MATHEMATICS OF COMPUTATION
Volume 67, Number 223, July 1998, Pages 1107–1120
S 0025-5718(98)00943-0

AN ALGORITHM FOR CONSTRUCTING A BASIS FOR C^r-SPLINE MODULES OVER POLYNOMIAL RINGS

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Abstract. Let $\square$ be a polyhedral complex embedded in the euclidean space E^d and $S^r(\square)$, $r \geq 0$, denote the set of all C^r-splines on $\square$. Then $S^r(\square)$ is an R-module where $R = E[x_1, \ldots, x_d]$ is the ring of polynomials in several variables. In this paper we state and prove the existence of an algorithm to write down a free basis for the above R-module in terms of obvious linear forms defining common faces of members of $\square$. This is done for the case when $\square$ consists of a finite number of parallelopipeds properly joined amongst themselves along the above linear forms.

1. Introduction

Let $\square$ be a polyhedral d-complex embedded in the euclidean space E^d, i.e., a compact subset of E^d subdivided into a finite collection of d-dimensional convex polyhedra which are properly joined (see [7] for details). Fix an integer $r \geq 0$. Let $C^r(\square)$ denote the set of real valued functions f defined on $\square$ such that $f \mid \sigma$ is in the polynomial ring $R = E[x_1, \ldots, x_d]$ for each d-face σ of $\square$ and f is r-times continuously differentiable on the whole of $\square$.

The elements of $C^r(\square)$, known as multivariate splines of smoothness r, have proved extremely useful in obtaining approximate solutions of partial differential equations by finite element methods. The set $C^r_k(\square)$ of all those C^r-splines which are of degree $\leq k$, form a real vector space. These vector spaces which are easily computable are usually taken as the approximant spaces for various suitable degrees k. Evidently these are finite dimensional. Determining their dimension as well as a basis having minimal support has been a very interesting and sometimes a difficult proposition of practical importance (see [5], [6], and [7]). The difficulty is caused by the fact that the vector space dimension of $C^r_k(\square)$ depends not only on the combinatorics but also on the geometry of $\square$.

In order to tackle the above "dimension problem", more algebraic approaches using the method of commutative algebra have been recently initiated by Haas [8], Billera and Rose ([3], [5], [6]). With pointwise operations of addition and multiplication the set $C^r(\square)$ forms a ring and the polynomial ring R is just a subring of $C^r(\square)$. Hence $C^r(\square)$ is an R-module in a natural way. It is easily seen that this module is finitely generated, torsion free and of rank equal to the number of d-faces of $\square$. The general question as to under what condition on d, r and $\square$,

Received by the editor December 15, 1994 and, in revised form, March 3, 1997.
1991 *Mathematics Subject Classification.* Primary 41A15, Secondary 13C10.
The first author was supported by the UGC research project no. F 8-5/94 (SR-I).
The second author was supported by the CSIR(JRF) no. 9/97(36)/92/EMR-I.

the R-module $C^r(\Box)$ is free, has been dealt with in [4]. The case when $C^r(\Box)$ is free is of practical importance in applications because in that case knowing a basis will easily determine general properties of a spline function on $\Box$. This would be especially useful when one can determine an algorithm for writing down basis elements of the R-modules $C^r(\Box)$ just by knowing the geometry of $\Box$ alone. It is proved by Billera and Rose [4] that, when $d = 2$, $C^r(\Box)$ is free over R iff $\Box$ is a manifold with boundary. However, even in this case there does not seem to be an algorithm for writing a basis even for especially simple rectangular grids.

The objective of this paper is to provide an algorithm for writing down a basis for the free R-module $C^r(\Box)$ where $\Box$ consists of a grid in the plane obtained by crossing a set of parallel lines (hyperplane in E^2) by another set of parallel lines. In particular, this includes the general polyhedral case of rectangular grids not covered in the case of a simplicial complex. More generally, we prove the existence of an algorithm for writing down an R-basis for $C^r(\Box)$ when $\Box$ consists of general parallelopipeds obtained by mutually intersecting affinely independent d-sets of parallel hyperplanes in E^d.

For the more familiar case when $\Box = \triangle$ is a simplicial complex, R. Haas [8] has studied the question of determining a free basis and a reduced free basis of the R-module $S^r(\triangle)$ for the case of planar cross-cut grids. As an application, using these R-module bases, she has also given techniques for deducing the vector space dimension of the spline space $S^r_k(\triangle)$ in certain cases.

2. Preliminaries

Let $\Box$ be imbedded in E^d. The fact that f is a C^r-spline on $\Box$ means (i) f is a globally C^r-function on $\Box$ and (ii) $f \mid \sigma$ is in the polynomial ring $R = E[x_1, \dots, x_d]$ for each d-face σ in $\Box$. The analytic condition that f is C^r on $\Box$ can be nicely translated into an algebraic condition (see [5]) as follows: Let $I(\sigma)$ denote the ideal of all polynomials in R which vanish on the face σ of $\Box$, and let $(I(\sigma))^r$ denote the r-fold product of $I(\sigma)$ with itself. Then we have

Algebraic Criterion. Let $f : \Box \to R$ be a piecewise polynomial function on $\Box$. Then $f \in C^r(\Box)$ iff for any two d-faces σ_1, σ_2 of $\Box$

$$f \mid \sigma_1 - f \mid \sigma_2 \in (I(\sigma_1 \cap \sigma_2))^{r+1}.$$

Let the faces of $\Box$ be linearly ordered in some manner, say $\sigma_1, \dots, \sigma_t$ where t is the number of d-faces of $\Box$. Then using the algebraic criterion, one can represent a spline $f \in C^r(\Box)$ as a t-tuple of polynomials $f = (f_1, \dots, f_t)$ where $f_i = f \mid \sigma_i$, $i = 1, \dots, t$, satisfying the condition that for each pair σ_i, σ_j of faces of $\Box$, $f_i - f_j \in (I(\sigma_i \cap \sigma_j))^{r+1}$. We will use this representation of splines in constructing a basis for the R-module $C^r(\Box)$ whenever it is free. We must emphasize that as pointed out by Billera and Rose [4], the freeness of R-module $C^r(\Box)$ depends not only on the combinatorics of $\Box$ but also on the geometry of $\Box$. However, when $d = 2$, $C^r(\Box)$ is free iff $\Box$ is a 2-dimensional manifold with boundary and therefore the freeness of $C^r(\Box)$ over R is independent of the geometry and is a combinatorial invariant. Writing a basis, however, and that too in terms of obvious linear forms defining $(d-1)$ faces of $\Box$, is a completely different problem of computational nature.

Let $\Box$ be a parallelogram in E^2 subdivided into four subparallelograms by two lines (hyperplanes in E^2) $l_1 = 0$, $l_2 = 0$, each parallel to a side of the parallelogram (see Fig. 1).

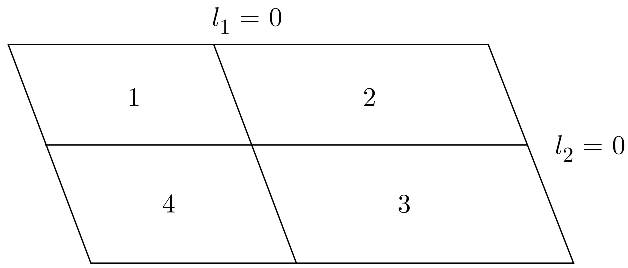

FIGURE 1

An R-basis for the R-module $C^r(\square)$ for this case was computed (Lemma 3.1 of [7]) to be the set consisting of four splines,

$$(1,1,1,1),(0,\tilde{l_1},\tilde{l_2},0),(0,0,\tilde{l_2},\tilde{l_2}),(0,0,\tilde{l_1}\tilde{l_2}).$$

Likewise, when a parallelopiped P in E^3 is subdivided into eight subparallelopipeds by planes $l_1 = 0$, $l_2 = 0$, $l_3 = 0$, each drawn parallel to the faces of P, then again, $C^r(\square)$ is free over R and a basis consisting of eight splines was constructed (ibid., Lemma 3.2). It was also indicated (ibid., Prop. 3.4) that this construction may be formulated in an algorithm for any d-dimensional parallelopiped in E^d subdivided into 2^d subparallelopipeds similar to the above special cases, but no proof was given. Here we extend the above constructions to the following general situations: Let L be a d-dimensional parallelopiped P in E^d which is subdivided into subparallelopipeds by drawing any finite number (not one) of hyperplanes in E^d, each one parallel to a side of P and let $\square$ denote the resulting d-complex. Then, for any $r \geq 0$, there is an algorithm to write down a basis for the free R-module $C^r(\square)$ just by inspection of the geometry of $\square$, i.e., each basis element can be expressed as a t-tuple (t is the number of d-faces of $\square$) of polynomials in which each tuple is a power of the linear form or their products which define these hyperplanes. As in the case of a simplicial complex (see [8]) the number of elements in any R-basis of $C^r(\square)$ would be 2^d—the number of maximal faces of $\square$.

3. A PARTICULAR ORDERING AND THE ALGORITHM

In the statement of our algorithm for writing an R-basis for $C^r(\square)$, the linear ordering of the faces of $\square$ is crucial and we explain it first. We consider the two-dimensional case in which a parallelogram P has been subdivided into $m.n$ subparallelograms by drawing $(m-1)$ lines $k_i = 0$, $i = 1, \dots, m-1$, parallel to one side and $(n-1)$ lines $l_j = 0$, $j = 1, \dots, n-1$, parallel to the other side of P (see Fig. 2).

Let $\square$ be the resulting 2-complex so obtained. Treating each face of $\square$ as an entry in an $m \times n$ matrix, we linearly order the first row in ascending order as 1st, 2nd, ... , nth. The last face of the second row is the $(n+1)$-th element, the last but one is the $(n+2)$-th element etc.; thus the first entry of the second row is the $2n$-th element. Next, the first element of the third row is the $(2n+1)$-th element, then the next one is the $(2n+2)$-nd, etc., the repeated last entry of the third row is the $3n$-th element, last element of the next following row is the $(3n+1)$-st element. In this manner we continue linearly ordering each face until the $(m.n)$-th element which would be either the first face of the last row or the last face of the last row

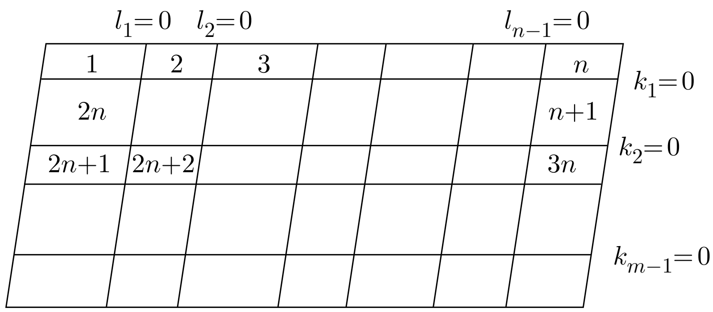

FIGURE 2

depending on whether m is even or odd respectively. We will refer to this linear ordering of the faces of $\square$ as the "snakelike" ordering.

With the above snakelike linear ordering of the $m.n$ faces of $\square$, we now explain our algorithm of writing down the basic $m.n$ C^r-splines as follows: Here $\tilde{k_i}$ and $\tilde{l_j}$ stand for $(k_i)^{r+1}$ and $(l_j)^{r+1}$ respectively. Our basic splines are:

$$b_1 = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \\ 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \end{bmatrix},$$

$$b_3 = \begin{bmatrix} 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \\ 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \end{bmatrix}, \dots, b_n = \begin{bmatrix} 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1} \\ 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1} \end{bmatrix},$$

$$b_{n+1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ \tilde{k_1} & \tilde{k_1} & . & . & . & \tilde{k_1} \\ \tilde{k_1} & \tilde{k_1} & . & . & . & \tilde{k_1} \\ . & . & . & . & . & . \\ \tilde{k_1} & \tilde{k_1} & . & . & . & \tilde{k_1} \end{bmatrix}, \quad b_{n+2} = \begin{bmatrix} 0 & . & . & . & 0 & 0 \\ \tilde{l}_{n-1}\tilde{k_1} & . & . & . & \tilde{l}_{n-1}\tilde{k_1} & 0 \\ \tilde{l}_{n-1}\tilde{k_1} & . & . & . & \tilde{l}_{n-1}\tilde{k_1} & 0 \\ . & . & . & . & . & . \\ \tilde{l}_{n-1}\tilde{k_1} & . & . & . & \tilde{l}_{n-1}\tilde{k_1} & 0 \end{bmatrix},$$

$$b_{n+3} = \begin{bmatrix} 0 & . & . & . & 0 & 0 & 0 \\ \tilde{l}_{n-2}\tilde{k_1} & . & . & . & \tilde{l}_{n-1}\tilde{k_1} & 0 & 0 \\ \tilde{l}_{n-2}\tilde{k_1} & . & . & . & \tilde{l}_{n-1}\tilde{k_1} & 0 & 0 \\ . & . & . & . & . & . & . \\ \tilde{l}_{n-2}\tilde{k_1} & . & . & . & \tilde{l}_{n-2}\tilde{k_1} & 0 & 0 \end{bmatrix}, \dots, b_{2n} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ \tilde{l_1}\tilde{k_1} & 0 & 0 & . & . & . & 0 \\ \tilde{l_1}\tilde{k_1} & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ \tilde{l_1}\tilde{k_1} & 0 & 0 & . & . & . & 0 \end{bmatrix},$$

$$b_{3n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ \tilde{k_2} & \tilde{k_2} & . & . & . & \tilde{k_2} \\ . & . & . & . & . & . \\ \tilde{k_2} & \tilde{k_2} & . & . & . & \tilde{k_2} \end{bmatrix}, \; b_{3n+1} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & \tilde{l_1}\tilde{k_2} & \tilde{l_1}\tilde{k_2} & . & . & . & \tilde{l_1}\tilde{k_2} \\ . & . & . & . & . & . & . \\ 0 & \tilde{l_1}\tilde{k_2} & \tilde{l_1}\tilde{k_2} & . & . & . & \tilde{l_1}\tilde{k_2} \end{bmatrix}, \dots,$$

$$b_{mn} = \begin{bmatrix} 0 & . & . & . & 0 & 0 \\ 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & \tilde{k}_{m-1}\tilde{l}_{n-1} \end{bmatrix}, \text{when } m \text{ is odd,}$$

$$= \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \tilde{k}_{m-1}\tilde{l}_1 & 0 & . & . & . & 0 \end{bmatrix}, \text{when } m \text{ is even.}$$

3.1. **The algorithm.** We let b_1 be the constant polynomial 1 on the whole of $\square$. To write b_2, we fill in zero at the first place, and in the first column also. Then we put $\tilde{l_1}$ at all the vacant places (we are crossing the line $l_1 = 0$ alone). To write b_3, we fill in zeros at the first two columns and we put $\tilde{l_2}$ at all the vacant places. Continue like this upto b_n. To write b_{n+1}, write zeros at all the preceding (in the snakelike ordering) n places and then put $\tilde{k_1}$ at all the vacant places. To write b_{n+2}, put zeros at the preceding $(n+1)$ places as well as in that column which precedes the $(n+2)$-th place in the snakelike ordering. Then put the product $\tilde{l}_{n-1}\tilde{k_1}$ at all the vacant places (we are crossing the lines $l_{n-1} = 0$ as well as $k_1 = 0$). Continue like this. In general, suppose we have written b_r as explained. Then to write b_{r+1} put zeros at the first r places (in the snakelike ordering). Also, put zeros in all those columns which precede the $(r+1)$-th entry of the matrix. While moving from the r-th place to the $(r+1)$-th place in the snakelike ordering if we are crossing a vertical line $l_j = 0$ alone, put $\tilde{l_j}$ at all the vacant places; if we are crossing a horizontal line $k_i = 0$ alone, we put $\tilde{k_i}$ at all the vacant places. However, if we are crossing a vertical line $l_j = 0$ and have already crossed a few horizontal lines, then assuming $k_i = 0$ was the last horizontal line that we have crossed, we put the product $\tilde{l_j}\tilde{k_i}$ at all the vacant places. This defines b_{r+1}. Thus we have defined b_r for all r, $0 \leq r \leq mn$.

Main results. First we settle the case of the 2-dimensional complex $\square$. We have,

3.2. **Theorem.** *Suppose a parallelogram region in E^2 is subdivided into mn parallelograms $m, n \geq 1$ by lines $k_i = 0$, $i = 1, 2, \ldots ., m-1$, parallel to one side and lines $l_j = 0$, $j = 1, 2, \ldots, n-1$, parallel to the other side (Fig. 2). Let the resulting 2-complex $\square$ be given the snakelike linear ordering. Then, for any $r \geq 0$ the set $B = b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_{mn}$ where b_i's are as written above, is an R-basis for the spline module $C^r(\square)$ over the polynomial ring $R = E[x_1, x_2]$.*

Proof. The proof is by induction on the number m of columns of the matrix like representation of the two-dimensional faces of $\square$. When $m = 1$, one easily checks that $(1, 1, ..., 1), (0, \tilde{l_1}, \tilde{l1}, \ldots, \tilde{l_1}), \ldots, (0, 0, \ldots, 0, \tilde{l}_{n-1})$ is an R-basis of $C^r(\square)$. Suppose $m > 1$ and the result is true for all 2-complexes having lesser number of rows than m. Suppose $\square$ has m rows. Since the number of zero entries in $B = \{b_1, b_2, \ldots, b_{mn}\}$ increase as we move along in the linear ordering, it is straightforward to see that the set B is linearly independent over R. We only use the fact that R is an integral domain and that if an element of R vanishes on the interior of a face of $\square$, then it vanishes on the whole of $\square$. Hence we have to show only the generating property of

B, i.e., we must show that every spline in $C^r(\square)$ is a linear combination of elements of B with coefficients in R.

For convenience, we arrange the mn elements of B in the form of an $m\times n$ matrix so that the snakelike ordering of the resulting matrix yields the set B. Thus,

$$B=\begin{bmatrix} e_{11} & . & . & . & e_{1n}\\ e_{21} & . & . & . & e_{2n}\\ . & . & . & . & .\\ . & . & . & . & .\\ e_{m1} & . & . & . & e_{mn}\end{bmatrix},$$

where $b_1=e_{11},\dots,b_n=e_{1n},b_{n+1}=e_{2n},\dots,b_{2n}=e_{21},\dots,b_{mn}=e_{mn}$.

Let

$$f=\begin{bmatrix} f_{11} & . & . & . & f_{1n}\\ f_{21} & . & . & . & f_{2n}\\ . & . & . & . & .\\ . & . & . & . & .\\ f_{m1} & . & . & . & f_{mn}\end{bmatrix},$$

be a C^r-spline on $\square$. We drop the last row of $\square$ and denote the resulting 2-complex by $\square'$ whose number of rows is less than m. By inductive hypothesis,

$$B'=\begin{bmatrix} e'_{11} & . & . & . & e'_{1n}\\ e'_{21} & . & . & . & e'_{2n}\\ . & . & . & . & .\\ . & . & . & . & .\\ e'_{m-1,1} & . & . & . & e'_{m-1,n}\end{bmatrix},$$

is a basis of $C^r(\square')$ where

$$e'_{11}=\begin{bmatrix} 1 & 1 & . & . & . & 1\\ 1 & 1 & . & . & . & 1\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ 1 & 1 & . & . & . & 1\end{bmatrix},\quad e'_{12}=\begin{bmatrix} 0 & \tilde{l}_1 & \tilde{l}_1 & . & . & . & \tilde{l}_1\\ 0 & \tilde{l}_1 & \tilde{l}_1 & . & . & . & \tilde{l}_1\\ . & . & . & . & . & . & .\\ . & . & . & . & . & . & .\\ 0 & \tilde{l}_1 & \tilde{l}_1 & . & . & . & \tilde{l}_1\end{bmatrix},$$

$$e'_{13}=\begin{bmatrix} 0 & 0 & \tilde{l}_2 & \tilde{l}_2 & . & . & . & \tilde{l}_2\\ 0 & 0 & \tilde{l}_2 & \tilde{l}_2 & . & . & . & \tilde{l}_2\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ 0 & 0 & \tilde{l}_2 & \tilde{l}_2 & . & . & . & \tilde{l}_2\end{bmatrix},\dots,e'_{1n}=\begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1}\\ 0 & 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1}\\ . & . & . & . & . & . & . & .\\ . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1}\end{bmatrix},$$

$$e'_{21}=\begin{bmatrix} 0 & 0 & . & . & . & 0\\ \tilde{l}_1\tilde{k}_1 & 0 & . & . & . & 0\\ \tilde{l}_1\tilde{k}_1 & 0 & . & . & . & 0\\ . & . & . & . & . & .\\ \tilde{l}_1\tilde{k}_1 & 0 & . & . & . & 0\end{bmatrix},\quad e'_{22}=\begin{bmatrix} 0 & 0 & 0 & . & . & . & 0\\ \tilde{l}_2\tilde{k}_1 & \tilde{l}_2\tilde{k}_1 & 0 & . & . & . & 0\\ \tilde{l}_2\tilde{k}_1 & \tilde{l}_2\tilde{k}_1 & 0 & . & . & . & 0\\ . & . & . & . & . & . & .\\ \tilde{l}_2\tilde{k}_1 & \tilde{l}_2\tilde{k}_1 & 0 & . & . & . & 0\end{bmatrix},$$

$$e'_{23}=\begin{bmatrix} 0 & 0 & 0 & 0 & . & . & . & 0\\ \tilde{l}_3\tilde{k}_1 & \tilde{l}_3\tilde{k}_1 & \tilde{l}_3\tilde{k}_1 & 0 & . & . & . & 0\\ \tilde{l}_3\tilde{k}_1 & \tilde{l}_3\tilde{k}_1 & \tilde{l}_3\tilde{k}_1 & 0 & . & . & . & 0\\ . & . & . & . & . & . & . & .\\ \tilde{l}_3\tilde{k}_1 & \tilde{l}_3\tilde{k}_1 & \tilde{l}_3\tilde{k}_1 & 0 & . & . & . & 0\end{bmatrix},\dots,e'_{2n}=\begin{bmatrix} 0 & 0 & . & . & . & 0\\ \tilde{k}_1 & \tilde{k}_1 & . & . & . & \tilde{k}_1\\ \tilde{k}_1 & \tilde{k}_1 & . & . & . & \tilde{k}_1\\ . & . & . & . & . & .\\ \tilde{k}_1 & \tilde{k}_1 & . & . & . & \tilde{k}_1\end{bmatrix},$$

etc. For the last row of e'_{ij}'s we have two cases:

Case 1. When m is odd

$$e'_{m-1,1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ \tilde{l}_1\tilde{k}_{m-2} & 0 & . & . & . & 0 \end{bmatrix},$$

$$e'_{m-1,2} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ \tilde{l}_2\tilde{k}_{m-2} & \tilde{l}_2\tilde{k}_{m-2} & 0 & . & . & . & 0 \end{bmatrix}, \ldots,$$

$$e'_{m-1,n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{k}_{m-2} & \tilde{k}_{m-2} & . & . & . & \tilde{k}_{m-2} \end{bmatrix}.$$

Case 2. When m is even

$$e'_{m-1,1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{k}_{m-2} & \tilde{k}_{m-2} & . & . & . & \tilde{k}_{m-2} \end{bmatrix},$$

$$e'_{m-1,2} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ 0 & \tilde{l}_1\tilde{k}_{m-2} & . & . & . & \tilde{l}_1\tilde{k}_{m-2} \end{bmatrix}, \ldots,$$

$$e'_{m-1,n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 & \tilde{l}_{n-1}\tilde{k}_{m-2} \end{bmatrix}.$$

Here e'_{ij}'s are obtained from e_{ij} by dropping the last row. Note that

$$\begin{bmatrix} f_{11} & . & . & . & f_{1n} \\ f_{21} & . & . & . & f_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{m-1,1} & . & . & . & f_{m-1,n} \end{bmatrix} \in C^r(\square').$$

Hence there exist polynomials α_{ij} $(i = 1, \dots, m-1; j = 1, \dots, n)$ such that

$$\begin{bmatrix} f_{11} & . & . & . & f_{1n} \\ f_{21} & . & . & . & f_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{m-1,1} & . & . & . & f_{m-1,n} \end{bmatrix} = \alpha_{11}e'_{11} + \alpha_{12}e'_{12} + \cdots + \alpha_{2n}e'_{2n} + \cdots + \alpha_{m-1,n}e'_{m-1,n}.$$

Now we extend each e'_{ij} to e_{ij} by adding the m-th rows as follows:

$$e_{11} = \begin{bmatrix} 1 & 1 & . & . & . & 1 \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & . & 1 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \\ 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \\ 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \\ 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \\ . & . & . & . & . & . & . \\ 0 & \tilde{l_1} & \tilde{l_1} & . & . & . & \tilde{l_1} \end{bmatrix},$$

$$e_{13} = \begin{bmatrix} 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \\ 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & \tilde{l_2} & \tilde{l_2} & . & . & . & \tilde{l_2} \end{bmatrix}, \dots, e_{1n} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & \tilde{l}_{n-1} \\ 0 & 0 & 0 & . & . & . & \tilde{l}_{n-1} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \tilde{l}_{n-1} \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & \tilde{l}_{n-1} \end{bmatrix},$$

$$e_{21} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ \tilde{l_1}\tilde{k_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ \tilde{l_1}\tilde{k_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ \tilde{l_1}\tilde{k_1} & 0 & . & . & . & 0 \end{bmatrix}, \quad e_{22} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ \tilde{l_2}\tilde{k_1} & \tilde{l_2}\tilde{k_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ \tilde{l_2}\tilde{k_1} & \tilde{l_2}\tilde{k_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ \tilde{l_2}\tilde{k_1} & \tilde{l_2}\tilde{k_1} & 0 & . & . & . & 0 \end{bmatrix},$$

$$e_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & . & . & . & 0 \\ \tilde{l_3}\tilde{k_1} & \tilde{l_3}\tilde{k_1} & \tilde{l_3}\tilde{k_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ \tilde{l_3}\tilde{k_1} & \tilde{l_3}\tilde{k_1} & \tilde{l_3}\tilde{k_1} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ \tilde{l_3}\tilde{k_1} & \tilde{l_3}\tilde{k_1} & \tilde{l_3}\tilde{k_1} & 0 & . & . & . & 0 \end{bmatrix}, \dots, e_{2n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ \tilde{k_1} & \tilde{k_1} & . & . & . & \tilde{k_1} \\ . & . & . & . & . & . \\ \tilde{k_1} & \tilde{k_1} & . & . & . & \tilde{k_1} \\ . & . & . & . & . & . \\ \tilde{k_1} & \tilde{k_1} & . & . & . & \tilde{k_1} \end{bmatrix},$$

etc. Corresponding to the last row of e_{ij}'s we have two cases:

Case 1. m is odd

$$e_{m-1,1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{l_1}\tilde{k}_{m-2} & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ \tilde{l_1}\tilde{k}_{m-2} & 0 & . & . & . & 0 \end{bmatrix},$$

$$e_{m-1,2} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \\ \tilde{l}_2\tilde{k}_{m-2} & \tilde{l}_2\tilde{k}_{m-2} & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ \tilde{l}_2\tilde{k}_{m-2} & \tilde{l}_2\tilde{k}_{m-2} & 0 & . & . & . & 0 \end{bmatrix}, \dots,$$

$$e_{m-1,n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{k}_{m-2} & \tilde{k}_{m-2} & . & . & . & \tilde{k}_{m-2} \\ . & . & . & . & . & . \\ \tilde{k}_{m-2} & \tilde{k}_{m-2} & . & . & . & \tilde{k}_{m-2} \end{bmatrix}.$$

Case 2. m is even

$$e_{m-1,1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{k}_{m-2} & \tilde{k}_{m-2} & . & . & . & \tilde{k}_{m-2} \\ . & . & . & . & . & . \\ \tilde{k}_{m-2} & \tilde{k}_{m-2} & . & . & . & \tilde{k}_{m-2} \end{bmatrix},$$

$$e_{m-1,2} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ 0 & \tilde{l}_1\tilde{k}_{m-2} & . & . & . & \tilde{l}_1\tilde{k}_{m-2} \\ . & . & . & . & . & . \\ 0 & \tilde{l}_1\tilde{k}_{m-2} & . & . & . & \tilde{l}_1\tilde{k}_{m-2} \end{bmatrix}, \dots,$$

$$e_{m-1,n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & \tilde{l}_{n-1}\tilde{k}_{m-2} \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \tilde{l}_{n-1}\tilde{k}_{m-2} \end{bmatrix}.$$

Then a straightforward calculation shows that

$$\sum_{i=1,j=1}^{m-1,n} \alpha_{ij} e_{ij} = \begin{bmatrix} f_{11} & . & . & . & f_{1n} \\ f_{21} & . & . & . & f_{2n} \\ . & . & . & . & . \\ f_{m-1,1} & . & . & . & f_{m-1,n} \\ . & . & . & . & . \\ f_{m-1,1} & . & . & . & f_{m-1,n} \end{bmatrix}.$$

Now we use the algebraic criterion separately for the two cases:

Case 1. When m is odd.

$f_{m,1} = f_{m-1,1} + \alpha_{m,1}\tilde{k}_{m-1}$ for some $\alpha_{m,1} \in R$,

$f_{m,2} = f_{m,1} + \beta_1 \tilde{l_1} = f_{m-1,2} + \beta_1' \tilde{k}_{m-1}$, for some $\beta_1, \beta_1' \in R$,

$\Rightarrow f_{m-1,1} + \alpha_{m,1}\tilde{k}_{m-1} + \beta_1 \tilde{l_1} = f_{m-1,2} + \beta_1'\tilde{k}_{m-1}$,

$\Rightarrow f_{m-1,2} + \gamma_1 \tilde{l_1} + \alpha_{m,1}\tilde{k}_{m-1} + \beta_1\tilde{l_1} = f_{m-1,2} + \beta_1'\tilde{k}_{m-1}$, for some $\gamma_1 \in R$,

$\Rightarrow (\beta_1' - \alpha_{m,1})\tilde{k}_{m-1} = (\beta_1 + \gamma_1)\tilde{l_1}$,

$\Rightarrow \beta_1' = \alpha_{m,1} + \alpha_{m,2}\tilde{l_1}$ for some $\alpha_{m,2} \in R$.

Hence, $f_{m,2} = f_{m-1,2} + \alpha_{m,1}\tilde{k}_{m-1} + \alpha_{m,2}\tilde{l_1}\tilde{k}_{m-1}$.

Continuing in this manner we get

$f_{m,n} = f_{m-1,n} + \alpha_{m,1}\tilde{k}_{m-1} + \alpha_{m,2}\tilde{l_1}\tilde{k}_{m-1} + \cdots + \alpha_{m,n}\tilde{l}_{n-1}\tilde{k}_{m-1}$.

Hence if we define

$$e_{m,1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{k}_{m-1} & \tilde{k}_{m-1} & . & . & . & \tilde{k}_{m-1} \end{bmatrix},$$

$$e_{m,2} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ 0 & \tilde{l_1}\tilde{k}_{m-1} & . & . & . & \tilde{l_1}\tilde{k}_{m-1} \end{bmatrix}, \ldots,$$

$$e_{m,n} = \begin{bmatrix} 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 0 \\ 0 & . & . & . & 0 & \tilde{l}_{n-1}\tilde{k}_{m-1} \end{bmatrix},$$

we find that

$$\begin{bmatrix} f_{11} & . & . & . & f_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{m,1} & . & . & . & f_{m,n} \end{bmatrix} = \sum_{i=1,j=1}^{m-1,n} \alpha_{ij}e_{ij} + \alpha_{m1}e_{m1} + \alpha_{m2}e_{m2} + \cdots + \alpha_{mn}e_{mn} = \sum_{i=1,j=1}^{m,n} \alpha_{ij}e_{ij}.$$

Case 2. When m is even.

$f_{m,n} = f_{m-1,n} + \alpha_{m,n}\tilde{k}_{m-1}$ for some $\alpha_{mn} \in R$,

$f_{m,n-1} = f_{m,n} + \beta_1\tilde{l}_{n-1} = f_{m-1,n-1} + \beta_1'\tilde{k}_{m-1}$ for some $\beta_1, \beta_1' \in R$,

$\Rightarrow f_{m-1,n} + \alpha_{m,n}\tilde{k}_{m-1} + \beta_1\tilde{l}_{n-1} = f_{m-1,n-1} + \beta_1'\tilde{k}_{m-1}$,

$\Rightarrow f_{m-1,n-1} + \gamma_1\tilde{l}_{n-1} + \alpha_{m,n}\tilde{k}_{m-1} + \beta_1\tilde{l}_{n-1} = f_{m-1,n-1} + \beta_1'\tilde{k}_{m-1}$, for some $\gamma_1 \in R$,

$\Rightarrow (\beta_1' - \alpha_{m,n})\tilde{k}_{m-1} = (\beta_1 + \gamma_1)\tilde{l}_{n-1}$,

$\Rightarrow \beta_1' = \alpha_{m,n} + \alpha_{m,n-1}\tilde{l}_{n-1}$ for some $\alpha_{m,n-1} \in R$.

Hence, $f_{m,n-1} = f_{m-1,n-1} + \alpha_{m,n}\tilde{k}_{m-1} + \alpha_{m,n-1}\tilde{l}_{n-1}\tilde{k}_{m-1}$.

Continuing in this manner we get
$f_{m,1} = f_{m-1,1} + \alpha_{m,n}\tilde{k}_{m-1} + \alpha_{m,n-1}\tilde{l}_{n-1}\tilde{k}_{m-1} + \cdots + \alpha_{m,1}\tilde{l}_1\tilde{k}_{m-1}$.
Hence if we define

$$e_{m,1} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{l}_1\tilde{k}_{m-1} & 0 & . & . & . & 0 \end{bmatrix},$$

$$e_{m,2} = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \\ \tilde{l}_2\tilde{k}_{m-1} & \tilde{l}_2\tilde{k}_{m-1} & 0 & . & . & . & 0 \end{bmatrix}, \ldots,$$

$$e_{m,n} = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 \\ \tilde{k}_{m-1} & \tilde{k}_{m-1} & . & . & . & \tilde{k}_{m-1} \end{bmatrix},$$

then we find that

$$\begin{aligned} \begin{bmatrix} f_{11} & . & . & . & f_{1n} \\ . & . & . & . & . \\ . & . & . & . & . \\ f_{m,1} & . & . & . & f_{m,n} \end{bmatrix} &= \sum_{i=1,j=1}^{m-1,n} \alpha_{ij}e_{ij} + \alpha_{m1}e_{m1} + \alpha_{m2}e_{m2} + \cdots + \alpha_{mn}e_{mn} \\ &= \sum_{i=1,j=1}^{m,n} \alpha_{ij}e_{ij}. \quad \square \end{aligned}$$

3.3. **Example.** For the case of Fig. 3 the R-basis in accordance with the above algorithm is given by:

$$b_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0 & \tilde{l}_1 & \tilde{l}_1 \\ 0 & \tilde{l}_1 & \tilde{l}_1 \\ 0 & \tilde{l}_1 & \tilde{l}_1 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 0 & 0 & \tilde{l}_2 \\ 0 & 0 & \tilde{l}_2 \\ 0 & 0 & \tilde{l}_2 \end{bmatrix},$$

$$b_4 = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{k}_1 & \tilde{k}_1 & \tilde{k}_1 \\ \tilde{k}_1 & \tilde{k}_1 & \tilde{k}_1 \end{bmatrix}, \quad b_5 = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{k}_1\tilde{l}_2 & \tilde{k}_1\tilde{l}_2 & 0 \\ \tilde{k}_1\tilde{l}_2 & \tilde{k}_1\tilde{l}_2 & 0 \end{bmatrix}, \quad b_6 = \begin{bmatrix} 0 & 0 & 0 \\ \tilde{k}_1\tilde{l}_1 & 0 & 0 \\ \tilde{k}_1\tilde{l}_1 & 0 & 0 \end{bmatrix},$$

$$b_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{k}_2 & \tilde{k}_2 & \tilde{k}_2 \end{bmatrix}, \quad b_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{l}_1\tilde{k}_2 & \tilde{l}_1\tilde{k}_2 \end{bmatrix}, \quad b_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{l}_2\tilde{k}_2 \end{bmatrix}.$$

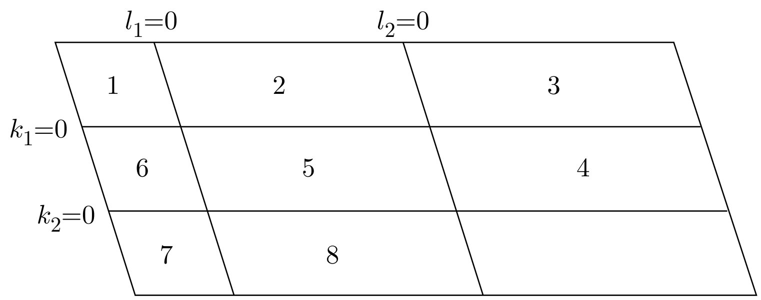

FIGURE 3

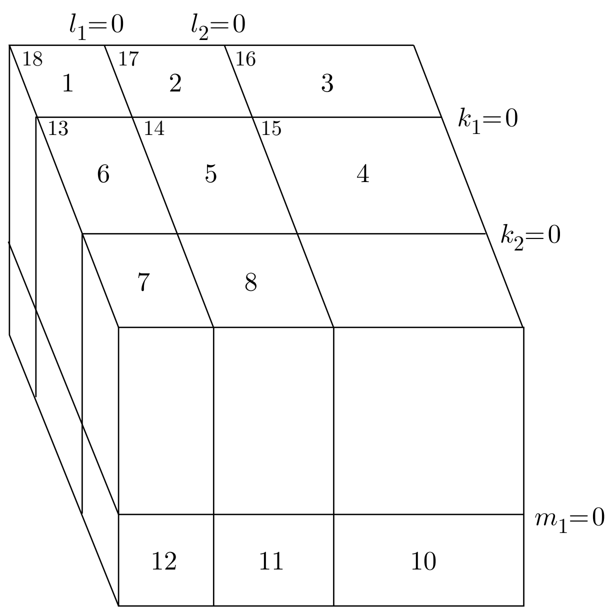

FIGURE 4

4. THE HIGHER-DIMENSIONAL CASES

Let us start with an example in the three-dimensional situation. Suppose a parallelopiped is subdivided into eighteen subparallelopipeds and is obtained by drawing hyperplanes $k_1 = 0$, $k_2 = 0$, $l_1 = 0$, $l_2 = 0$ and $m_1 = 0$ in E^3 (see Fig. 4).

Suppose $\square$ is the 3-dimensional subcomplex so obtained. To extend the snakelike linear ordering on the eighteen faces we linearly order the nine faces above the plane $m_1 = 0$ as in the two-dimensional case. From ninth place we go down to the tenth place just below the ninth one and then cover the second level according to the snakelike linear ordering, i.e., first move along a row in the second level, then along the next row but backward, then along the third row until we have enumerated the second level completely. If there is yet another level along the z-axis, we go down to the third level and extend the snakelike linear ordering in the obvious way. Returning to our example (Fig. 4), we first explain the algorithm of writing a basis using a linear ordering just described. Start with the first element b_1 by filling in 1 in all the eighteen places of the $3 \times 3 \times 2$ matrix. To write b_2 put zero in the first place and then fill in zeros in the first column including all the levels below the first column; then put $\tilde{l_1}$ at all the vacant places (we are crossing the plane $l_1 = 0$ in the linear ordering). We fill all the entries of the first level according to the 2-dimensional case putting zeros in the second level below the zeros of the first level. This will give us the first nine elements of the basis. To write the tenth basis element, we put zeros at all the preceding places (i.e., on the whole of the first level), and then put $\tilde{m_1}$ at all the vacant places. To write b_{11}, we put zeros in all the preceding ten places, also put zeros in the column of the second level which has tenth entry in it, then we fill all vacant places by $\tilde{l_2}\tilde{m_1}$ (we are crossing $l_2 = 0$ and $m_1 = 0$ both). We continue like this as in the first level remembering to include factor $\tilde{m_1}$ everywhere in the second level. Since it is unmanageable to write the three-dimensional matrices, we write below all the eighteen basis elements of the particular example of Fig. 4 according to the linear ordering described above. These are :

$b_1 = (1, 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, 1)$,
$b_2 = (0, \tilde{l_1}, \tilde{l_1}, \tilde{l_1}, \tilde{l_1}, 0, 0, \tilde{l_1}, \tilde{l_1}, \tilde{l_1}, \tilde{l_1}, 0, 0, \tilde{l_1}, \tilde{l_1}, \tilde{l_1}, \tilde{l_1}, 0)$,
$b_3 = (0, 0, \tilde{l_2}, \tilde{l_2}, 0, 0, 0, 0, \tilde{l2}, \tilde{l_2}, 0, 0, 0, 0, \tilde{l_2}, \tilde{l_2}, 0, 0)$,
$b_4 = (0, 0, 0, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, \tilde{k_1}, 0, 0, 0)$,
$b_5 = (0, 0, 0, 0, \tilde{l_2}\tilde{k_1}, \tilde{l_2}\tilde{k_1}, \tilde{l_2}\tilde{k_1}, \tilde{l_2}\tilde{k_1}, 0, 0, \tilde{l_2}\tilde{k_1}, \tilde{l_2}\tilde{k_1}, \tilde{l_2}\tilde{k_1}, \tilde{l_2}\tilde{k_1}, 0, 0, 0, 0)$,
$b_6 = (0, 0, 0, 0, 0, \tilde{l_1}\tilde{k_1}, \tilde{l_1}\tilde{k_1}, 0, 0, 0, 0, \tilde{l_1}\tilde{k_1}, \tilde{l_1}\tilde{k_1}, 0, 0, 0, 0, 0)$,
$b_7 = (0, 0, 0, 0, 0, 0, \tilde{k_2}, \tilde{k_2}, \tilde{k_2}, \tilde{k_2}, \tilde{k_2}, \tilde{k_2}, 0, 0, 0, 0, 0, 0)$,
$b_8 = (0, 0, 0, 0, 0, 0, 0, \tilde{l_1}\tilde{k_2}, \tilde{l_1}\tilde{k_2}, \tilde{l_1}\tilde{k_2}, \tilde{l_1}\tilde{k_2}, 0, 0, 0, 0, 0, 0, 0)$,
$b_9 = (0, 0, 0, 0, 0, 0, 0, 0, \tilde{l_2}\tilde{k_2}, \tilde{l_2}\tilde{k_2}, 0, 0, 0, 0, 0, 0, 0, 0)$,
$b_{10} = (0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}, \tilde{m_1}, \tilde{m_1}, \tilde{m_1}, \tilde{m_1}, \tilde{m_1}, \tilde{m_1}, \tilde{m_1}, \tilde{m_1})$,
$b_{11} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{l_2}\tilde{m_1}, \tilde{l_2}\tilde{m_1}, \tilde{l_2}\tilde{m_1}, \tilde{l_2}\tilde{m_1}, 0, 0, \tilde{l_2}\tilde{m_1}, \tilde{l_2}\tilde{m_1})$,
$b_{12} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{l_1}\tilde{m_1}, \tilde{l_1}tildem_1, 0, 0, 0, 0, \tilde{l_1}\tilde{m_1})$,
$b_{13} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}\tilde{k_2}, \tilde{m_1}\tilde{k_2}, \tilde{m_1}\tilde{k_2}, \tilde{m_1}\tilde{k_2}, \tilde{m_1}\tilde{k_2}, \tilde{m_1}\tilde{k_2})$,
$b_{14} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}\tilde{k_2}, \tilde{l_1}, \tilde{m_1}\tilde{k_2}, \tilde{l_1}, \tilde{m_1}\tilde{k_2}, \tilde{l_1}, \tilde{m_1}\tilde{k_2}, \tilde{l_1}, 0)$,
$b_{15} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}\tilde{l_2}\tilde{k_1}, \tilde{m_1}\tilde{l_2}\tilde{k_1}, 0, 0)$,
$b_{16} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}\tilde{k_1}, \tilde{m_1}\tilde{k_1}, \tilde{m_1}\tilde{k_1},)$,
$b_{17} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}\tilde{k_1}\tilde{l_2}, \tilde{m_1}\tilde{k_1}\tilde{l_2})$,
$b_{18} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \tilde{m_1}\tilde{l_1}\tilde{k_1})$.

Now, with this example, it is clear what would be the basis in the case when we have $m.n.p$ number of parallelopipeds by intersecting m, n and p parallel hyperplanes in E^3. A simple but long calculation can be exhibited to prove (we have done it to satisfy ourselves) the above statement for the particular example having 18 parallelopipeds in the three-dimensional case. The general case of three or higher dimensions can be stated and proved using induction, as in the two-dimensional case (Theorem 3.1). We omit the lengthy proof (we saw the two-dimensional case) and give only the statement of the general result.

4.1. **Theorem.** *Let P be a d-dimensional parallelopiped in E^d. We subdivide P into $n_1.n_2.\cdots.n_d$ number of subparallelopipeds by drawing hyperplanes in E^d parallel to the sides of P. Suppose $\Box$ denotes the resulting d-complex in E^d. Then, with the snakelike linear ordering on the faces of $\Box$ and for any $r \geq 0$, there is an algorithm to write down a basis consisting of $n_1.n_2.\cdots.n_d$ number of C^r-splines* *on* $\square$ *which will form a basis for the spline module* $C^r(\square)$ *over the polynomial ring* $R = E[x_1, \ldots, x_d]$. $\square$

Acknowledgement

The authors are thankful to the referee for his suggestions which improved the exposition of the paper.

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