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UPPER BOUNDS FOR RESIDUES OF DEDEKIND ZETA FUNCTIONS AND CLASS NUMBERS OF CUBIC AND QUARTIC NUMBER FIELDS

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ABSTRACT. Let K be an algebraic number field. Assume that $\zeta_K(s)/\zeta(s)$ is entire. We give an explicit upper bound for the residue at $s = 1$ of the Dedekind zeta function $\zeta_K(s)$ of K. We deduce explicit upper bounds on class numbers of cubic and quartic number fields.

1. INTRODUCTION

Let K be an algebraic number field of degree $m = r_1 + 2r_2 > 1$, where r_1 is the number of real places of K and r_2 is the number of complex places of K. Let κ_K be the residue at $s = 1$ of the Dedekind zeta function function $\zeta_K(s)$ of K. Let d_K be the absolute value of the discriminant of K. Let h_K be its class number. Then (see [Lan, Chapter XIII, Section 3, Theorem 2]):

$$h_K = \frac{w_K\sqrt{d_K}}{2^{r_1}(2\pi)^{r_2}\mathrm{Reg}_K}\kappa_K, \tag{1}$$

where $w_K \geq 2$ is the number of complex roots of unity in K and Reg_K is the regulator of K. To get upper bounds on h_K we need lower bounds on Reg_K (e.g., see [Sil]) and upper bounds on κ_K (e.g., see [Lou00]). If K is a real quadratic number field, then

$$h_K \leq \frac{1}{2}\sqrt{d_K} \tag{2}$$

([Le] and [Ram, Corollary 2]); if K is a real cyclic cubic number field, then

$$h_K \leq \frac{2}{3}\sqrt{d_K} \tag{3}$$

(see [MP], and use [Lou93] instead of [MP, Lemme 3.2] to obtain that this bound is valid for real cyclic cubic number fields of not necessarily prime discriminants). With $e = \exp(1)$, it is known that

$$\kappa_K \leq \left(\frac{e\log d_K}{2(m-1)}\right)^{m-1} \tag{4}$$

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([Lou00, Theorem 1] and [Lou01, Theorem 1]). If K is **abelian**, we have a better bound:

$$\kappa_K \leq \left(\frac{\log d_K + m\lambda_K}{2(m-1)}\right)^{m-1}, \tag{5}$$

where $\lambda_K = 0$ if K is real and $\lambda_K = 5/2 - \log 6$ if K is imaginary (use [Ram, Corollary 1] and notice that if K is imaginary, then $m/2$ of the m characters in the group of primitive Dirichlet characters associated with K are odd). For some **totally real** number fields, an improvement on (4) is known (see [Lou01, Theorem 2]): if K ranges over a family of totally real number fields of a given degree $m > 1$ for which $\zeta_K(s)/\zeta(s)$ is entire, there exists C_m (computable) such that $d_K \geq C_m$ implies

$$\kappa_K \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!} \leq \frac{1}{\sqrt{2\pi(m-1)}}\left(\frac{e\log d_K}{2(m-1)}\right)^{m-1}. \tag{6}$$

It is known that $\zeta_K(s)/\zeta(s)$ is entire if K is normal (see [MM, Chapter 2, Theorem 3]), or if the Galois group of its normal closure is solvable (see [Uch], [vdW] and [MM, Chapter 2, Corollary 4.2]), e.g., for any cubic or quartic number field. This paper generalizes (6) to not necessarily totally real number fields:

Theorem 1. *Let r_1 and r_2 be given, with $r_1+2r_2 \geq 3$. There exists d_{r_1,r_2} effectively computable such that for any number field K of degree $m = r_1 + 2r_2$ with r_1 real places and r_2 complex places, we have*

$$\kappa_K \leq \frac{\log^{m-1} d_K}{2^{m-1}(m-1)!}, \tag{7}$$

provided that (i) $d_K \geq d_{r_1,r_2}$ and (ii) that $\zeta_K(s)/\zeta(s)$ is entire.

For given r_1 and r_2, we will explain how to use any mathematical software, we use Maple, to compute such a d_{r_1,r_2}. It appears that for the small values of $r_1 + 2r_2 = m$, say for $3 \leq m \leq 6$, this bound (7) holds true with no restriction on the size of d_K (in fact, we have an even better bound, see Theorem 3), the reason being that these computed d_{r_1,r_2}'s are less than or equal to the least discriminants of number fields of degree $m = r_1 + 2r_2 \leq 6$ with r_1 real places and r_2 complex places. However, even in the simplest situation where we assume that K is totally real, we could not in [Lou05] obtain beforehand a $C > 0$ such that (7) holds true for K's of root-discriminants $\rho_K = d_K^{1/m}$ greater than C.

Set

$$\gamma = \lim_{m\to\infty}\Big(\sum_{k=1}^{m}\frac{1}{k} - \log m\Big) = 0.57721\cdots$$

(Euler's constant) and

$$\lambda_{r_2,m} = 2 + r_2\log 4 - (m-1)(\log(4\pi) - \gamma).$$

Since $\lambda_{r_2,m} < 0$ for $m \geq 3$, Theorem 1 follows from the bound

$$\kappa_K \leq \frac{\big(\log d_K + \lambda_{r_2,m}\big)^{m-1}}{2^{m-1}(m-1)!} + O_{r_2,m}(\log^{m-3} d_K), \tag{8}$$

where the implied constants are effective and depend on r_2 and m only. To prove (8), we generalize the method introduced in [Lou96]. Set

$$\gamma(1) = \lim_{m\to\infty}\Big(\sum_{k=1}^{m}\frac{\log k}{k} - \frac{1}{2}\log^2 m\Big) = -0.07281\cdots$$

and

$$\mu_{r_2,m} = 3 + r_2\pi^2/12 - (m-1)(\pi^2/8 - \gamma^2 - 2\gamma(1)).$$

The error term in (8) is less than or equal to zero if $\mu_{r_2,m} > 0$ and d_K is large enough. Now, $\mu_{r_2,m} > 0$ if and only if we are in one of the following cases:

Table 1

m	r_2	$\lambda_{r_2,m}$	$\mu_{r_2,m}$	$d_K \geq$
2	0	$2+\gamma-\log(4\pi) = 0.04619\cdots$	$1.95384\cdots$	3
2	1	$2+\gamma-\log\pi = 1.43248\cdots$	$2.77631\cdots$	3
3	0	$2+2\gamma-2\log(4\pi) = -1.90761\cdots$	$0.90769\cdots$	146
3	1	$2+2\gamma-2\log(2\pi) = -0.52132\cdots$	$1.73015\cdots$	4
4	1	$2+3\gamma-\log(16\pi^3) = -2.47513\cdots$	$0.68400\cdots$	75100
4	2	$2+3\gamma-\log(4\pi^3) = -1.08883\cdots$	$1.50647\cdots$	35
5	2	$2+4\gamma-4\log(2\pi) = -3.04264\cdots$	$0.46031\cdots$	$21\cdot 10^{10}$
6	3	$2+5\gamma-\log(16\pi^5) = -3.61015\cdots$	$0.23662\cdots$	$21\cdot 10^{31}$

It will follow that we have a pleasingly explicit bound:

Theorem 2. *Assume that we are in one of the eight cases of Table* 1*. Then,*

$$\kappa_K \leq \frac{\big(\log d_K + \lambda_{r_2,m}\big)^{m-1}}{2^{m-1}(m-1)!},$$

provided that d_K is large enough, as given in the last column of Table 1*.*

The results in [Lou93] and [Lou96] are the case $m = 2$ of Theorem 2 above. (However, in the quadratic case we have an even better bound (see [Ram]).) Finally, by taking constants slightly less than these $\lambda_{r_2,m}$, we have a the fully explicit following result where we do not have any restriction on d_K (compare with Theorem 1):

Theorem 3. *Let K be a number field of degree $m \in \{2,3,4,5,6\}$ for which $\zeta_K(s)/\zeta(s)$ is entire. Then,*

$$\kappa_K \leq \frac{(\log d_K + \lambda)^{m-1}}{2^{m-1}(m-1)!},$$

where λ is as in Table 2*:*

TABLE 2

m	$r_2 = 0$	$r_2 = 1$	$r_2 = 2$	$r_2 = 3$
2	0.04620	1.43249		
3	−1.74865	−0.52132		
4	−2.94863	−2.07896	−1.08883	
5	−4.21779	−3.29415	−2.41877	
6	−5.49315	−4.55901	−3.64104	−2.76490

Corollary 4. *If K is a totally real cubic number field, then*

$$h_K \leq \frac{1}{2}\sqrt{d_K}. \tag{9}$$

If K is a totally real quartic number field which contains no quadratic subfield, then

$$h_K \leq \frac{5\sqrt{10}}{24}\sqrt{d_K}. \tag{10}$$

We refer to [Dai] for examples of number fields with very large class numbers.

2. Proof of the bound (8)

We adapt [Lou00, Proof of Theorem 7]. Let K be a number field of degree $m = r_1 + 2r_2 > 1$. Assume that $\zeta_K(s)/\zeta(s)$ is entire. Set $A_{K/\mathbf{Q}} = \sqrt{d_K/4^{r_2}\pi^{m-1}}$,

$$\Gamma_{K/\mathbf{Q}}(s) = \Gamma^{r_1-1}(s/2)\Gamma^{r_2}(s) = \frac{2^{r_2(s-1)}}{\pi^{r_2/2}}\Gamma^{r_1+r_2-1}(s/2)\Gamma^{r_2}((s+1)/2)$$

(notice that $r_1 + r_2 - 1 \geq 0$ and $r_2 \geq 0$) and

$$F_{K/\mathbf{Q}}(s) = A_{K/\mathbf{Q}}^s \Gamma_{K/\mathbf{Q}}(s)(\zeta_K(s)/\zeta(s)).$$

Then, $F_{K/\mathbf{Q}}(s)$ is entire and $F_{K/\mathbf{Q}}(s) = F_{K/\mathbf{Q}}(1-s)$. Let

$$S_{K/\mathbf{Q}}(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_{K/\mathbf{Q}}(s)x^{-s}\mathrm{d}s \quad (c > 1 \text{ and } x > 0) \tag{11}$$

denote the inverse Mellin transform of $F_{K/\mathbf{Q}}(s)$. Then,

$$S_{K/\mathbf{Q}}(x) = \frac{1}{x}S_{K/\mathbf{Q}}(\frac{1}{x}) \tag{12}$$

(notice that $F_{K/\mathbf{Q}}(s)$ is entire, shift the vertical line of integration $\Re(s) = c > 1$ in (11) leftwards to the vertical line of integration $\Re(s) = 1 - c < 0$, then use the functional equation $F_{K/\mathbf{Q}}(1-s) = F_{K/\mathbf{Q}}(s)$ to come back to the vertical line of integration $\Re(s) = c > 1$). For $\Re(s) > 1$,

$$F_{K/\mathbf{Q}}(s) = \int_0^\infty S_{K/\mathbf{Q}}(x)x^s\frac{\mathrm{d}x}{x}$$

is the Mellin transform of $S_{K/\mathbf{Q}}(x)$. Using (12), we obtain

$$F_{K/\mathbf{Q}}(s) = \int_1^\infty S_{K/\mathbf{Q}}(x)(x^s + x^{1-s})\frac{\mathrm{d}x}{x} \tag{13}$$

on the whole complex plane. Now, write $\zeta_K(s)/\zeta(s) = \sum_{n\geq 1} a_{K/\mathbf{Q}}(n)n^{-s}$ and $\zeta^{m-1}(s) = \sum_{n\geq 1} a_{m-1}(n)n^{-s}$ ($\Re(s) > 1$). Then, $|a_{K/\mathbf{Q}}(n)| \leq a_{m-1}(n)$ (see [Lou01, (55)]) and

$$S_{K/\mathbf{Q}}(x) = \sum_{n\geq 1} a_{K/\mathbf{Q}}(n)H_{K/\mathbf{Q}}(nx/A_{K/\mathbf{Q}}),$$

where

$$H_{K/\mathbf{Q}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma_{K/\mathbf{Q}}(s)x^{-s}\mathrm{d}s.$$

Since $H_{K/\mathbf{Q}}(x) > 0$ for $x > 0$ (see [Lou01, Theorem 20]) [1]), we have

$$|S_{K/\mathbf{Q}}(x)| \leq \sum_{n\geq 1} a_{m-1}(n)H_{K/\mathbf{Q}}(nx/A_{K/\mathbf{Q}}).$$

Plugging this into (13), we obtain

$$\begin{aligned}
\frac{\sqrt{d_K}}{(2\pi)^{r_2}}\kappa_K = F_{K/\mathbf{Q}}(1) &= \int_1^\infty S_{K/\mathbf{Q}}(x)(1+1/x)\mathrm{d}x \\
&\leq \sum_{n\geq 1} a_{m-1}(n)\int_1^\infty H_{K/\mathbf{Q}}(nx/A_{K/\mathbf{Q}})(1+1/x)\mathrm{d}x \\
&= \sum_{n\geq 1} a_{m-1}(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\int_1^\infty (nx/A_{K/\mathbf{Q}})^{-s}(1+1/x)\mathrm{d}x\right)\Gamma_{K/\mathbf{Q}}(s)\mathrm{d}s \\
&= \sum_{n\geq 1} a_{m-1}(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{1}{s-1}+\frac{1}{s}\right)\Gamma_{K/\mathbf{Q}}(s)(n/A_{K/\mathbf{Q}})^{-s}\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{1}{s-1}+\frac{1}{s}\right)\Gamma_{K/\mathbf{Q}}(s)\zeta^{m-1}(s)A_{K/\mathbf{Q}}^s\mathrm{d}s.
\end{aligned}$$

Therefore, we have

$$\kappa_K \leq I_K(s) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f_K(s)\mathrm{d}s \quad (c>1), \tag{14}$$

where

$$f_K(s) = \tilde{\Gamma}^{r_2}(s)\Lambda^{m-1}(s)\left(\frac{1}{s-1}+\frac{1}{s}\right)d_K^{(s-1)/2},$$

$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\tilde{\Gamma}(s) = \Gamma((s+1)/2)/\big(\Gamma(s/2)/\Gamma(1/2)\big)$. Recall that $\Lambda(s)$ has only two poles, both simple, at $s=1$ and $s=0$, and satisfies the functional equation $\Lambda(s) = \Lambda(1-s)$. Moreover, $1/\Gamma(s/2)$ is entire whereas $\Gamma((s+1)/2)$ has a simple pole at each odd negative integer. It follows that $f_K(s)$ has a pole of order $m>1$ at $s=1$, a pole of order $m-r_2 = r_1+r_2 \geq 1$ at $s=0$, and a pole of order $r_2 \geq 0$ at each negative odd integer. Now, as in [Lou01, Page 1207], in the range $\sigma_1 \leq \sigma \leq \sigma_2$ and $|t| \geq 1$, we have $\tilde{\Gamma}(\sigma+it) = O(\sqrt{|t|})$ and there exists $M \geq 0$ such that $\Lambda(\sigma+it) = O(|t|^M e^{-\pi|t|/4})$. Hence, we are allowed to shift in (14) the

[1]Notice the misprints in [Lou00, page 273, line 1] and [Lou01, Theorem 20] where one should read

$$(M_1 \star M_2)(x) = \int_0^\infty M_1(x/t)M_2(t)\frac{\mathrm{d}t}{t}.$$

vertical line of integration $\Re(s)=c>1$ leftwards to the vertical line of integration $\Re(s)=1/2$. We pick up one residue and obtain:

$$\kappa_K \le \mathrm{Res}_{s=1}(f_K(s)) + I_K(1/2) = \mathrm{Res}_{s=1}(f_K(s)) + O_{r_2,m}(d_K^{-1/4}). \tag{15}$$

The bound (8) now follows from Lemma 5 below.

3. Computation of some residues

To prove Theorems 2 and 3, we need a better approximation to $I_K(s)$. By shifting in (14) the vertical line of integration $\Re(s)=c>1$ leftwards to the vertical line of integration $\Re(s)=-2$, we pick up three residues and we obtain:

$$\kappa_K \le \mathrm{Res}_{s=1}(f_K(s)) + \mathrm{Res}_{s=0}(f_K(s)) + \mathrm{Res}_{s=-1}(f_K(s)) + I_K(-2), \tag{16}$$

where $\mathrm{Res}_{s=1}(f_K(s))$ is a polynomial of degree $m-1$ in $\log d_K$ with real coefficients, $\sqrt{d_K}\mathrm{Res}_{s=0}(f_K(s))$ is a polynomial of degree r_1+r_2-1 in $\log d_K$ with real coefficients, and $d_K\mathrm{Res}_{s=-1}(f_K(s))$ is a polynomial of degree r_2-1 in $\log d_K$ with real coefficients. This section is devoted to computing these residues.

Lemma 5. *Set*

$$f_{k,l}(s) = \tilde\Gamma^k(s)\Lambda^l(s)\left(\frac{1}{s-1}+\frac{1}{s}\right)e^{(s-1)X}.$$

Then, $\mathrm{Res}_{s=1}(f_{k,l}(s))$ *is a polynomial of degree l in X with real coefficients and*

$$\mathrm{Res}_{s=1}(f_{k,l}(s)) = \frac{\big(X+A_{k,l}\big)^l}{l!} \quad (l=1),$$
$$\mathrm{Res}_{s=1}(f_{k,l}(s)) = \frac{\big(X+A_{k,l}\big)^l}{l!} - C_{k,l}\frac{X^{l-2}}{(l-2)!} \quad (l=2)$$

and

$$\mathrm{Res}_{s=1}(f_{k,l}(s)) = \frac{\big(X+A_{k,l}\big)^l}{l!} - C_{k,l}\frac{X^{l-2}}{(l-2)!} + O_{k,l}(X^{l-3}) \quad (l\ge 3),$$

where

$$A_{k,l} = \big(2+k\log 4 - l(\log(4\pi)-\gamma)\big)/2$$

and

$$C_{k,l} = \big(3+k\pi^2/12 - l(\pi^2/8-\gamma^2-2\gamma(1))\big)/2.$$

Proof. We have $\tilde\Gamma(s)=1+a(s-1)+b(s-1)^2+O((s-1)^3)$, with $a=\log 2$ and $b=(\log^2 2-\pi^2/12)/2$, and $(s-1)\Lambda(s)=1+c(s-1)+d(s-1)^2+O((s-1)^3)$, with

$$c=-\frac{\log(4\pi)-\gamma}{2} \text{ and } d=\frac{2(\log(4\pi)-\gamma)^2+\pi^2-8\gamma^2-16\gamma(1)}{16}. \tag{17}$$

For $k\ge 0$ and $l\ge 0$, it holds that

$$\begin{aligned}&\big(1+az+bz^2+O(z^3)\big)^k\big(1+cz+dz^2+O(z^3)\big)^l\big(1+z-z^2+O(z^3)\big)\\ &\qquad = 1+A_{k,l}z+B_{k,l}z^2+O(z^3),\end{aligned}$$

where $A_{k,l}=ka+lc+1$ and $B_{k,l}=klac+k\big(b+\frac{k-1}{2}a^2\big)+l\big(d+\frac{l-1}{2}c^2\big)+ka+lc-1$.

Hence, the desired results hold true with

$$C_{k,l} = A_{k,l}^2/2 - B_{k,l} = \big(k(a^2-2b) + l(c^2-2d) + 3\big)/2.$$

We have $a^2 - 2b = \pi^2/12$ and $c^2 - 2d = \gamma^2 + 2\gamma(1) - \pi^2/8$. □

Lemma 6. *Set $r = l - k$, let $f_{k,l}(s)$ and $A_{k,l}$ be as in Lemma 5, and set*

$$C'_{k,l} = \big(3 - k\pi^2/12 - l(\pi^2/8 - \gamma^2 - 2\gamma(1))\big)/2.$$

If $r = 0$ or $r = 1$, then

$$\mathrm{Res}_{s=0}(f_{k,l}(s)) = (-1)^l(\pi/2)^k \frac{\big(X - A_{k,l}\big)^r}{r!} e^{-X}.$$

If $r = 2$, then

$$\mathrm{Res}_{s=0}(f_{k,l}(s)) = (-1)^l(\pi/2)^k \left(\frac{\big(X - A_{k,l}\big)^r}{r!} - C'_{k,l}\frac{X^{r-2}}{(r-2)!}\right) e^{-X}.$$

If $r \geq 3$, then

$$\mathrm{Res}_{s=0}(f_{k,l}(s)) = (-1)^l(\pi/2)^k \left(\frac{\big(X - A_{k,l}\big)^r}{r!} - C'_{k,l}\frac{X^{r-2}}{(r-2)!} + O_{k,l}(X^{r-3})\right) e^{-X}.$$

Proof. Here, $\tilde{\Gamma}(s) = \frac{\pi s}{2}\big(1 - as + bs^2 + O(s^3)\big)$, with $a = \log 2$ and $b = (\log^2 2 + \pi^2/12)/2$, and $s\Lambda(s) = -\big(1 - cs + ds^2 + O(s^3)\big)$, with c and d as in (17). □

Lemma 7. *Let $f_{k,l}(s)$ be as in Lemma 5. We have*

$$\mathrm{Res}_{s=-1}(f_{1,l}(s)) = \frac{3}{2}\big(\frac{\pi}{6}\big)^l e^{-2X}.$$

Lemma 8. *It holds that*

$$|I_K(-2)| \leq \frac{5}{4\pi^2}\frac{\Gamma(r_2/2+1)}{3^m}\left(\frac{14}{m-1}\right)^{r_2/2+1} d_K^{-3/2}.$$

Proof. Using

$$|\tilde{\Gamma}(-2+it)| = \left(\frac{4+t^2}{1+t^2}\frac{\pi t}{2}\tanh(\frac{\pi t}{2})\right)^{1/2} \leq \sqrt{2\pi|t|}$$

and

$$|\Lambda(-2+it)| = |\Lambda(3-it)| \leq \frac{\zeta(3)}{\pi^{3/2}}|\Gamma((3-it)/2)| = \frac{\zeta(3)}{2\pi}\sqrt{\frac{1+t^2}{\cosh(\pi t/2)}} \leq \frac{1}{3}e^{-\pi|t|/7},$$

we obtain:

$$\begin{aligned} d_K^{3/2}|I_K(-2)| &\leq \frac{5}{6\pi}\int_0^\infty |\tilde{\Gamma}(-2+it)|^{r_2}|\Lambda(-2+it)|^{m-1}\mathrm{d}t \\ &\leq \frac{5}{2\pi 3^m}\int_0^\infty (2\pi t)^{r_2/2}e^{-\pi(m-1)t/7}\mathrm{d}t, \end{aligned}$$

and the desired bound. □

Table 3. Minimal discriminants

m	$r_2=0$	$r_2=1$	$r_2=2$	$r_2=3$
2	5	3		
3	49	23		
4	725	275	117	
5	14641	4511	1609	
6	300125	92779	28037	9747

4. Proof of Theorems 2 and 3, and contents of Tables 1 and 2

We use (16), the previous lemmas and Table 3 above (see [Odl]).

1. If K is a real quadratic field, then

$$\kappa_K \leq \frac{\log d_K + 2 + \gamma - \log(4\pi)}{2} - \frac{\log d_K - (2+\gamma-\log(4\pi))}{2\sqrt{d_K}} + \frac{35}{18\pi^2 d_K^{3/2}}$$

is less than or equal to $\bigl(\log d_K + 2 + \gamma - \log(4\pi)\bigr)/2$ for $d_K \geq 3$.

2. If K is an imaginary quadratic field, then

$$\kappa_K \leq \frac{\log d_K + 2 + \gamma - \log\pi}{2} - \frac{\pi}{2\sqrt{d_K}} + \frac{\pi}{4d_K} + \frac{35\sqrt{14\pi}}{36\pi^2 d_K^{3/2}}$$

is less than or equal to $\bigl(\log d_K + 2 + \gamma - \log\pi\bigr)/2$ for $d_K \geq 3$.

3. If K is a totally real cubic number field, then

$$\begin{aligned}\kappa_K \leq\ & \frac{\bigl(\log d_K + 2 + 2\gamma - 2\log(4\pi)\bigr)^2}{8} - (3/2+\gamma^2+2\gamma(1)-\pi^2/8)\\ &+\frac{\bigl(\log d_K - (2+2\gamma-2\log(4\pi))\bigr)^2}{8\sqrt{d_K}} - \frac{3/2+\gamma^2+2\gamma(1)-\pi^2/8}{\sqrt{d_K}} + \frac{35}{108\pi^2 d_K^{3/2}}\end{aligned}$$

is less than or equal to $\bigl(\log d_K + 2 + 2\gamma - 2\log(4\pi)\bigr)^2/8$ for $d_K \geq 146$, and less than or equal to $\bigl(\log d_K - 1.74865)^2/8$ for $d_K \geq 49$.

4. If K is a not totally real cubic number field, then

$$\begin{aligned}\kappa_K \quad \leq \quad & \frac{\bigl(\log d_K + 2 + 2\gamma - 2\log(2\pi)\bigr)^2}{8} - (3/2+\gamma^2+2\gamma(1)-\pi^2/12)\\ &+\frac{\pi}{4\sqrt{d_K}}\bigl(\log d_K - 2 - 2\gamma + 2\log(2\pi)\bigr) + \frac{\pi^2}{24 d_K} + \frac{35\sqrt{7\pi}}{216\pi^2 d_K^{3/2}}\end{aligned}$$

is less than or equal to $\bigl(\log d_K + 2 + 2\gamma - 2\log(2\pi)\bigr)^2/8$ for $d_K \geq 4$.

5. The other cases are easily dealt with by using any software for symbolic computation, e.g., Maple, to compute the residues which appear in (16).

5. Proof of Corollary 4

1. Let K be a totally real cubic field. Then, $\mathrm{Reg}_K \geq \frac{1}{16}\log^2(d_K/4)$ (see [Cus, Theorem 1] or [Nak, Section 2.3]). Hence,

$$h_K = \frac{\sqrt{d_K}}{4\mathrm{Reg}_K}\kappa_K \leq \frac{(\log d_K - 1.74865)^2}{2\log^2(d_K/4)}\sqrt{d_K} \leq \frac{1}{2}\sqrt{d_K}.$$

2. Let K be a totally real quartic number field which contains no real quadratic subfield. Then, $\mathrm{Reg}_K \geq \frac{1}{80\sqrt{10}}\log^3 d_K$ (see [Cus, Theorem 2]). By (7) (see also [Lou01, Theorem 2, point 3]), we have $\kappa_K \leq \frac{1}{48}\log^3 d_K$. Hence, by (1), we obtain

$$h_K = \frac{\sqrt{d_K}}{8\mathrm{Reg}_K}\kappa_K \leq \frac{5\sqrt{10}}{24}\sqrt{d_K}.$$

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