MATHEMATICS OF COMPUTATION
Volume 65, Number 216
October 1996, Pages 1613–1619

ON THE ZEROS OF THE RAMANUJAN τ-DIRICHLET SERIES IN THE CRITICAL STRIP

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ABSTRACT. We describe computations which show that each of the first 12069 zeros of the Ramanujan τ-Dirichlet series of the form $\sigma + it$ in the region $0 < t < 6397$ is simple and lies on the line $\sigma = 6$. The failures of Gram's law in this region are also noted. The first 5018 zeros and 2228 successive zeros beginning with the 20001st zero are also calculated. The distribution of the normalized spacing of the zeros is examined and it appears to be that of the eigenvalues of random matrices. These comptuations are done with a Berry-Keating formula for the τ-Dirichlet series and evaluated using *Mathematica*™.

1. INTRODUCTION

The Ramanujan τ-function [5] is defined in terms of its generating function

$$g(z) = z\prod_{k=1}^{\infty}(1-z^k)^{24} = \sum_{n=1}^{\infty}\tau_n z^n. \tag{1}$$

We consider the associated Dirichlet series

$$f(s) = \sum_{n=1}^{\infty}\tau_n n^{-s}, \tag{2}$$

which is also given by the integral

$$f(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1} g(e^{-x})\,dx. \tag{3}$$

This Dirichlet series has been studied by several authors, notably [4, 9, 10]. We have the functional equations

$$z^6 g(e^{-2\pi z}) = (\frac{1}{z})^6 g(e^{-2\pi\frac{1}{z}}) \tag{4}$$

and

$$(2\pi)^s\Gamma(6-s)f(6-s) = (2\pi)^{-s}\Gamma(6+s)f(6+s). \tag{5}$$

Received by the editor August 26, 1991 and, in revised form, January 8, 1993 and January 10, 1995.

1991 *Mathematics Subject Classification.* Primary 11M41, 65A05.

Key words and phrases. Riemann hypothesis, L-functions, Ramanujan conjecture, Ramanujan τ-Dirichlet series.

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As with the Riemann ζ-function, we can use the functional equation for f to split f into the product of two functions

$$f(6+it) = Z(t)e^{-i\vartheta(t)}, \tag{6}$$

where

$$Z(t) = \Gamma(6+it)f(6+it)(2\pi)^{-it}\sqrt{\frac{\sinh(\pi t)}{\pi t(1+t^2)(4+t^2)(9+t^2)(16+t^2)(25+t^2)}}, \tag{7}$$

$$\vartheta(t) = \frac{-i}{2}\log\frac{\Gamma(6+it)}{\Gamma(6-it)} - t\log(2\pi), \tag{8}$$

and the branch of the logarithm in the formula for $\vartheta(t)$ is chosen so that $\vartheta(0) = 0$ and $\vartheta(t)$ is continuous for real t. The functions Z and ϑ are even and odd, respectively. Moreover, we have asymptotically for large t, $|\arg t| < \pi/2$,

$$\begin{aligned}\vartheta(t) \quad &= \quad t\log\frac{t}{2\pi} - t + \frac{11\pi}{4} - \frac{181}{12t} + \frac{26999}{360t^3} - \frac{1115101}{1260t^5} + \frac{23237999}{1680t^7} \\ &\quad -\frac{295081381}{1188t^9} + \frac{1742885234309}{360360t^{11}} - \frac{15472974061}{156t^{13}} + \cdots .\end{aligned} \tag{9}$$

Now, just as in the case of the Riemann ζ-function, because $Z(t)$ is real for real t (which corresponds to the critical line), we can search for zeros on the critical line by finding sign changes in $Z(t)$. Moreover, we have that the number of zeros in the critical strip $\sigma + it$ where $5.5 < \sigma < 6.5$ and $0 < t < T$ is given by

$$N(T) = \frac{1}{\pi}(\vartheta(T) + \operatorname{Im}\log f(6+iT)), \tag{10}$$

where the branch of $\log f(6+iT)$ is chosen so as to make $\log f$ continuous along the polygonal path from 12 to $12+iT$ to $6+iT$. Thus we can count the number of zeros in the critical strip and, because $N(T)$ must be an integer, we have a second check on the errors in $\vartheta(T)$ and $f(6+iT)$.

As with the ζ-function, we define the Gram points g_n to be the solutions to

$$\vartheta(g_n) = n\pi, \tag{11}$$

where n is an integer. "Gram's law", which says that the sign of $Z(g_n)$ should be $(-1)^n$, works nearly all of the time. Thus, finding sign changes costs on average only slightly more than a single evaluation of Z. A Gram point g_n is called "good" if $(-1)^n Z(g_n) > 0$, otherwise it is called "bad". A Gram block of length k is an interval $g_n < t < g_{n+k}$ where g_n and g_{n+k} are good Gram points, but $g_{n+1}, \ldots, g_{n+k-1}$ are bad Gram points. "Rosser's rule" says that each Gram block of length k contains k zeros.

2. Evaluation of the Dirichlet series

Neither the integral expression for $f(s)$, (3), nor a formula based on this expression but expressed in terms of the incomplete gamma function are effective for calculating $f(s)$ far up in the critical strip. The problem is severe cancellation of digits: (empirically) about $t/2$ digits are lost when evaluating $f(6+it)$ using these formulae. Repeatedly applying summation by parts to (2), (cf. [10]), also known as Abel summation, works somewhat better, although huge tables of partial sums of τ_n must be stored and high-precision arithmetic must still be employed. The first 642 zeros (up to $t = 571.756\ldots$) were calculated to about 35 digits using this method.

For large t, the most effective way to evaluate $Z(t)$ is with the asymptotic Berry-Keating formula (cf. [1]):

$$Z(t) \sim Z_0(t,K) + Z_3(t,K) + Z_4(t,K) + \cdots, \tag{12}$$

where

$$Z_0(t,K) = 2\mathrm{Re}\sum_{n=1}^{\infty}\left(\frac{\tau_n}{n^6}e^{i(\vartheta(t)-t\log\,n)}\frac{1}{2}\mathrm{erfc}\left(\frac{\xi(n,t)}{Q(K,t)}\sqrt{t/2}\right)\right), \tag{13}$$

$$\begin{aligned} Z_m(t,K) \quad &= \quad \frac{2}{\sqrt{\pi}}(t/2)^{m/2}\mathrm{Re}\left(\frac{(-i)^m b_m(t)}{Q^m(K,t)}\sum_{n=1}^{\infty}\left(\frac{\tau_n}{n^6}e^{i(\vartheta(t)-t\log\,n)}\right.\right. \\ &\quad \left.\left.\cdot\exp\left(\frac{-t\xi^2(n,t)}{2Q^2(K,t)}\right)H_{m-1}\left(\frac{\xi(n,t)}{Q(K,t)}\sqrt{t/2}\right)\right)\right) \qquad (m \geq 3), \end{aligned} \tag{14}$$

$$\begin{aligned} \sum_{m=3}^{\infty} z^m b_m(t) \quad &= \quad \exp\left(i(\vartheta(z+t)-\vartheta(t)-z\vartheta'(t))-\frac{1}{2}z^2\vartheta''(t)\right)-1 \\ &= \quad \exp\left(i\sum_{k=3}^{\infty}z^k\,\mathrm{Im}\frac{i^k\psi^{(k-1)}(6+it)}{k!}\right), \end{aligned} \tag{15}$$

$$\xi(n,t) = \log\,n - \vartheta'(t), \tag{16}$$

and

$$Q^2(K,t) = K^2 - it\vartheta''(t). \tag{17}$$

It should be noted that while convergence for the ζ-function begins near $n = \sqrt{t/(2\pi)}$, convergence for the τ-Dirichlet series does not begin until $n \approx t/(2\pi)$. This is a result of the fact that the dominant terms in the expansions for $\theta(t)$ and $\vartheta(t)$ differ by a factor of 2.

3. Results

Although the zeros calculated with the Berry-Keating formula agree very well with the zeros calculated using Abel summation, because the formula is asymptotic and actually diverges, results based on it cannot be regarded as truly rigorous. Nevertheless, we are confident that the values for the zeros are correct to within 0.000001. The programming and the evaluation of the zeros was all done using *Mathematica*™ on SPARC and NeXT workstations of modest speed. A single evaluation of $Z(t)$ near $t = 10000$ took on the order of 6 minutes. Because of the high cost of evaluation of $Z(t)$, the zeros were found by finding the appropriate zero of the polynomial that interpolates some 10 nearby values of $Z(t)$, evaluating $Z(t)$ at that zero, adding the new value to the data being interpolated, and iterating until $|Z(t)| < 0.00000001$. Using this "glorified secant" method, each zero costs about 3 evaluations of $Z(t)$.

We found that each of the first 12069 zeros of the Ramanujan τ-Dirichlet series of the form $\sigma + it$ in the region $0 < t < 6397$ is simple and lies on the line $\sigma = 6$. In this range Gram's law fails 897 times and is correct 11172 times. In the region

$9877.7 < t < 10822.6$ there are 2228 zeros, all of which are simple and lie on the critical line. In this range Gram's law fails 223 times. Rosser's rule was not observed to fail. In Table 1 we present counts of the various types of Gram blocks encountered.

TABLE 1. Number of Gram blocks of various types among (A) the first 12068 Gram intervals and (B) 2228 Gram intervals beginning with the 20000th interval

length	*zero pattern*	*count A*	*count B*
1	(1)	10323	1800
2	(0, 2)	403	90
2	(2, 0)	397	98
3	(0, 1, 2)	16	4
3	(2, 1, 0)	17	7
3	(0, 3, 0)	14	5
4	(0, 1, 3, 0)	1	1

Although an exhaustive search for extrema of $Z(t)$ was not performed, large extrema seem to increase in a way consistent with a Lindelöf hypothesis. Table 2 gives examples of large and small extrema, respectively.

TABLE 2. Large and small extrema of $Z(t)$

t	$Z(t)$
238.53	-6.432
256.45	6.220
296.44	7.648
468.82	-7.433
773.14	-10.124
885.75	10.211
921.85	-11.309
2046.30	11.523
2526.42	-12.587
2997.87	12.990
3927.96	13.901
4438.90	-13.935
5840.54	15.527
10358.02	-15.627

t	$Z(t)$
243.83	-0.0931887
325.66	0.0664191
381.27	0.0577698
625.83	0.0278474
2152.88	-0.0270708
10006.36	0.0154048
10297.21	0.0209262
10429.43	0.0036954
10487.58	-0.0066925

We also located the first 5018 zeros and the 2228 zeros between $t = 9877.7$ and $t = 10822.6$, i.e., the 20001st through the 22228th zeros. The spacing between successive zeros $6 + i\gamma_n$ and $6 + i\gamma_{n+1}$ was normalized to be

$$\delta_n = (\gamma_{n+1} - \gamma_n)\frac{\log(\gamma_{n+1}\gamma_n/(2\pi)^2)}{2\pi}$$

and central moments of the normalized spacing were calculated. Note that this normalization is slightly different from that of [7] in that it attempts to address a slight bias for small γ_n. The results are presented in Table 3, where the moments for the distribution associated with a Gaussian unitary ensemble (GUE) are included for comparison.

TABLE 3. Moments of $\delta_n - 1$ for (A) the first 5017 zero-pairs and (B) 2227 zero-pairs beginning with the 20001st pair

k	A	B	GUE
2	0.1371	0.1530	0.1800
3	0.0128	0.0203	0.0380
4	0.0520	0.0663	0.1013
5	0.0158	0.0276	0.0656
6	0.0320	0.0500	0.1110
7	0.0182	0.0385	0.1243
8	0.0269	0.0569	0.1969
9	0.0216	0.0610	0.2902
10	0.0276	0.0860	0.4881

Figures 1 and 2 show the pair correlation of the zeros of $Z(t)$ for the two sets of zeros. The solid lines in both figures are the GUE prediction $y = 1 - ((\sin \pi x)/(\pi x))^2$. See [7] for further details of this common type of plot.

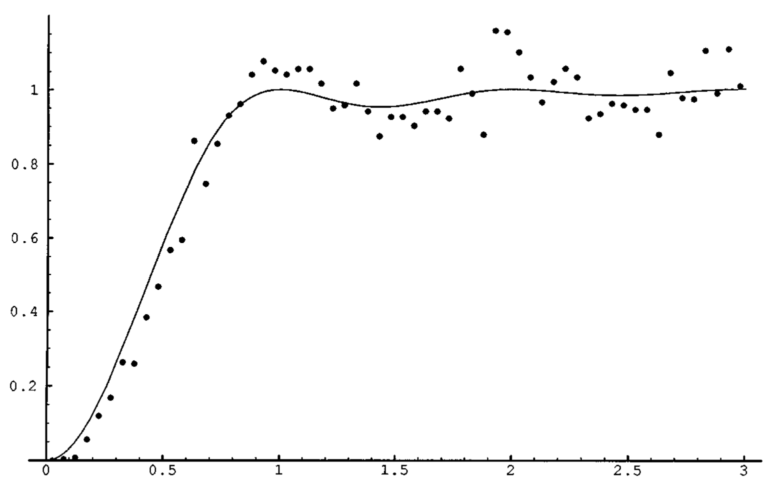

FIGURE 1. Pair correlation function: first 5018 zeros

We further investigated an implication of the Riemann hypothesis for the τ-Dirichlet function $f(s)$. This investigation was analogous to that of [10]. We first define

$$\xi(s) = (2\pi)^{-12s}\Gamma(12s)f(12s). \tag{18}$$

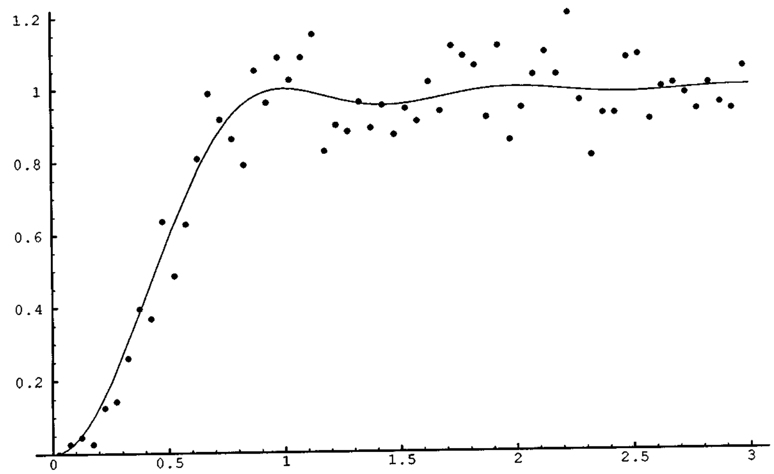

FIGURE 2. Pair correlation function: zeros 20001 through 22228

By an argument essentially the same as that for Riemann's ξ-function (cf. [2, pp. 39-47]) we can show that

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{12s}{\rho}), \tag{19}$$

where the product is over all of the zeros of f and

$$\xi(0) = \xi(1) = (2\pi)^{-12}\Gamma(12)f(12) = 0.010486273129241\ldots. \tag{20}$$

Furthermore,

$$\xi(s) = \int_1^\infty (z^{12s} + z^{12-12s}) \frac{g(e^{-2\pi z})}{z}\, dz. \tag{21}$$

Consider now the coefficients λ_k, where

$$\log\left(\frac{\xi(1/s)}{\xi(1)}\right) = \sum_{k=0}^{\infty} \lambda_k (1-s)^k, \tag{22}$$

$$\lambda_0 = 0, \tag{23}$$

$$\lambda_k = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \binom{k-1}{j-1} \sigma_j \quad (\text{for } k \geq 1), \tag{24}$$

$$\lambda_k = \frac{1}{k} \sum_{\rho} \left[1 - \left(\frac{\rho}{\rho - 12}\right)^k\right]. \tag{25}$$

It is clear from (25) that the Riemann hypothesis for f implies that $\lambda_k > 0$ for all positive k. As in [6], if we assume the Riemann hypothesis f, and further that the zeros are very evenly distributed, we can show that

$$\lambda_k \approx 12 \log k - 12(\log \frac{6}{\pi} + \gamma - 1). \tag{26}$$

The first 800 values of λ_k were found and they agree rather well with the above approximation.

4. Conclusions

The calculations for this study were done using *Mathematica*™. While this system proved to be quite useful for preliminary investigation and algorithm design, it is estimated that the investigation could be extended by several orders of magnitude if the algorithm were written in a low-level language and run on a fast computer. None of the calculations using the Berry-Keating formula required more precision than that provided by double-precision arithmetic, so considerable speedup can be expected.

The Berry-Keating method appears to be quite general and is likely extendable to many Dirichlet series such as those described in [3] and [10].

References

1. M. B. Berry and J. P. Keating, *A new asymptotic representation for $\zeta(\frac{1}{2}+it)$ and quantum spectral determinants*, Proc. R. Soc. Lond. A **437**, (1992), 151-173. MR **93j:**11057
2. H. M. Edwards, Riemann's zeta function, Academic Press, New York, 1974. MR **57:**5922
3. H. R. P. Ferguson, R. D. Major, K. E. Powell, and H. G. Throolin, *On Zeros of Mellin Transforms of $SL_2(\mathbf{Z})$ Cusp Forms*, Math. Comp. **42** (1984), 241-255. MR **85e:**11036
4. A. Guthmann, *Ramanujans Tau-Funktion*, unpublished, June, 1988.
5. G. H. Hardy, *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, Chelsea, New York, 1959. MR **21:**4881
6. J. B. Keiper, *Power series expansions of Riemann's ξ function*, Math. Comp. **58**(1992), 765-773. MR **92f:**11116
7. A. M. Odlyzko, *On the Distribution of Spacings Between Zeros of the Zeta Function*, Math. Comp. **48** (1987), 273-308. MR **88d:**11082
8. A. M. Odlyzko, *The 10^{20}-th Zero of the Riemann Zeta function and 70 Million of its Neighbors*, unpublished.
9. R. Spira, *Calculation of the Ramanujan τ-Dirichlet Series*, Math. Comp. **27**(1973), 379-385. MR **48:**5337
10. H. Yoshida, *On Calculations of Zeros of L-functions Related With Ramanujan's Discriminant Function on the Critical Line*, J. Ramanujan Math. Soc. **3** (1988), 87-95. MR **90b:**11044
11. H. Yoshida, *On Calculations of Zeros of Various L-functions*, to appear.