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PRECISE AND FAST COMPUTATION OF THE GENERAL COMPLETE ELLIPTIC INTEGRAL OF THE SECOND KIND

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ABSTRACT. We developed an efficient procedure to evaluate two auxiliary complete elliptic integrals of the second kind $B(m)$ and $D(m)$ by using their Taylor series expansions, the definition of Jacobi's nome, and Legendre's relation. The developed procedure is more precise than the existing ones in the sense that the maximum relative errors are 1-3 machine epsilons, and it runs drastically faster; around 5 times faster than Bulirsch's `cel2` and 16 times faster than Carlson's R_F and R_D.

1. INTRODUCTION

1.1. **Complete elliptic integrals of first and second kind.** The complete elliptic integrals of the first and the second kind $K(m)$ and $E(m)$ appear in various fields of mathematical physics and engineering [1, 6]. Consult with the references of our recent work [17] for practical examples in astrophysics and celestial mechanics such as the expression of gravitational potential and the acceleration vector caused by a uniform ring. The integrals are defined as

$$K(m) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}, \quad E(m) \equiv \int_0^{\pi/2} \left(\sqrt{1-m\sin^2\theta}\right) d\theta. \tag{1.1}$$

They are the special value of the incomplete elliptic integrals of the first and the second kind, $F(\varphi|m)$ and $E(\varphi|m)$ as

$$K(m) = F\left(\frac{\pi}{2}\middle| m\right), \quad E(m) = E\left(\frac{\pi}{2}\middle| m\right), \tag{1.2}$$

where the incomplete integrals are defined as

$$F(\varphi|m) \equiv \int_0^{\varphi} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}, \quad E(\varphi|m) \equiv \int_0^{\varphi} \left(\sqrt{1-m\sin^2\theta}\right) d\theta. \tag{1.3}$$

Refer to the formulas 110.02, 110.03, 110.06, and 110.07 of [6]. We frequently cite the formulas of this handbook throughout the present article. Then we abbreviate the references to its formulas as BF110.02 hereafter. The complete integrals are real-valued when $m \le 1$. Their special values are

$$K(0) = E(0) = \frac{\pi}{2}, \quad K(1) = +\infty, \quad E(1) = 1. \tag{1.4}$$

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The values for negative parameters are reduced to those for positive ones as

$$K(m) = \frac{K(m_r)}{\sqrt{m_c}}, \quad E(m) = \sqrt{m_c}\, E(m_r), \quad (m < 0) \tag{1.5}$$

where

$$m_c \equiv 1 - m, \quad m_r \equiv \frac{-m}{1-m}, \tag{1.6}$$

are the complementary and the reduced parameters, respectively. The latter satisfies the condition $0 < m_r < 1$ when $m < 0$. These transformation formulas are obtained as the special cases of the imaginary modulus transformation formulas of the incomplete integrals. Refer to the first and second forms of BF160.02. These features enable us to consider only the cases when $0 < m < 1$.

1.2. **Auxiliary complete elliptic integrals.** Rather important are not the pair of $K(m)$ and $E(m)$ but a pair of auxiliary complete elliptic integrals of the second kind [2]:

$$B(m) \equiv \int_0^{\pi/2} \frac{\cos^2\theta d\theta}{\sqrt{1 - m\sin^2\theta}}, \quad D(m) \equiv \int_0^{\pi/2} \frac{\sin^2\theta d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{1.7}$$

They are linearly related with $K(m)$ and $E(m)$ as

$$K(m) = B(m) + D(m), \quad E(m) = B(m) + m_c D(m). \tag{1.8}$$

The special values are computed as

$$B(0) = D(0) = \frac{\pi}{4}, \quad B(1) = 1, \quad D(1) = +\infty. \tag{1.9}$$

The values for negative parameters are alternatively reduced to those for positive ones as

$$B(m) = \frac{D(m_r)}{\sqrt{m_c}}, \quad D(m) = \frac{B(m_r)}{\sqrt{m_c}} \quad (m < 0). \tag{1.10}$$

These are derived from equations (1.5) and (1.8). Therefore we can restrict ourselves to the case $0 < m < 1$. The auxiliary integrals are needed in computing the derivatives of $K(m)$ and $E(m)$ without cancellation as

$$\frac{dK(m)}{dm} = \frac{B(m)}{2m_c}, \quad \frac{dE(m)}{dm} = \frac{-D(m)}{2}. \tag{1.11}$$

These are derived from BF710.00 and BF710.02. Also, $B(m)$ or $D(m)$ is required in the evaluation of the corresponding incomplete integrals:

$$B(\varphi|m) \equiv \int_0^{\varphi} \frac{\cos^2\theta d\theta}{\sqrt{1 - m\sin^2\theta}}, \quad D(\varphi|m) \equiv \int_0^{\varphi} \frac{\sin^2\theta d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{1.12}$$

This is because their computations for arbitrary value of φ are transformed to those for the case $0 < \varphi < \pi/2$ by the amplitude reduction formulas

$$B(\varphi \pm j\pi|m) = B(\varphi|m) \pm 2jB(m), \quad D(\varphi \pm j\pi|m) = D(\varphi|m) \pm 2jD(m). \tag{1.13}$$

These are derived from those of $F(\varphi|m)$ and $E(\varphi|m)$ given in BF113.02.

1.3. **Auxiliary incomplete elliptic integrals.** The auxiliary incomplete elliptic integrals are required in computing the partial derivatives of $F(\varphi|m)$ and $E(\varphi|m)$ with respect to the parameter m without cancellation as

$$\left(\frac{\partial F(\varphi|m)}{\partial m}\right)_{\varphi} = \frac{1}{2m_c}\left[B(\varphi|m) - \left(\frac{\sin\varphi\cos\varphi}{\sqrt{1-m\sin^2\varphi}}\right)\right], \tag{1.14}$$

$$\left(\frac{\partial E(\varphi|m)}{\partial m}\right)_{\varphi} = \frac{-D(\varphi|m)}{2}. \tag{1.15}$$

Refer to BF710.07 and BF710.09. The former derivative is related with the partial derivatives of Jacobian elliptic functions with respect to m as

$$\left(\frac{\partial \mathrm{sn}(u|m)}{\partial m}\right)_{u} = \mathrm{cn}(u|m)\mathrm{dn}(u|m)\left(\frac{\partial F(\varphi|m)}{\partial m}\right)_{\varphi}, \tag{1.16}$$

$$\left(\frac{\partial \mathrm{cn}(u|m)}{\partial m}\right)_{u} = -\mathrm{sn}(u|m)\mathrm{dn}(u|m)\left(\frac{\partial F(\varphi|m)}{\partial m}\right)_{\varphi}, \tag{1.17}$$

$$\left(\frac{\partial \mathrm{dn}(u|m)}{\partial m}\right)_{u} = -m\mathrm{sn}(u|m)\mathrm{cn}(u|m)\left(\frac{\partial F(\varphi|m)}{\partial m}\right)_{\varphi}, \tag{1.18}$$

$$\left(\frac{\partial \mathrm{am}(u|m)}{\partial m}\right)_{u} = \mathrm{dn}(u|m)\left(\frac{\partial F(\varphi|m)}{\partial m}\right)_{\varphi}. \tag{1.19}$$

Refer to BF710.50 through BF710.53. These derivatives are necessary in applying the method of variation of constant to the perturbed rotational motion of a rigid body [14, 15].

1.4. **Cancellation problem.** One may claim that the combination of $K(m)$ and $E(m)$ is sufficient for the purposes discussed in the previous subsection. Either $B(m)$ or $D(m)$ is computable from the pair of $K(m)$ and $E(m)$ as

$$B(m) = \frac{E(m) - m_c K(m)}{m}, \quad D(m) = \frac{K(m) - E(m)}{m}. \tag{1.20}$$

These are obtained by solving equation (1.8) with respect to $B(m)$ and $D(m)$. Nevertheless, these expressions face a severe loss of information in numerical computation. This is true especially when m is small. Such a situation frequently occurs when we discuss the rotational motions of rigid bodies under weak torques. Good examples are those of solid planets, satellites, and asteroids in the solar system. The typical values of m is of the order of 10^{-7} in these cases [13]. Figure 1 shows the parameter dependence of the logarithms of relative errors of $B(m)$ computed by two methods: equation (1.20) using the given values of $K(m)$ and $E(m)$ and Bulirsch's `cel2` [2]. Here the function `cel2` is defined as

$$\texttt{cel2}\,(k_c, a, b) \equiv \int_0^{\infty}\left(\frac{a + b\xi^2}{(1+\xi^2)\sqrt{(1+\xi^2)(1+k_c^2\xi^2)}}\right)d\xi. \tag{1.21}$$

It can directly evaluate $B(m)$ and $D(m)$ as

$$B(m) = \texttt{cel2}\,(k_c, 1, 0), \quad D(m) = \texttt{cel2}\,(k_c, 0, 1), \tag{1.22}$$

where

$$k_c \equiv \sqrt{m_c}, \tag{1.23}$$

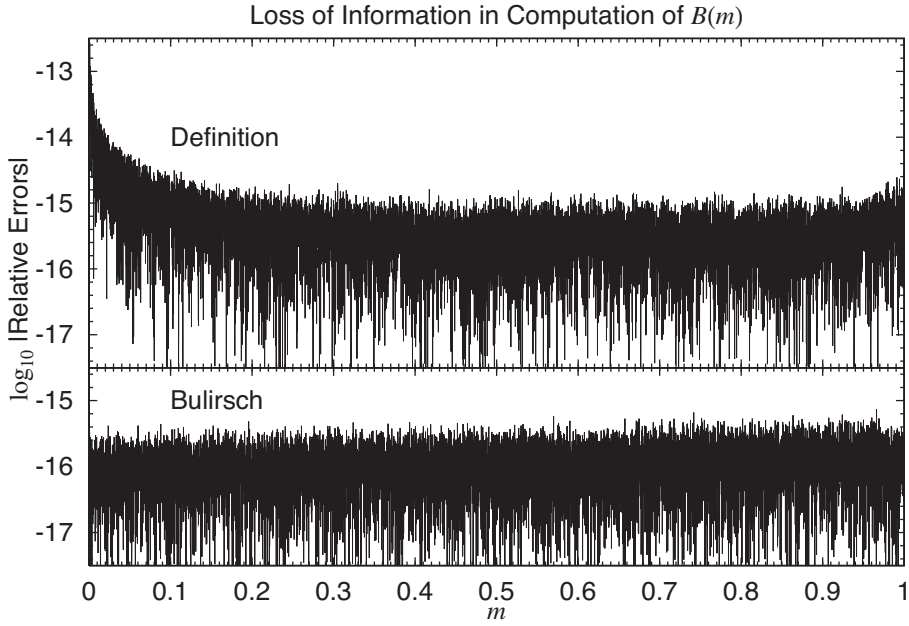

FIGURE 1. Loss of Information in Computation of $B(m)$. Shown are the relative errors of two methods to compute $B(m)$ in the double precision environment; the defining formula given in equation (1.20) and Bulirsch's `cel2`.

is the complementary modulus. The errors shown in the figure are measured as the differences from the quadruple precision computations prepared by `qcel2`. It is the quadruple precision extension of `cel2`. The accuracy of the quadruple precision computation is confirmed at the level of 10^{-33}. This was done by random comparison with the extremely high precision computation in 40 digits by Mathematica [22]. We computed the values of $K(m)$ and $E(m)$ by Cody's Chebyshev approximate formulas of Hastings type [20, 10, 11, 12] in preparing the graph of the former method. We confirm that the manner of round-off error increase is independent on the procedure to compute $K(m)$ and $E(m)$. A similar result as shown in Figure 1 is obtained for $D(m)$. The reason of the significant increase of round-off errors in case of the defining formulas is clear. This is because equation (1.20) is of the form of the difference of two positive quantities of similar magnitude when $m \sim 0$. Figure 2 illustrates the parameter dependence of $K(m)$, $E(m)$, $B(m)$, and $D(m)$. It is obvious that $m_c \sim 1$ when m is small. Then $E(m)$ and $m_c K(m)$ become of a similar magnitude. This produces round-off errors in evaluating $B(m)$. Similarly, $K(m)$ and $E(m)$ are roughly the same when $m \sim 0$. This is the cause of information loss in computing $D(m)$.

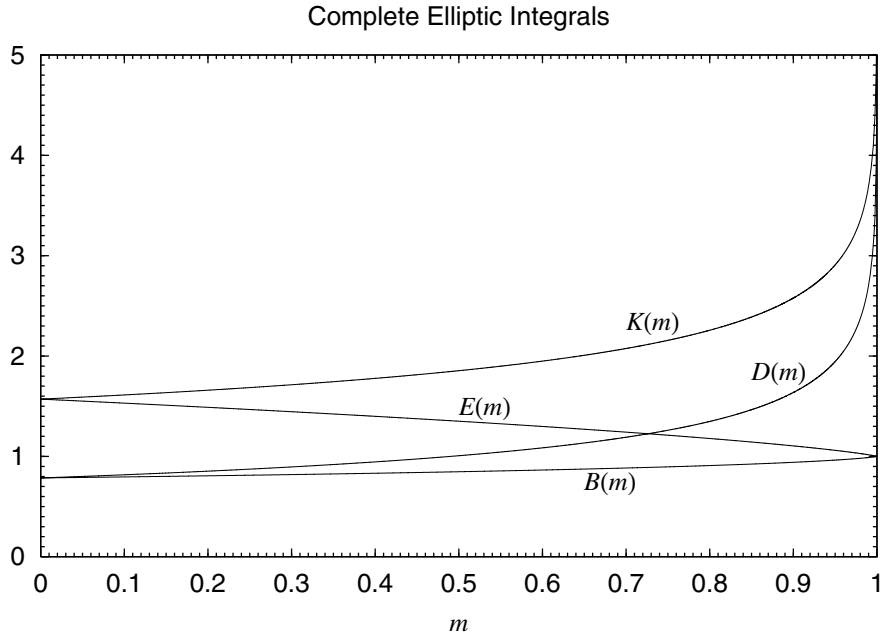

FIGURE 2. Parameter Dependence of Complete Elliptic Integrals.

1.5. **Comparison of existing methods.** We need specific procedures in order to compute $B(m)$ and/or $D(m)$ with ignorable round-off errors. There exist two methods: Bulirsch's `cel2` described already and Carlson's R_F and R_D [7, 8]. The latter functions are defined as

$$R_F(x,y,z) \equiv \frac{1}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)(t+y)(t+z)}}, \tag{1.24}$$

$$R_D(x,y,z) \equiv \frac{3}{2}\int_0^\infty \frac{dt}{\sqrt{(t+x)^3(t+y)(t+z)}}, \tag{1.25}$$

and related with $F(\varphi|m)$ and $D(\varphi|m)$ as

$$F(\varphi|m) = \sin\varphi R_F\left(\cos^2\varphi, 1-m\sin^2\varphi, 1\right), \tag{1.26}$$

$$D(\varphi|m) = \left(\frac{\sin^3\varphi}{3}\right) R_D\left(\cos^2\varphi, 1-m\sin^2\varphi, 1\right). \tag{1.27}$$

Therefore, we can use them in computing $B(m)$ and $D(m)$ as

$$B(m) = K(m) - D(m), \quad K(m) = R_F\left(0, m_c, 1\right), \quad D(m) = \frac{R_D\left(0, m_c, 1\right)}{3}. \tag{1.28}$$

Numerical comparison with the higher precision computations revealed that both of these methods are almost always precise. The exception is the case of $B(m)$ computed by Carlson's functions if $m \sim 1$. Figures 3 and 4 depict the parameter dependence of the relative errors of the two methods to compute $B(m)$ and $D(m)$ in the double precision environment. The precision loss of $B(m)$ in Carlson's case when $m \sim 1$ is due to its expression shown in the above. There $B(m)$ is expressed as the difference of two positive quantities $K(m)$ and $D(m)$. Both $K(m)$ and $D(m)$

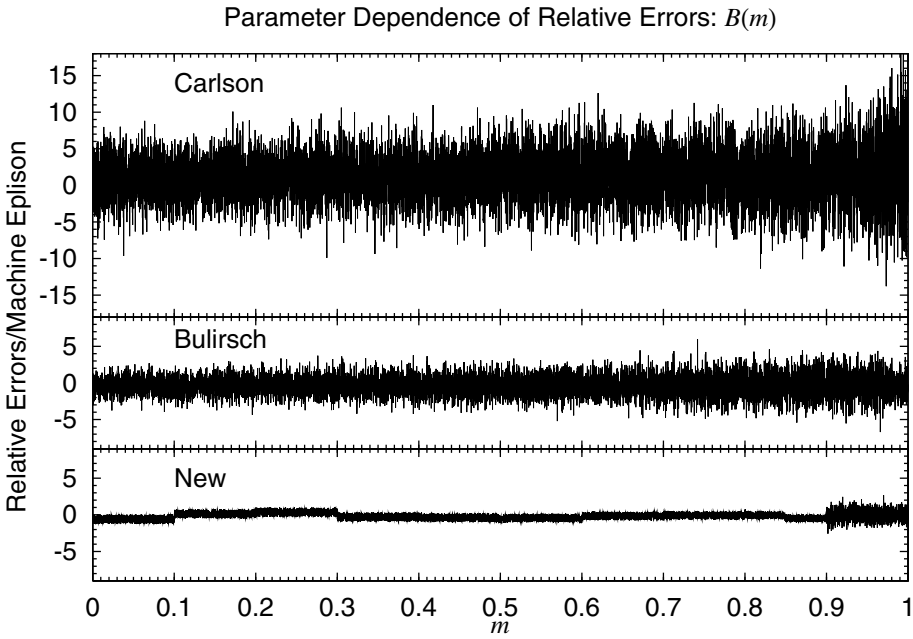

FIGURE 3. Parameter Dependence of Relative Errors of $B(m)$. Shown are the relative errors of three methods to compute $B(m)$ in the double precision environment; Carlson's R_F and R_D, Bulirsch's `cel2`, and the new method presented in §2.

go to infinity logarithmically when $m \to 1$. See Figure 2 again. Then the magnitude of $K(m)$ and $D(m)$ become similar. Thus Carlson's form faces a cancellation. This means that `cel2` is more appropriate than R_F and R_D in the sense of computational precision. Table 1 lists the averaged CPU time of these methods. We measured the CPU times as simple means of the results for $2^{24} - 1$ equally spaced grid points in the standard domain of the parameter $0 < m < 1$ and scaled by that to compute the sine function in the double precision environment. All the computation codes were written in Fortran 77/90, compiled by the Intel Visual Fortran 8.0, and executed at a PC with Intel Core Duo processor under Windows XP. The table clearly shows that `cel2` runs 2.9-3.5 times faster than the combination of R_F and R_D. This mainly owes to the difference in the speed of convergence of the main algorithm. Bulirsch's `cel2` is based on the Landen transformation in combination with the arithmetic-geometric mean [2, 3, 4, 5]. It is of quadratic convergence. Namely the number of significant digits is doubled by each iteration. Meanwhile Carlson's functions rely on the duplication theorem [7, 8]. It is of linear convergence. Then the number of significant digits increases by a same amount in each application. Thus the higher the desired relative precision is, the faster `cel2` becomes than R_F and R_D. This is observed from the ratios of their CPU times in the single and double precision environments. Therefore, `cel2` is more preferable than the combination of R_F and R_C also in terms of the computational speed. Actually, `cel2` runs around three times faster than the pair of R_F and R_C already in the single precision environment. Nevertheless, it is also true that even `cel2` requires

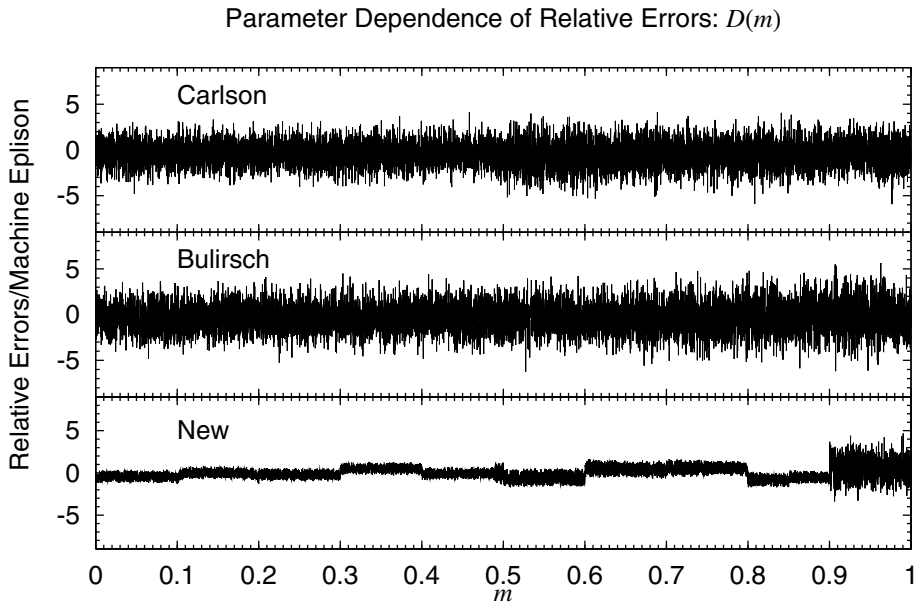

FIGURE 4. Parameter Dependence of Relative Errors of $D(m)$. Same as Figure 3 but for $D(m)$.

TABLE 1. Averaged CPU Times to Compute $B(m)$ and $D(m)$ Simultaneously. The unit of CPU time is that to compute the double precision sine function.

Method	Procedures	Single	Double
Carlson	R_F & R_D	18.0	53.5
Bulirsch	`cel2`	6.2	15.5
New	`elbd`	1.1	3.2

a significant amount of computational time, say 6-16 times more than that of the sine function. This is still a large computational labor as we consider the frequent needs to evaluate the integrals in practical applications.

1.6. **Introduction of new method.** Recently we developed a new method to compute $K(m)$ and/or $E(m)$ in [17]. Hereafter, we cite it as Paper I. The key techniques used there are the utilization of Taylor series expansion and the combination of the defining relation of Jacobi's nome and Legendre's relation. This is a continuation of our trials to accelerate the procedures to compute the complete and incomplete elliptic integrals and the Jacobian elliptic functions [19, 16, 17, 18]. The new method is sufficiently precise and significantly faster than the existing procedures to compute $K(m)$ and/or $E(m)$ including Cody's method as well as Innes' classic formulation [21]. None of the procedures discussed in Paper I including ours is not suitable to compute $B(m)$ and/or $D(m)$ as we saw in Figure 1. Then we

TABLE 2. Necessary Minimum Order of Taylor Expansion Polynomials of $B(m)$ and $D(m)$

	Single		Double	
m	J_B	J_D	J_B	J_D
$(0.0, 0.1]$	5	5	11	11
$(0.1, 0.2]$	5	5	11	11
$(0.2, 0.3]$	6	6	12	12
$(0.3, 0.4]$	6	6	12	13
$(0.4, 0.5]$	6	7	12	15
$(0.5, 0.6]$	6	8	13	16
$(0.6, 0.7]$	7	8	15	17
$(0.7, 0.8]$	9	10	18	20
$(0.8, 0.85]$	7	8	14	17
$(0.85, 0.9]$	9	10	18	20

adapt our approach in Paper I to the computation of $B(m)$ and $D(m)$. The resulting method is significantly more precise than the existing procedures. See Figures 3 and 4 again. The errors of the new method are 1-3 machine epsilons at most. Meanwhile, the new method runs much faster than `cel2`. See Table 1 again. In this short article, we will present the detail of the new method to compute $B(m)$ and $D(m)$ simultaneously.

2. New method

Let us consider computing the auxiliary complete elliptic integrals $B(m)$ and $D(m)$. Refer to Paper I for the basic background of the present approach. Once the auxiliary integrals are computed, we can calculate $K(m)$ and $E(m)$ from them without cancellation by using equation (1.8).

2.1. **Case of small parameter.** When $0 < m \le 0.9$, we evaluate $B(m)$ and $D(m)$ by their piecewise approximate polynomials derived from their Taylor series expansions. In constructing the piecewise polynomials, we divide the given domain $(0.0, 0.9]$ into 10 subdomains; $(0.0, 0.1]$, $(0.1, 0.2]$, $(0.2, 0.3]$, $(0.3, 0.4]$, $(0.4, 0.5]$, $(0.5, 0.6]$, $(0.6, 0.7]$, $(0.7, 0.8]$, $(0.8, 0.85]$, and $(0.85, 0.9]$. In each subdomain, the approximate polynomials are written as

$$B(m) \approx \sum_{j=0}^{J_B} B_j \, (m - m_0)^j \,, \quad D(m) \approx \sum_{j=0}^{J_D} D_j \, (m - m_0)^j \,. \tag{2.1}$$

The necessary minimum order of the polynomials J_B and J_D are listed in Table 2 for the single and double precision environment. We obtained them by numerical comparison with the quadruple precision computations. We set the reference value of m in each subdomain m_0 as the center values of the corresponding subdomains. Namely, they are 0.05, 0.15, 0.25, 0.35, 0.45, 0.55, 0.65, 0.75, 0.825, and 0.875. Meanwhile, the coefficients B_j and D_j themselves are explicitly given in Tables 3 through 12. We obtained them by Mathematica [22] with a command like

```
Series[(EllipticE[m] - (1 - m)EllipticK[m])/m, {m, 0.05, 12}]]
```

This gives the coefficients B_j in Table 3.

TABLE 3. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.0 < m \leq 0.1$. The coefficients are expressed with a few more digits than necessary in order to avoid round-off errors in the implementation.

j	B_j	D_j
0	0.790401413584395132	0.800602040206397048
1	0.102006266220019155	0.313994477771767757
2	0.039878395558551461	0.205913118705551955
3	0.021737136375982167	0.157744346538923994
4	0.013960979767622058	0.130595077319933092
5	0.009892518822669142	0.113308474489758567
6	0.007484612400663336	0.101454199173630195
7	0.005934625664295474	0.092918784207297437
8	0.004874249053581664	0.086565380148168087
9	0.004114606930310886	0.081727984665103014
10	0.003550452989196177	0.077990665729107038
11	0.003119229959988475	0.075080426851268007

TABLE 4. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.1 < m \leq 0.2$

j	B_j	D_j
0	0.801024064452844894	0.834232667811735098
1	0.110695344529634015	0.360495281619098276
2	0.047348746716993718	0.262379664114505869
3	0.028484367255041423	0.223723944518094276
4	0.020277811444003597	0.206447811775681053
5	0.015965005853099119	0.199809440876486856
6	0.013441320273553635	0.199667451603795275
7	0.011871565736951440	0.204157558868236842
8	0.010868363672485521	0.212387467960572375
9	0.010231587232710565	0.223948914061499360
10	0.009849585546666211	0.238708097425597860
11	0.009656606347153765	0.256707203545463756

TABLE 5. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.2 < m \leq 0.3$

j	B_j	D_j
0	0.812597772919920493	0.873152581892675550
1	0.121109617945510113	0.420622230667770216
2	0.057293376831239877	0.344231061559450379
3	0.038509451602167328	0.331133021818721762
4	0.030783430301775233	0.345277285052808412
5	0.027290564934732527	0.377945322150393392
6	0.025916369289445199	0.427378012464553881
7	0.025847203343361799	0.494671744307822406
8	0.026740923539348855	0.582685115665646201
9	0.028464314554825705	0.695799207728083165
10	0.030995446237278954	0.840018401472533403
11	0.034384369179940976	1.023268503573606061
12	0.038738002072493936	1.255859085136282496

TABLE 6. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.3 < m \leq 0.4$

j	B_j	D_j
0	0.825323557983515895	0.919027039242097348
1	0.133862116083687790	0.501002159288247514
2	0.071011293597988675	0.468831270566456863
3	0.054178477417387376	0.517714227776400015
4	0.049451744948102993	0.620843391317303107
5	0.050222196224107476	0.782364393786869723
6	0.054742913171830353	1.019114535076102913
7	0.062746257927001699	1.359345202748496052
8	0.074669881043476886	1.845717302358827942
9	0.091480845177733472	2.541071703153920729
10	0.114705092110997824	3.537404655208041337
11	0.146571132581439876	4.969296002977425930
12	0.190257137333846284	7.033822870030031126
13		10.02004322503447140

TABLE 7. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.4 < m \leq 0.5$

j	B_j	D_j
0	0.839479570270612971	0.974404366546369673
1	0.149916440306396336	0.613246805394160910
2	0.090831935819428835	0.671096669502166996
3	0.080347033483341786	0.870727620185086140
4	0.085638440500470454	1.229542231202690761
5	0.101954725932990372	1.826605967544420569
6	0.130574811533616015	2.806934530997762740
7	0.176105076358849928	4.418789329084028134
8	0.246835164402955447	7.083236057478765325
9	0.356424476867718855	11.51508812055758294
10	0.527002562230102743	18.93151118599927464
11	0.794389634259304750	31.41199693820496388
12	1.216762532429718021	52.52072945457582854
13		88.38485473506529806
14		149.5663744939804784
15		254.3179084310411743

TABLE 8. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.5 < m \leq 0.6$

j	B_j	D_j
0	0.855469615156419991	1.043455295115133534
1	0.170896072689739584	0.779625721928504850
2	0.121335229026948226	1.029742360932067582
3	0.128201883574947410	1.622037223411353130
4	0.164687281451527560	2.787989531185347620
5	0.237418908749381742	5.048381487372069147
6	0.369208104716495452	9.463277611943484295
7	0.605658733847927717	18.18148994942766790
8	1.033705561557812744	35.58098059117916870
9	1.818988489363267885	70.63393546191445013
10	3.279377651273850938	141.8285800834330593
11	6.029888380717536331	287.4487512501321663
12	11.26979685557794172	587.1153846499230762
13	21.35457785038283450	1207.065435225480616
14		2495.588727248664223
15		5184.692429394806441
16		10817.21333690413275

TABLE 9. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.6 < m \leq 0.7$

j	B_j	D_j
0	0.873920061848643136	1.133678336575733166
1	0.199814057482376946	1.048643173729970391
2	0.172769615878015213	1.753465041198464516
3	0.228106913284202167	3.523182726803385513
4	0.370468141118071220	7.749476413813974582
5	0.679271252884820555	17.98645005585073306
6	1.348008496681757302	43.25591634623261333
7	2.827670976853820704	106.6815344540960170
8	6.179468250123914084	268.0984865731174340
9	13.93568601034281150	683.6241148502898048
10	32.21892928105972203	1763.497085219187407
11	76.00696295922610103	4592.374753831163809
12	182.3214490877540696	12053.44101904888928
13	443.5150764411264816	31846.66302074208170
14	1091.854722902838829	84621.22135905680802
15	2715.765866403819588	225956.4231829078900
16		605941.5172817588600
17		1631082.599539268321

TABLE 10. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.7 < m \leq 0.8$

j	B_j	D_j
0	0.895902820924731621	1.260612826574911614
1	0.243140003766786662	1.548665638082676581
2	0.273081875594105532	3.553669411871607615
3	0.486280007533573324	9.900444676104398756
4	1.082747437228230918	30.32056661745247199
5	2.743445290986452500	98.18025865888308915
6	7.555817828670234627	329.7710104345570550
7	22.05194082493752427	1136.655989742890393
8	67.15640644740229408	3993.834335746229798
9	211.2722537881770962	14242.72958655527085
10	681.9037843053270682	51394.75729168872096
11	2246.956231592536517	187246.7029146231521
12	7531.483865999711792	687653.0923753899027
13	25608.51260130241579	2542385.535653982270
14	88140.74740089604971	9453781.219347490272
15	306564.4242098446591	35328363.01797091708
16	1076036.077811072194	132593262.3833930149
17	3807218.502573632648	499544968.1840548215
18	13566382.24422139551	1888409347.294438724
19		7160267534.478937192
20		27223307946.96339622

TABLE 11. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.8 < m \leq 0.85$

j	B_j	D_j
0	0.915922052601931494	1.402200569110579095
1	0.294714252429483394	2.322205897861749447
2	0.435776709264636140	7.462158366466719683
3	1.067328246493644239	29.43506890797307903
4	3.327844118563268085	128.1590924337895775
5	11.90406004445092906	591.0807036911982326
6	46.47838820224626394	2830.546229607726377
7	192.7556002578809477	13917.76431889392230
8	835.3356299261900064	69786.10525163921228
9	3743.124548343029103	355234.1420341879635
10	17219.07731004063094	1830019.186413931054
11	80904.60401669850158	9519610.812032515607
12	386808.3292751742460	49920868.75574849454
13	1876487.670110449342	263567700.9826023474
14	9216559.908641567755	1399645765.120061119
15		7469935792.837635005
16		40041555958.35610574
17		215463066814.4966654

TABLE 12. Coefficients of Taylor Expansion Polynomials of $B(m)$ and $D(m)$: $0.85 < m \leq 0.9$

j	B_j	D_j
0	0.931906061029524828	1.541690112721819084
1	0.348448029538453861	3.379176214579645449
2	0.666809178846938248	14.94058385670236672
3	2.210769135708128663	81.91773929235074881
4	9.491765048913406881	497.4900546551479866
5	47.09304791027740853	3205.184010234846235
6	255.9200460211233087	21457.32237355321926
7	1480.029532675805408	147557.0156564174712
8	8954.040904734313578	1035045.290185256525
9	56052.48220982686950	7371922.334832212125
10	360395.7241626000917	53143443.95142401142
11	2367539.415273216078	386882347.5795976313
12	15829949.57277684102	2839458401.528033778
13	107415809.3278511100	20982661229.43898942
14	738058546.0239595692	155961775401.7662418
15	5126022002.555101497	1165096220419.884791
16	35935340655.02416589	8742012983013.913805
17	253988125761.2812212	65847254626723.66919
18	1808180007145.359570	497679873706243.4393
19		3773018634056605.405
20		28682631948378196.60

TABLE 13. Polynomial Coefficients of Maclaurin Series Expansion of B^*. The terms up to $j = 5$ and $j = 13$ are enough for the single and double precision computations, respectively.

j	B^*_X	B^*_0
0	0	+1
1	−1/4	−1/4
2	−1/32	+3/64
3	−3/256	+3/128
4	−25/4096	+665/49152
5	−245/65536	+3437/393216
6	−1323/524288	+15981/2621440
7	−7623/4194304	+188287/41943040
8	−184041/134217728	+129334777/37580963840
9	−4601025/4294967296	+327273375/120259084288
10	−29548805/34359738368	+19096474969/8658654068736
11	−193947611/274877906944	+631505527133/346346162749440
12	−2591845347/4398046511104	+2224154230753/1451355348664320
13	−35156056117/70368744177664	+181962561086453/139330113471774720

TABLE 14. Polynomial Coefficients of Maclaurin Series Expansion of D^*. The terms up to $j = 5$ and $j = 12$ are enough for the single and double precision computations, respectively.

j	D^*_X	D^*_0
0	+1/2	−1
1	−1/8	0
2	−3/128	+5/128
3	−5/512	+31/1536
4	−175/32768	+2365/196608
5	−441/131072	+10409/1310720
6	−4851/2097152	+117929/20971520
7	−14157/8388608	+2458621/587202560
8	−2760615/2147483648	+194646309/60129542144
9	−8690825/8589934592	+5577961675/2164663517184
10	−112285459/137438953472	+363577654297/173173081374720
11	−370263621/549755813888	+632563423193/362838837166080
12	−19870814327/35184372088832	+102453646108723/69665056735887360

2.2. **Case of large parameter.** If $0.9 < m < 1$, we first evaluate two modifications of the integrals:

$$B^*(m) \equiv mB(m), \quad D^*(m) \equiv mD(m), \tag{2.2}$$

by using the definition of Jacobi's nome and Legendre's relation. Then we compute $B(m)$ and $D(m)$ as

$$B(m) = \frac{B^*(m)}{m}, \quad D(m) = \frac{D^*(m)}{m}. \tag{2.3}$$

Let us explain the details to compute $B^*(m)$ and $D^*(m)$. We begin with Jacobi's nome $q(m)$. It is defined as a function of m by way of $K(m)$ as

$$q(m) \equiv \exp\left(\frac{-\pi K(1-m)}{K(m)}\right). \tag{2.4}$$

Refer to BF1050.01. By solving this definition with respect to $K(1-m)$ and changing the variable from m to $m_c = 1 - m$, we obtain an expression of $K(m)$ in terms of $q(m_c)$ and $K(m_c)$ as

$$K(m) = \left(\frac{K(m_c)}{\pi}\right)\left[-\log q(m_c)\right]. \tag{2.5}$$

The Maclaurin series expansion of $q(m)$ is given as

$$q(m) = \frac{m}{16}\left(1 + \frac{m}{2} + \cdots\right). \tag{2.6}$$

Refer to BF901.00. Then we split the right-hand side of equation (2.5) into the sum of regular and logarithmically singular parts as

$$K(m) = K_0 + K_X X, \tag{2.7}$$

where

$$K_0 \equiv \frac{K(m_c)}{\pi}\left[-\log\left(\frac{16q(m_c)}{m_c}\right)\right], \quad K_X \equiv \frac{K(m_c)}{\pi}, \quad X \equiv -\log\left(\frac{m_c}{16}\right). \tag{2.8}$$

Both K_0 and K_X are regular around $m_c = 0$ as

$$K_0 \approx -\left(\frac{m_c}{4} + \frac{21m_c^2}{128}\right), \quad K_X \approx \frac{1}{2} + \frac{m_c}{8}. \tag{2.9}$$

Meanwhile Legendre's relation is a formula on $K(m)$, $K(m_c)$, $E(m)$, and $E(m_c)$. It is expressed as

$$E(m)K(m_c) + K(m)E(m_c) - K(m)K(m_c) = \frac{\pi}{2}. \tag{2.10}$$

Refer to BF110.10. We obtain an expression of $E(m)$ from this as

$$E(m) = \left[1 - \left(\frac{E(m_c)}{K(m_c)}\right)\right]K(m) + \frac{\pi}{2K(m_c)}. \tag{2.11}$$

Substitute the expression of $K(m)$ provided in equation (2.7) into this. Then we obtain a similar expression of $E(m)$ in terms of X as

$$E(m) = E_0 + E_X X, \tag{2.12}$$

where

$$E_0 \equiv \frac{\pi}{2K(m_c)} + \left[1 - \left(\frac{E(m_c)}{K(m_c)}\right)\right]K_0, \quad E_X \equiv \left[1 - \left(\frac{E(m_c)}{K(m_c)}\right)\right]K_X. \tag{2.13}$$

Again, both E_0 and E_X are regular around $m_c = 0$ as

$$E_0 \approx 1 - \frac{m_c}{4}, \quad E_X \approx \frac{m_c}{4} + \frac{3m_c^2}{32}. \tag{2.14}$$

The expressions of the complete integrals using the logarithm when $m \sim 1$ have been well known from early days. Refer to Article 78 of [9], though the way of derivation is quite different. Later Hastings adopted them as a base to obtain the efficient approximations of $K(m)$ and $E(m)$ in the early days of modern computers [20]. He used $Y \equiv -\log m_c$ as the singular variable as

$$K(m) = K_1 + K_X Y, \quad E(m) = E_1 + E_X Y. \tag{2.15}$$

Then he derived the Chebyshev polynomial approximation of K_X, E_X, and the new coefficients

$$K_1 \equiv K_0 + K_X \log 16, \quad E_1 \equiv E_0 + E_X \log 16, \tag{2.16}$$

in the domain $0 \le m_c < 1$. This is in order to construct a uniformly approximating formulation in low precision purposes [1]. This pioneer work was extended by Cody to the case of higher precisions [10, 11, 12]. Let us go further. We obtain similar expressions of $B^*(m)$ and $D^*(m)$ from equations (2.2), (2.7), and (2.12) as

$$B^*(m) = B_0^* + B_X^* X, \quad D^*(m) = D_0^* + D_X^* X, \tag{2.17}$$

where B_0^*, B_X^*, D_0^*, and D_X^* are defined in terms of K_0, K_X, E_0, and E_X as

$$B_0^* \equiv E_0 - m_c K_0, \quad B_X^* \equiv E_X - m_c K_X, \quad D_0^* \equiv K_0 - E_0, \quad D_X^* \equiv K_X - E_X. \tag{2.18}$$

They are all regular around $m_c = 0$ as

$$B_0^* \approx 1 - \frac{m_c}{4}, \quad B_X^* \approx -\left(\frac{m_c}{4} + \frac{m_c^2}{32}\right), \tag{2.19}$$

$$D_0^* \approx -\left(1 - \frac{51m_c^2}{128}\right), \quad D_X^* \approx \frac{1}{2} - \frac{m_c}{8}. \tag{2.20}$$

Some low order coefficients of these approximate polynomials are listed in Tables 13 and 14. They are directly obtained by using Mathematica [22] with a command such as

$$\texttt{Series}[(\texttt{EllipticE}[\texttt{x}] - \texttt{x}\,\texttt{EllipticK}[\texttt{x}])/\texttt{Pi}, \{\texttt{x}, 0, 20\}]$$

This gives the coefficients of B_X^* in Table 13. Numerical comparison with the original definitions reveal that the necessary minimum order of the polynomials is 5 and 13 or 12 in the single and the double precision environments, respectively. Let us summarize the procedure when $0.9 < m < 1$. We first compute B_0^*, B_X^*, D_0^*, and D_X^* by their approximate polynomials in terms of m_c, next evaluate X by calling a logarithm function once, then calculate $B^*(m)$ and $D^*(m)$, and finally obtain $B(m)$ and $D(m)$ by a simple division by m.

2.3. **Cost and performance.** We will compare the computational cost and performance of the new method with the existing ones. Let us examine the computing precision first. Figures 3 and 4 already illustrate the comparison of relative errors of $B(m)$ and $D(m)$, respectively. Obviously, the new method is the most precise. We prepared Figure 5 in order to see the error distribution of the new method integral by integral. The figure shows that the relative errors of all four complete elliptic integrals are less than the machine epsilon in the single precision environment. A

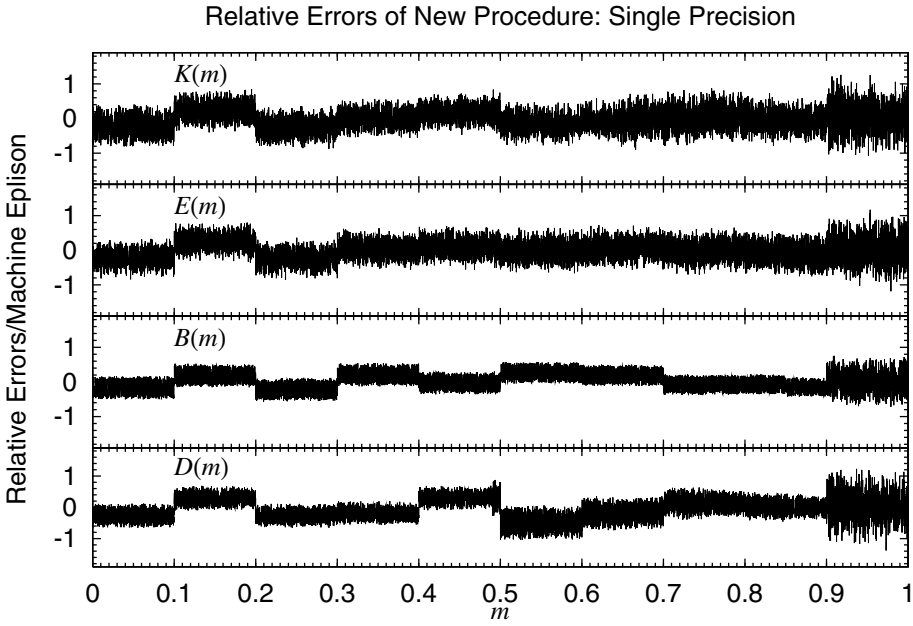

FIGURE 5. Relative Errors of Four Complete Elliptic Integrals Computed by New Procedure in Single Precision.

similar plot confirms that the maximum relative errors in the double precision environment are around 3 machine epsilons in the region $0.9 < m < 1$. The comparison in computational speed is already given in Table 1. The new method calculates two polynomials of the orders 5 to 10 and 11 to 20 in the single and double precision environment, respectively. When $m > 0.9$, it calls one logarithm and one division in addition. Thus the computational labor is much smaller than those of `cel2`, R_F, or R_D. This was easily seen in Table 1.

3. Conclusion

By adopting the same approach when we developed a fast method to compute $K(m)$ and/or $E(m)$ in Paper I [17], we created a new method to calculate the auxiliary complete elliptic integrals $B(m)$ and $D(m)$. The core technique is the combination of the Taylor series expansions of the integrals, the definition of Jacobi's nome, and Legendre's relation. The new method is significantly more precise than Bulirsch's `cel2` and Carlson's R_F and R_D in the sense that the magnitude of relative errors is less than 1-3 machine epsilons. This is mainly due to the simplicity of the algorithm and the small number of arithmetic operations required. The new method needs the evaluation of one or two polynomials of the order 5-20 by Horner's method and a call of the logarithm function provided by a standard mathematical library when $m > 0.9$. Thanks to the effectiveness of the policy of divide-and-rule, the new procedure is drastically faster than the existing procedures. It runs 4.8 times faster than `cel2` and 16.8 times faster than the pair of R_F and R_D in the double precision computation. These acceleration factors change as 5.6 and 16.4 in the

single precision environment. The obtained procedure will be useful in developing an efficient method to compute the auxiliary incomplete elliptic integrals $B(\varphi|m)$ and $D(\varphi|m)$. Finally, we caution the readers that the new method is tailor-made and *not* scalable. All its parameters including the choice of subdomains, the orders of truncated polynomials, and the Taylor series coefficients have to be adapted for specific numeric processors with a higher precision as the extended precision expressed as `real*10` in terms of Fortran or for higher precision environments as that of quadruple (`real*16`) precision. Here we presented the results only for the single (`real*4`) and the double (`real*8`) precision environments. Also, the new procedure assumes the availability of a mathematical library of the logarithmic function. Meanwhile Bulirsch's `cel2` is scalable. One has only to prepare an approximation of π constant. Also, it requires only the square root function apart from standard arithmetics operations. The Fortran routine of the new procedure `elbd` is available from the author upon request.

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