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ON THE HIGHLY ACCURATE SUMMATION OF CERTAIN SERIES OCCURRING IN PLATE CONTACT PROBLEMS

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Abstract. The infinite series $R_p = \sum_{k=1}^{\infty} (2k-1)^{-p}\, x^{2k-1}, 0 < 1-x \ll 1$, $p = 2$ or 3, and the related series

$$C(x,b,2) = \sum_{k=1}^{\infty} (2k-1)^{-2} \cosh(2k-1)x/\cosh(2k-1)b, \quad 0 < 1 - x/b \ll 1,$$

$$S(x,b,3) = \sum_{k=1}^{\infty} (2k-1)^{-3} \sinh(2k-1)x/\cosh(2k-1)b,$$

are of interest in problems concerning contact between plates and unilateral supports. This article will re-examine a previously published result of Baratella and Gabutti for R_p, and will present new, rapidly convergent, series for $C(x,b,2)$ and $S(x,b,3)$.

1. Introduction

When $0 < x \le 1/2$ the infinite series

$$R_p = \sum_{k=1}^{\infty} (2k-1)^{-p}\, x^{2k-1}, \qquad p = 2 \text{ or } 3\,, \tag{1}$$

are easy to sum to high accuracy using direct summation and a suitable computer algebra package. For example, when either series is approximated by the sum of its first r terms the error, E_r, satisfies

$$E_r(x) = \sum_{k=r+1}^{\infty} \frac{x^{2k-1}}{(2k-1)^p} \;\le\; \left(\frac{1}{2}\right)^{2r+1} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2r+1+2k)^p} \;<\; \left(\frac{1}{2}\right)^{2r-1} \frac{1}{3\,(2r+1)^p}.$$

When, however, $0 < 1 - x << 1$ the series R_p are not easy to sum. The series

$$\begin{aligned} C(x,b,2) &= \sum_{k=1}^{\infty} (2k-1)^{-2} \cosh(2k-1)x/\cosh(2k-1)b\,, \qquad 0 < x/b < 1, \\ S(x,b,3) &= \sum_{k=1}^{\infty} (2k-1)^{-3} \sinh(2k-1)x/\cosh(2k-1)b \end{aligned}$$

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are, likewise, easy to sum when $b\,(1-x/b)$ is not small. Suppose, for example, that the series $C(x,b,2)$ is truncated after r terms. The error is then

$$\begin{aligned}\hat{E}_c(r) &= \sum_{k=r+1}^{\infty} \frac{1}{(2k-1)^2} \cosh((2k-1)\,x)/\cosh((2k-1)\,b)\\ &< \sum_{k=r+1}^{\infty} \frac{1}{(2k-1)^2}\left[e^{-(2k-1)b(1-x/b)} + e^{-(2k-1)b(1+x/b)}\right]\\ &< \sum_{k=r+1}^{\infty} \frac{\left(e^{-b(1-x/b)}\right)^{2k-1}}{(2k-1)^2} + \sum_{k=r+1}^{\infty} \frac{\left(e^{-b}\right)^{2k-1}}{(2k-1)^2}\\ &< \frac{1}{(2r+1)^2}\left[\frac{e^{-(2r+1)b(1-x/b)} + e^{-(2r+1)b}}{1-e^{-2b(1-x/b)}}\right].\end{aligned}$$

Hence, when $b(1-x/b)$ is not small this series, too, is easy to sum to high accuracy by use of direct summation.

The series $R_p(x)$, $p=2,3$, $0<1-x<<1$, and $C(x,b,2)$, $S(x,b,3)$, $0<1-x/b<<1$, are of interest in problems concerning contact between plates and unilateral supports, [4], and the problem of their summation to high accuracy has attracted the attention of several numerical analysts [2], [3], [5], [7]. This article will address the same problem.

1.1. **Previous work.** The series were first discussed in [5], where the method of summation is based on Plana's summation formula (see, for example [8], pp. 145-46). The accuracy of this initial attempt at the problem is inferior to later attempts.

In [7], the method is to substitute the result

$$\frac{1}{(k-1/2)^p} = \int_0^{\infty} e^{-kt}\, e^{t/2}\, \frac{t^{p-1}}{(p-1)!}\, dt\,, \qquad (k \ge 1) \tag{2}$$

in R_p, interchange the order of summation and integration, and sum the new (geometric) series in k; hence the series R_p is transformed to the integral [1]

$$R_p(x) = \frac{x}{2^{p-1}(p-1)!}\int_0^1 \frac{[-\ln\, u^2]^{p-1}}{[1-(xu)^2]}\, du \tag{3}$$

which is evaluated numerically by use of Gaussian quadrature.

In [2] the identity $-\ln\, u^2 \equiv \ln\, x^2 - \ln\,(xu)^2$ and the binomial theorem are used to transform (3) to

$$R_p(x) = \frac{x}{2^{p-1}(p-1)!}\sum_{m=0}^{p-1}\binom{p-1}{m}(\ln\, x^2)^{p-1-m}\, T_m(x)$$

where

$$T_m(x) = \int_0^1 \frac{[-\ln\,(x\,u)^2]^m}{[1-(xu)^2]}\, du\,.$$

[1]In [2], [3] and [7] the series are taken to be $R_p(z)$, where z is a complex variable; in this article we take $R_p = R_p(x)$ where x is a real variable.

By use of the expansion

$$(-\ln u)^m = \sum_{k=0}^{\infty}(-1)^k S_{k+m}^{(m)} \frac{m!}{(m+k)!}(1-u)^{k+m}, \qquad 0 < u \leq 1,$$

where $S_{k+m}^{(m)}$ are the Stirling numbers of the first kind, approximations, with accompanying error bounds, are obtained for the integrals $T_m(x)$. These are valid for $0 \leq x \leq 1$.

In [3] the series $R_p, p = 2, 3,$ are treated as special cases of Legendre's Chi function and are essentially expressed as rapidly convergent alternating power series in $\xi = -\ln(x)/\pi$; when these power series are truncated after r terms, where r is even, the error is close to the first term not taken, which is close to

$$\frac{-\pi\, \xi^{2r+3}}{(2r+2)(2r+3)},$$

in the case when $p = 2$. As we shall show, these remarkable series provide the most efficient means of summing the series R_p when $1/2 \leq x < 1$.

In the case of the series $C(x, b, 2)$ and $S(x, b, 3)$, the method given in [7] leads to infinite series (of integrals) which converge rapidly except when $0 < b << 1$—a deficiency which is resolved in [2] by use of a transformed series which converges like a geometric series with ratio $1/3$ for all $b \geq 0$.

In this article accurate bounds will be derived for the errors which occur when the series $R_p, p = 2,\ 3,$ as transformed in [3] are truncated after r terms and new series will be presented for $C(x, b, 2)$ and $S(x, b, 3)$; further, the errors which result when the latter series are truncated after r terms will be examined and it will be shown that the error is close to

$$2b\, \frac{([1 - x/b]/2)^{2r+3}}{(2r+2)\,(2r+3)}$$

in the case of $C(x, b, 2)$ and that it is smaller than this in the case of $S(x, b, 3)$.

2. The series $R_p(x)$, $p = 2, 3$, as transformed by Boersma and Dempsey

As the series $R_p(x)$ are easily summed when $0 < x < 1/2$, intuition suggests that when they are transformed to series in the new variable $(1 - x)$ the transformed series will be easy to sum when $1/2 \leq x < 1$. This is indeed the case, but the resulting series can be shown to converge more slowly in the region of interest than do the series presented in [3]. [2] These are

[2]It can, for example, be shown that

$$R_2(x) = \frac{\pi^2}{8} + \frac{1}{2}\left(\ln x\, \ln\left[\frac{1+x}{1-x}\right] - \sum_{k=1}^{\infty}\frac{(1-x)^{k+1}}{k+1}\, b_{k-1} - \sum_{k=1}^{\infty}\frac{(1-x)^k}{k^2}\right),$$

where $0 < x < 1$ and

$$b_{k-1} = \sum_{i=0}^{k-1}\frac{1}{(k-i)\,2^{i+1}}, \qquad\qquad k = 1, 2 \ldots .$$

This result which, when each infinite series is truncated after r terms has an error, E_{R2}, satisfying

$$0 > E_{R2} > -\frac{(1-x)^{r+1}}{4x}\left[\frac{1-x}{(r+2)} + \frac{2}{(r+1)^2}\right],$$

should be compared to Procedure 3 of [2].

$$
\begin{aligned}
R_2(x) \quad &= \quad \frac{\pi^2}{8} - \frac{\pi\,\xi}{2}\,[1+\ln(2)-\ln(\pi\,\xi)] \\
&- \quad \pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,(1-2^{-2k+1})}{(2k)\,(2k+1)} \zeta(2k)\,\xi^{2k+1}\,, \qquad 0 \le \xi \le 1\,,
\end{aligned} \tag{4}
$$

and

$$
\begin{aligned}
R_3(x) \quad &= \quad \frac{7}{8}\,\zeta(3) - \frac{\pi^3}{8}\,\xi + \frac{\pi^2}{4}\,\xi^2\,\,[3/2+\ln(2)-\ln(\pi\,\xi)] \\
&+ \quad \pi^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,(1-2^{-2k+1})}{(2k)\,(2k+1)\,(2k+2)} \zeta(2k)\,\xi^{2k+2}\,, \qquad 0 \le \xi \le 1\,,
\end{aligned} \tag{5}
$$

where $\xi = -\ln(x)/\pi$.

2.1. **Computation of $R_2(x)$ and $R_3(x)$.** When the infinite series in (4) is truncated after r terms the error, E_{R2}, is given by

$$
E_{R2} = \pi \sum_{k=r+1}^{\infty} \frac{(-1)^k (1-2^{-2k+1})}{(2k)(2k+1)}\,\zeta(2k)\,\xi^{2k+1}, \qquad 0 \le \xi \le 1\,, \tag{6}
$$

where ([1], 23.1.15)

$$
1 \,<\, \zeta(2k+2) = \frac{(-1)^k (2\pi)^{2k+2} B_{2k+2}}{2\,(2k+2)!} < \frac{1}{(1-2^{-2k-1})}. \tag{7}
$$

The terms in (4) alternate in sign and steadily decrease to zero as $k \to \infty$. Bounds for the error when the series is approximated by its first r terms are easily obtained from Leibnitz's theorem for alternating series which, when r is even, gives the result

$$
\frac{-\pi\,\xi^{2r+3}}{(2r+2)\,(2r+3)} < E_{R2} < 0\,. \tag{8}
$$

The upper bound in this result may, as will now be demonstrated, be improved on.

When r is **even** and the terms of the series (6) are grouped in pairs, starting with the first two terms, the sum of each pair is less than zero and the pth pair, T_p say, $p = 1, 2, \ldots$, satisfies the inequality

$$
T_p < -\pi\,\xi^{2r+3+4p} \left[\frac{(1-2^{(-2r-1-4p)})}{(2r+2+4p)\,(2r+3+4p)} - \frac{\xi^2}{(2r+4+4p)\,(2r+5+4p)} \right].
$$

Summing all such pairs, and noting that $T_p < 0$, we see that

$$
\begin{aligned}
E_{R2} &< -\pi\,\xi^{2r+3} \sum_{k=1}^{\infty} \frac{(1-2^{-2r-1-4k}-\xi^2)\,\xi^{4(k-1)}}{(2r-2+4k)(2r-1+4k)} \\
&= \; -\pi\,\xi^{2r+3} \sum_{k=0}^{\infty} \frac{(1-2^{-2r-5-4k}-\xi^2)\,\xi^{4k}}{(2r+2+4k)(2r+3+4k)}\,.
\end{aligned}
$$

Further, use of results like

$$
Q = \sum_{k=0}^{\infty} \frac{\xi^{2r+3+4k}}{(2r+2+4k)} = \xi \sum_{k=0}^{\infty} \int_0^{\xi} \xi^{2r+1+4k}\,d\xi = \xi \int_0^{\xi} \frac{\xi^{2r+1}}{(1-\xi^4)}\,d\xi
$$

lead to

$$
E_{R2} < -\pi\,(1-\xi^2) \int_0^{\xi} \frac{u^{2r+1}}{(1-u^4)}(\xi-u)\,du + 4\,\pi \int_0^{\frac{\xi}{2}} \frac{u^{2r+1}}{(1-u^4)}(\xi/2-u)\,du,
$$

from which result and (8) we obtain the inequality

(9)
$$\frac{-\pi\,\xi^{2r+3}}{(2r+2)\,(2r+3)} < E_{R2} < \frac{-\pi\,\xi^{2r+3}}{(2r+2)\,(2r+3)}\left[1-\xi^2-\frac{1}{(1-(\xi/2)^4)\,2^{2r+1}}\right].$$

which is valid for all **even** $r > 1$.

When $r = 6$ and $x = 0.9$ we find that

$$-1.142486 \times 10^{-24} < E_{R2} < -1.141061 \times 10^{-24}$$

whereas calculation gives $E_{R2} = -1.141425 \times 10^{-24}$.

In the case of $R_3(x)$, we can similarly show that when the infinite series in (5) is truncated after r terms the error, E_{R3}, satisfies, for **even** r, the inequality

$$\frac{\pi^2\,\xi^{2r+4}}{(2r+2)\,(2r+3)\,(2r+4)}\left[1-\xi^2-\frac{1}{(1-(\xi/2)^4)\,2^{2r+1}}\right] < E_{R3} < \frac{\pi^2\,\xi^{2r+4}}{(2r+2)\,(2r+3)\,(2r+4)}.$$

When $r = 6$ and $x = 0.9$:

$$7.513925 \times 10^{-27} < E_{R3} < 7.523306 \times 10^{-27}$$

whereas calculation gives $E_{R3} = 7.517046 \times 10^{-27}$.

3. The series $C(x, b, 2)$ and $S(x, b, 3)$

These are the series specifically referred to in [5] (equations (13a) and (13b), respectively).

3.1. $C(x, b, 2)$, $0 < b - x \ll b$. Consider the following boundary value problem: solve for T

$$\begin{aligned} T_{xx} &+ T_{yy} = 0, \qquad 0 \le x \le \pi, \quad 0 \le y \le b;\\ T(x,0) &= x; \quad T_y(x,b) = 0;\\ T_x(0,y) &= 0; \quad T_x(\pi,y) = 0\,. \end{aligned}$$

By expanding in terms of the eigenfunctions $\cos nx$, it is easy to show that

$$T(x,y) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cosh[(2n-1)\,(b-y)]}{\cosh[(2n-1)\,b]}\,\frac{\cos(2n-1)x}{(2n-1)^2}\,.$$

Hence

(10)
$$\begin{aligned} T(\pi,y) &= \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\,\frac{\cosh[(2n-1)\,(b-y)]}{\cosh[(2n-1)\,b]}\\ &= \frac{\pi}{2} + \frac{4}{\pi}\,C(b-y,b,2)\,. \end{aligned}$$

Now put $\hat{T} = T - x$ and solve

$$\begin{aligned} \hat{T}_{xx} &+ \hat{T}_{yy} = 0, \qquad 0 \le x \le \pi, \quad 0 \le y \le b;\\ \hat{T}(x,0) &= 0; \quad \hat{T}_y(x,b) = 0;\\ \hat{T}_x(0,y) &= -1; \quad \hat{T}_x(\pi,y) = -1\,. \end{aligned}$$

The eigenfunctions are now $\sin[(2n-1)\pi y/2b]$ and

$$\hat{T}(x,y) = \frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi y/2b]}{(2n-1)^2} \cdot \Big(\tanh[(2n-1)\pi^2/(4b)]\,\cosh[(2n-1)\frac{\pi x}{2b}] - \sinh[(2n-1)\frac{\pi x}{2b}]\Big)\,.$$

Hence

(11)

$$\begin{aligned}\hat{T}(\pi,y) &= -\frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin[(2n-1)\,\pi y/(2b)]\ \tanh[(2n-1)\pi^2/(4b)]\\ &= -\frac{8b}{\pi^2}\Bigg[\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi y/2b]}{(2n-1)^2}\\ &\qquad -\sum_{n=1}^{\infty}\big(1-\tanh[(2n-1)\pi^2/(4b)]\big)\,\frac{\sin[(2n-1)\pi y/2b]}{(2n-1)^2}\Bigg]\,.\end{aligned}$$

If we now substitute for $\sum_{n=1}^{\infty}\,\sin[(2n-1)\,\pi y/(2b)]/(2n-1)^2$ from equation (3.4) of [3] we obtain, on eliminating T from equations (10) and (11), the result

$$\begin{aligned}C(x,b,2) &= \frac{\pi^2}{8} - \Big[1-\ln(\frac{\mu_1}{2})\Big]\,\frac{b}{\pi}\,\mu_1\\ &\quad -\frac{2b}{\pi}\,\mu_1\sum_{k=1}^{\infty}(-1)^k\frac{[2^{2k-1}-1]\,B_{2k}}{2k\,(2k+1)!}\,{\mu_1}^{2k}\\ &\quad +\frac{2b}{\pi}\sum_{k=1}^{\infty}\big(1-\tanh[(2k-1)\pi^2/(4b)]\big)\,\frac{\sin[(2k-1)\mu_1]}{(2k-1)^2}\,,\end{aligned}\tag{12}$$

where $\mu_1 = (1-x/b)\,\pi/2$ and B_{2k} denotes the Bernoulli polynomial of order $(2k)$, as defined in [1]. The series for $S(x,b,3)$ can be similarly transformed by integrating (12) with respect to x, from x to b.

4. Computation of $C(x,b,2)$ and $S(x,b,3)$

When each of the infinite series in (12) and in the corresponding result for $S(x,b,3)$ are approximated by the sum of their first r terms errors, $E_c(r)$ and $E_s(r)$, respectively, are introduced. In this section we shall present bounds for these.

4.1. **Computation of $C(x,b,2)$.** From equation (12),

$$E_c(r) \,=\, b\,(1-x/b)\,G + \frac{2b}{\pi}\,H\,,\tag{13}$$

where

$$\begin{aligned}G &= \sum_{k=r+1}^{\infty}(-1)^{k+1}\frac{[2^{2k-1}-1]\,B_{2k}}{2k\,(2k+1)!}\Big[\frac{\pi}{2}\,\Big(1-\frac{x}{b}\Big)\Big]^{2k},\\ H &= \sum_{k=r+1}^{\infty}\Big(1-\tanh[(2k-1)\frac{\pi^2}{4\,b}]\Big)\,\frac{\sin[(2k-1)(1-x/b)\,\pi/2]}{(2k-1)^2}\,.\end{aligned}$$

Hence, by use of (7),

$$\sum_{k=r+1}^{\infty} \frac{[(1-x/b)/2]^{2k}}{2k\,(2k+1)} - \sum_{k=r+1}^{\infty} \frac{[(1-x/b)/4]^{2k}}{k\,(2k+1)} < G < \sum_{k=r+1}^{\infty} \frac{[(1-x/b)/2]^{2k}}{2k\,(2k+1)}\,,$$

which leads to

$$\begin{aligned} &\frac{([1-x/b]/2)^{2r+2}}{(2r+2)(2r+3)} \left[1 - \frac{(1/2)^{2r+1}}{\left(1-(1-x/b)^2/16\right)}\right] \\ &\qquad < G < \frac{((1-x/b)/2)^{2r+2}}{(2r+2)(2r+3)\left(1-(1-x/b)^2/4\right)}\,. \end{aligned} \tag{14}$$

In the case of H we have

$$| \, H \, | < \sum_{k=r+1}^{\infty} \frac{(1-\tanh[(2k-1)\pi^2/(4\,b)])}{(2k-1)^2} < (\frac{\pi^2}{8}-1)\left(1-\tanh[(2r+1)\frac{\pi^2}{4b}]\right)\,. \tag{15}$$

Hence, when each of the infinite series in (12) are truncated after r terms the error, $E_c(r)$, satisfies

$$U \; < \; E_c(r) \; < \; V\,,$$

where

$$U = \frac{2b([1-x/b]/2)^{2r+3}}{(2r+2)\,(2r+3)} \left[1 - \frac{(1/2)^{2r+1}}{\left(1-(1-x/b)^2/16\right)}\right] - \alpha\left(1-\tanh[(2r+1)\frac{\pi^2}{4\,b}]\right)$$

$$V = \frac{2b([1-x/b]/2)^{2r+3}}{(2r+2)\,(2r+3)} \left[\frac{1}{1-(1-x/b)^2/4}\right] + \alpha\left(1-\tanh[(2r+1)\frac{\pi^2}{4\,b}]\right)$$

and $\alpha = (\pi^2/8-1)\,(2b)/\pi$. When $r=5$, $b=0.1$ and $x=0.09$ we find that

$$1.564239 \times 10^{-20} < E_c(5) < 1.568926 \times 10^{-20}\,,$$

whereas calculation of $E_c(5)$ gives $E_c(5) = 1.567536663 \times 10^{-20}$.

4.2. **Computation of $S(x,b,3)$.** From equation (12) it can be shown that

$$\begin{aligned} S(x,b,3) &= \frac{\pi^2\,x}{8} - \frac{b\,x}{4}\,[2-x/b]\left(\frac{3}{2}-\ln\frac{\pi}{4}\right) - \frac{b^2}{4}\left(1-\frac{x}{b}\right)^2 \ln(1-\frac{x}{b}) \\ &+ \frac{4b^2}{\pi^2}\sum_{k=1}^{r}(-1)^k \frac{[2^{2k-1}-1]\,B_{2k}}{2k\,(2k+2)!}\left[\frac{\pi}{2}(1-\frac{x}{b})\right]^{2k+2} \\ &+ \frac{4b^2}{\pi^2}\sum_{k=1}^{r}\left(1-\tanh[(2k-1)\frac{\pi^2}{4\,b}]\right)\frac{\cos\mu_k}{(2k-1)^3} + b^2\,A + E_s(r)\,, \end{aligned} \tag{16}$$

where, for $k = 1, 2, \ldots$, $\mu_k = (2k-1)\,(1-x/b)\,\pi/2$ and

$$A = \sum_{k=1}^{\infty} (-1)^{k+1}\left(\frac{\pi}{2}\right)^{2k} \frac{[2^{2k-1}-1]\,B_{2k}}{2k\,(2k+2)!}\,. \tag{17}$$

[3] The error term, $E_s(r)$, satisfies

$$E_s(r) \;=\; -b^2\,(1-x/b)^2\,\bar{G} \;+\; 4b^2\,\bar{H}/\pi^2\,,$$

where

$$\begin{aligned}\bar{G} &= \sum_{k=r+1}^{\infty}(-1)^{k+1}\,\frac{[2^{2k-1}-1]\,B_{2k}}{2k\,(2k+2)!}\,\Big[\frac{\pi}{2}\,(1-\frac{x}{b})\Big]^{2k},\\ \bar{H} &= \sum_{k=r+1}^{\infty}\Big(1-\tanh[(2k-1)\frac{\pi^2}{4\,b}]\Big)\,\frac{\cos\left[(2k-1)(1-x/b)\,\pi/2\right]}{(2k-1)^3}\,.\end{aligned}$$

It can be shown that

$$\bar{U} \;<\; E_s(r) \;<\; \bar{V}\,, \tag{18}$$

where

$$\bar{U} = -\frac{4b^2\,([1-x/b]/2)^{2r+4}}{(2r+2)(2r+3)(2r+4)}\left[\frac{1}{1-(1-x/b)^2/4}\right]-\frac{b^2}{\pi^2}\left[1-\tanh[(2r+1)\frac{\pi^2}{4\,b}]\right],$$

$$\begin{aligned}\bar{V} = -&\frac{4b^2\,([1-x/b]/2)^{2r+4}}{(2r+2)(2r+3)(2r+4)}\left[1-\frac{(1/2)^{2r+1}}{\Big(1-(1-x/b)^2/16\Big)}\right]\\ &+\frac{b^2}{\pi^2}\left[1-\tanh[(2r+1)\frac{\pi^2}{4\,b}]\right].\end{aligned}$$

When $r=5,\quad b=0.1,\quad x=0.09$, we find that

$$-1.120662\times 10^{-23} < E_s(5) < -1.117314\times 10^{-23}\,,$$

whereas calculation gives $E_s(5) = -1.119409\times 10^{-23}$.

5. Conclusion

In this article we have shown that when either of the series (4), (5) are truncated after r terms, the magnitude of the resulting error in the sum of the series will be close to (and will not exceed)

$$\frac{\pi}{(2r+2)\,(2r+3)}\left(\frac{-\ln(x)}{\pi}\right)^{2r+3} \approx \frac{\pi\,{\epsilon_1}^{2r+3}}{(2r+2)\,(2r+3)}\,,$$

where $0<\epsilon_1=(1-x)/\pi << 1$.

Further, when the series in (12) are truncated after r terms the error in the sum of the series will be close to

$$\frac{2b\,{\epsilon_2}^{2r+3}}{(2r+2)\,(2r+3)}$$

when $0<\epsilon_2=(1-x/b)/2<<1$, and $b=O(1)$, at most; and the error in $S(x,b,3)$ will be smaller than this.

Besides being of interest to the numerical analyst ([2], [3], [5], [7]) and to the specialist in unilateral plate contact problems, [4], the series $C(x,b,2)$ and $S(x,b,3)$ occur elsewhere in applied mathematics—for example, in connection with the torsion of a prismatic bar of rectangular cross-section where, with $b^*=(\pi c)/(2b)$,

[3] We note that the series for the number A converges like $\sum_k 2^{-2k}/k^3$,

$C(y\,(2b^*)/c, b^*, 2)$ occurs in the expression for the shear stress on the boundaries $x = \pm b/2$, of the bar, $-b/2 \leq x \leq b/2,\ \ -c/2 \leq y \leq c/2$ (see, for example, [6], pp. 244-45).

In any such application the use of the alternative series discussed in this article should prove worthwhile.

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