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REMARKS ON THE SCHOOF-ELKIES-ATKIN ALGORITHM

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ABSTRACT. Schoof's algorithm computes the number m of points on an elliptic curve E defined over a finite field $\mathbb{F}_q$. Schoof determines m modulo small primes ℓ using the characteristic equation of the Frobenius of E and polynomials of degree $O(\ell^2)$. With the works of Elkies and Atkin, we have just to compute, when ℓ is a "good" prime, an eigenvalue of the Frobenius using polynomials of degree $O(\ell)$. In this article, we compute the complexity of Müller's algorithm, which is the best known method for determining one eigenvalue and we improve the final step in some cases. Finally, when ℓ is "bad", we describe how to have polynomials of small degree and how to perform computations, in Schoof's algorithm, on x-values only.

1. Introduction

Let E be an elliptic curve defined over the finite field $\mathbb{F}_q$ of large characteristic p. The set of $\mathbb{F}_q$-points of E, denoted $E(\mathbb{F}_q)$, is a finite abelian group [20].

In 1985, Schoof [17] gave a deterministic polynomial-time algorithm for computing $\#E(\mathbb{F}_q)$. The algorithm determines the characteristic equation of the Frobenius π of E, acting on the ℓ-torsion points $E[\ell]$ of E, for ℓ prime. But, working on $E[\ell]$ uses computations on polynomials modulo the ℓ-th division polynomial f_ℓ, and this is not practical, due to the size of f_ℓ.

In 1991, Elkies [10] showed how to perform computations in the kernel of an isogeny of degree ℓ, by computing a factor of degree $d = (\ell - 1)/2$ of f_ℓ. This idea works for nearly half the primes ℓ, called *Elkies primes*. For such an ℓ, the algorithm has just to compute an eigenvalue of π acting on $E[\ell]$.

Atkin [1] had given in 1988 the *sort and match* method used now for "bad" primes ℓ. Then he made the algorithm practical for very large finite fields [2] and the method became the SEA (for Schoof-Elkies-Atkin) algorithm.

For the last improvements in this scope, see [5], [6] and [12] and for the case p small, see [7] and the implementation in [13].

In this article we compute, for an Elkies prime ℓ, the complexity of the best asymptotic method used for computing an eigenvalue of π over $E[\ell]$ and we show then how to avoid, in some cases, the computation with y-coordinates of points. Finally, for a bad prime ℓ, we explain how to obtain a proper factor of f_ℓ and show then how to avoid again, in Schoof's algorithm, computations with y-coordinates of points.

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These results have enabled Morain [15] to compute $\#E(\mathbb{F}_p)$, for p prime of 500 digits (this is the actual record).

2. The SEA algorithm

2.1. **Elliptic curve over $\mathbb{F}_q$.** Let E be a non-supersingular elliptic curve given by an affine equation $\mathcal{F}(x,y)=0$ where

$$\mathcal{F}(x,y)=y^2+a_1xy+a_3y-(x^3+a_2x^2+a_4x+a_6)$$

with the a_i's in $\mathbb{F}_q$.

The set $E(\mathbb{F}_q)=\{(x,y)\in\mathbb{F}_q\times\mathbb{F}_q,\mathcal{F}(x,y)=0\}\cup\{O_E\}$ is an abelian group and the law, denoted $\oplus$, has $O_E=[0:1:0]$ as neutral element. We denote by f_n the n-th division polynomial in x. The degree of f_n is $(n^2-1)/2$ if n is odd. The group of n-torsion points, $E[n]=\{P\in E(\bar{\mathbb{F}}_q)\mid nP=O_E\}$ can be represented by $\mathbb{F}_q[x,y]/(f_n(x),\mathcal{F}(x,y))$ (see [18]).

The morphism $\pi: E(\bar{\mathbb{F}}_q)\to E(\bar{\mathbb{F}}_q)$, $(x,y)\mapsto(x^q,y^q)$ of E satisfies $\pi^2-t\pi+q=0$ over $E(\bar{\mathbb{F}}_q)$, with $t\in\mathbb{Z}$, satisfying $|t|\le 2\sqrt{q}$. Recall : $\#E(\mathbb{F}_q)=q+1-t$. When ℓ is an odd prime number (see [6] for $\ell=2$), we consider the restriction π_ℓ of π to $E[\ell]$, which satisfies $\pi_\ell^2-\tau\pi_\ell+k=0$ over $E[\ell]$ with $t\equiv\tau \bmod \ell$ and $q\equiv k \bmod \ell$. Now, if $\ell\ne p$, $E[\ell]\cong\mathbb{Z}/\ell\mathbb{Z}\times\mathbb{Z}/\ell\mathbb{Z}$, so we can view $E[\ell]$ as a vector space over $\mathbb{F}_\ell$ and $x^2-\tau x+k$ as the characteristic equation of π_ℓ. We denote by $G_1,G_2,\dots,G_{\ell+1}$ the $(\ell+1)$ cyclic subgroups of $E(\bar{\mathbb{F}}_q)$, of order ℓ.

2.2. **The SEA algorithm.** Schoof [17] determines $\#E(\mathbb{F}_q)=q+1-t$ by searching for a match among the ℓ equations $(x^{q^2},y^{q^2})\oplus k(x,y)=\tilde\tau(x^q,y^q),\ 0\le\tilde\tau\le\ell-1$, over $E[\ell]$.

Elkies works in the kernel G_i of one of the $\ell+1$ isogenies $E\xrightarrow{\varrho_i}E_i, 1\le i\le\ell+1$, of degree ℓ. When $D=\tau^2-4k$ is a square modulo ℓ the eigenvalues of π_ℓ are in $\mathbb{F}_\ell$ and ℓ is called an Elkies prime. Hence, in this case, the eigenspaces are $\mathbb{F}_q$-rational and the corresponding isogenies are defined over $\mathbb{F}_q$ and if we let $E[\ell]_\lambda$ be an eigenspace with P_λ a generator, we have $h_\ell(x)=\prod_{i=1}^d(x-x(iP_\lambda))\in\mathbb{F}_q[x]$ and

$$E[\ell]_\lambda=\langle P_\lambda\rangle=\mathbb{F}_q[x,y]/(h_\ell(x),\mathcal{F}(x,y)).$$

Let $\Phi_\ell(x,y)=0$ be the canonical equation of the modular curve $X_0(\ell)$ (see [2], [15] for a simpler equation). We know that ℓ is an Elkies prime if and only if $\Phi_\ell(j(E),x)=0$ has a root in $\mathbb{F}_q$.

For $p\ne 2,3$ and ℓ an Elkies prime, the formulas of Atkin [2],[15] give, from a root of $\Phi_\ell(j(E),x)=0$ in $\mathbb{F}_q$, the value of $p_1=\sum_{i=1}^d x(iP_\lambda)$ and the coefficients of the corresponding E_i. So [10], one can compute the $p_k=\sum_{i=1}^d x^k(iP_\lambda)$ for $1\le k\le d$ and hence h_ℓ by Newton's formula if $\ell\ll p$.

If $p=2$ or 3 or $\ell\approx p$ see Couveignes' work [7] and also [13].

Once h_ℓ is known, we have to search a match among the $\ell-1$ equations $(x^q,y^q)=\tilde\lambda(x,y)\ ,\ 1\le\tilde\lambda\le\ell-1$, over $E[\ell]_\lambda$.

If D is not a square modulo ℓ, then ℓ is called an *Atkin prime* and the G_i's are $\mathbb{F}_{p^e}$-rational where e is the smallest integer n for which π_ℓ^n is in $\mathbb{F}_\ell$.

3. Looking for one eigenvalue

3.1. **Computing λ mod ℓ.** We compute the complexity of the algorithm of Müller [16] which computes λ mod ℓ. Müller uses an integer $k_{opt}\approx\lceil\sqrt{d}\rceil$ such that for all

λ in $\mathbb{F}_\ell^*$, there are integers i, j with $1 \le i, j \le k_{opt}$ such that $\lambda \equiv \pm i/j \bmod \ell$. So, he compares $j(x^q, y^q)$ and $i(x, y)$ using division polynomials, which means comparisons of rational functions.

The elementary operation is taken to be the cost of one multiplication of two elements in $\mathbb{F}_q$. Let $M(d)$ be the number of operations needed to compute the multiplication of two polynomials of degree d (see [11]).

Proposition 1. *Müller's method takes $O(M(d)\log q) + O(\sqrt{d}M(d)) + O(d^2)$ operations and $O(d\sqrt{d})$ space.*

Proof. The computation of $x^q \bmod h_\ell(x)$ requires $O(M(d)\log q)$ operations and the computation of the k_{opt} first division polynomials requires $O(\sqrt{d}M(d))$ operations.

The $x(j(x^q, y^q))$ are computed using the recursive formulae of division polynomials in x^q (see [8]). This requires $O(\sqrt{d}M(d))$ operations, which is more efficient than modular compositions $x(j(x, y)) \circ x^q$ (see [4], [19]).

To compare two rational functions modulo $h_\ell(x)$ in $\mathbb{F}_q[x]$, one can test the match using a random linear map [16] and then verify polynomial equality. So, the comparisons of coordinates takes $O(2M(d)) + O(2d^2)$ operations. □

3.2. **The sign of λ mod ℓ.** Suppose we have integers i, j such that $j\pi_\ell = \pm i$ over $E[\ell]_\lambda$, where ℓ is an Elkies prime. We have $\lambda \equiv \pm\lambda_0 \bmod \ell$ with $\lambda_0 \equiv ij^{-1} \bmod \ell$. For $\mu \in \mathbb{F}_{\ell^2}^*$, we call *semi-order* of μ, denoted $s(\mu)$, the order of μ in $\mathbb{F}_{\ell^2}^*/(\pm 1)$.

- If $p \ne 2$, E has an equation of the form $y^2 = \mathcal{G}(x) := x^3 + a_2x^2 + a_4x + a_6$.

Theorem 1. *Let h_ℓ be the factor of f_ℓ corresponding to λ and g_ℓ be a factor of degree $s(\lambda_0)$ of h_ℓ and let r be* Resultant$(g_\ell, \mathcal{G})$*. Then $\lambda = \lambda_0^{s(\lambda_0)}(\frac{r}{q})\lambda_0$. When $\ell \equiv 3 \bmod 4$, one can take $g_\ell = h_\ell$.*

Proof. For $s(\lambda_0)$ odd, we have $\pi_\ell^{s(\lambda_0)} = \pm\lambda_0^{s(\lambda_0)} Id$ over $E[\ell]_\lambda$. If $\pi_\ell^{s(\lambda_0)} = Id$, then $E[\ell]_\lambda \subset E(\mathbb{F}_{q^{s(\lambda_0)}})$; hence, for all P in $E[\ell]_\lambda$, $\mathcal{G}(x(P))$ is a square in $\mathbb{F}_{q^{s(\lambda_0)}}$, and since $\prod_{i=1}^{s(\lambda_0)}(\mathcal{G}(x_i)) = r$, with x_i the roots of g_ℓ, r is a square in $\mathbb{F}_q$. Whereas, if $\pi_\ell^{s(\lambda_0)} = -Id$ over $E[\ell]_\lambda$, then $(\frac{r}{q}) = -1$. □

Note that, if $\ell \equiv 1 \bmod 4$, then one can compute λ_0 using h_ℓ and then determine λ, if $s(\lambda_0) = s(\pm\lambda_0)$ is odd, using a factor of h_ℓ.

- If $p = 2$, let $y^2 + xy = x^3 + B$ (with $B \in \mathbb{F}_{2^m}$) be an equation of E (see [14]).

Proposition 2. *Let h_ℓ be a factor of f_ℓ corresponding to $\lambda = \pm\lambda_0$. If h_ℓ has a factor $g_\ell = x^{s(\lambda_0)} - \tilde{s}_1 x^{s(\lambda_0)-1} + \cdots + (-1)^{s(\lambda_0)}\tilde{s}_{s(\lambda_0)}$ of odd degree $s(\lambda_0)$, then*

$$\lambda = \begin{cases} \lambda_0^{s(\lambda_0)}\lambda_0 & \textit{if} \;\; \mathrm{Tr}(\tilde{s}_1 + B(\tilde{s}_{s(\lambda_0)-1}^2 - 2\tilde{s}_{s(\lambda_0)}\tilde{s}_{s(\lambda_0)-2})/\tilde{s}_{s(\lambda_0)}^2) = 0, \\ -\lambda_0^{s(\lambda_0)}\lambda_0, & \textit{otherwise.} \end{cases}$$

When $\ell \equiv 3 \bmod 4$, one can take $g_\ell = h_\ell$.

Proof. The equation $X^2 + X = \gamma$ has a root in an extension $\mathbb{F}_{2^n}$ if and only if $\mathrm{Tr}(\gamma) = 0$ (see [9]). Hence the points of $E[\ell]_\lambda = \langle P = (x, y)\rangle$ are in $\mathbb{F}_{q^{s(\lambda_0)}}$ if and only if $\mathrm{Tr}(\gamma_i) = 0$, where $x_i = x(iP)$ and $\gamma_i = x_i + B/x_i^2$. Finally, computing $\sum_{i=1}^{s(\lambda_0)} \mathrm{Tr}(\gamma_i)$ gives the desired result. □

4. Elkies' method for Atkin primes

4.1. **Computing a factor of f_ℓ.** Assume that $q = p$ prime, $\neq 2, 3$ and that ℓ is an Atkin prime with $\ell \ll p$.

The $(\ell+1)$ curves E_i are defined over $\mathbb{F}_{p^e}$, hence f_ℓ has a factor of degree d over $\mathbb{F}_{p^e}$ and so by conjugation we can find a factor of degree ed over $\mathbb{F}_p$.

First, we compute a monic irreducible factor $M_\ell(x)$ of degree e of $\Phi_\ell(j(E), x)$ in $\mathbb{F}_p[x]$. We denote by x_i, $i = 1, 2, \ldots, e$, the roots of $M_\ell(x) = 0$ in $\mathbb{F}_{p^e}$. Then, in $\mathbb{F}_p[x]/M_\ell(x)$, we determine ed polynomials $p_k(x) = \sum_{j=0}^{e-1} a_{j,k} x^j$ of degree $e-1$, (see [2], [10]) and since, for $1 \leq k \leq ed$, we have

$$p_k \stackrel{\text{def}}{=} \sum_{i=1}^{e} p_k(x_i) = \sum_{i=1}^{e} (\sum_{j=0}^{e-1} a_{j,k} x_i^j) = \sum_{j=0}^{e-1} a_{j,k} (\sum_{i=1}^{e} x_i^j) = \sum_{j=0}^{e-1} a_{j,k} \tilde{p}_j$$

with $\tilde{p}_j = \sum_{i=1}^{e} x_i^j$ computed from the symmetric functions of $M_\ell(x)$, a factor of degree ed of f_ℓ can be computed.

Example. We consider the elliptic curve $y^2 = x^3 + 2x + 41$ over $\mathbb{F}_{59}$ with $j = 31$. We determine a factor of the division polynomial f_5 of E. Over $\mathbb{F}_{59}$, $x^3 + 41x^2 + 45x + 32$ is a factor of $\Phi_5(x, 31)$. We obtain

$p_1(x)$	$56x^2 + 31x + 41$	$p_4(x)$	$16x^2 + 11x + 6$
$p_2(x)$	$46x^2 + 22x + 26$	$p_5(x)$	$51x^2 + 41x + 17$
$p_3(x)$	$21x^2 + 20x + 39$	$p_6(x)$	$34x^2 + 41x$

And $p_0 = 2$, $p_1 = 38$, $p_2 = 28$, $p_3 = 22$, $p_4 = 7$, $p_5 = 38$, $p_6 = 21$, hence $x^6 + 21x^5 + 13x^3 + 10x^2 + 3x + 55$ is a factor of f_5 over $\mathbb{F}_{59}$.

4.2. **Computing t mod ℓ.** We show how, when ℓ is an Atkin prime, we can test the equation $\pi_\ell^2 + k = \tilde{\tau}\pi_\ell$ in $\tilde{\tau}$ by computing only x-coordinates of points. We recall first that if

$$(x_1, y_1) \oplus (x_2, y_2) = (x_3, y_3) \quad \text{and} \quad (x_1, y_1) \ominus (x_2, y_2) = (x_4, y_4),$$

then we have

$$(x_3 + x_4)(x_1 - x_2)^2 = S(x_1, x_2) \quad \text{and} \quad x_3 x_4 (x_1 - x_2)^2 = P(x_1, x_2)$$

with

$$S(x_1, x_2) = (x_1 + x_2)(a_1 a_3 + 2a_4 + 2x_1 x_2) + x_1 x_2 (a_1^2 + 4a_2) + 4a_6 + a_3^2,$$

and

$$P(x_1, x_2) = (x_1 x_2 - a_4)(x_1 x_2 - a_4 - a_1 a_3) - (x_1 + x_2 + a_2)(a_3^2 + 4a_6) - a_1^2 a_6.$$

So the values x_3 and x_4 are solutions of the quadratic equation $E(X) = NX^2 - SX + P$ with $N(x_1, x_2) = (x_1 - x_2)^2$.

Following Müller's idea, we introduce the integers i, j and k_{opt} with the equation $i\pi_\ell^2 + ik = j\pi_\ell$. We search a value j for which $x(j\pi_\ell)$ is a root of $E(X) = 0$ given by $S(x_i^{q^2}, x_{ik})$, $P(x_i^{q^2}, x_{ik})$ and $N(x_i^{q^2}, x_{ik})$.

Indeed, if $x(i\pi_\ell^2 + ik) = x(j\pi_\ell)$, then, for some τ_0, $\pi_\ell^2 + k = \pm\tau_0\pi_\ell$ over $E[\ell]$, so $\tau \equiv \pm\tau_0 \bmod \ell$. Whereas, if $x(i\pi_\ell^2 - ik) = x(j\pi_\ell)$, then $\pi_\ell^2 - k = \pm\tau_0\pi_\ell$ and $\pi_\ell = 2k/(\tau \pm \tau_0)$, which is impossible since ℓ is an Atkin prime.

Hence, we avoid the computation of y^{q^2} and y^q and obtain $t \equiv \pm\tau_0 \bmod \ell$.

4.3. **The sign of $t \bmod \ell$.** Since π_ℓ satisfies the equation $x^2 - \tau x + k = 0$, we have $\pi_\ell^n = Q_n \pi_\ell + P_n$ with P_n and Q_n some polynomials in τ and k. We have $P_n = -kQ_{n-1}$ and moreover the polynomial Q_n contains only even powers of τ if n is odd and only odd powers otherwise [3]. On the other hand, $\pi_\ell^e = P_e$ and the value of e does not depend on the sign of τ. Hence, when e is odd, we have $P_e(\pm\tilde{\tau}, k) = \pm P_e(\tilde{\tau}, k)$, so $\pi_\ell^e = \pm P_e(\tau_0, k)$. Let w_0 be $P_e(\tau_0, k)$.

Proposition 3. *Assume $p \neq 2$, e odd; let h_ℓ be a factor of degree ed of f_ℓ, g_ℓ be a factor of degree $es(w_0)$ of h_ℓ and r be* Resultant$(g_\ell, \mathcal{G})$. *Then, when $s(w_0)$ is odd, we have $t \equiv (\frac{r}{q}) w_0^{s(w_0)} \tau_0 \bmod \ell$. When $\ell \equiv 3 \bmod 4$, one can take $g_\ell = h_\ell$.*

Proof. We have $\pi_\ell^e = \pm w_0 Id$ over $E[\ell]$; hence, if $s(w_0)$ is odd, then $\pi_\ell^{es(w_0)} = \pm w_0^{s(w_0)} Id$ over $E[\ell]$ and, if d is odd, then $\pi_\ell^{ed} = \pm w_0^d Id = \pm(\frac{w_0}{\ell}) Id$ over $E[\ell]$. □

From $\pi_\ell^2 = \tau_0 \pi_\ell + k$, we easily compute $w_0 = P_e(\tau_0, k)$. The decomposition type of h_ℓ is determined by computing $s(w_0)$.

Example. Let us consider the curve $y^2 = x^3 + 4312x + 9167$ over $\mathbb{F}_{12853}$. If $\ell = 19$, then we have $e = 5$ and using a factor h_{19} of degree 45 of f_{19} we obtain $t \equiv \pm 7 \bmod 19$. We compute $r = \text{Resultant}(x^3 + 4312x + 9167, h_{19}) = 11226$; since $(\frac{r}{p}) = 1$ and $w_0 = P_5(7, 9) = 4$, we have $t \equiv 7 \bmod 19$.

If $\ell = 13$, then $e = 7$ and $\tau_0 = 5$. Since $w_0 = P_7(5, 9) = 10$, and $s(10) = 3$, the polynomial h_{13} has an irreducible factor g_{13} of degree 21. We obtain $r = \text{Resultant}(x^3 + 4312x + 9167, g_{13}) = 9515$ and $(\frac{r}{p}) = -1$, so we have $t \equiv 5 \bmod 13$.

Proposition 4. *Let h_ℓ be a factor of degree ed of f_ℓ. If $p = 2$ and e is odd, then, when $s(w_0)$ is odd, we have*

$$\tau = \begin{cases} w_0^{s(w_0)} \tau_0 & \textit{if} \ \ \text{Tr}(\tilde{s}_1 + B(\tilde{s}_{es(w_0)-1}^2 - 2\tilde{s}_{es(w_0)} \tilde{s}_{es(w_0)-2}) / \tilde{s}_{es(w_0)}^2) = 0, \\ -w_0^{s(w_0)} \tau_0 & \textit{otherwise}, \end{cases}$$

with $\tilde{s}_i$ the symmetric functions of a factor g_ℓ of h_ℓ of degree $es(w)$. When $\ell \equiv 3 \bmod 4$, one can take $g_\ell = h_\ell$.

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