}{x - x_k}} \qquad \text{for} \qquad x \notin \boldsymbol{x}_n \qquad [6],$$

where the prime indicates that the first and last entries of the sum are halved. It is true that $\boldsymbol{e}_T(x)^\top \boldsymbol{f} = \boldsymbol{e}(T^{-1}(x))^\top \boldsymbol{g}$ for any choice of T^{-1}. In this case, the barycentric formula is only useful for $x \in \mathbb{I}$.

Computation of $\mathcal{P}_{\mathbb{I}}$. We begin with the case of computing $\mathcal{P}_{\mathbb{I}} f$. In this case, we have a choice of $\mathcal{P}_{\mathbb{I}} f$, determined by its behaviour at ± 1. In the following theorem, if $\check{f}_0 = 0$ and $c = 0$, then the solution is bounded at ± 1. Otherwise, we can prescribe that $\mathcal{P}_{\mathbb{I}} f$ is bounded at either ± 1 (but not both) by choosing $c = \mp \check{f}_0$.

Theorem 5.1. *Suppose f is $\mathcal{C}^1[\mathbb{I}]$ and its first derivative has bounded variation. Let $g(z) = f(T(z))$ and $\Phi = \mathcal{P}_{\mathbb{U}} g$. For constants $c \in \mathbb{C}$,*

$$\mathcal{P}_{\mathbb{I}} f(x) = \frac{\Phi(T_+^{-1}(x)) + \Phi(T_-^{-1}(x))}{2} - \frac{x\check{f}_0 + c}{2\sqrt{x+1}\sqrt{x-1}}.$$

For $x \in \mathbb{I}$,

$$\mathcal{P}^+_{\mathbb{I}} f(x) = \frac{1}{2}\left[\Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x)) + \mathrm{i}\frac{\check{f}_0 x + c}{2\sqrt{1-x^2}}\right],$$

$$\mathcal{P}^-_{\mathbb{I}} f(x) = \frac{1}{2}\left[\Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x)) - \mathrm{i}\frac{\check{f}_0 x + c}{2\sqrt{1-x^2}}\right].$$

Proof. It follows from the hypotheses that g is $\mathcal{C}^1[\mathbb{U}]$ and its first derivative has bounded variation. Let $u = [\Phi(T_+^{-1}(x)) + \Phi(T_-^{-1}(x))]/2$. As p approaches $x \in \mathbb{I}$ from above, $T_+^{-1}(p)$ approaches $T_\downarrow^{-1}(x)$ from the interior and $T_-^{-1}(p)$ approaches $T_\uparrow^{-1}(x)$ from the exterior. As p approaches $\mathbb{I}$ from below, $T_+^{-1}(p)$ approaches $T_\uparrow^{-1}(x)$ from the interior and $T_-^{-1}(p)$ approaches $T_\downarrow^{-1}(x)$ from the exterior. In other words,

$$2u^+(x) = \Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x)) \quad \text{and} \quad 2u^-(x) = \Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x)).$$

Combining the above with the fact that

$$\Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x)) = g(T_\uparrow^{-1}(x)) = f(x),$$
$$\Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x)) = g(T_\downarrow^{-1}(x)) = f(x),$$

we obtain

$$\begin{aligned} u^+(x) + u^-(x) &= \frac{1}{2}\left[\Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x)) + \Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x))\right] \\ &= f(x). \end{aligned}$$

We have not quite computed $\mathcal{P}_{\mathbb{I}} f$:

$$u(\infty) = \frac{\Phi(0) + \Phi(\infty)}{2} = \frac{\hat{g}_0}{2} = \frac{\check{f}_0}{2}.$$

Let

$$\psi(x) = \frac{x\check{f}_0 + c}{2\sqrt{x+1}\sqrt{x-1}}.$$

By the properties of the principal branch of the square root, we find that

$$\psi^+(x) = -\mathrm{i}\frac{x\check{f}_0 + c}{2\sqrt{1-x^2}} \qquad \text{and} \qquad \psi^-(x) = \mathrm{i}\frac{x\check{f}_0 + c}{2\sqrt{1-x^2}}.$$

Clearly, $\psi^+ + \psi^- = 0$, whilst $\psi(\infty) = \frac{\check{f}_0}{2}$. Thus $u - \psi$ decays at ∞ and satisfies

$$u^+ - \psi^+ + u^- - \psi^- = u^+ + u^- = f.$$

The fact that this expression includes all possible solutions which satisfy a Hölder condition follows from Section 84 in [19], which gives an expression for the class of all such solutions, in terms of a contour integral. □

We therefore obtain the following, stable approximation:

Definition 5.2. Let $\boldsymbol{g} = (f(x_1), \ldots, f(x_{n-1}), f(x_n), f(x_{n-1}), \ldots, f(x_2))^\top$. If $x \in \mathbb{C}\backslash\mathbb{I}$ and $c \in \mathbb{C}$, then, we define

$$\mathcal{P}_{\mathbb{I},n} f(x) = \frac{1}{2}\left[\boldsymbol{e}_+(T_+^{-1}(x))^\top + \boldsymbol{e}_-(T_-^{-1}(x))^\top\right]\boldsymbol{g} - \frac{x\check{f}_0^n + c}{2\sqrt{x+1}\sqrt{x-1}}.$$

If $x \in \mathbb{I}$, then we define

$$\mathcal{P}_{\mathbb{I},n}^+ f(x) = \frac{1}{2}\left[\boldsymbol{e}_+(T_\downarrow^{-1}(x))^\top + \boldsymbol{e}_-(T_\uparrow^{-1}(x))^\top\right]\boldsymbol{g} + \mathrm{i}\frac{\check{f}_0^n x + c}{2\sqrt{1-x^2}},$$
$$\mathcal{P}_{\mathbb{I},n}^- f(x) = \frac{1}{2}\left[\boldsymbol{e}_+(T_\uparrow^{-1}(x))^\top + \boldsymbol{e}_-(T_\downarrow^{-1}(x))^\top\right]\boldsymbol{g} - \mathrm{i}\frac{\check{f}_0 x + c}{2\sqrt{1-x^2}}.$$

As in the unit circle case, the convergence rate of this approximation is an immediate consequence of the convergence of $\boldsymbol{e}(x)^\top\boldsymbol{g}$ to g. We omit the details, but it clearly depends on the differentiability and analyticity of $f(\cos\theta)$. This is similar to the connection between the convergence rate of the Clenshaw–Curtis quadrature and the analyticity of f in E_ρ [25].

For a vector $\boldsymbol{g} = (g_1, \ldots, g_{2n-2})^\top$, let

$$\boldsymbol{g}_\downarrow = (g_1, \ldots, g_n)^\top \quad \text{and} \quad \boldsymbol{g}_\uparrow = (g_n, \ldots, g_{2n-2}, g_0)^\top.$$

These are the values of $\boldsymbol{g}$ corresponding to the points of $\boldsymbol{z}_n$ lying in $\mathbb{U}_\downarrow$ and $\mathbb{U}_\uparrow$, respectively. We could approximate $\mathrm{i}\mathcal{H}_n^{-1} f$ by its values at Chebyshev points plus the unbounded term:

$$\boldsymbol{e}_T(x)^\top \left(\boldsymbol{g}_\uparrow^- + \boldsymbol{g}_\downarrow^+ - \boldsymbol{g}_\downarrow^- - \boldsymbol{g}_\uparrow^+\right) + \mathrm{i}\frac{\check{f}_0^n x + c}{\sqrt{1-x^2}}, \tag{5.1}$$

where $\boldsymbol{g}_\uparrow^\pm = (\boldsymbol{g}^\pm)_\uparrow$ and $\boldsymbol{g}_\downarrow^\pm = (\boldsymbol{g}^\pm)_\downarrow$. However, $\boldsymbol{g}^\pm$ are not necessarily symmetric, hence a square root singularity is introduced upon projecting onto the interval. Thus (5.1) does not converge rapidly, unlike $\mathcal{P}_n^+ f - \mathcal{P}_n^- f$. On the other hand, the values of the bounded part at the Chebyshev points themselves, which equal $\boldsymbol{g}_\uparrow^- + \boldsymbol{g}_\downarrow^+ - \boldsymbol{g}_\downarrow^- - \boldsymbol{g}_\uparrow^+$, converge rapidly.

Computation of $\mathcal{M}_\mathbb{I} f$. For $g(z) = f(T(z))$, note that $-g(z)\mathrm{sgn}\arg z$ is equivalent to projecting f to the lower half circle, and $-f$ to the upper half circle. If $f(\pm 1) \neq 0$, then $g(z)\mathrm{sgn}\arg z$ has a jump at ± 1, hence does not satisfy any Hölder condition. Therefore, for now, we assume that $f(\pm 1) = 0$. In a similar manner to Theorem 4.1, we obtain the following result:

Theorem 5.3. *Suppose that $f(\pm 1) = 0$ and f is $\mathcal{C}^1[\mathbb{I}]$ and its first derivative has bounded variation. Let $g(z) = f(T(z))$ and $\Phi = -\mathcal{M}_\mathbb{U} g(z)\mathrm{sgn}\arg z$. Then*

$$\mathcal{M}_\mathbb{I} f(x) = \frac{\Phi(T_+^{-1}(x)) + \Phi(T_-^{-1}(x))}{2}.$$

Furthermore, for $x \in \mathbb{I}$,

$$\mathcal{M}_\mathbb{I}^+ f(x) = \frac{\Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x))}{2}$$

and

$$\mathcal{M}_\mathbb{I}^- f(x) = \frac{\Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x))}{2}.$$

Proof. Let $u = [\Phi(T_+^{-1}(x)) + \Phi(T_-^{-1}(x))]/2$ and $h(z) = -g(z)\,\mathrm{sgn}\arg z$. Now

$$2u^+(x) = \Phi^+(T_\downarrow^{-1}(x)) + \Phi^-(T_\uparrow^{-1}(x)) \quad \text{and} \quad 2u^-(x) = \Phi^+(T_\uparrow^{-1}(x)) + \Phi^-(T_\downarrow^{-1}(x)).$$

Therefore,

$$u^+(x) - u^-(x) = \frac{1}{2}\left[h(T_\downarrow^{-1}(x)) - h(T_\uparrow^{-1}(x))\right] = \frac{1}{2}\left[f(x) + f(x)\right] = f(x).$$

Furthermore,

$$u(\infty) = \frac{\Phi(0) + \Phi(\infty)}{2} = \frac{\hat{h}_0}{2} = 0,$$

since h is symmetric. Thus $\mathcal{M}_\mathbb{I} f = u$. ☐

Given the values of a function at Chebyshev–Lobatto points, we can successfully approximate $\mathcal{M}_\mathbb{I}$ and $\mathcal{H}_\mathbb{I}$:

Definition 5.4. Let $\tilde{\boldsymbol{g}} = (0, f(x_2), \ldots, f(x_{n-1}), 0, -f(x_{n-1}), \ldots, -f(x_2))^\top$. If $x \in \mathbb{C}\backslash\mathbb{I}$, then we define

$$\mathcal{M}_{\mathbb{I},n} f(x) = \frac{1}{2}\left[\boldsymbol{e}_+(T_+(x))^\top - \boldsymbol{e}_-(T_-(x))^\top\right]\tilde{\boldsymbol{g}}.$$

If $x \in \mathbb{I}$, then we define

$$\mathcal{M}^+_{\mathbb{I},n} f(x) = \frac{1}{2}\left[\boldsymbol{e}_+(T_\downarrow^{-1}(x))^\top - \boldsymbol{e}_-(T_\uparrow^{-1}(x))^\top\right]\tilde{\boldsymbol{g}},$$
$$\mathcal{M}^-_{\mathbb{I},n} f(x) = \frac{1}{2}\left[\boldsymbol{e}_+(T_\uparrow^{-1}(x))^\top - \boldsymbol{e}_-(T_\downarrow^{-1}(x))^\top\right]\tilde{\boldsymbol{g}}.$$

If $f(x) = \tilde{f}(x)\sqrt{1-x^2}$, then sgn $\theta f(\cos\theta)\sqrt{1-\cos^2\theta} = \tilde{f}(\cos\theta)\sin\theta$, hence the differentiability of $g(z)\mathrm{sgn}\arg z$ depends on the differentiability of $\tilde{f}$. If the first ρ derivatives of f vanish at both endpoints, then $f(\cos\theta)$ is $\mathcal{C}^\rho$ at zero and π. Thus we still achieve superalgebraic convergence whenever $f \in \mathcal{C}^\infty[-1,1]$, and f and all its derivatives go to zero at the endpoints.

It might seem odd that computing $\mathcal{H}^{-1}$ is easier than computing $\mathcal{H}$. This is the opposite of the analytic development in [1, 19], where the goal is to express the solution to the Riemann–Hilbert problem as a contour integral. But this is a manifestation of (1.7), since, if f is $\mathcal{C}^\infty[\mathbb{I}]$, we can expand it in terms of the basis U_k and $\frac{f(x)}{\sqrt{1-x^2}}$ in terms of the basis $\frac{T_k(x)}{\sqrt{1-x^2}}$ efficiently.

The formula (1.8) means that it is possible to compute $\mathcal{H}f$ spectrally fast for the case where f itself is $\mathcal{C}^\infty[\mathbb{I}]$ as well, even when it does not go to zero at ± 1. We can find a related formula for the computation of $\mathcal{M}f$. We first solve the moment problem:

Lemma 5.5. *Define*

$$\psi_0(z) = \frac{2}{\mathrm{i}\pi}\begin{cases}\operatorname{arctanh} z & \text{for } |z| < 1,\\ \operatorname{arctanh}\frac{1}{z} & \text{for } |z| > 1,\end{cases}$$

$$\mu_m(z) = \sum_{j=1}^{\left\lfloor\frac{m+1}{2}\right\rfloor}\frac{z^{2j-1}}{2j-1} = \operatorname{arctanh} z - \frac{1}{2}z^{2\left\lfloor\frac{m+1}{2}\right\rfloor+1}\phi\left(z^2, 1, \frac{1}{2} + \left\lfloor\frac{m+1}{2}\right\rfloor\right),$$

where ϕ is the Lerch transcendental function [4]*. Then*

$$\mathcal{M}_{\mathbb{U}} z^m \mathrm{sgn}\arg z = \psi_m(z) \quad \textit{for}$$

$$\psi_m(z) = z^m\left[\psi_0(z) - \frac{2}{\mathrm{i}\pi}\begin{cases}\mu_{-m-1}(z) & \textit{for } m < 0\\ \mu_m(1/z) & \textit{for } m > 0\end{cases}\right]$$

Proof. Note that, for $z \in \mathbb{U}$,

$$\psi_0^+(z) - \psi_0^-(z) = \frac{2}{\mathrm{i}\pi}\left[\operatorname{arctanh} z - \operatorname{arctanh}\frac{1}{z}\right] = \mathrm{sgn}\ \arg z.$$

Since $\operatorname{arctanh}\frac{1}{z} \to 0$ as $z \to \infty$, it follows that

$$\mathcal{M}_{\mathbb{U}}\,\mathrm{sgn}\ \arg z = \psi_0(z).$$

A solution to the problem $u^+ - u^- = z^m \mathrm{sgn}\ \arg z$ which does not respect the boundary condition at ∞ is

$$z^m\mathcal{M}\,\mathrm{sgn}\ \arg z = z^m\psi_0(z).$$

But we know that $\operatorname{arctanh} z$ has the following Taylor series:

$$\operatorname{arctanh} z = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots \qquad [2].$$

The terms up to $\mathcal{O}(z^m)$ can be written as $\mu_m(z)$, where the expression in terms of ϕ follow from its series definition.

Consider the case where $m < 0$. Now $\mu_{-m-1}(z)$ is at most an $(-m-1)$-degree polynomial, hence $z^m \mu_{-m-1}(z)$ decays as $z \to \infty$, and ψ_m is analytic at ∞. On the other hand, by the definition of μ,

$$\operatorname{arctanh} z - \mu_{-m-1}(z) = \mathcal{O}\big(z^{-m}\big)\,, \qquad z \to 0;$$

therefore, ψ_m is also analytic at zero. Finally, for $z \in \mathbb{U}$,

$$\begin{aligned}\psi_m^+ - \psi_m^- &= \frac{2}{\mathrm{i}\pi} z^m \left[\operatorname{arctanh} z - \mu_{-m-1}(z) - \operatorname{arctanh}\frac{1}{z} + \mu_{-m-1}(z)\right] \\ &= \frac{2}{\mathrm{i}\pi} z^m \left[\operatorname{arctanh} z - \operatorname{arctanh}\frac{1}{z}\right] = z^m \operatorname{sgn} \arg z.\end{aligned}$$

Therefore, $\mathcal{M} z^m \operatorname{sgn} \arg z = \psi_m(z)$.

By the exact same logic, when $m > 0$, $\mathcal{M} z^m \operatorname{sgn} \arg z = \psi_m(z)$. □

Remark. Though $\psi_m(z)$ is the arctanh function plus a polynomial in z, for z off the unit circle, $\psi_m(z)$ can potentially be computed more accurately and efficiently using methods to compute ϕ. One approach is to use the method developed in [3], or the built-in MATHEMATICA routine. Possibly, a more efficient and accurate method could be based on the integral representation

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} \mathrm{e}^{-at}}{1 - z\mathrm{e}^{-t}}\, \mathrm{d}t.$$

In our case this becomes, for $a = \frac{1}{2} + \left\lfloor \frac{m+1}{2} \right\rfloor$,

$$\phi\big(z^2, 1, a\big) = \int_0^\infty \frac{\mathrm{e}^{-at}}{1 - z^2 \mathrm{e}^{-t}}\, \mathrm{d}t = \frac{1}{a} \int_0^\infty \frac{\mathrm{e}^{-t}}{1 - z^2 \mathrm{e}^{-t/a}}\, \mathrm{d}t.$$

As m becomes large, the integrand rapidly converges to one, and hence Gauss–Laguerre quadrature is very effective. As it is outside the scope of this paper, we simply treat each ψ_m as a black box special function, which we compute to machine precision using the polynomial representation with sufficient extra precision arithmetic.

Using this equation for the moments of f, we obtain the following:

Theorem 5.6. *If f is $\mathcal{C}^1[\mathbb{I}]$ and its first derivative has bounded variation, then*

$$\mathcal{M}_{\mathbb{I}} f(x) = -\frac{1}{4} \sum_{k=0}^\infty \check{f}_k \left[\psi_k(T_+^{-1}(x)) + \psi_k(T_-^{-1}(x)) + \psi_{-k}(T_+^{-1}(x)) + \psi_{-k}(T_-^{-1}(x))\right],$$

$$\begin{aligned}\mathcal{M}_{\mathbb{I}} f(x) \underset{x \to -1}{\sim} &-\frac{1}{2\mathrm{i}\pi} f(-1) \left[\log(-x-1) - \log 2\right] \\ &+ \frac{1}{\mathrm{i}\pi} \sum_{k=0}^\infty \check{f}_k (-1)^k \left[\mu_{k-1}(-1) + \mu_k(-1)\right],\end{aligned}$$

$$\mathcal{M}_{\mathbb{I}} f(x) \underset{x \to 1}{\sim} \frac{1}{2\mathrm{i}\pi} f(1) \left[\log(x-1) - \log 2\right] + \frac{1}{\mathrm{i}\pi} \sum_{k=0}^\infty \check{f}_k \left[\mu_{k-1}(1) + \mu_k(1)\right],$$

and, for $x \in \mathbb{I}$,

$$\mathcal{M}_{\mathbb{I}}^{+} f = -\frac{1}{4}\sum_{k=0}^{\infty} \check{f}_k \Big[\psi_k^{+}(T_{\downarrow}^{-1}(x)) + \psi_k^{-}(T_{\uparrow}^{-1}(x)) + \psi_{-k}^{+}(T_{\downarrow}^{-1}(x)) + \psi_{-k}^{-}(T_{\uparrow}^{-1}(x))\Big]$$

$$(5.2) \qquad = -\frac{2}{\mathrm{i}\pi} f(x) \operatorname{arctanh} T_{\downarrow}^{-1}(x) + \frac{1}{2\mathrm{i}\pi}\sum_{k=0}^{\infty} \check{f}_k \Big\{ T_{\downarrow}^{-1}(x)^k \Big[\mu_k\Big(T_{\uparrow}^{-1}(x)\Big) + \mu_{k-1}\Big(T_{\uparrow}^{-1}(x)\Big)\Big] + T_{\uparrow}^{-1}(x)^k \Big[\mu_k\Big(T_{\downarrow}^{-1}(x)\Big) + \mu_{k-1}\Big(T_{\downarrow}^{-1}(x)\Big)\Big]\Big\},$$

$$\mathcal{M}_{\mathbb{I}}^{-} f = -\frac{1}{4}\sum_{k=0}^{\infty} \check{f}_k \Big[\psi_k^{+}(T_{\uparrow}^{-1}(x)) + \psi_k^{-}(T_{\downarrow}^{-1}(x)) + \psi_{-k}^{+}(T_{\uparrow}^{-1}(x)) + \psi_{-k}^{-}(T_{\downarrow}^{-1}(x))\Big]$$

$$(5.3) \qquad = -\frac{2}{\mathrm{i}\pi} f(x) \operatorname{arctanh} T_{\uparrow}^{-1}(x) + \frac{1}{2\mathrm{i}\pi}\sum_{k=0}^{\infty} \check{f}_k \Big\{ T_{\downarrow}^{-1}(x)^k \Big[\mu_k\Big(T_{\uparrow}^{-1}(x)\Big) + \mu_{k-1}\Big(T_{\uparrow}^{-1}(x)\Big)\Big] + T_{\uparrow}^{-1}(x)^k \Big[\mu_k\Big(T_{\downarrow}^{-1}(x)\Big) + \mu_{k-1}\Big(T_{\downarrow}^{-1}(x)\Big)\Big]\Big\}.$$

Proof. Let $g(z) = f(T(z))$. By expanding g in terms of its Laurent series, we define

$$\Phi(z) = -\mathcal{M}_{\mathbb{U}} g(z) \operatorname{sgn} \arg z = -\sum_{k=-\infty}^{\infty} \hat{g}_k \psi_k(z).$$

Let $\xi = T_{+}^{-1}(x)$. Then define

(5.4)

$$u(x) = \frac{\Phi(T_{+}^{-1}(x)) + \Phi(T_{-}^{-1}(x))}{2} = -\frac{1}{2}\sum_{k=-\infty}^{\infty} \hat{g}_k \left[\psi_k(\xi) + \psi_k(\xi^{-1})\right]$$

$$= -\frac{1}{4}\sum_{k=0}^{\infty} \check{f}_k \left[\psi_k(\xi) + \psi_k(\xi^{-1}) + \psi_{-k}(\xi) + \psi_{-k}(\xi^{-1})\right]$$

$$= -\frac{1}{2\mathrm{i}\pi}\sum_{k=0}^{\infty} \check{f}_k \Big\{ 2(\xi^k + \xi^{-k}) \operatorname{arctanh} \xi - \xi^k \left[\mu_k(\xi^{-1}) + \mu_{k-1}(\xi^{-1})\right] + \xi^{-k}\left[\mu_k(\xi) + \mu_{k-1}(\xi)\right]\Big\}.$$

By the same logic as Theorem 5.3, we know that u satisfies $u^+ - u^- = f$ and $u(\infty) = 0$. We must demonstrate that u is also analytic off the unit interval.

We first prove that this sum does indeed converge. Because each ψ_k is analytic off the unit circle, we know the partial sums must take their maximum on the unit circle. Using (5.4) with $\xi = T_{\downarrow}^{-1}(x) \neq \pm 1$ and the fact that

$$\frac{1}{2}\sum_{k=0}^{\infty} \check{f}_k \left[\xi^k + \xi^{-k}\right] = \sum_{k=0}^{\infty} \check{f}_k T_k(x) = f(x),$$

we obtain that u^+ is equal to (5.2), assuming we can split the sum into two, or in other words, as long as the sum converges absolutely. By the same logic, u^- is equivalent to (5.3), subject to the same conditions.

From the summation definition of μ, we know it takes its maximum at $z = 1$, so that, for $z \in \mathbb{U}$,

$$\left|z^k\left[\mu_k(z)+\mu_{k-1}(z)\right]+z^{-k}\left[\mu_k(z)+\mu_{k-1}(z)\right]\right|$$
$$\leq 4\sum_{j=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor}\frac{1}{2j-1}\leq 2\log\left(-1+2\left\lfloor\frac{k+1}{2}\right\rfloor\right).$$

Thus absolute convergence is assured whenever

$$\sum_{k=0}^{\infty}\left|\check{f}_k\right|\log\left(-1+2\left\lfloor\frac{k+1}{2}\right\rfloor\right)$$

converges. It is sufficient that $\check{f}_k = \mathcal{O}(k^{-2})$, which is true whenever $f \in \mathcal{C}^1[\mathbb{I}]$ with f' having bounded variation.

We have yet to prove the behaviour of u at the endpoints. But the behaviour as $x \to 1$ follows from (5.4) and the fact that

$$\begin{aligned}
2\operatorname{arctanh}T_+^{-1}(x) &= \log(1+T_+^{-1}(x))-\log(1-T_+^{-1}(x))\\
&\sim \log 2-\log\left(\sqrt{2(x-1)}+\mathcal{O}(x-1)\right)\\
&\sim \log 2-\frac{1}{2}\log 2-\frac{1}{2}\left[\log|x-1|+\mathrm{i}\arg(x-1)\right]\\
&= \frac{\log 2-\log(x-1)}{2}.
\end{aligned}$$

Similar logic proves the expression as $x \to -1$. □

We thus obtain the following approximations:

Definition 5.7. For $\boldsymbol{f} = f(\boldsymbol{x}_n)$ and $\Psi = (2\psi_0, \psi_1+\psi_{-1}, \ldots, \psi_{n-1}+\psi_{1-n})$, define

$$\mathcal{M}_{\mathbb{I},n}f = -\frac{1}{4}\left[\Psi(T_+^{-1}(x))+\Psi(T_-^{-1}(x))\right]\check{\boldsymbol{f}}$$

and, for

$$\boldsymbol{\mu} = \left(\mu_0(z^{-1}), z\left[\mu_1(z^{-1})+\mu_0(z^{-1})\right], \ldots, z^{n-1}\left[\mu_{n-1}(z^{-1})+\mu_{n-2}(z^{-1})\right]\right),$$

define

$$\begin{aligned}
\mathcal{M}^+_{\mathbb{I},n}f &= -\frac{2}{\mathrm{i}\pi}\boldsymbol{e}_T(x)^\top\boldsymbol{f}\operatorname{arctanh}T_\downarrow^{-1}(x)+\frac{1}{2\mathrm{i}\pi}\left[\boldsymbol{\mu}(T_\downarrow^{-1}(x))+\boldsymbol{\mu}(T_\uparrow^{-1}(x))\right]\check{\boldsymbol{f}},\\
\mathcal{M}^-_{\mathbb{I},n}f &= -\frac{2}{\mathrm{i}\pi}\boldsymbol{e}_T(x)^\top\boldsymbol{f}\operatorname{arctanh}T_\uparrow^{-1}(x)+\frac{1}{2\mathrm{i}\pi}\left[\boldsymbol{\mu}(T_\downarrow^{-1}(x))+\boldsymbol{\mu}(T_\uparrow^{-1}(x))\right]\check{\boldsymbol{f}}.
\end{aligned}$$

Consider the Hilbert transform of e^x over $\mathbb{I}$, which we can find in closed form:

$$\frac{\mathrm{e}^x}{\pi}[\mathrm{Ei}(1-x)-\mathrm{Ei}(-1-x)] \qquad [16].$$

Figure 3 demonstrates the effectiveness of approximating this function by $2\mathrm{i}\mathcal{M}_{\mathbb{I},n}f$ throughout the complex plane and $\mathrm{i}(\mathcal{M}^+_{\mathbb{I},n}f+\mathcal{M}^-_{\mathbb{I},n}f)$ on the unit interval.

Note that $\boldsymbol{\mu}$ is a vector of polynomials, hence $\left[\boldsymbol{\mu}(T_\downarrow^{-1}(x))+\boldsymbol{\mu}(T_\uparrow^{-1}(x))\right]\check{\boldsymbol{f}}$ itself is a polynomial. Depending on the application, it might be more suitable to represent this term by its values at Chebyshev points. Denote the operator which shifts a

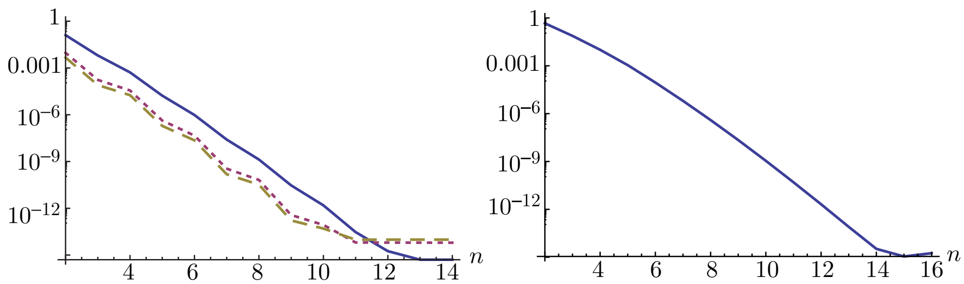

FIGURE 3. The maximum error in approximating the Hilbert transform of e^x by $2i\mathcal{M}_{\mathbb{I},n}f$ over 200 evenly spaced points on circles of radii 2 (left graph, plain), 25 (left graph, dotted) and 50 (left graph, dashed) and by $i(\mathcal{M}^+_{\mathbb{I},n}f + \mathcal{M}^-_{\mathbb{I},n}f)$ at 200 evenly spaced points on the unit interval (right graph)

vector to the left by S, padding on the right by zeros. Determine the vector $\boldsymbol{h}$ such that

$$\check{\boldsymbol{h}} = \begin{pmatrix} 2 & & & \\ & 4 & & \\ & & \ddots & \\ & & & 4 \end{pmatrix} \sum_{j=1}^{(n+1)/2} \frac{1}{2j-1} S^{2j-1} \check{\boldsymbol{f}}$$

using the inverse discrete cosine transform. Then

$$-i\mathcal{H}_{\mathbb{I}} f \approx -2 \frac{\operatorname{arctanh} T_\downarrow^{-1}(x) + \operatorname{arctanh} T_\uparrow^{-1}(x)}{i\pi} \boldsymbol{e}_T(x)^\top \boldsymbol{f} + \frac{1}{i\pi} \boldsymbol{e}_T(x)^\top \boldsymbol{h}.$$

This expression follows by rearranging the order of sums, using the definition of $\boldsymbol{\mu}$.

Constructing $\boldsymbol{h}$ requires $\mathcal{O}(n \log n)$ operations, as the summation is equivalent to multiplication by a Toeplitz matrix. Therefore, it will outperform the Gaussian and Clenshaw–Curtis quadrature methods, which require $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2 \log n)$ operations, respectively, to determine $\mathcal{H}f$ at $\boldsymbol{x}_n$. Moreover, these quadrature methods cannot handle the blow-up near the endpoints of the integration interval.

6. Half line and real line revisited

We orient the half line $\mathbb{R}^+ = [0, \infty)$ from the origin to ∞. We can map the interval to the half line using

$$H(x) = \frac{1+x}{1-x}.$$

We then obtain the following result:

Theorem 6.1. *Suppose that $r(y) = \frac{\alpha}{y} + \mathcal{O}\left(\frac{1}{y^2}\right)$ as $y \to \infty$ and r is $\mathcal{C}^1[\mathbb{R}^+]$ and its first derivative has bounded variation. Let $g(z) = r(H(Tz))$ and $\varphi = \mathcal{P}_{\mathbb{U}} g$. Then,*

$$\mathcal{P}_{\mathbb{R}^+} r(y) = \frac{\varphi(T_+^{-1}(H^{-1}(y))) + \varphi(T_-^{-1}(H^{-1}(y)))}{2}$$

and, for $\Phi = \mathcal{M}_{\mathbb{I}} f$,

$$\mathcal{M}_{\mathbb{R}^+} r(y) = \Phi(H^{-1}(y)) - \Phi^+(1),$$

Proof. Let

$$u(y) = \frac{\varphi(T_+^{-1}(H^{-1}(y))) + \varphi(T_-^{-1}(H^{-1}(y)))}{2}.$$

By the same logic as Theorem 5.1,

$$u^+(y) + u^{-1}(y) = \frac{g(T_\downarrow^{-1}(H^{-1}(y))) + g(T_\uparrow^{-1}(H^{-1}(y)))}{2} = r(y).$$

Furthermore,

$$2u^\pm(\infty) = \varphi^+(1) + \varphi^-(1) = g(1) = r(H(T(1))) = r(\infty) = 0,$$

hence

$$\mathcal{P}_{\mathbb{R}^+} r(y) = u(y).$$

Now consider the second part of the theorem, where we define $\Phi = \mathcal{M}_{\mathbb{I}} f$. Since $f(1) = r(\infty) = 0$, the logarithmic term in Theorem 5.6 vanishes, and we are left with

$$\Phi^+(1) = \Phi^-(1) = \frac{1}{\mathrm{i}\pi} \sum_{k=0}^{\infty} \check{f}_k \left[\mu_{k-1}(1) + \mu_k(1)\right].$$

Therefore, $\Phi - \Phi^+(1)$ goes to zero at one, and hence

$$\mathcal{M}_{\mathbb{R}^+} r(y) = \Phi(H^{-1}(y)) - \Phi^+(1). \qquad \square$$

We thus obtain the following approximations:

Definition 6.2. Let $f(x) = r(H(x))$ and $g(z) = f(T(z))$. If $y \in \mathbb{C}\backslash\mathbb{R}^+$, then define

$$\mathcal{P}_{\mathbb{R}^+,n} r(y) = \frac{1}{2}[\mathcal{P}_{\mathbb{U},n} g(T_+^{-1}(H^{-1}(y))) + \mathcal{P}_{\mathbb{U},n} g(T_-^{-1}(H^{-1}(y)))]$$

and

$$\mathcal{M}_{\mathbb{R}^+,n} r(y) = \mathcal{M}_{\mathbb{I},n} f(H^{-1}(y)) - \frac{1}{\mathrm{i}\pi}\boldsymbol{\mu}(1)\check{\boldsymbol{f}}.$$

If $y \in \mathbb{R}^+$, then define

$$\mathcal{P}^+_{\mathbb{R}^+,n} r(y) = \frac{1}{2}[\mathcal{P}^+_{\mathbb{U},n} g(T_\downarrow^{-1}(H^{-1}(y))) + \mathcal{P}_{\mathbb{U},n} g(T_\uparrow^{-1}(H^{-1}(y)))],$$
$$\mathcal{P}^-_{\mathbb{R}^+,n} r(y) = \frac{1}{2}[\mathcal{P}^+_{\mathbb{U},n} g(T_\uparrow^{-1}(H^{-1}(y))) + \mathcal{P}_{\mathbb{U},n} g(T_\downarrow^{-1}(H^{-1}(y)))],$$

and

$$\mathcal{M}^\pm_{\mathbb{R}^+,n} r(y) = \mathcal{M}^\pm_{\mathbb{I},n} f(H^{-1}(y)) - \frac{1}{\mathrm{i}\pi}\boldsymbol{\mu}(1)\check{\boldsymbol{f}}.$$

Similar to Section 4, the convergence rate of this approximation depends on the differentiability of f in $\mathbb{R}^+$, and the existence of an asymptotic expansion of f at ∞. When applied to computing the Hilbert transform over $\mathbb{R}^+$, it converges more rapidly than the method proposed in [9], which requires exponential decay at infinity to achieve superalgebraic convergence.

Computation over two half lines. Let $q(y) = r(-y)$. It is clear that $\mathcal{M}_{\mathbb{R}^-} r(y) = -\mathcal{M}_{\mathbb{R}^+} q(-y)$, where $\mathbb{R}^-$ is oriented from $-\infty$ to zero. Because it is analytic everywhere off $\mathbb{R}^+$, $\mathcal{M}^+_{\mathbb{R}^+} f - \mathcal{M}^-_{\mathbb{R}^+} f = 0$ along $(-\infty, 0)$. Likewise, $\mathcal{M}^+_{\mathbb{R}^-} f - \mathcal{M}^-_{\mathbb{R}^-} f = 0$ on $[0, \infty)$. Therefore,

$$\mathcal{M}_{\mathbb{R}} f = (\mathcal{M}_{\mathbb{R}^+} + \mathcal{M}_{\mathbb{R}^-}) f.$$

Whereas Section 4 required that f had the same asymptotic series at $\pm\infty$ to obtain fast convergence, by splitting the real line into two half lines, we can drop this requirement. Furthermore, computation of $\mathcal{M}_{\mathbb{R}^\pm} f$ only requires that f is $\mathcal{C}^1[\mathbb{R}^\pm]$ and its first derivative has bounded variation for each choice of $\pm$ separately, hence does not require continuity at zero.

There is one concern which must be addressed, however. We know that $\mathcal{M}_{\mathbb{R}} f$ is bounded, whereas $\mathcal{M}_{\mathbb{R}^+}$ and $\mathcal{M}_{\mathbb{R}^-}$ have singularities at zero. But we find that $H^{-1}(y) = -1 + 2y + \mathcal{O}(y^2)$, therefore, for $p(x) = r(-H(x))$ and $f(x) = r(H(x))$,

$$\begin{aligned}
\mathcal{M}_{\mathbb{R}^-} r(y) \underset{y\to 0}{\sim}\ & \frac{1}{2\mathrm{i}\pi} r(0) \log y \\
& - \frac{1}{\mathrm{i}\pi} \sum_{k=0}^{\infty} \check{p}_k \left\{ (-1)^k \left[\mu_{k-1}(1) + \mu_k(1)\right] - \mu_{k-1}(1) - \mu_k(1) \right\}, \\
\mathcal{M}_{\mathbb{R}^+} r(y) \underset{y\to 0}{\sim}\ & -\frac{1}{2\mathrm{i}\pi} r(0) \log(-y) \\
& + \frac{1}{\mathrm{i}\pi} \sum_{k=0}^{\infty} \check{f}_k \left\{ (-1)^k \left[\mu_{k-1}(1) + \mu_k(1)\right] - \mu_{k-1}(1) - \mu_k(1) \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{M}_{\mathbb{R}} r(y) &= \mathcal{M}_{\mathbb{R}^+} r(y) + \mathcal{M}_{\mathbb{R}^-} r(y) \\
&\underset{y\to 0}{\sim} \frac{1}{2\mathrm{i}\pi} r(0)(\log y - \log(-y)) \\
&\qquad + \frac{1}{\mathrm{i}\pi} \sum_{k=0}^{\infty} \left(\check{f}_k - \check{p}_k\right) \left\{ (-1)^k \left[\mu_{k-1}(1) + \mu_k(1)\right] - \mu_{k-1}(1) - \mu_k(1) \right\} \\
&= \frac{1}{2\pi} r(0)(\arg y - \arg(-y)) + \cdots = \frac{1}{2} r(0) \begin{cases} 1 & \text{for } \Im y > 1, \\ -1 & \text{for } \Im y < 1 \end{cases} + \cdots,
\end{aligned}$$

whose limit is bounded at zero from both above and below the real line.

Defining $\mathcal{M}_{\mathbb{R}^-,n} r(y) = -\mathcal{M}_{\mathbb{R}^+,n} q(-y)$, we obtain $\mathcal{M}_{\mathbb{R}} r \approx \mathcal{M}_{\mathbb{R}^-,n} r + \mathcal{M}_{\mathbb{R}^+,n} r$. This approximation is more computationally intensive then the method proposed in Section 4: it takes $\mathcal{O}(n^2)$ operations to compute at the $2n-1$ points $\pm H(\boldsymbol{x}_n)$, versus $\mathcal{O}(n \log n)$ operations to evaluate at the $2n-1$ points $R(\boldsymbol{z}_{2n-1})$. However, it has the benefit of obtaining spectral convergence for a more general class of functions. For example, consider the function

$$r(y) = \frac{\arctan y}{y},$$

whose Hilbert transform is

$$\mathcal{H} r(y) = -\frac{\log(1+y^2)}{2y} \qquad [16].$$

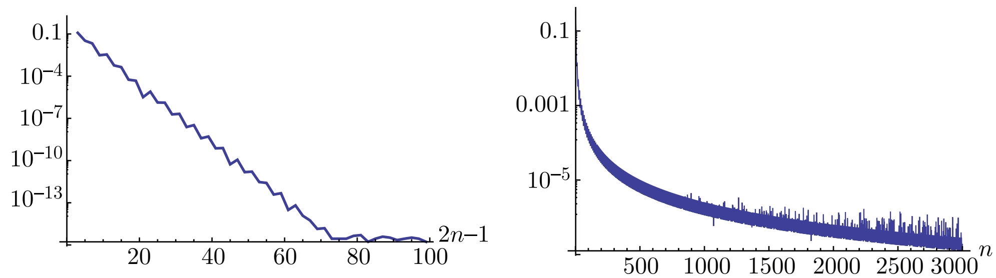

FIGURE 4. The maximum error over the points $R(\ddagger_{10})$ of (6.1) (left graph) and (4.2) (right graph) for approximating the Hilbert transform of $\frac{\arctan y}{y}$. Note that the value on the bottom axis corresponds to the total number of function evaluations

Unlike the example in Section 4, this does not have the same asymptotic series at $\pm\infty$. Figure 4 demonstrates that, for $\boldsymbol{r} = r(H(\boldsymbol{x}_n))$ and $\boldsymbol{p} = r(-H(\boldsymbol{x}_n))$,

$$\mathrm{i}\begin{cases} \left(\mathcal{M}_{\mathbb{R}^+,n} + \mathcal{M}^+_{\mathbb{R}^-,n} + \mathcal{M}^-_{\mathbb{R}^-,n}\right) r(z) & \text{for } z < 0, \\ \left(\mathcal{M}^+_{\mathbb{R}^+,n} + \mathcal{M}^-_{\mathbb{R}^+,n} + \mathcal{M}_{\mathbb{R}^-,n}\right) r(z) & \text{for } z > 0, \\ 2(\boldsymbol{\mu}(1) - \boldsymbol{\mu}(-1))(\check{\boldsymbol{p}} - \check{\boldsymbol{r}}) & \text{for } z = 0, \end{cases} \tag{6.1}$$

is a superalgebraic convergent approximation of $\mathcal{H}r$, whereas (4.2) converges only algebraically.

Without delving into details, we note that this approach also applies to combination of intervals, and even combination of intervals and half lines. Thus we can successfully compute the Hilbert transform over $\mathbb{R}$ of any piecewise smooth function with finitely many pieces. As a result, this approach could potentially be used to construct an adaptive scheme for computing the Hilbert transform.

7. FUTURE WORK

The solution to the gravity wave equation is periodic as $t \to \infty$, thus we need to compute the oscillatory Hilbert transform

$$\mathcal{H}[f(t)\mathrm{e}^{\mathrm{i}\omega g(t)}].$$

An asymptotic expansion for such transforms was found in [28] and a method based on series acceleration methods was constructed in [17]. In recent years, there has been significant progress on the computation of oscillatory integrals of the form

$$\int_{-1}^{1} f(t)\mathrm{e}^{\mathrm{i}\omega g(t)}\,\mathrm{d}t,$$

using asymptotic information as $\omega \to \infty$ [14, 21, 22]. It remains to be seen if such methods can be generalized for the oscillatory Hilbert transform.

Consider the matrix-valued Riemann–Hilbert problem of finding an analytic function $\Phi : \mathbb{C}\backslash\Gamma \to \mathbb{C}^{d\times d}$ such that

$$\Phi_+(z) = \Phi_-(z)G(z) \quad \text{for} \quad z \in \Gamma, \quad \text{and} \quad \Phi(\infty) = I, \tag{7.1}$$

where $G : \Gamma \to \mathbb{C}^{d\times d}$. This can be reformulated as an inhomogenuous linear equation:

$$\mathcal{L}U = U_+ - U_- G = G - I \qquad \text{and} \qquad U(\infty) = 0,$$

where $\Phi = U + I$. The linear operator $\mathcal{L}$ maps functions analytic in $\mathbb{C}\backslash\Gamma$ to functions defined in Γ. But $\mathcal{M}$ can be viewed as a one-to-one map from Γ to $\mathbb{C}\backslash\Gamma$. This applies equally well to matrix-valued functions in a componentwise manner. Thus by combining the two linear operators as $\mathcal{LM}$, we obtain a linear operator that maps the class of Hölder continuous matrix-valued functions on Γ to itself. In other words, solving $\mathcal{LM}V = G - I$ on Γ means that $U = \mathcal{M}V$ solves the original matrix-valued Riemann–Hilbert problem. Immediately, this opens up the possibility of constructing spectral methods for solving (7.1).

A particular example is the homogeneous Painlevé II transcendental, which can be expressed as a matrix-valued Riemann–Hilbert problem on a domain consisting of six rays originating at zero:

$$\left\{ z \in \mathbb{C} : \arg z = \frac{\pi}{6} + \frac{\pi}{3}(k-1) \quad \text{for} \quad k = 1, \ldots, 6 \right\} \qquad [11].$$

For other Painlevé equations, the domain consists of a combination of circles, intervals, arcs and rays. Just as we combined $\mathcal{M}$ on two half lines to determine $\mathcal{M}$ on the real line, we propose that it is possible to combine multiple half lines and circles to compute $\mathcal{M}$ on such domains. We fully develop this approach for the computation of homogeneous Painlevé II transcendentals in [23].

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