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ON ϕ–AMICABLE PAIRS

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Abstract. Let $\phi(n)$ denote Euler's totient function, i.e., the number of positive integers $< n$ and prime to n. We study pairs of positive integers (a_0, a_1) with $a_0 \leq a_1$ such that $\phi(a_0) = \phi(a_1) = (a_0 + a_1)/k$ for some integer $k \geq 1$. We call these numbers *ϕ–amicable pairs with multiplier k*, analogously to Carmichael's multiply amicable pairs for the σ–function (which sums all the divisors of n).

We have computed all the ϕ–amicable pairs with larger member $\leq 10^9$ and found 812 pairs for which the greatest common divisor is squarefree. With any such pair infinitely many other ϕ–amicable pairs can be associated. Among these 812 pairs there are 499 so-called primitive ϕ–amicable pairs. We present a table of the 58 primitive ϕ–amicable pairs for which the larger member does not exceed 10^6. Next, ϕ–amicable pairs with a given prime structure are studied. It is proved that a relatively prime ϕ–amicable pair has at least twelve distinct prime factors and that, with the exception of the pair $(4, 6)$, if one member of a ϕ–amicable pair has two distinct prime factors, then the other has at least four distinct prime factors. Finally, analogies with construction methods for the classical amicable numbers are shown; application of these methods yields another 79 primitive ϕ–amicable pairs with larger member $> 10^9$, the largest pair consisting of two 46-digit numbers.

1. Introduction

Let $\phi(n)$ be Euler's totient function. The pair (a_0, a_1) with $1 < a_0 \leq a_1$ is called *ϕ–amicable with multiplier k* if

$$\phi(a_0) = \phi(a_1) = \frac{a_0 + a_1}{k} \text{ for some integer } k \geq 1. \tag{1}$$

Since $\phi(n) < n$, we cannot have $k = 1$. To see that in fact $k > 2$, notice that if $k = 2$, then

$$\frac{a_0 + a_1}{2} > \frac{\phi(a_0) + \phi(a_1)}{2} = \phi(a_0).$$

If $a_0 = a_1 = a$, we have the equation $\phi(a) = 2a/k = a/l$ provided that k is even. This is known [7] to have the (only) solutions $a = 2^\alpha$ for $l = 2$ and $a = 2^\alpha 3^\beta$ for $l = 3$. If k is odd, $k = pk'$ say, where p is an odd prime, then $p \mid a$, $a = p^\gamma b$ with $\gcd(p, b) = 1$, and the equation easily reduces to the form $\phi(b) = b/l$ (where $l = k'(p-1)/2$). We assume from now that $a_0 < a_1$.

An analogous definition for the σ–function was given by Carmichael [4, p. 399], who called two positive integers a_0 and a_1 a *multiply amicable pair* if $\sigma(a_0) =$

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$\sigma(a_1) = l(a_0 + a_1)$ for some positive integer l. For $l = 1$, we obtain the "classical" amicable pairs, like $(220, 284) = (2^2 5 \cdot 11, 2^2 71)$, which was known already to the ancient Greeks. Mason [8] gives various multiply amicable pairs for $l = 2$ and $l = 3$.

This paper is organized as follows. In Section 2 the results are presented of an exhaustive computation of all the ϕ–amicable pairs with larger member $\leq 10^9$. From the pairs found one readily sees that if (a_0, a_1) is a ϕ–amicable pair such that $p^n \mid a_0$ and $p^n \mid a_1$ for some prime p and positive integer n, then p^n may be replaced by p^{n+m} for any positive integer m, yielding infinitely many other ϕ–amicable pairs. The smallest such example is $(2^2, 2 \cdot 3)$ inducing the pairs $(2^{n+1}, 2^n 3)$, $n = 1, 2, \ldots$. So-called *primitive* ϕ–amicable pairs are introduced next, i.e., pairs (a_0, a_1) which cannot be generated from a smaller ϕ–amicable pair with some substitution like the above one. There are 499 primitive ϕ–amicable pairs $\leq 10^9$.

In Section 3 ten basic properties of primitive ϕ–amicable pairs are given, most of them being used in the sequel. In Section 4 we describe several theoretical results which were partly suggested by our numerical results. For example, we prove that there are finitely many primitive ϕ–amicable pairs with a given number of different prime factors, and we discuss the number of different prime factors in the members of a relatively prime ϕ–amicable pair. In Section 5, finally, analogies with construction methods for ordinary amicable pairs are derived. Application yields another 79 primitive ϕ–amicable pairs in addition to those found with our exhaustive search.

Very few results are proved in this paper. For full details, see [5].

A ϕ–amicable pair (a_0, a_1) with multiplier k is denoted sometimes by $(a_0, a_1; k)$. The notation $a \parallel b$ denotes that a is a unitary divisor of b, that is, $\gcd(a, b/a) = 1$. We shall regularly use the following well-known properties of Euler's ϕ–function: $\phi(n) < n$ for $n > 1$; if $d < n$ and $d \mid n$, then $\phi(d) < \phi(n)$; ϕ is multiplicative; if p is a prime and m a positive integer, then $\phi(p^m) = p^{m-1}(p-1)$; $\phi(n)/n = \prod_{p \mid n}(p-1)/p$.

2. Exhaustive computation of ϕ-amicable pairs

We have computed a complete list of ϕ–amicable pairs with larger member $\leq 10^9$ as follows. Suppose a_1 is given, and we wish to test if it is the larger member of a ϕ–amicable pair (a_0, a_1). We test if there is an integer k such that, for $a_0 = k\phi(a_1) - a_1$, $\phi(a_0) = \phi(a_1)$. If $a_0 \leq \phi(a_1)$, $\phi(a_0) < \phi(a_1)$. Therefore, we must have $\phi(a_1) < a_0 < a_1$, so that lower and upper bounds for the admissible values of k are given by

$$1 + \frac{a_1}{\phi(a_1)} < k < \frac{2a_1}{\phi(a_1)}. \tag{2}$$

For all a_1 with $2 \leq a_1 \leq 10^9$ we computed $\phi(a_1)$ and for k in the range (2): $a_0 = k\phi(a_1) - a_1$ and $\phi(a_0)$. If $\phi(a_0) = \phi(a_1)$, (a_0, a_1) is a ϕ–amicable pair with multiplier k.

Inspection of these pairs suggested the following two propositions which are easily proved with the help of the defining equations (1). These propositions express rules to generate ϕ–amicable pairs from other ϕ–amicable pairs by multiplying or dividing both members of a pair by some prime p.

Proposition 1. *Let $(a_0, a_1; k)$ be given.*

a. *If p is a prime with $p \mid \gcd(a_0, a_1)$, then we also have $(pa_0, pa_1; k)$.*

b. *If p is a prime with $p^2 \mid \gcd(a_0, a_1)$, then we also have $(a_0/p, a_1/p\,; k)$.*

Proposition 2. *Let* $(a_0, a_1; k)$ *be given.*

a. *If* p *is a prime with* $p \nmid a_0a_1$ *and* $p-1 \mid k$*, then we also have*

$$(pa_0, pa_1; kp/(p-1)).$$

b. *If* p *is a prime with* $p \parallel \gcd(a_0, a_1)$ *and* $p \mid k$*, then we also have*

$$(a_0/p, a_1/p\,; k(p-1)/p).$$

Remark. When we reduce a ϕ–amicable pair by applying (possibly repeatedly) Proposition 1b, we arrive at a pair with $p \parallel \gcd(a_0, a_1)$. Now it is easy to see that if $p^c \parallel a_i$ $(c \geq 2)$, then $p^{c-1} \mid a_{1-i}$, and that, if $p^2 \parallel a_i$ and $p \parallel a_{1-i}$, it is impossible to have $p \mid k$ $(i = 0$ or $1)$. It follows that the condition $p \parallel \gcd(a_0, a_1)$ in Proposition 2b effectively may be replaced by $p \parallel a_0$, $p \parallel a_1$.

Propositions 1 and 2 suggest the definition of a minimal pair from which no smaller pairs can be generated. We call these pairs *primitive* ϕ–amicable pairs.

Definition. A ϕ–amicable pair $(a_0, a_1; k)$ is called primitive if $\gcd(a_0, a_1)$ is squarefree, and if $\gcd(a_0, a_1, k) = 1$. [1]

By applying Propositions 1b and 2b we see that any non-primitive pair can be reduced to a primitive pair. For example,

$$\begin{cases} 2\cdot 3^2 5\cdot 31 \\ 2\cdot 3^3 5\cdot 11 \\ \quad k=8 \end{cases} \xrightarrow{\text{Prop. 1b}} \begin{cases} 2\cdot 3\cdot 5\cdot 31 \\ 2\cdot 3^2 5\cdot 11 \\ \quad k=8 \end{cases} \xrightarrow{\text{Prop. 2b}} \begin{cases} 3\cdot 5\cdot 31 \\ 3^2 5\cdot 11 \\ \ k=4 \end{cases},$$

where the third pair is a primitive ϕ–amicable pair. In the other direction, starting with the primitive ϕ–amicable pair $(3\cdot 7\cdot 71\cdot 193, 3\cdot 7^2 11\cdot 13\cdot 17; 4)$:

$$\begin{cases} 3\cdot 7\cdot 71\cdot 193 \\ 3\cdot 7^2 11\cdot 13\cdot 17 \\ \quad k=4 \end{cases} \xrightarrow{\text{Prop. 2a}} \begin{cases} 3\cdot 5\cdot 7\cdot 71\cdot 193 \\ 3\cdot 5\cdot 7^2 11\cdot 13\cdot 17 \\ \quad k=5 \end{cases} \xrightarrow{\text{Prop. 1a}} \begin{cases} 3^2\cdot 5\cdot 7\cdot 71\cdot 193 \\ 3^2\cdot 5\cdot 7^2 11\cdot 13\cdot 17 \\ \quad k=5 \end{cases}.$$

By applying Proposition 2a with $p = 2$ to the second pair in this chain, we obtain a pair with multiplier $k = 10$ (which turns out to be the *smallest* pair with this multiplier).

In Tables 1–2 the 58 primitive ϕ–amicable pairs $(a_0, a_1; k)$ with $a_1 \leq 10^6$ are listed, for increasing values of a_1. The last column gives pairs $b, \overline{k}$ for which $(ba_0, ba_1; \overline{k})$ is a non-primitive ϕ–amicable pair—but with squarefree greatest common divisor—obtained by (possibly repeated) application of Proposition 2a.

The total number of primitive ϕ–amicable pairs with larger member $\leq 10^9$ is 499. They occur with multipliers 3, 4, 5, and 7, and frequencies 109, 158, 144, and 88, respectively. We have applied Proposition 2a to these pairs (where possible repeatedly), and generated 475 non-primitive pairs with multipliers 5, 6, 7, 8, 9, 10, and 11, and frequencies 34, 109, 20, 158, 109, 34, and 11, respectively. Of these 475 non-primitive pairs, 313 have larger member $\leq 10^9$. So there are 812 ϕ–amicable pairs $(a_0, a_1; k)$ with $a_1 \leq 10^9$ for which $\gcd(a_0, a_1)$ is squarefree.

[1] Report [5] gives a slightly different definition which is equivalent to the one given here.

TABLE 1. The first 30 primitive ϕ–amicable pairs of the 58 with larger member $\leq 10^6$.

#	a_0	a_1	k	$b, \overline{k}$
1	$4 = 2^2$	$6 = 2 \cdot 3$	5	
2	$78 = 2 \cdot 3 \cdot 13$	$90 = 2 \cdot 3^2 5$	7	
3	$465 = 3 \cdot 5 \cdot 31$	$495 = 3^2 5 \cdot 11$	4	2, 8
4	$438 = 2 \cdot 3 \cdot 73$	$570 = 2 \cdot 3 \cdot 5 \cdot 19$	7	
5	$609 = 3 \cdot 7 \cdot 29$	$735 = 3 \cdot 5 \cdot 7^2$	4	2, 8
6	$1158 = 2 \cdot 3 \cdot 193$	$1530 = 2 \cdot 3^2 5 \cdot 17$	7	
7	$2530 = 2 \cdot 5 \cdot 11 \cdot 23$	$3630 = 2 \cdot 3 \cdot 5 \cdot 11^2$	7	
8	$3685 = 5 \cdot 11 \cdot 67$	$4235 = 5 \cdot 7 \cdot 11^2$	3	2, 6; 6, 9
9	$3934 = 2 \cdot 7 \cdot 281$	$4466 = 2 \cdot 7 \cdot 11 \cdot 29$	5	
10	$5475 = 3 \cdot 5^2 73$	$6045 = 3 \cdot 5 \cdot 13 \cdot 31$	4	2, 8
11	$5978 = 2 \cdot 7^2 61$	$6622 = 2 \cdot 7 \cdot 11 \cdot 43$	5	
12	$7525 = 5^2 7 \cdot 43$	$7595 = 5 \cdot 7^2 31$	3	2, 6; 6, 9
13	$11925 = 3^2 5^2 53$	$13035 = 3 \cdot 5 \cdot 11 \cdot 79$	4	2, 8
14	$12207 = 3 \cdot 13 \cdot 313$	$17745 = 3 \cdot 5 \cdot 7 \cdot 13^2$	4	2, 8
15	$21035 = 5 \cdot 7 \cdot 601$	$22165 = 5 \cdot 11 \cdot 13 \cdot 31$	3	2, 6; 6, 9
16	$19815 = 3 \cdot 5 \cdot 1321$	$22425 = 3 \cdot 5^2 13 \cdot 23$	4	2, 8
17	$22085 = 5 \cdot 7 \cdot 631$	$23275 = 5^2 7^2 19$	3	2, 6; 6, 9
18	$21189 = 3 \cdot 7 \cdot 1009$	$27195 = 3 \cdot 5 \cdot 7^2 37$	4	2, 8
19	$40334 = 2 \cdot 7 \cdot 43 \cdot 67$	$42826 = 2 \cdot 7^2 19 \cdot 23$	5	
20	$59045 = 5 \cdot 7^2 241$	$61915 = 5 \cdot 7 \cdot 29 \cdot 61$	3	2, 6; 6, 9
21	$66795 = 3 \cdot 5 \cdot 61 \cdot 73$	$71445 = 3 \cdot 5 \cdot 11 \cdot 433$	4	2, 8
22	$60726 = 2 \cdot 3 \cdot 29 \cdot 349$	$75690 = 2 \cdot 3^2 5 \cdot 29^2$	7	
23	$65945 = 5 \cdot 11^2 109$	$76615 = 5 \cdot 7 \cdot 11 \cdot 199$	3	2, 6; 6, 9
24	$70422 = 2 \cdot 3 \cdot 11^2 97$	$77418 = 2 \cdot 3^2 11 \cdot 17 \cdot 23$	7	
25	$73486 = 2 \cdot 7 \cdot 29 \cdot 181$	$77714 = 2 \cdot 7^2 13 \cdot 61$	5	
26	$70334 = 2 \cdot 11 \cdot 23 \cdot 139$	$81466 = 2 \cdot 7 \cdot 11 \cdot 23^2$	5	
27	$89745 = 3 \cdot 5 \cdot 31 \cdot 193$	$94575 = 3 \cdot 5^2 13 \cdot 97$	4	2, 8
28	$94666 = 2 \cdot 11 \cdot 13 \cdot 331$	$103334 = 2 \cdot 7 \cdot 11^2 61$	5	
29	$87591 = 3 \cdot 7 \cdot 43 \cdot 97$	$105945 = 3 \cdot 5 \cdot 7 \cdot 1009$	4	2, 8
30	$109298 = 2 \cdot 7 \cdot 37 \cdot 211$	$117502 = 2 \cdot 7^2 11 \cdot 109$	5	

The smallest pair with multiplier 11 comes from primitive pair number 136:

$$13290459 = 3 \cdot 7 \cdot 13 \cdot 89 \cdot 547,\ 14385189 = 3 \cdot 7 \cdot 13 \cdot 23 \cdot 29 \cdot 79,\ k = 4,$$

by multiplication of both members by $b = 110$ (i.e., by applying Proposition 2a successively with $p = 2$, 5, and 11). Furthermore, with the factors $b = 2, 5, 10$, pair number 136 gives three other pairs with multipliers 8, 5, and 10, respectively. The complete list of 499 primitive ϕ–amicable pairs with larger member $\leq 10^9$ is given in [5]. For each primitive pair $(a_0, a_1; k)$ the pairs $(b, \overline{k})$ are given for which $(ba_0, ba_1; \overline{k})$ is a ϕ–amicable pair, obtained by (possibly repeated) application of Proposition 2a.

It is easy to show that pairs with multipliers 3 and 4, and with squarefree greatest common divisor, must be primitive. All the pairs with multiplier 6, 8, 9, and 10 which we found are non-primitive, but we do not know whether there exist *primitive*

TABLE 2. The final 28 primitive ϕ–amicable pairs of the 58 with larger member $\leq 10^6$.

#	a_0	a_1	k	b, $\overline{k}$
31	$119434 = 2 \cdot 7 \cdot 19 \cdot 449$	$122486 = 2 \cdot 7 \cdot 13 \cdot 673$	5	
32	$164486 = 2 \cdot 7 \cdot 31 \cdot 379$	$175714 = 2 \cdot 7^2 11 \cdot 163$	5	
33	$211002 = 2 \cdot 3 \cdot 11 \cdot 23 \cdot 139$	$214038 = 2 \cdot 3^2 11 \cdot 23 \cdot 47$	7	
34	$239775 = 3 \cdot 5^2 23 \cdot 139$	$245985 = 3 \cdot 5 \cdot 23^2 31$	4	2, 8
35	$326325 = 3 \cdot 5^2 19 \cdot 229$	$330315 = 3 \cdot 5 \cdot 19^2 61$	4	2, 8
36	$267486 = 2 \cdot 3 \cdot 109 \cdot 409$	$349410 = 2 \cdot 3 \cdot 5 \cdot 19 \cdot 613$	7	
37	$287763 = 3 \cdot 7 \cdot 71 \cdot 193$	$357357 = 3 \cdot 7^2 11 \cdot 13 \cdot 17$	4	5, 5; 2, 8; 10, 10
38	$350405 = 5 \cdot 11 \cdot 23 \cdot 277$	$378235 = 5 \cdot 11 \cdot 13 \cdot 23^2$	3	2, 6; 14, 7; 6, 9
39	$367114 = 2 \cdot 11^2 37 \cdot 41$	$424886 = 2 \cdot 7 \cdot 11 \cdot 31 \cdot 89$	5	
40	$335814 = 2 \cdot 3 \cdot 97 \cdot 577$	$438330 = 2 \cdot 3 \cdot 5 \cdot 19 \cdot 769$	7	
41	$363486 = 2 \cdot 3 \cdot 29 \cdot 2089$	$455010 = 2 \cdot 3 \cdot 5 \cdot 29 \cdot 523$	7	
42	$441035 = 5 \cdot 7 \cdot 12601$	$466165 = 5 \cdot 7 \cdot 19 \cdot 701$	3	2, 6; 6, 9
43	$444414 = 2 \cdot 3 \cdot 17 \cdot 4357$	$531330 = 2 \cdot 3 \cdot 5 \cdot 89 \cdot 199$	7	
44	$545566 = 2 \cdot 7^2 19 \cdot 293$	$558194 = 2 \cdot 7 \cdot 13 \cdot 3067$	5	
45	$494054 = 2 \cdot 11 \cdot 17 \cdot 1321$	$561946 = 2 \cdot 7 \cdot 11 \cdot 41 \cdot 89$	5	
46	$344810 = 2 \cdot 5 \cdot 29^2 41$	$564630 = 2 \cdot 3 \cdot 5 \cdot 11 \cdot 29 \cdot 59$	7	
47	$442686 = 2 \cdot 3 \cdot 89 \cdot 829$	$577410 = 2 \cdot 3 \cdot 5 \cdot 19 \cdot 1013$	7	
48	$546675 = 3 \cdot 5^2 37 \cdot 197$	$582285 = 3 \cdot 5 \cdot 11 \cdot 3529$	4	2, 8
49	$573545 = 5 \cdot 7^2 2341$	$605815 = 5 \cdot 7 \cdot 19 \cdot 911$	3	2, 6; 6, 9
50	$614185 = 5 \cdot 11 \cdot 13 \cdot 859$	$621335 = 5 \cdot 11^2 13 \cdot 79$	3	2, 6; 14, 7; 6, 9
51	$489363 = 3 \cdot 7^2 3329$	$628845 = 3 \cdot 5 \cdot 7 \cdot 53 \cdot 113$	4	2, 8
52	$624358 = 2 \cdot 7^2 23 \cdot 277$	$650762 = 2 \cdot 7 \cdot 23 \cdot 43 \cdot 47$	5	
53	$722502 = 2 \cdot 3^2 11 \cdot 41 \cdot 89$	$755898 = 2 \cdot 3 \cdot 11 \cdot 13 \cdot 881$	7	
54	$756925 = 5^2 13 \cdot 17 \cdot 137$	$809795 = 5 \cdot 7 \cdot 17 \cdot 1361$	3	2, 6; 6, 9
55	$793914 = 2 \cdot 3 \cdot 11 \cdot 23 \cdot 523$	$813846 = 2 \cdot 3 \cdot 11^2 19 \cdot 59$	7	
56	$806386 = 2 \cdot 7 \cdot 239 \cdot 241$	$907214 = 2 \cdot 7 \cdot 11 \cdot 43 \cdot 137$	5	
57	$886006 = 2 \cdot 11 \cdot 17 \cdot 23 \cdot 103$	$909194 = 2 \cdot 11^2 13 \cdot 17^2$	5	
58	$898835 = 5 \cdot 7 \cdot 61 \cdot 421$	$915565 = 5 \cdot 7^2 37 \cdot 101$	3	2, 6; 6, 9

pairs with such a multiplier. Notice that we found both primitive and non-primitive pairs with multipliers 5 and 7. The smallest non-primitive pairs with multipliers 5 and 7 are

$$1438815 = 3 \cdot 5 \cdot 7 \cdot 71 \cdot 193,\ 1786785 = 3 \cdot 5 \cdot 7^2 11 \cdot 13 \cdot 17,\ k = 5,$$

and

$$4905670 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 277,\ 5295290 = 2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23^2,\ k = 7,$$

respectively; their “mother pairs” are numbers 37 and 38 in Table 2, respectively.

From each ϕ–amicable pair (a_0, a_1) with $\gcd(a_0, a_1) > 1$ it is possible to generate infinitely many others with Proposition 1a. For example, from the primitive pair $(2 \cdot 5 \cdot 11 \cdot 23, 2 \cdot 3 \cdot 5 \cdot 11^2, 7)$ (number 7 in Table 1) we generate the non-primitive pairs:

$$(2^{i_1+1} 5^{i_2+1} 11^{i_3+1} 23, 2^{i_1+1} 3 \cdot 5^{i_2+1} 11^{i_3+2}, 7),\ i_1, i_2, i_3 \geq 0,\ i_1 + i_2 + i_3 > 0.$$

3. Basic properties of primitive ϕ–amicable pairs

In this section we list ten basic properties of primitive ϕ–amicable pairs. Most of the proofs are omitted. They are simple exercises, or see [5].

Let (a_0, a_1) be a primitive ϕ-amicable pair with $a_0 < a_1$, and let p be a prime.

B1 *We cannot have $a_0 = 2$ or $a_0 = 3$.*

B2[2] *For $i = 0$ or 1, if $p^2 \mid a_i$, then $p^2 \parallel a_i$ and $p \parallel a_{1-i}$.* *Proof:* If $p^3 \mid a_i$, then $k\phi(a_i) = kp^2\phi(a_i/p^2) = p^3(a_i/p^3) + a_{1-i}$, so $p^2 \mid a_{1-i}$ which would make (a_0, a_1) non-primitive. So $p^2 \parallel a_i$. From this, it follows in a similar way that $p \parallel a_{1-i}$.

B3 *For $i = 0$ or 1, if $4 \mid a_i$, then $(a_0, a_1) = (4, 6)$.*

B4 *a_0 is even if and only if a_1 is even.*

B5 *a_1 is not prime.* *Proof:* This is true for any a_0, a_1 satisfying $1 < a_0 < a_1$ and $\phi(a_0) = \phi(a_1)$, for then $\phi(a_1) < a_0$. If $a_1 = p$, then $\phi(a_1) = p - 1 < a_0 < p = a_1$, which is impossible.

B6 *For $i = 0$ or 1, a_i cannot be a perfect square, or twice a perfect square, except if $(a_0, a_1) = (4, 6)$.*

B7 *We cannot have $a_0 \mid a_1$.*

B8 *For at least one prime p, $p \parallel a_0a_1$.* *Proof:* Let $i = 0$ or 1. If the result is not true, then for all primes q dividing a_0a_1 we have either $q \parallel a_i$ and $q \parallel a_{1-i}$, or $q^2 \parallel a_i$ and $q \parallel a_{1-i}$. Cancellation of factors $(q-1)$ from both sides of the equation $\phi(a_0) = \phi(a_1)$ then leads to a denial of the Fundamental Theorem of Arithmetic, since $a_0 \neq a_1$.

B9 *If a_0 and a_1 are squarefree, then $3 \mid a_0$ if and only if $3 \mid a_1$.*

B10 *If (a_0, a_1) has multiplier $k = 3$, then the smallest prime divisor of a_0a_1 is at least 5. If (a_0, a_1) has multiplier $k = 4$, then a_0a_1 is odd.* *Proof:* From (1) it follows that

$$k = \frac{a_0}{\phi(a_0)} + \frac{a_1}{\phi(a_1)}. \tag{3}$$

Let $i = 0$ or 1. If a_i is even, then by **B4** also a_{1-i} is even, so that $a_i/\phi(a_i) > 2$ for $i = 0, 1$, which contradicts (3) if $k \leq 4$. If $3 \mid a_i$, then $(a_0 + a_1)/3$ can only be integral if $3 \mid a_{1-i}$. This implies that $a_i/\phi(a_i) > 3/2$ for $i = 0, 1$, contradicting (3) if $k = 3$.

4. ϕ–amicable pairs with a given prime structure

Property **B5** states that there exists no ϕ–amicable pair for which the larger member is a prime. Here, we shall study pairs with a given prime structure more generally. First we have the following general finiteness result:

Theorem 1. *There are only finitely many primitive ϕ–amicable pairs with a given number of different prime factors.*

Proof. This proof is inspired by analogous results of Borho [1] for ordinary and unitary amicable pairs. First we notice that if the total number t of different prime factors of a primitive ϕ–amicable pair (a_0, a_1) is prescribed, then there are only finitely many values of k, r, and s with $r + s = t$ for which

$$k = \frac{a_0}{\phi(a_0)} + \frac{a_1}{\phi(a_1)} = \prod_{i=1}^{r} \frac{p_i}{p_i - 1} + \prod_{j=1}^{s} \frac{q_j}{q_j - 1}$$

[2] In [5], **B2** is given in the weaker form: *if $p^2 \parallel a_i$, then $p \parallel a_{1-i}$*, and similarly for **B3**.

can hold. Therefore we are done if we can show that the equation

$$k = \frac{x_1}{x_1 - 1} \cdots \frac{x_r}{x_r - 1} + \frac{y_1}{y_1 - 1} \cdots \frac{y_s}{y_s - 1} \tag{4}$$

has finitely many solutions in integers ≥ 2, for given k, r, and s. Here, we should realize that the only solutions of (4) that can actually correspond to a primitive ϕ–amicable pair are those for which all the x_i's and y_j's are prime. In those cases in which $x_{i_1} = y_{j_1} = p$, say, with corresponding $a_0 = \prod_{i=1}^{r} x_i$ and $a_1 = \prod_{j=1}^{s} y_j$, also $(a_0 p, a_1)$, and $(a_0, a_1 p)$ are potential primitive ϕ–amicable pairs (leading to the same equation (4)). However, only at most one of these three can actually be a primitive ϕ–amicable pair (if one of them satisfies the first equality in (1), the others do not).

Suppose (4) has infinitely many solutions. Let $z = (z_1, \ldots, z_t)$ be a solution of (4) and denote by $z^{(1)}, z^{(2)}, \ldots,$ an infinite sequence of different solutions. Then this contains an infinite subsequence in which the last component is non-decreasing, i.e., without loss of generality we may assume $z_t^{(1)} \leq z_t^{(2)} \leq \ldots.$ This reasoning can be repeated for the components $t-1, t-2, \ldots, 1$, so that after t subsequence-transitions we finally have a (still) infinite subsequence of different solutions ordered in such a way that each component of one solution is not greater than the corresponding component of the next solution. However, since the right-hand side of (4) is monotonically decreasing in each of its variables, if $z^{(1)}$ is a solution, then $z^{(2)}$ cannot be a solution. This is a contradiction. □

The next theorem gives an upper bound for the smallest odd prime divisor of a ϕ–amicable pair, as a function of the multiplier and the maximum of the numbers of different prime factors in the two members.

Theorem 2. *Let $(a_0, a_1; k)$ be a ϕ–amicable pair, where a_0 and a_1 have r and s different prime divisors, respectively. Let P be the smallest odd prime divisor of $a_0 a_1$ and $m = \max\{r, s\}$. Then*

$$P \leq \frac{k + 4m - 8}{k - 4} \quad \textit{with } k \geq 5, \textit{ if the pair is even,} \tag{5}$$

and

$$P \leq \frac{k + 2m - 2}{k - 2} \quad \textit{with } k \geq 3, \textit{ if the pair is odd.} \tag{6}$$

Proof. We have

$$k = \prod_{p \mid a_0} \frac{p}{p-1} + \prod_{q \mid a_1} \frac{q}{q-1} \quad (p, q \text{ primes}).$$

If both a_0 and a_1 are even, **B10** implies that $k \geq 5$. Furthermore, since consecutive primes differ at least by 1, it follows that

$$k \leq 2 \prod_{i=1}^{r-1} \frac{P + i - 1}{P + i - 2} + 2 \prod_{i=1}^{s-1} \frac{P + i - 1}{P + i - 2} \leq 4 \prod_{i=1}^{m-1} \frac{P + i - 1}{P + i - 2}.$$

Cancellation in the last product yields

$$k \leq 4 \frac{P + m - 2}{P - 1},$$

and the result follows. The proof is similar if both a_0 and a_1 are odd. □

Remark. Using the fact that consecutive odd primes differ at least by 2, we can show that $P \le (2k^2+32m-64)/(k^2-16)$ if the pair is even (which is sharper than (5) for $k < 4(m-2)$) and $P \le (2k^2+8m-8)/(k^2-4)$ if the pair is odd (which is sharper than (6) for $k < 2(m-1)$).

The next theorem deals with ϕ–amicable pairs (a_0, a_1) in which $\gcd(a_0, a_1) = 1$. A corollary concerns ϕ–amicable pairs of the form (p, a) in which p is a prime number with $p < a$.

Theorem 3. *If* $(a_0, a_1; k)$ *is a* ϕ*-amicable pair with* $\gcd(a_0, a_1) = 1$*, then*

$$k - 1 < \frac{a_0 a_1}{\phi(a_0 a_1)} < \frac{k^2}{4}, \tag{7}$$

and $a_0 a_1$ *has at least twelve different prime factors.*

Proof. Recall that, for any real $x > 0$, $x + (1/x) \ge 2$, with equality only when $x = 1$. Since $\gcd(a_0, a_1) = 1$, we have, from (1),

$$\frac{\phi(a_0 a_1)}{a_0 a_1} = \frac{\phi(a_0)\phi(a_1)}{a_0 a_1} = \frac{(a_0+a_1)^2}{k^2 a_0 a_1} = \frac{1}{k^2}\left(\frac{a_0}{a_1} + 2 + \frac{a_1}{a_0}\right) \ge \frac{4}{k^2}.$$

Since $a_0 < a_1$, we have the right-hand inequality in (7). For the left-hand inequality, we note simply that $(a_i/\phi(a_i)) - 1 > 0$ for $i = 0, 1$. Then

$$k = \frac{a_0}{\phi(a_0)} + \frac{a_1}{\phi(a_1)} < \frac{a_0 a_1}{\phi(a_0)\phi(a_1)} + 1 = \frac{a_0 a_1}{\phi(a_0 a_1)} + 1.$$

To prove that $a_0 a_1$ has at least twelve different prime factors, we set $A = a_0 a_1$, $F(A) = A/\phi(A) = \prod_{p|A} p/(p-1)$, and ω equal to the number of different prime factors of A. Let $A = p_1 p_2 \ldots p_\omega$, with $p_1 < p_2 < \cdots < p_\omega$. We require the following observations. (i) By **B2**, **B4** and **B9**, A is squarefree and not divisible by 2 or 3. (ii) Using (1), if p and q are primes with $p \mid A$ and $q \bmod p = 1$, then $q \nmid A$. (iii) Since $\phi(a_0) = \phi(a_1)$ and $\phi(a_0 a_1) = \phi(a_0)\phi(a_1)$, $\phi(A)$ is a perfect square.

Suppose $k \ge 4$. We have $p_1 \ge 5$, $p_2 \ge 7$, $\ldots$. If $\omega \le 32$, then

$$F(A) \le F(5 \cdot 7 \cdot 11 \cdot \ldots \cdot 139) < 2.994,$$

(there being 32 primes from 5 to 139, inclusive) and we have a contradiction of the left-hand inequality in (7). So $\omega \ge 33$ when $k \ge 4$.

Now suppose that $k = 3$. It is relatively easy to show that $\omega \ge 11$. To show that in fact $\omega \ge 12$, we assume that $\omega = 11$ and show this to be untenable. The earlier calculations (omitted here) show that we must have $5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \mid A$. We cannot have $p_8 \ge 73$, for in that case $p_1 = 5, \ldots, p_7 = 37$, $p_8 \ge 73$, $p_9 \ge 83$, $p_{10} \ge 89$, $p_{11} \ge 97$, and

$$F(A) \le F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 73 \cdot 83 \cdot 89 \cdot 97) < 1.997,$$

contradicting the left-hand inequality in (7). Therefore, $p_8 = 59$ or 67.

If $p_8 = 67$, then $p_9 = 73$, since

$$F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 83 \cdot 89 \cdot 97) < 1.9992,$$

and $p_{10} = 83$ or 89, since

$$F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 73 \cdot 97 \cdot 107) < 1.999.$$

If $p_{10} = 89$, then $p_{11} \in \{97, 107, 109\}$, since

$$F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 73 \cdot 89 \cdot 163) < 1.995.$$

In all three cases, since $23 \cdot 67 \cdot 89 \mid A$, we have $11^3 \parallel \phi(A)$ so $\phi(A)$ is not a perfect square. If $p_{10} = 83$, then $p_{11} \in \{89, 97, 107, 109\}$, since

$$F(5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 37 \cdot 67 \cdot 73 \cdot 83 \cdot 163) < 1.996.$$

In these cases, since $83 \mid A$, we have $41 \parallel \phi(A)$.

The proof continues in this way, on the assumption next that $p_8 = 59$. $\square$

Corollary. *If p is prime, $p < a$ and (p, a) is a ϕ-amicable pair, then a has at least twelve different prime factors.*

The proof makes use of the calculations in the proof of Theorem 3, and the fact that, when $k = 3$, $p - 1 = \phi(a) = (p + a)/3$ from which $2\phi(a) = a + 1$.

The odd numbers > 1 we know satisfying the equation $a + 1 = 2\phi(a)$ are $a = 3$, $3 \cdot 5$, $3 \cdot 5 \cdot 17$, $3 \cdot 5 \cdot 17 \cdot 257$, $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537$, $3 \cdot 5 \cdot 17 \cdot 353 \cdot 929$, and $3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 83623937$ (cf. [6, Problem B37]), but for all of them $3 \mid a$, so that these cannot be a member of a ϕ–amicable pair (p, a) with multiplier 3. In Section 5 we will see that solutions of the equation $a + 1 = 2\phi(a)$ sometimes can help to generate new ϕ–amicable pairs.

The following propositions show how from a number a satisfying $a + 1 = 2\phi(a)$ other numbers with that property can be found.

Proposition 3. *If $a+1 = 2\phi(a)$ and if $q = a+2$ is a prime number, then $aq+1 = 2\phi(aq)$.*

Proposition 4. *Let $a+1 = 2\phi(a)$ and write $a^2+a+1 = D_1D_2$ with $0 < D_1 < D_2$. If both $q = a+1+D_1$ and $r = a+1+D_2$ are prime numbers, then $aqr+1 = 2\phi(aqr)$.*

We have also proved the following theorem in which one of the members of a ϕ–amicable pair has precisely two distinct prime factors.

Theorem 4. *Except for the pair $(4, 6)$, if one member of a ϕ-amicable pair has exactly two distinct prime factors, then the other member has at least four distinct prime factors.*

The proof is largely computational.

5. Analogies with amicable pairs

We have determined some analogies with amicable pairs in the construction of ϕ–amicable pairs with a given prime structure. Assume, for example, that a_0 and a_1 have the form $a_0 = ar$, $a_1 = apq$, where p, q, and r are distinct primes not dividing a. Examples are the pairs numbered 4 and 9 in Table 1. Substitution in (1) yields after some simple calculations:

$$r = (p - 1)(q - 1) + 1 \text{ and } (cp - d)(cq - d) = a(c + d), \tag{8}$$

where

$$c = k\phi(a) - 2a \text{ and } d = k\phi(a) - a. \tag{9}$$

It is convenient now to choose a and k such that c is a small positive number. For example, if we choose $a = 5 \cdot 7$ and $k = 3$, then $c = 2$ and $d = 37$. The second equation in (8) then reduces to:

$$(2p - 37)(2q - 37) = 1365 = 3 \cdot 5 \cdot 7 \cdot 13. \tag{10}$$

Writing the right-hand side as 1×1365 and equating with the two factors in the left-hand side yields $p = 19$ and $q = 701$, and, from the first equation in (8), $r = 12601$, p, q and r being primes. This gives the ϕ–amicable pair

$$441035 = 5 \cdot 7 \cdot 12601, \quad 466165 = 5 \cdot 7 \cdot 19 \cdot 701, \quad k = 3, \tag{11}$$

which is number 42 in Table 2. Other ways of writing the right-hand side of (10) as a product of two factors do not yield success. The two pairs of this form in Table 1 are obtained by choosing $a = 6$, $k = 7$ (number 4), and $a = 14$, $k = 5$ (number 9); both cases have $c = 2$.

For $c = 2, 4, 6, 8$, and 10, we have computed all the numbers a with $2 \le a \le 10^5$ for which $(c + 2a)/\phi(a)$ is an integer (called k in (9)). We found 76 solutions, and for each of them, we checked whether there are primes p, q, and r satisfying (8). As a result we only found five primitive ϕ–amicable pairs of the form (ar, apq), and they all occur in our list of 499 primitive ϕ–amicable pairs below 10^9 (namely, as numbers 4, 9, 42, 109, and 148). In the case $c = 2$, $k = 4$, the first equation in (9) reduces to $1 = 2\phi(a) - a$, which occurs in the Corollary to Theorem 3. For the three largest solutions given below this Corollary (the other four have $a \le 10^5$), we also checked (9), and we found the 24-digit primitive ϕ–amicable pair with multiplier $k = 4$:

$$\begin{aligned} 643433053433705010822135 &= 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 7694364698739721, \\ 643433068822434241053705 &= 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929 \cdot 64231061 \cdot 119791963. \end{aligned} \tag{12}$$

This construction method is analogous to similar methods known for (ordinary) amicable numbers. The simplest example is the so-called *rule of Thābit ibn Qurrah* [3] which constructs amicable pairs of the form (apq, ar) where a is a power of 2:

If the three numbers $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$, *and* $r = 9 \cdot 2^{2n-1} - 1$ *are prime numbers, then* $2^n pq$ *and* $2^n r$ *form an amicable pair.*

This rule yields amicable pairs for $n = 2, 4$, and 7, but for no other values of $n \le 20,000$ [2].

The ϕ–amicable pairs which we have constructed so far are squarefree (if we choose a to be squarefree). We also tried to find pairs of the form $a_1 = as^2r$, $a_2 = aspq$, where s is a prime not dividing a. Similarly to the above derivation, we found the two ϕ–amicable pairs

$$2609115 = 3 \cdot 5 \cdot 31^2 181, \quad 2747685 = 3 \cdot 5 \cdot 31 \cdot 19 \cdot 311, \quad k = 4, \tag{13}$$

$$11085135 = 3 \cdot 5 \cdot 31^2 769, \quad 11770545 = 3 \cdot 5 \cdot 31 \cdot 17 \cdot 1489, \quad k = 4. \tag{14}$$

From the pair (11), two more pairs with squarefree greatest common divisor can be generated with the help of Proposition 2a, namely with $p = 2$, and (next) $p = 3$; from each of the pairs (12), (13) and (14), one more such pair can be generated with Proposition 2a, $p = 2$.

With amicable pairs of the form (au, ap) where p is a prime and $\gcd(a, p) = 1$, it is often possible to associate many other amicable pairs of the form (auq, ars) with the following rule [10, Theorem 2]:

Let (au, ap) *be a given amicable pair, where* p *is a prime with* $\gcd(a, p) = 1$ *and let* $C = (p + 1)(p + u)$. *Write* $C = D_1 D_2$ *with* $0 < D_1 < D_2$. *If the three integers* $r = p + D_1$, $s = p + D_2$, *and* $q = u + r + s$ *are primes not dividing* a, *then* (auq, ars) *is also an amicable pair.*

At present, we know 319 amicable pairs of the required form (au, ap), and for almost all of them the number C has extremely many divisors. Consequently, many

TABLE 3. The 79 primitive ϕ–amicable pairs generated by applying Theorem 5 to the 38 primitive ϕ–amicable pairs $\leq 10^9$ which are of the form (ap, au) with p a prime, $\gcd(p, a) = 1$.

#	a	p	u	k	# D_1	the D_1's
4	6	73	95	7	3	16, 24, 36
8	55	67	77	3	1	16
9	14	281	319	5	2	32, 140
14	39	313	455	4	1	256
17	35	631	665	3	2	30, 486
26	506	139	161	5	1	138
38	1265	277	299	3	1	54
42	35	12601	13319	3	5	40, 810, 4032, 9450, 13440
50	715	859	869	3	2	78, 192
65	1518	829	851	7	5	24, 108, 168, 184, 630
89	273	14561	16159	4	6	32, 512, 1040, 1536, 3840, 7280
109	285	29641	30263	4	2	520, 9880
140	465	31249	33263	4	4	4, 378, 13392, 43008
141	598	21529	25991	5	1	4290
148	255	79561	80183	4	5	340, 1536, 11232, 17238, 97920
168	602	57793	63167	5	7	336, 774, 5418, 7840, 8428, 16254, 43008
203	285	199501	203699	4	10	240, 1536, 2128, 2880, 14250, 25536, 161280, 178752, 220500, 238336
225	6	10266913	13689215	7	2	653184, 1586304
267	6	18196993	24262655	7	4	86016, 180936, 5677056, 17842176
318	935	269281	283679	3	3	1350, 61440, 345600
332	6578	44851	45149	5	4	120, 720, 1170, 1656
365	506	642529	754271	5	3	270, 133860, 451632
409	5478	87649	96031	7	3	73920, 86592, 95284
488	2485	365509	375803	3	2	139392, 274912

thousands of new amicable pairs have been found with this rule, including several large pairs [9, Lemma 1].

Such a rule also exists for ϕ–amicable pairs. We have the following result.

Theorem 5. *Let $(ap, au; k)$ be a given ϕ–amicable pair, where p is a prime with $\gcd(a, p) = 1$ (notice that a and u need not be coprime), and let $C = (p-1)(p+u)$. Write $C = D_1 D_2$ with $0 < D_1 < D_2$. If the three integers $r = p + D_1$, $s = p + D_2$, and $q = u + r + s$ are primes not dividing a, then $(ars, auq; k)$ is also a ϕ–amicable pair.*

This result has been applied to those primitive ϕ–amicable pairs with larger member $\leq 10^9$ which are of the required form, and to the large pair (12).

From the pairs with larger member $\leq 10^9$, 79 primitive ϕ–amicable pairs were generated. Table 3 gives the rank number of the "mother" pair of the form (ap, au) in column 1, the values of a, p, u, and k in columns 2–5, the number of primitive ϕ–amicable pairs generated from this mother pair in column 6, and the "successful" values of D_1 in column 7. To any listed value of D_1 corresponds a primitive ϕ–amicable pair of the form (ars, auq) with $r = p + D_1$, $s = p + (p-1)(p+u)/D_1$, and $q = u + r + s$, with the same multiplier as the mother pair. The three pairs generated by mother pair number 4 have larger member $\leq 10^6$, and occur as numbers 36, 40,

TABLE 4. Six primitive ϕ–amicable pairs with larger member $> 10^6$ and $\leq 10^9$, found with Theorem 5 from pairs 8, 9, 14, 17 and 26 in Table 1.

from #	a_0	a_1	k
8	$3017465 = 5 \cdot 11 \cdot 83 \cdot 661$	$3476935 = 5 \cdot 7 \cdot 11^2 821$	3
9	$8729014 = 2 \cdot 7 \cdot 421 \cdot 1481$	$9918986 = 2 \cdot 7 \cdot 11 \cdot 29 \cdot 2221$	5
9	$24236842 = 2 \cdot 7 \cdot 313 \cdot 5531$	$27523958 = 2 \cdot 7 \cdot 11 \cdot 29 \cdot 6163$	5
14	$27716559 = 3 \cdot 13 \cdot 569 \cdot 1249$	$40334385 = 3 \cdot 5 \cdot 7 \cdot 13^2 2273$	4
26	$61531118 = 2 \cdot 11 \cdot 23 \cdot 277 \cdot 439$	$71445682 = 2 \cdot 7 \cdot 11 \cdot 23^2 877$	5
17	$90348545 = 5 \cdot 7 \cdot 1117 \cdot 2311$	$95264575 = 5^2 7^2 19 \cdot 4093$	3

and 47 (with $D_1 = 36$, 24, and 16, respectively) in Table 2. Furthermore, among the 79 pairs, there are six with larger member between 10^6 and 10^9 (also found with our exhaustive search). We list them in Table 4.

From pair (12) we found eight new large primitive ϕ–amicable pairs. The values of D_1 in Theorem 5 leading to these eight pairs are:

$$73914531840,\ 76666855680,\ 7394851553280,\ 123635643997056,$$
$$193847579836416,\ 865200857636352,\ 3982965255818208,\ 4194831919218688.$$

All pairs have multiplier $k = 4$. In the largest pair (with $D_1 = 73914531840$) both members have 46 decimal digits:

$$a_0 = 1030754714216455355643689856057807107041652655,$$
$$a_1 = 1030754738868600773437177012460443629727125585,$$

and $a_0 = ars$, $a_1 = auq$, with $a = 3 \cdot 5 \cdot 17 \cdot 353 \cdot 929$,

$$r = 7694438613271561, \quad s = 160194569901409981543 3,$$
$$u = 64231061 \cdot 119791963, \quad q = 1601961087817595849737.$$

Hence, in addition to the 499 primitive ϕ–amicable pairs with larger member $\leq 10^9$ we have found with the help of the method described in the beginning of this section and with Theorem 5, another 79 primitive ϕ–amicable pairs with larger member $> 10^9$.

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