AN APPROXIMATE INERTIAL MANIFOLDS APPROACH TO POSTPROCESSING THE GALERKIN METHOD FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. In a recent paper we have introduced a postprocessing procedure for the Galerkin method for dissipative evolution partial differential equations with periodic boundary conditions. The postprocessing technique uses approximate inertial manifolds to approximate the high modes (the small scale components) in the exact solutions in terms of the Galerkin approximations, which in this case play the role of the lower modes (large scale components). This procedure can be seen as a defect-correction technique. But contrary to standard procedures, the correction is computed only when the time evolution is completed. Here we extend these results to more realistic boundary conditions. Specifically, we study in detail the two-dimensional Navier-Stokes equations subject to homogeneous (nonslip) Dirichlet boundary conditions. We also discuss other equations, such as reaction-diffusion systems and the Cahn-Hilliard equations.

1. INTRODUCTION

The Navier-Stokes (NS) equations (see Section 2), in a smooth bounded domain Ω, with nonslip Dirichlet boundary conditions, can be written as an abstract dissipative evolution equation

$$(1) \qquad \frac{du}{dt} + \nu A u + R(u) = f,$$

in an appropriate Hilbert space H with norm $|\cdot|$ (see Section 2) (*cf.* [6], [23], [38]). Here $A : D(A) \subset H \to H$ is a densely defined, unbounded, self-adjoint and positive operator with compact inverse and contains the higher-order spatial derivatives, $\nu > 0$ is a scalar, and $R : D(R) \subset H \to H$ is a nonlinear map which gathers lower order spatial derivatives and nonlinear terms.

The eigenfunctions $\{w_1, w_2, \dots\}$ of A with the associated eigenvalues $0 < \lambda_1 \leq \lambda_2, \dots$ form a complete orthonormal set in H. Let $H_m = \mathrm{span}\,\{w_1, w_2, \dots, w_m\}$, P_m the orthogonal projection of H onto H_m and $Q_m = I - P_m$. Every solution u of (1) can be decomposed uniquely into

$$(2) \qquad u = p + q, \qquad p = P_m u, \quad q = Q_m u.$$

The solution u of (1) can be approximated by the Galerkin approximation, $u_m(t) \in H_m$, obtained by integrating the standard Galerkin method (SGM) which

is given by the system of ordinary differential equations

$$(3) \qquad \frac{du_m}{dt} + \nu A u_m + P_m R(u_m) = P_m f, \qquad u_m(0) = P_m u(0).$$

Notice that since u_m is sought in H_m, the Galerkin error $u - u_m$ will never be smaller that the best-approximation error $u - p = q$; that is,

$$(4) \qquad |u - u_m| \geq |u - p| = |q|.$$

Thus, for a better approximation of u, we propose the following algorithm to improve the SGM. We call it the *Postprocessing Galerkin Method* (PPGM).

Postprocessing Galerkin algorithm. Suppose that the solution u is wanted at time $T > 0$. Then,

1. Compute the Galerkin approximation $u_m(T)$ by integrating (3) with respect to time.
2. Solve the linear elliptic problem $\nu A q_m = Q_m(f - R(u_m(T)))$.
3. Add q_m to $u_m(T)$ in order to obtain the new approximation $u_m(T) + q_m$ to $u(T)$.

We will see that the convergence rate of the new approximation is better than that of the SGM, or, broadly speaking, that $|u(T) - (u_m(T) + q_m)| = o(|q|)$. Moreover, this improvement is obtained at a small computational cost, so that the new method is computationally more efficient than the SGM (see Figure 2 below). We note that in practice, the whole of q_m is never computed, and it is replaced by an adequate truncation $P_{m'} q_m$ with $m' > m$ sufficiently large.

The PPGM was introduced and analyzed in [21]. The analysis in [21] relies heavily on the properties of Fourier expansions and the fact that complex exponentials form an algebra; therefore we restricted our treatment in [21] to dissipative evolution PDEs subject to periodic boundary conditions. This may inspire doubts about the PPGM improving the SGM when more realistic boundary conditions are imposed. This question becomes more relevant in view of recent results like those in [4], where, thanks to a clever algorithm for approximating the eigenfunctions and eigenvalues of the Stokes operator A, actual computations of (3) are carried out on complex geometries. In the present paper we analyze the method independently of the properties of the eigenfunctions w_j. We present here a new analysis with much simpler ideas than those presented in [21]. These simpler ideas have allowed us to extend the PPGM to more technical situations [18], [22]. Although for simplicity we concentrate on the Dirichelet (nonslip) boundary conditions and the NS equations, the techniques used here are easily adapted to other situations.

The PPGM is reminiscent of classical defect-correction techniques for steady state problems. To better appreciate this fact, suppose that we are dealing with the steady state problem, so that instead of (1) and (3) we have

$$(5) \qquad \nu A v + R(v) = f, \qquad v \in H,$$

$$(6) \qquad \nu A v_m + P_m R(v_m) = P_m f, \qquad v_m \in H_m,$$

respectively. Thus, in this case the PPGM method would be:

1′. Find the Galerkin approximation $v_m \in H_m$ by solving (6).
2′. Find $z_m \in Q_m H$ by solving the linear elliptic problem $\nu A z_m = Q_m(f - R(v_m))$.
3′. Add z_m to v_m in order to obtain the new approximation $v_m + z_m \in H$ to u.

Thus, the whole technique amounts to solving in step $1'$ the full nonlinear problem in the finite dimensional space H_m, where nonlinearities are more easily treated, and then computing a correction on its orthogonal complement (or, in practice, an adequate truncation of it) in step $2'$. Notice that in step $2'$, we just compute the high-frequency component z_m of the solution $\tilde{v} = v_m + z_m$ of the (more familiar) linear elliptic problem

$$(7) \qquad \nu A\tilde{v} = f - R(v_m).$$

Furthermore, once $\tilde{v}$ is computed, a better approximation $\tilde{\tilde{v}}$ can be computed by replacing v_m by $\tilde{v}$ in the nonlinear term R in (7). The process can be further iterated until a prescribed accuracy is reached, or one can use Newton's iteration to converge faster to v. Notice also that solving the steady state problem in step $2'$ instead of (7) is feasible due to the fact that we are dealing with spectral methods and both P_m and Q_m commute with A. For finite-element methods, one should use (7) instead of step $2'$ [11], [22], [43], [44].

Defect-correction techniques, or the related two-grid, two-level or multilevel methods, are a well established technique for nonlinear *steady* state problems (see e.g. [3], [11], [31], [43], [44], and the references cited therein). That is, the three steps $1'$–$3'$ are nothing new. For *evolution* problems, similar defect-correction techniques are used on the steady state problems that arise when doing implicit time-stepping.

More recently there has been renewed interest in multilevel methods for evolution problems (see e.g. [2], [7], [10], [32], [33], [34]). Part of the recent interest in multilevel methods for evolution problems arose from studies initiated by [14], [16], [13] and [41] in connection with inertial manifolds (IM) [15] and their approximations. In fact, the multilevel methods developed from this approach are not called that, but rather nonlinear Galerkin methods (NLG) [13], [26], [32]. It was through this approach that we arrived at the PPGM.

It may be clarifying then to summarize the main facts of this approach. To do this, notice that, using (2) and the fact that both P_m and Q_m commute with A, equation (1) can be rewritten as the coupled system

$$(8) \qquad \frac{dp}{dt} \quad + \quad \nu Ap + P_m R(p + q) = P_m f, \qquad p \in H_m,$$

$$(9) \qquad \frac{dq}{dt} \quad + \quad \nu Aq + Q_m R(p + q) = Q_m f, \qquad q \in Q_m H.$$

It is shown in [14] that, under certain circumstances which are motivated by the dynamical systems approach, the q terms other than νAq in (9) can be considered negligible as compared to the other terms. Thus, the authors of [14] suggested the following approximation model for the high-frequency component q of u:

$$(10) \qquad q \approx \Phi_1(p) = (\nu A)^{-1} Q_m(f - R(p)).$$

It is noteworthy that in [16] the authors used a totally different technique, which is also motivated by a dynamical systems approach, which led them to a model similar to (10). Notice the similarity of the above approximation with step $2'$ of the defect-correction technique (we will discuss the differences later). In [14] it is also shown that $\Phi_1(p)$ is a good approximation to q, in the sense that $|q - \Phi_1(p)| = o(|q|)$ (see Section 2 for a more precise statement). Based on this fact, the NLG method was developed. In the NLG method, the exact solution u of (1) is approximated by

$y_m(t) + \Phi_1(y_m(t))$, where $y_m \in H_m$ is obtained by solving

(11)
$$\frac{dy_m}{dt} + \nu A y_m + P_m R(y_m + \Phi_1(y_m)) = P_m f, \qquad y_m(0) = P_m u(0),$$

$$\Phi_1(y_m) = (\nu A)^{-1} Q_m (f - R(y_m)),$$

or certain variants of the above equation which preserve the dissipative nature of (1) (*cf.* [27], [10]).

The NLG error satisfies $|u(t) - (y_m(t) + \Phi_1(y_m(t)))| \approx |q - \Phi_1(p)| = o(|q|)$ [10], which (recall (4)) should imply a better computational performance than the SGM. However, the continuous update of $\Phi_1(y_m)$ along the time evolution is so computationally costly that the NLG, in spite of its $o(|q|)$ error, is generally less efficient than the SGM.

Let us clarify this point now with an example from [21]. Consider the 2-dimensional NS equations (see Section 2) in the vorticity-streamfunction formulation

(12)
$$\frac{\partial \omega}{\partial t} - \nu \Delta \omega + \nabla \omega \times \nabla \Psi = g,$$

$$-\Delta \Psi = \omega,$$

in the spatial domain $\Omega = [0, 2\pi]^2$, with $\nu = 0.01$ and subject to periodic boundary conditions. Here the vorticity $\omega = (\nabla \times u) \cdot \mathbf{k}$, the velocity $u = \nabla \times (\Psi \mathbf{k})$, the streamfunction is Ψ and $g = \nabla \times f$. We set the forcing term g as $g(x, y, t) = f_1(x, y)(2 + \cos(t))/3$, where

$$f_1(x, y) = \begin{cases} (1 + \cos(4r_+))^2/8, & \text{if } r_+ < \pi/4, \\ -(1 + \cos(4r_-))^2/8, & \text{if } r_- < \pi/4, \\ 0, & \text{otherwise,} \end{cases}$$

$r_\pm = |x + iy - (\pi(1 + i) \pm e^{i\theta}\pi/2)|$, and $\theta = 0.7$. The function f_1 represents stirring the fluid in opposite directions at the locations $\pi(1 + i) \pm e^{i\theta}\pi/2$. Let us fix the vorticity ω at $t = 0$ by $\omega(x, y, 0) = \omega_0(x, y)$, where $\omega_0 = (\nu \Delta)^{-1} f_1$.

Figures 1 and 2 show the errors of the different methods when they use several values N of Fourier modes in each spatial direction. The results correspond to runs from $t = 40\pi$ to $T = 44\pi$. Results corresponding to the SGM are marked with asterisks and joined by continuous lines, and those of the NLG are marked by small circles and joined by dotted lines. Since in the present example ω is in the Sobolev spaces H^σ for all $\sigma < 13/2$, following suggestions in [41], Φ_1 was truncated to $N^{13/11}$ Fourier modes in each spatial direction (see also [10], [28] and [37]).

Figure 1 is a convergence diagram where the errors in the vorticity ω committed by the methods, measured in the H^1 norm, are plotted versus the number N of Fourier modes in each spatial direction. The slopes of least-squares fits to the last four points of each method are shown in Figure 1. We see that whereas in the SGM the errors decay like $N^{-5.5}$, in the NLG they decay like $N^{-6.5}$. Thus, the NLG has a better convergence rate than the SGM, because its error decays faster with the number of modes ($m = N^2$ in this example) than the SGM.

However, efficiency (ratio of cost to error) is another matter. In Figure 2, the same errors as in Figure 1 are plotted versus the smallest amount of computing time (in seconds) that the methods needed to achieve those errors. It is clear from Figure 2 that the NLG takes more than twice the computing time of the SGM for
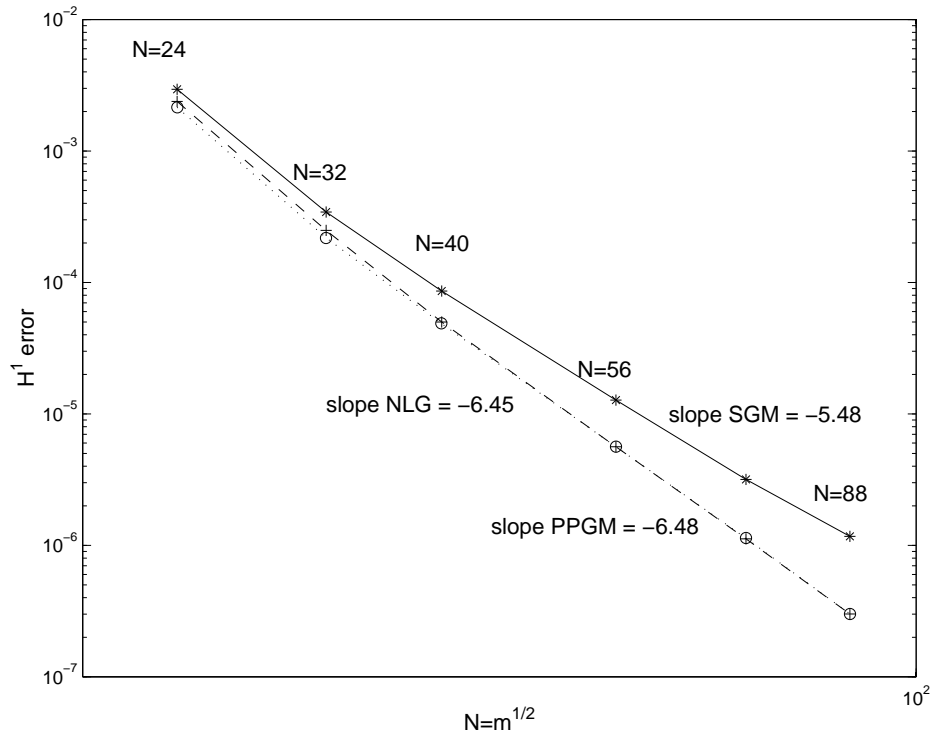
FIGURE 1. Convergence diagram; ∗ SGM, + PPGM, ∘ NLG

any error. This can be seen by drawing any horizontal line (i.e., selecting a desired accuracy); then, its intersection with the continuous line of the SGM is on the right of its intersection with the dotted line of the NLG by a factor of more than two (i.e., to achieve that desired accuracy, the SGM needs less than half the computing time of the NLG).

The poor practical performance of the NLG led us to develop the PPGM. The aim was to obtain the better accuracy of the NLG at the cost of the cheaper SGM. This, as shown in Figures 1 and 2, is achieved by the PPGM, whose results are represented by + sings and joined by dashed lines. In, Figure 1, the NLG and the PPGM commit virtually the same errors for a given N (∘ and + signs are plotted on almost the same locations). Furthermore, in Figure 2, the + sings of the PPGM and the ∗ of the SGM are (almost) on the same vertical lines (i.e., same cost) but since the PPGM error is smaller, it turs out to be more efficient. Again, this can be checked by noticing that the broken line joining + sings is the leftmost (i.e., the PPGM needs less computing time to reach a given accuracy).

Two final comments are pertinent in this example. First, following standard practice with spectral Fourier methods, nonlinear terms were approximated by interpolation, in what is known as pseudospectral methods; we made sure that the aliasing error arising from this practice was not dominant (i.e., that the errors were the same as those of the more costly pure spectral methods). Second, we were very careful in checking the sources of error and cost to avoid reaching wrong conclusions
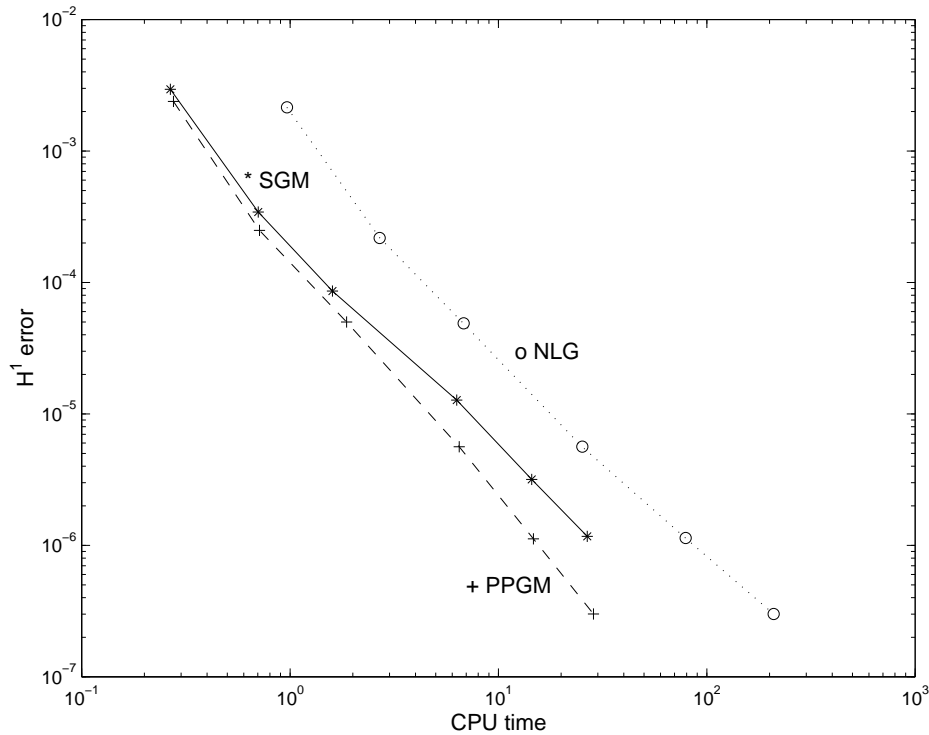
FIGURE 2. Efficiency diagram

on the relative efficiency of the methods (see [19], [20]). We refer the reader to [21], where full details of the methods used are provided, and where L^2 errors are shown.

Let us turn to the relation of these methods with defect-correction techniques. Notice that both the NLG (11) and the PPGM in steps 1–3 are defect-correction methods for *evolution* problems: the corrections $\Phi(y_m)$ in the NLG and q_m in the PPGM are obtained through solving linear elliptic problems. Notice also that, for the correction, both the NLG and the PPGM use the Foias-Manley-Temam mapping Φ_1 in (10), since q_m in step 3 is $q_m = \Phi_1(u_m(T))$. Let us then look more closely to the relation between Φ_1 and the standard defect-correction approach for *steady state* problems in steps $1'$–$3'$. Observe that on going from (9) to (10), the dq/dt term is dropped. Thus, in the steady state case $2,'$ we have the full forcing (source) term f of the original problem (5), whereas in the evolution case (10), only $Q_m f$ instead of $Q_m f - dq/dt$ is present. This implies that while in $1'$–$3'$, continuing iteration of the process might bring better and better approximations, this is not (necessarily) so in (10) unless some approximation to dq/dt is included. Procedures for doing this can be found in [8], [9], [10], [39] and [40]. Our practical experience with them [20] inspired us to use only Φ_1 in (10). For the possibility of using Newton's or another iteration method see, for instance, [16], [26], [35] and [41].

Observe also that in the PPGM the correction q_m is computed only once, as opposed to more standard defect-correction practices for evolution problems like

the NLG or the method presented in [7], where the correction is computed at every time step over the interval $[0, T]$.

A quick explanation of the $o(|q|)$ error of the PPGM may make things clearer. For linear problems with constant coefficients u_m is p, so that the error $p - u_m = 0$. However, $u - u_m = q$, which is not negligible (and hence (4)). For nonlinear problems, we show here that $|p - u_m| = o(|q|)$, and this allows $\Phi_1(u_m)$ to be a good approximation to $\Phi_1(p) \approx q$. In fact, since Φ_1 is Lipschitz-continuous (see, e.g., [10]), then $|\Phi_1(u_m) - \Phi_1(p)| \leq C |u_m - p|$. Thus, by adding $\pm\Phi_{(p)}$, for the error we have that

$$|u_m + \Phi_1(u_m) - p - q| \leq (1 + C) |u_m - p| + |\Phi_1(p) - q| = o(|q|).$$

We perform the analysis summarized here for both L^2 and H^1 norms.

We point out that in [14], the graph of Φ_1, that is, $\{p + \Phi_1(p) \mid p \in H_m\}$, was called an approximate inertial manifold (AIM) (see also [16], [41]), in reference to the IMs, where $q = \Phi(p)$ for some mapping Φ, which is sometimes known to exist but is hardly ever known explicitly. We do not discuss here whether AIMs approximate IMs or not. We simply use Φ_1 and use the term AIM that other authors coined before us.

We remark that our results are not restricted to time-independent forcing terms f in (1), and are valid for Hölder-continuous (in time) f (see Remark 2 in Section 3). We also show how to extend our analysis to dissipative equations other than the NS equations, such as reaction-diffusion systems and the Cahn-Hilliard equations. We note that there are in the literature AIMs other than the Φ_1 used here [8], [9], [14], [16], [17], [30], [38], [39], [41]. Our results hold also for the Euler-Galerkin approximate inertial manifold which was introduced in [16] (see also [13] and [26]).

The rest of the paper is organized as follows. In Section 2 some standard preliminary material is introduced. Section 3 is devoted to studying the PPGM for the NS equations. Finally, in Section 4, the extension of the results to other equations is discussed.

2. PRELIMINARY RESULTS

We consider the NS equations

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= \nu\Delta u + F, \\ \operatorname{div}(u) &= 0, \end{aligned}$$

in a smooth bounded domain $\Omega \subset R^2$, subject to the homogeneous (nonslip) Dirichlet boundary condition $u|_{\partial\Omega} = 0$. In the rest of the paper, we use the spaces

$$H = \left\{ u \in L^2(\Omega)^2 \mid \operatorname{div}(u) = 0, \, u \cdot n|_{\partial\Omega} = 0 \right\}$$

and the space

$$V = \left\{ u \in H_0^1(\Omega)^2 \mid \operatorname{div}(u) = 0 \right\}.$$

Let Π be the orthogonal L^2 projection $\Pi : L^2(\Omega)^2 \to H$. Projecting onto H, the NS equations become equation (1) with $A = -\Pi\Delta$, $R(u) = B(u, u) = \Pi[(u \cdot \nabla)u]$, and $f = \Pi F$ (see, e.g., [6]). It is well known that then $D(A^{1/2}) = V$. Following the notation of [6], we denote by $(\cdot, \cdot)$ the inner product in H (i.e., in $L^2(\Omega)^2$), and by $|\cdot|$ and $\|\cdot\|$ the norms

$$|u| = \|u\|_{L^2(\Omega)^2}, \qquad \|v\| = \left| A^{1/2}v \right|,$$

in H and V respectively. We will also denote by $\|\cdot\|_\infty$ the $L^\infty(\Omega)$ norm. For k an integer, we consider the Sobolev space $H^k = H^k(\Omega)$ with its usual norm rescaled to

$$(13) \qquad \|u\|_{H^k} = \left(\sum_{|\alpha| \le k} \lambda_1^{k-|\alpha|} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

This norm is readily extended to $H^k(\Omega)^2$. We will use the inequalities

$$(14) \qquad \|\omega\|_{H^k(\Omega)^2} \le c_1 \left| A^{k/2}\omega \right|, \quad \forall \omega \in D(A^{k/2}), \qquad k = 1, 2, 3,$$

where, throughout this paper, $c, c_1, c_2, \ldots$ will denote dimensionless constants. (See [6], pp. 36-41, on the regularity requirements on the domain Ω for (14) to be valid).

We recall the Brézis-Gallouet [5] inequality in two dimensions:

$$(15) \qquad \|u\|_{L^\infty(\Omega)} \le c_2 \|u\|_{H^1} \left(1 + \log \frac{\|u\|_{H^2}}{\lambda_1^{1/2} \|u\|_{H^1}} \right)^{1/2}, \qquad \forall u \in H^2.$$

This inequality is also valid for $u \in H^2(\Omega)^2$, and as a result we have

$$(16) \qquad \|v\|_\infty \le c_3 \|v\| \, L_m, \qquad \forall v \in H_m,$$

where

$$L_m = \left(1 + \log \frac{\lambda_m}{\lambda_1} \right)^{1/2}.$$

The bilinear form B satisfies the following skew property:

$$(17) \qquad (B(u,v), w) = -(B(u,w), v) \qquad \text{for all} \quad u, v, w \in V$$

(see, for instance, [6], p. 53). Moreover, for any δ with $1/2 < \delta < 1$ we have

$$(18) \qquad |A^{-\delta}(R(u) - R(v))| \le L \, |u - v|, \qquad \forall u, v \in V,$$

with $L = c_4 |A^{1/2}(u+v)|$ (see, e.g., [6], p. 55).

We recall the following property of the NS equations. There exist constants M_0 and M_1 such that for any solution u of (1), there is a time $T_0 = T_0(|u(0)|, |f|)$ such that

$$(19) \qquad |u(t)| \le M_0, \qquad |A^{1/2}u(t)| \le M_1, \qquad t \ge T_0.$$

Here and below, $M, M_0, M_1, \ldots$ and $K, K_0, K_1, \ldots$ denote constants that may depend on the data of the problem (i.e., ν, f, $|u(0)|$ and λ_1). A simple modification to the argument given in [14] allows us to choose the time T_0 large enough so that

$$(20) \qquad |q(t)| \; \le \; K_0 \frac{L_m}{\lambda_{m+1}}, \qquad K_0 = \frac{c_5}{\nu} \left(\frac{|Q_m f|}{L_m} + M_1^2 \right),$$

$$(21) \qquad \|q(t)\| \; \le \; K_1 \frac{L_m}{\lambda_{m+1}^{1/2}}, \qquad K_1 = \frac{c_6}{\nu} \left(\frac{|Q_m f|}{L_m} + M_1^2 + \frac{M_0 M_1^2}{\nu L_m} \right),$$

for every m and for every $t \ge T_0$. In particular these estimates hold, for all $-\infty < t < \infty$, for any solution in the global attractor. We now summarize some results concerning Φ_1 and the rates of convergence of the SGM and the NLG. First of all, we notice that the convergence properties of the discretizations of (1) depend on the approximation capabilities of the space where the solution is sought. For the SGM, the solution u_m is in H_m, so that, for example, the L^2 error cannot be better

than the L^2 norm of $u - P_m u = u - p = q$. In fact, if (19), (20) and (21) are satisfied for $T_0 = 0$ then

$$(22) \qquad |u(t) - u_m(t)| \leq C(t) \frac{L_m}{\lambda_{m+1}}, \quad \text{and} \quad \|u(t) - u_m(t)\| \leq C(t) \frac{L_m}{\lambda_{m+1}^{1/2}}$$

(see, for instance, [10] and [36]). Moreover, the above estimate (22) is sharp, as has been demonstrated by an example in [41]. For the NLG, the solution $y_m + \Phi_1(y_m)$ lies in the manifold $\mathcal{M}_1 = \text{graph}(\Phi_1)$. The L^2-distance and the H^1-distance of the solution u of (1) to this manifold are $|q - \Phi_1(p)|$ and $\|q - \Phi_1(p)\|$, respectively. These quantities can be bounded [14] by

$$(23) \qquad |q - \Phi_1(p)| \leq K_3 \frac{L_m}{\lambda_{m+1}^{3/2}}, \qquad \|q - \Phi_1(p)\| \leq K_3 \frac{L_m}{\lambda_{m+1}},$$

with $K_3 = K_3(\nu^{-1}, |f|, \lambda_1, M_1^3)$. One can then prove that

$$(24) \qquad |u(t) - y_m(t) - \Phi_1(y_m(t))| \leq C(t) \frac{L_m^2}{\lambda_{m+1}^{3/2}},$$

$$(25) \qquad \|u(t) - y_m(t) - \Phi_1(y_m(t))\| \leq C(t) \frac{L_m^2}{\lambda_{m+1}}.$$

3. MAIN RESULTS

This section is devoted to proving the following theorem.

Theorem 1. *Let $T > 0$ be fixed. Let u be a solution of the NS equations on $[T_0, T]$ such that (19), (20) and (21) are satisfied for $T_0 = 0$. Then, there exists a constant $C = C(T, M_1)$ such that for any $t \in (0, T)$ the solution $u_m(t)$ of the standard Galerkin method, equation (3), satisfies*

$$(26) \qquad |p(t) - u_m(t)| \leq C \frac{L_m^4}{\lambda_{m+1}^{3/2}},$$

$$(27) \qquad \left| u(t) - \left(u_m(t) + \Phi_1(u_m(t)) \right) \right| \leq C \frac{L_m^4}{\lambda_{m+1}^{3/2}}.$$

$$(28) \qquad \|p(t) - u_m(t)\| \leq C \frac{L_m^4}{\lambda_{m+1}},$$

$$(29) \qquad \left\| u(t) - \left(u_m(t) + \Phi_1(u_m(t)) \right) \right\| \leq C \frac{L_m^4}{\lambda_{m+1}}.$$

The factor L_m^4 can be replaced by a constant (see Remark 1 below).

Before proving this theorem, some remarks may be helpful. We first show that, as we mentioned in the Introduction, (27)–(29) are implied by (26). In fact, notice that since $u = p + q$, we can write $u - (u_m + \Phi_1(u_m)) = (p - u_m) + (q - \Phi_1(u_m))$, which, on adding $\pm\Phi_1(p)$, can be expressed as

$$(30) \qquad u - (u_m + \Phi_1(u_m)) = (p - u_m) + (q - \Phi_1(p)) + (\Phi_1(p) - \Phi_1(u_m)).$$

The second term on the right hand side above is (recall (23)) $O(L_m \lambda_{m+1}^{-3/2})$. The last term, since Φ_1 is Lipschitz-continuous [10], can be bounded as

$$(31) \qquad |\Phi_1(p) - \Phi_1(u_m)| \leq l\,|p - u_m|, \quad \|\Phi_1(p) - \Phi_1(u_m)\| \leq l\,\|p - u_m\|,$$

where l can be made arbitrary small by choosing m large enough. It is then clear that (27) follows from (26) and (23). Similarly, if we take into account that $\|p - u_m\| \leq \lambda_{m+1}^{1/2} |p - u_m|$, (28) is a straightforward from (26), and by the same argument as above we get (29).

The proof of (26) given below will be divided into a number of separate intermediate results. We now comment on the main ideas, to facilitate the reading. This proof is done by stability plus consistency arguments, reminiscent of finite-difference analysis, rather than by the standard error analysis typical of spectral and finite-element methods. This allows us to study only quantities in H_m, for example $u_m - p$, rather than in quantities H like $u - u_m$, and exploit the finite-dimensionality in certain inequalities.

The first step in our proof of (26) is the *stability* of the SGM. This is no novelty, and is presented in Theorem 2 for convenience of the reader. By stability we mean that we bound the error $|p - u_m|$ in terms of the quantity

$$(32) \qquad W = \max_{0 \leq s \leq T} \left| \int_0^s e^{-\nu(s-r)A} P_m G(r) \, dr \right|,$$

which is a convenient norm of the residual $P_m G$, where

$$G = B(p, q) + B(q, p) + B(q, q).$$

Notice that $P_m G$ equals $(d/dt)p + \nu A p + P_m B(p, p) - P_m f$; that is, it is the residual or truncation error obtained when u_m is replaced by p in the "discrete" equation (3) satisfied by u_m.

The second step in the proof of (26) is *consistency*, that is, to show that (32) is of order $L_m^4 \lambda_{m+1}^{-3/2}$. Consistency is presented in Theorem 3. From this theorem and the stability of Theorem 2, the main result of this section, Theorem 1, follows.

We now comment on the proof of Theorem 3 (consistency), which is prepared by three lemmas. Observe that we want to show that (32) is $O(L_m^4 \lambda_{m+1}^{-3/2})$. Since G is only of the size of $\|q\|$ $(= O(\lambda_{m+1}^{-1/2}))$ ([6], p. 50), in order to get a better rate we must exploit the smoothing effect of $e^{-\nu(s-r)A}$ in (32) by using the inequality

$$\left| e^{\nu(t-s)A} P_m G(s) \right| \leq \left\| A e^{\nu(t-s)P_m A} \right\|_{\mathcal{L}(H_m)} \left| A^{-1} P_m G(s) \right|.$$

(Here and in the sequel, $\|\cdot\|_{\mathcal{L}(X)}$ denotes the operator norm in the Hilbert space X). Lemmas 1 and 2 below show that $\left| A^{-1} P_m G(s) \right| = O(L_m^2 \lambda_{m+1}^{-3/2})$. This result is not valid for $\left| A^{-1} G(s) \right|$, so that the finite-dimensional space H_m plays a key role here. After Lemmas 1 and 2, W in (32) can be bounded by

$$W \leq K L_m^2 \lambda_{m+1}^{-3/2} \int_0^T \left\| A e^{\nu(T-s)P_m A} \right\|_{\mathcal{L}(H_m)} ds.$$

Lemma 3 then shows that $\int_0^t \left\| A e^{\nu(T-s)P_m A} \right\|_{\mathcal{L}(H_m)} dr \leq c L_m^2$, and consistency follows. Notice that again we have exploited the finite dimension of H_m, since whereas the operator norm $\left\| A e^{\nu(T-s)P_m A} \right\|_{\mathcal{L}(H_m)}$ is integrable in $[0, T]$, the norm $\left\| A e^{\nu(T-s)A} \right\|_{\mathcal{L}(H)}$ is $(\nu(T-s))^{-1}$ and, hence, nonintegrable.

Theorem 2. *Let $\delta \in (1/2, 1)$ be fixed, and let $T > 0$ also be fixed. Let u be a solution of the NS equations such that (19), (20) and (21) are satisfied for $T_0 = 0$.*

Then, there exists a constant $C = C(T, M_1, \delta)$ such that for any $t \in [0, T]$ the solution u_m of the standard Galerkin method satisfies

$$(33) \qquad |p(t) - u_m(t)| \leq C\left(|p(0) - u_m(0)| + \max_{0 \leq s \leq T}\left|\int_0^s e^{-\nu(s-r)A}P_m G(r)\,dr\right|\right),$$

where

$$p(t) = P_m u(t) \qquad and \qquad G(r) = R(u(r)) - R(p(r)).$$

Proof. Let us set $e = p - u_m$. Subtracting (3) from (8), we have

$$\frac{d}{dt}e = -\nu A e + P_m(R(u_m) - R(p)) - P_m G.$$

Hence,

$$\begin{aligned} e(t) &= e^{-\nu t A}e(0) + \int_0^t e^{-\nu(t-s)A}P_m(R(u_m(s)) - R(p(s)))\,ds \\ &\quad - \int_0^t e^{-\nu(t-s)A}P_m G(s)\,ds. \end{aligned}$$

Taking norms and using (18) and the fact that $\|u_m(t)\| \leq M_1$ for $t \geq T_0$ (see [6], p. 77), we obtain

$$|e(t)| \leq |e^{-\nu t A}e(0)| + \frac{c_4 M_1}{\nu^\delta}\int_0^t \frac{\delta^\delta}{(t-s)^\delta}L\,|e(s)|\,ds + \left|\int_0^t e^{-\nu(t-s)A}P_m G(s)\,ds\right|.$$

Now (33) follows from a generalized Gronwall inequality ([24], p. 6). $\qquad\square$

Lemma 1. *There exist a constant c such that for any solution u of the NS equations satisfying (19), (20) and (21) for $T_0 = 0$, the following bounds hold:*

$$(34) \qquad \left|A^{-1}P_m B(p, q)\right|, \left|A^{-1}P_m B(q, p)\right| \leq cM_1 L_m\,\|q\|_{H^{-1}(\Omega)^2},$$

$$(35) \qquad \left|A^{-1}P_m B(q, q)\right| \leq cL_m\,|q|^2.$$

Proof. For $u, v \in V$ we have

$$(36) \qquad \left|A^{-1}P_m B(u, v)\right| = \max_{\substack{\xi \in H_m \\ |\xi|=1}}\left|\left(A^{-1}B(u, v), \xi\right)\right| = \max_{\substack{\xi \in H_m \\ |\xi|=1}}\left|\left(B(u, v), A^{-1}\xi\right)\right|$$

$$= \max_{\substack{\xi \in H_m \\ |\xi|=1}}\left|\left(B(u, A^{-1}\xi), v)\right)\right|,$$

where for the last equality we have used the skew property (17).

Let us put $\omega = A^{-1}\xi$ (notice that $\omega \in C^\infty(\Omega) \cap D(A)$). We now replace u and v in (36) by p and q respectively. Since $\mathrm{div}(q) = 0$,

$$(B(p, \omega), q) = (q, p \cdot \nabla\omega).$$

Moreover, since $p \in V$ and $\omega \in C^\infty(\Omega) \cap D(A))$, then $p \cdot \nabla\omega \in H_0^1(\Omega)^2$, and as a result of the above we have

$$\begin{aligned} |(B(p, \omega), q)| = |(q, p \cdot \nabla\omega)| &\leq \|q\|_{H^{-1}(\Omega)^2}\,\|p \cdot \nabla\omega\|_{H_0^1(\Omega)^2} \\ &\leq c\,\|q\|_{H^{-1}(\Omega)^2}\,\|\nabla(p \cdot \nabla\omega)\|_{L^2(\Omega)^2} \\ (37) \qquad &\leq c\,\|q\|_{H^{-1}(\Omega)^2}\left(\|p\|_\infty\,\|\omega\|_{H^2(\Omega)^2} + \|\nabla\omega\|_\infty\,\|\nabla p\|_{L^2(\Omega)^2}\right) \end{aligned}$$

Applying the Brézis-Gallouet inequality (15) to $\nabla\omega$, we get

$$\|\nabla\omega\|_\infty \leq c\,\|\nabla\omega\|_{H^1(\Omega)^2}\left(1 + \log\frac{\|\nabla\omega\|_{H^2(\Omega)^2}}{\lambda_1^{1/2}\,\|\nabla\omega\|_{H^1(\Omega)^2}}\right)^{1/2}.$$

Since $\|\nabla\omega\|_{H^k(\Omega)^2} \leq \|\omega\|_{H^{k+1}(\Omega)^2}$, and using (14) and the fact that $\left|A^{1/2}\omega\right| = \|\nabla\omega\|_{L^2(\Omega)^2}$, we can bound

$$\frac{\|\nabla\omega\|_{H^2(\Omega)^2}}{\lambda_1^{1/2}\|\nabla\omega\|_{H^1(\Omega)^2}} \leq \frac{\|\nabla\omega\|_{H^2(\Omega)^2}}{\lambda_1\|\nabla\omega\|_{L^2(\Omega)^2}} \leq \frac{\|\omega\|_{H^3(\Omega)^2}}{\lambda_1\left|A^{1/2}\omega\right|} \leq c\frac{\left|A^{3/2}\omega\right|}{\lambda_1\left|A^{1/2}\omega\right|}.$$

Hence, we obtain

$$\|\nabla\omega\|_\infty \leq c\left|A\omega\right|L_m.$$

Then, from (37), and using (16) to bound $\|p\|_\infty$, we get

$$|(B(p,\omega),q)\| \leq c'L_m\|p\|\left|A\omega\right|\|q\|_{H^{-1}(\Omega)^2}.$$

Since $|A\omega| = 1$ and $\|p\| \leq M_1$, we conclude that

(38) $$\left|A^{-1}P_mB(p,q)\right| \leq cL_mM_1\|q\|_{H^{-1}(\Omega)^2}.$$

For $\left|A^{-1}B(q,p)\right|$, arguing as before, we only have to bound $(p, q \cdot \nabla\omega)$. Let $q = (q^1, q^2)^T$; then we observe that

$$(p, q \cdot \nabla\omega) = \int_\Omega q^1 p \cdot \partial_x\omega\,dxdy + \int_\Omega q^2 p \cdot \partial_y\omega\,dxdy.$$

Let us estimate the first term on the right hand side from above. As before, since $p \in V$ and $\omega \in C^\infty(\Omega) \cap D(A)$ we have $p \cdot \partial_x\omega \in H_0^1(\Omega)$. Therefore,

$$\left|\int_\Omega q^1 p \cdot \partial_x\omega\,dxdy\right| \leq c\left\|q^1\right\|_{H^{-1}(\Omega)}\|\nabla(p \cdot \partial_x\omega)\|_{L^2(\Omega)^2}.$$

By using arguments similar to the ones above, especially applying the Brézis-Gallouet inequality (15) twice, and the fact that $p, \omega \in H_m$, we obtain

$$\|\nabla(p \cdot \partial_x\omega)\|_{L^2(\Omega)^2} \leq \|\nabla p\|_{L^2(\Omega)^2}\|\partial_x\omega\|_\infty + \|p\|_\infty\|\nabla\partial_x\omega\|_{L^2(\Omega)^2}$$

$$\leq cL_m\Big(\|p\|\left|A\omega\right| + \|p\|\|\omega\|_{H^2(\Omega)^2}\Big)$$

$$\leq c'L_m\|p\|\left|A\omega\right| \leq c'L_m\|p\|,$$

because $|A\omega| = 1$. The term $\int_\Omega q^2 p \cdot \partial_y\omega\,dxdy$ can be treated similarly, and we conclude that

$$\left|A^{-1}P_mB(q,p)\right| \leq cL_m\|p\|\|q\|_{H^{-1}(\Omega)^2},$$

which, together with (38), readily leads to (34), since $\|p\| \leq M_1$.

To prove (35), let us apply (36) with $u = v = q$ and use (17) to obtain

$$\left|A^{-1}P_mB(q,q)\right| \leq \max_{\substack{\xi\in H_m\\|\xi|=1}}\left|\left(B(q, A^{-1}\xi), q\right)\right| \leq |q|^2\max_{\substack{\xi\in H_m\\|\xi|=1}}\left\|\nabla A^{-1}\xi\right\|_\infty \leq cL_m\left|q\right|^2,$$

where in the last step we have applied the Brézis-Gallouet inequality (15) to $\left\|\nabla A^{-1}\xi\right\|_\infty$ and used the fact that $\xi \in H_m$. $\qquad\square$

Lemma 2. *There exist a constant $c > 0$ such that for any $q \in V$ the following bound holds:*

(39) $$\|q\|_{H^{-1}(\Omega)^2} \leq c\left|A^{-1/2}q\right|.$$

Proof. Notice that $\|q\|_{H^{-1}(\Omega)^2} = \sup\{|(q,v)| \mid v \in H_0^1(\Omega)^2, \|v\|_{H_0^1(\Omega)^2} = 1\}$. But

$$|(q,v)| = |(q,\Pi v)| = \left|(AA^{-1}q, \Pi v)\right| = \left|(\Delta A^{-1}q, \Pi v)\right| = \left|(\nabla A^{-1}q, \nabla \Pi v)\right|$$

$$\leq \left|\nabla A^{-1}q\right| |\nabla \Pi v| = \left|A^{1/2}A^{-1}q\right| |\nabla \Pi v| = \left|A^{-1/2}q\right| |\nabla \Pi v|,$$

where we recall that Π is the orthogonal projection of L^2 onto H. Since (see Remark 1.10 in [6], p. 9) $\Pi : H_0^1 \to H^1$ is continuous, (39) follows. $\square$

Lemma 3. *Let* $g : [0,T] \to H_m$ *be such that*

$$\max_{0 \leq t \leq T} \left|A^{-1}g(t)\right| \leq C.$$

Then for any $t \in [0,T]$,

(40) $$\left|\int_0^t e^{-(t-s)\nu A}g(s)\,ds\right| \leq \frac{L_m^2}{\nu}C.$$

Proof. We start by noticing that

$$\int_0^t e^{-(t-s)\nu A}g(s)\,ds = \frac{1}{\nu}\int_0^t \nu A e^{-(t-s)\nu A}A^{-1}g(s)\,ds.$$

Since $g(s) \in H_m$ for $s \in [0,T]$, we have

$$Ae^{-(t-s)\nu A}A^{-1}g(s) = P_m A e^{-(t-s)\nu P_m A}A^{-1}g(s),$$

so that

(41) $$\left|\int_0^t e^{-(t-s)\nu A}g(s)\,ds\right| \leq C\frac{1}{\nu}\int_0^t \left\|\nu P_m A e^{-(t-s)\nu P_m A}\right\|_{\mathcal{L}(H_m)}\,ds.$$

In the sequel, we change variables $r = t - s$ in the integral above. Observe that

$$\left\|\nu P_m A e^{-r\nu P_m A}\right\|_{\mathcal{L}(H_m)} = \max_{1 \leq j \leq m} \nu\lambda_j e^{-\nu\lambda_j r} \leq \max_{z \in [\nu\lambda_1, \nu\lambda_m]} z e^{-rz}$$

$$= \begin{cases} \nu\lambda_m e^{-\nu\lambda_m r} & (r \leq 1/(\nu\lambda_m)), \\ \dfrac{e^{-1}}{r} & (1/(\nu\lambda_m) \leq r \leq 1/(\nu\lambda_1)), \\ \nu\lambda_1 e^{-\nu\lambda_1 r} & (r \geq 1/(\nu\lambda_1)). \end{cases}$$

In order to estimate the right hand side of (41) we decompose the interval $[0,t]$ as the union of

$$I_1 = [0, \frac{1}{\nu\lambda_m}] \cap [0,t], \qquad I_2 = [\frac{1}{\nu\lambda_m}, \frac{1}{\nu\lambda_1}] \cap [0,t], \qquad I_3 = [\frac{1}{\nu\lambda_1}, t] \cap [0,t].$$

We then have

$$\int_{I_1} \left\|\nu P_m A e^{-r\nu P_m A}\right\|_{\mathcal{L}(H_m)}\,dr \leq \int_{I_1} \nu\lambda_m e^{-\nu\lambda_m r}\,dr \leq 1 - \frac{1}{e},$$

$$\int_{I_2} \left\|\nu P_m A e^{-r\nu P_m A}\right\|_{\mathcal{L}(H_m)}\,dr \leq \frac{1}{e}\int_{I_2}\frac{1}{r}\,dr \leq \frac{L_m^2}{e},$$

and

$$\int_{I_3} \left\|\nu P_m A e^{-r\nu P_m A}\right\|_{\mathcal{L}(H_m)}\,dr \leq \int_{I_3} \nu\lambda_1 e^{-\nu\lambda_1 r}\,dr \leq \frac{1}{e} - e^{-\lambda_1 t}.$$

Combine the above with (41) to obtain (40). $\square$

Theorem 3. *Suppose that the conditions of Theorem 2 hold. Then there exists a constant $c > 0$ such that*

$$(42) \qquad \max_{t \geq 0} \left| \int_0^t e^{-\nu(t-s)A} P_m G(s) \, ds \right| \leq \frac{L_m^2}{\nu} \left(L_m M_1 \left| A^{-1/2} q \right| + c L_m \left| q \right|^2 \right)$$

$$\leq \frac{K_0 L_m^4}{\nu \lambda_{m+1}^{3/2}} \left(M_1 + \frac{c L_m K_0}{\lambda_{m+1}^{1/2}} \right).$$

Proof. Let us recall that

$$P_m G = P_m B(p,q) + P_m B(q,p) + P_m B(q,q).$$

Then (42) follows immediately from (20), (21), and Lemmas 1, 2 and 3.   □

Remark 1. The leading term $\nu^{-1} L_m^4 K_0 M_1 \lambda_{m+1}^{-3/2}$ on the right hand side of (42) can be replaced by $\nu^{-2} L_m^2 M_1^3 K_0 \lambda_{m+1}^{-3/2}$ in the following way. Let us denote $g(s) = B(p,q) + B(q,p)$. Using integration by parts we can write

$$(43) \qquad \int_0^t e^{-\nu(t-s)A} P_m g(s) \, ds = \frac{1}{\nu} \left(A^{-1} P_m g(t) - e^{-\nu t A} A^{-1} P_m g(0) \right)$$

$$- \frac{1}{\nu} \int_0^t e^{-\nu(t-s)A} A^{-1} P_m \frac{dg(s)}{ds} \, ds.$$

Notice that no term $A e^{\nu(t-s)A}$ appears now, so that the L_m^2 factor arising from Lemma 3 will not be present in the corresponding bound. Then the result follows from the the fact that for $t \geq T_0$,

$$(44) \qquad \left\| \frac{du}{dt}(t) \right\| \leq c \frac{M_1^2}{\nu} M_1, \qquad \left| \frac{dq}{dt}(t) \right| \leq c \frac{M_1^2}{\nu} K_0 \frac{L_m}{\lambda_{m+1}}.$$

(Here, we are assuming that $|f|$ is not too small; see [14] for a more precise statement of (44)). Notice that the above estimate (44) holds for all $t \in (-\infty, \infty)$ for solutions in the global attractor. Let us remark, however, that this approach provides a more favorable estimate than (42) only if m is sufficiently large so that $L_m > M_1^2/\nu$.

Furthermore, all terms L_m can be removed from (42) at the price of replacing all constants depending on M_1 by constants depending on $M_2 = \max_{t \geq T_0} |Au|$, which is several orders of magnitude bigger than M_1.

Remark 2. Theorem 1 is not restricted to time-independent forcing terms. If $f = f(t)$ satisfies $f_\infty = \max_{t \geq 0} |f(t)| < +\infty$ and for some $\theta \geq 1/2$

$$|f(t_1) - f(t_2)| \leq L_f |t_1 - t_2|^\theta, \qquad \forall t_1, t_2 \geq 0,$$

then it can be shown (see [29]) that (23) holds (inequality (23) was first proved in [14] for time-independent forces f). Since f plays no role in the results in this section, they are also valid for $f = f(t)$.

It is important to notice that for time-dependent forces, the estimates (44) do not necessarily hold (except for f analytic in a strip of the complex plane around the positive real time axis). Hence the approach provided by (43) is of no use, and Lemma 3 has to be used.

4. Further extensions

The fact that the postprocessed Galerkin method has the same rate of convergence as the NLG with $\Phi_{app} = \Phi_1$ is not an exclusive property of the NS equations. It applies also to other two-dimensional (and three-dimensional) dissipative PDEs. In fact, none of the properties of the solutions of the NS equations that have played a key role in our analysis are exclusive to the NS equations, nor are the three lemmas in the previous section applicable only to the NS equations.

Let us review first the properties of the NS equations that we have used. These are (19) (i.e., dissipativity) and (14) (i.e., the continuity of the embeddings of $D(A^\alpha)$ into the corresponding Sobolev spaces). Notice that we have used (14) in order to have some control on the L^∞ norms of both p and u_m (and their spatial derivatives) in the nonlinear terms. To avoid the presence of $|Au|$ in our error bounds we relied heavily on the Brézis-Gallouet inequality (15). This works nicely for two-dimensional problems; however, for three-dimensional problems we will have to use the estimate

$$(45) \qquad M_2 = \max_{t \geq T_0} |Au|$$

instead.

As for the lemmas in the previous section, Lemma 3 is valid even for sectorial operators, while Lemma 2 depends on the boundary conditions and the relation between Sobolev spaces and the the fractional powers of A. Lemma 1, however, depends on the particular nonlinearity of the equation.

We show now how to extend the results of the previous section to reaction-diffusion (RD) equations of the form

$$(46) \qquad u_t - \nu\Delta u + R(u) = f,$$

where R is a polynomial nonlinearity

$$(47) \qquad R(u) = \sum_{j=1}^{2k+1} a_j u^j, \qquad a_{2k+1} > 0.$$

We consider the equation on a bounded domain Ω in R^d, $d \leq 3$, subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. Notice that the operator A is $A = -\Delta$. We refer the reader to, for instance, [38] for the properties of the solutions of this equation.

For the RD equation (46), the mapping Φ_1 possesses the following approximation property [8]. For $u = p + q$ satisfying (19) and (45)

$$(48) \qquad |q - \Phi_1(p)| \leq \frac{K_4}{\lambda_{m+1}^2},$$

where K_4 depends on M_2 (in fact, on M_2^{2k}). All the bounds that we obtain here for the RD equation will depend on this constant M_2, rather than on M_1.

We present a version of Lemma 1 for the RD equation.

Lemma 4. *There exists a constant c such that for any solution u of the RD equation (46) such that (19) and (45) are satisfied for $T_0 \geq 0$, the following bound holds:*

$$(49) \qquad \left|A^{-1}(R(u) - R(p))\right| \leq K M_1^{k-1/2} M_2^{k+1/2} \left|A^{-1}q\right|.$$

Proof. Reasoning as in Lemma 1, we have to bound $\left|(R(u) - R(p), A^{-1}\xi)\right|$ with $|\xi| = 1$. Notice that $R(u) - R(p)$ is a linear combination of terms that are either q, or $q^l p^n$ with $l \geq 1$ and $2 \leq l + n \leq 2k + 1$. Then for $\omega = A^{-1}\xi$ we denote $v = q^{l-1}p^n\omega$ and $Q = A^{-1}q$. Since $D(A)$ is an algebra, and since q, p and ω belong to $D(A)$, then $\nabla v = 0$ on $\partial\Omega$. As a result we have

$$(q^l p^n, \omega) = (q, v) = -(\text{div}(\nabla Q), v) = (\nabla Q, \nabla v) = -(Q, \Delta v),$$

where, for the last equality, we have used the fact that $Q\nabla v = 0$ on $\partial\Omega$. Then, if we show that $|\Delta v| \leq KM_1^{k-1/2}M_2^{k+1/2}|\xi|$, (49) will follow. Assume for simplicity that $l = 1$ (the case $l \geq 1$ is treated similarly). Then

$$(50) \qquad \Delta v = n\omega p^{n-1}\Delta p + n(n-1)\omega p^{n-2}\nabla p \cdot \nabla p + 2np^{n-1}\nabla p \cdot \nabla\omega + p^n\Delta\omega.$$

We bound each term on the right hand side above separately. Take the first term. Using Agmon's inequality [1], [38] in three dimensions

$$(51) \qquad \|f\|_\infty \leq c\|f\|_{H^1}^{1/2}\|f\|_{H^2}^{1/2}, \qquad \forall f \in H^2(\Omega),$$

and (13), we have

$$\begin{aligned}\left|\omega p^{n-1}\Delta p\right| &\leq \|\omega\|_\infty \|p\|_\infty^{n-1}|\Delta p| \leq cM_1^{\frac{n-1}{2}}M_2^{\frac{n+1}{2}}\|\omega\|_{H^1}^{1/2}\|\omega\|_{H^2}^{1/2} \\ &\leq c\lambda_1^{-1/4}M_1^{k-1/2}M_2^{k+1/2}|\xi|,\end{aligned}$$

where for the last inequality we have used the fact that $\|\omega\|_{H^1} \leq \lambda_1^{-1/2}\|\omega\|_{H^2}$. For the first term on the right hand side of (50) we write

$$\begin{aligned}|p^n\Delta\omega| &\leq \|p\|_\infty^n|\Delta\omega| \leq c\|p\|_{H^1}^{n/2}\|p\|_{H^2}^{n/2}|\xi| \leq c\|p\|_{H^1}^{1/2}M_1^{\frac{n-1}{2}}M_2^{n/2}|\xi| \\ &\leq c\lambda_1^{-1/4}\|p\|_{H^2}^{1/2}M_1^{\frac{n-1}{2}}M_2^{n/2}|\xi| \leq c\lambda_1^{-1/4}M_1^{k-1/2}M_2^{k+1/2}|\xi|.\end{aligned}$$

For the third term on the right hand side of (50) we first notice that, using Hölder's inequality and Sobolev's inequalities in three dimensions, we have the bound

$$\begin{aligned}|\nabla p \cdot \nabla\omega| &\leq \|\nabla p\|_{L^3(\Omega)^2}\|\nabla\omega\|_{L^6(\Omega)^2} \leq c\|\nabla p\|_{H^{1/2}(\Omega)^2}\|\nabla\omega\|_{H^1(\Omega)^2} \\ &\leq c\lambda_1^{-1/4}\|p\|_{H^2}\|\omega\|_{H^2},\end{aligned}$$

so that

$$\begin{aligned}\left|p^{n-1}\nabla p \cdot \nabla\omega\right| &\leq \|p\|_\infty^{n-1}|\nabla p \cdot \nabla\omega| \leq c\lambda_1^{-1/4}M_2^{\frac{n+1}{2}}M_1^{\frac{n-1}{2}}|\xi| \\ &\leq c\lambda_1^{-1/4}M_1^{k-1/2}M_2^{k+1/2}|\xi|.\end{aligned}$$

Since the second term on the right hand side of (50) can be treated similarly, the proof is complete. □

After this lemma, if we want to show that, for the RD equation, the solution u_m of the SGM satisfies $|p - u_m| = O(L_m^2/\lambda_{m+1}^2)$, we can proceed exactly as in the case of the two-dimensional NS equations. For the three-dimensional case, one always has to replace the Brézis-Gallouet inequality (16) by Agmon's inequality (51).

Remark 3. The results on the RD equation can be also extended with minor modifications to the Cahn-Hilliard equation

$$u_t + \nu\Delta^2 u + \Delta R(u) = f$$

(R as in (47)) with boundary conditions

$$\partial_n u = \partial_n \Delta u = 0,$$

where ∂_n denotes normal derivative (see [38]). If $A = -\Delta$, then $|p - u_m| = O(L_m^2/\lambda_{m+1}^3)$; that is, the same order of convergence (except for the L_m^2 term) as the approximation order of Φ_1 [8].

Acknowledgments

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