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ON THE COUPLING OF BEM AND FEM FOR EXTERIOR PROBLEMS FOR THE HELMHOLTZ EQUATION

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Abstract. This paper deals with the coupled procedure of the boundary element method (BEM) and the finite element method (FEM) for the exterior boundary value problems for the Helmholtz equation. A circle is selected as the common boundary on which the integral equation is set up with Fourier expansion. As a result, the exterior problems are transformed into nonlocal boundary value problems in a bounded domain which is treated with FEM, and the normal derivative of the unknown function at the common boundary does not appear. The solvability of the variational equation and the error estimate are also discussed.

1. Introduction

The purpose of this paper is to couple BEM and FEM for the numerical solution of the exterior boundary value problems

$$\triangle u + k^2 u = 0 \quad \text{in} \quad \Omega, \tag{1.1a}$$

$$u = u_0(x, y) \quad \text{on} \quad \Gamma \tag{1.1b}$$

and

$$\triangle u + k^2 u = 0 \quad \text{in} \quad \Omega, \tag{1.2a}$$

$$\frac{\partial u}{\partial n} = u_n(x, y) \quad \text{on} \quad \Gamma \tag{1.2b}$$

with the Sommerfeld radiation condition

$$\frac{\partial u}{\partial r} - iku = o(r^{-1/2}) \qquad \text{as} \quad r \to \infty \tag{1.3}$$

uniformly for all directions, where Ω is an unbounded domain in the plane R^2 with boundary Γ which is a closed smooth curve, $\mathrm{Im}(k) > 0, i = \sqrt{-1}, r = \sqrt{x^2 + y^2}$ and n is the outer normal to Γ.

The coupled procedure in [1] and [2] are based upon the direct BEM in which the boundary integral equations come from Green's formula, and the unknowns on the common boundary involve $\frac{\partial u}{\partial n}$ as well as u. Feng and Yu (cf. [3], [4]) developed an integral equation for the Laplace equation with a circular boundary by using Green's function. Here we also take a circle as the common boundary,

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but an integral equation which is called the Dirichlet to Neumann (DtN) boundary condition in [5] is obtained by Fourier expansion. As a result, only u is considered unknown on the circle. Such a procedure is advantageous from a numerical point of view. In Section 4, we give a convergence analysis which is different from one for $k = 0$ in [5] and $k > 0$ in [6].

2. Integral equation on a circle

In order to obtain an integral equation on a circle, assume the boundary Γ in this section is a circle of radius R whose centre is the origin of coordinates. In the polar coordinates system r, θ in the plane, the equation (1.1a) and the boundary conditions (1.1b) and (1.2b) become, respectively,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + k^2 u = 0, \tag{2.1}$$

$$u(R, \theta) = u_0(\theta), \tag{2.2}$$

and

$$\frac{\partial u}{\partial n}(R, \theta) = u_n(\theta). \tag{2.3}$$

From the periodicity of the boundary conditions, the solution of equation (2.1) is a 2π-periodic function for θ and can be expressed as a Fourier series. We find that the function

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n H_{|n|}^{(1)}(kr) e^{in\theta}, \qquad r > R, \tag{2.4}$$

satisfies equation (2.1) and the radiation condition (1.3), where $H_n^{(1)}(z)$ denotes the Hankel function of the first kind

$$H_n^{(1)}(z) = J_n(z) + iY_n(z), \qquad n = 0, 1, 2, \ldots, \tag{2.5}$$

and $J_n(z)$ and $Y_n(z)$ being the Bessel functions of the first and second kind, respectively:

$$J_n(z) = \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(j+n)!} (\frac{z}{2})^{2j+n}, \qquad n = 0, 1, 2, \ldots, \tag{2.6}$$

$$\begin{aligned} Y_n(z) = & \frac{2}{\pi} J_n(z) \ln \frac{z}{2} - \frac{1}{\pi} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} (\frac{z}{2})^{2j-n} \\ & - \frac{1}{\pi} \sum_{j=0}^{\infty} (-1)^j \frac{1}{j!(j+n)!} [\psi(n+j+1) \\ & + \psi(j+1)] (\frac{z}{2})^{2j+n}, \qquad n = 0, 1, 2, \ldots, \end{aligned} \tag{2.7}$$

with

$$|\arg(z)| < \pi, \quad \psi(1) = -\gamma, \quad \psi(m) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{m-1}, \quad m \geq 2,$$

in which $\gamma = 0.5772156649\ldots$ is Euler's constant. The coefficient c_n in (2.4) will be determined from the boundary conditions.

Substituting the boundary condition (2.2) into (2.4), we have

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n G_{|n|}(kR, kr) e^{in\theta}, \qquad r > R, \tag{2.8}$$

where a_n is the Fourier coefficient for $u_0(\theta)$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} u_0(\theta)e^{-in\theta}d\theta, \qquad G_n(x,y) = H_n^{(1)}(y) \,/\, H_n^{(1)}(x).$$

From (2.6) and (2.7) we find that the asymptotic behavior of $H_n^{(1)}(z)$ is given by

$$H_n^{(1)}(z) = -i\frac{(n-1)!}{\pi}\Big(\frac{z}{2}\Big)^{-n}\Big(1+\frac{1}{n-1}\Big(\frac{z}{2}\Big)^2 + O\Big(\frac{1}{n^2}\Big)\Big) \quad \text{as } n \to \infty \tag{2.9}$$

and furthermore

$$G_n(kR, kr) = \Big(\frac{R}{r}\Big)^n\Big(1+O\Big(\frac{1}{n}\Big)\Big) \quad \text{as } n \to \infty. \tag{2.10}$$

Hence the series in (2.8) is absolutely and uniformly convergent on any closed interval in (R,∞) and can be differentiated term by term, and the function $u(r,\theta)$ in (2.8) is indeed the solution for the Dirichlet problem (1.1a), (2.2) with the radiation condition (1.3). Introducing the integral operator P defined by

(2.11)

$$Pv(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\Big[G_0(kR,kr) + 2\sum_{n=1}^{\infty} G_n(kR,kr)\cos n(\theta-\theta')\Big]v(\theta')d\theta', \quad r > R,$$

the formula (2.8) can be rewritten as

$$u(r,\theta) = Pu_0(\theta), \qquad r > R. \tag{2.12}$$

Differentiating (2.12) for r and letting $r \to R+0$, we get a boundary integral equation

$$Ku(R,\theta) = \frac{\partial u}{\partial n}(R,\theta), \tag{2.13}$$

where K is the boundary integral operator defined by

$$Kv(\theta) = \frac{k}{2\pi}\int_0^{2\pi}\Big[H_0(kR) + 2\sum_{n=1}^{\infty} H_n(kR)\cos n(\theta-\theta')\Big]v(\theta')d\theta' \tag{2.14}$$

with

$$H_n(z) = -\frac{dH_n^{(1)}(z)}{dz}\Big/ H_n^{(1)}(z) = \frac{H_{n+1}^{(1)}(z)}{H_n^{(1)}(z)} - \frac{n}{z}, \qquad n = 0,1,2,\ldots. \tag{2.15}$$

From (2.9), we find that

$$H_n(z) = \frac{n}{z}\Big(1 - \frac{2}{n(n-1)}\big(\frac{z}{2}\big)^2 + O\big(\frac{1}{n^4}\big)\Big) \quad \text{as } n \to \infty \tag{2.16}$$

and the series in (2.14) is divergent. Since

$$\sum_{n=1}^{\infty}\frac{1}{n}\cos n\theta = -\ln\Big|2\sin\frac{\theta}{2}\Big|, \qquad \theta \neq 2m\pi,$$

and since in the theory of generalized functions ([7])

$$\sum_{n=1}^{\infty} n\cos n\theta = -\frac{1}{4\sin^2\frac{\theta}{2}}, \qquad \theta \neq 2m\pi,$$

it follows that the integral kernel in (2.14) contains a singularity of the $(\theta-\theta')^{-2}$ type and the integral should be considered as the finite part of the Hadamard hypersingular integral ([3], [4], [8]).

Similarly, we can obtain the solution for the Neumann problem (1.2a),(2.3),(1.3)

$$u(r,\theta)=\frac{1}{2\pi k}\int_0^{2\pi}\left[\frac{G_0(kR,kr)}{H_0(kR)}+2\sum_{m=1}^{\infty}\frac{G_m(kR,kr)}{H_m(kR)}\cos m(\theta-\theta')\right]u_n(\theta')d\theta',\quad r\geq R,$$

and another boundary integral equation

$$\frac{1}{2\pi k}\int_0^{2\pi}\left[\frac{1}{H_0(kR)}+2\sum_{m=1}^{\infty}\frac{1}{H_m(kR)}\cos m(\theta-\theta')\right]\frac{\partial u}{\partial n}(R,\theta')d\theta'=u(R,\theta)\tag{2.17}$$

whose kernel contains an integrable singularity of the $\ln|\theta-\theta'|$ type.

3. Coupled procedure and variational formulation

Now we discuss the coupled procedure of BEM and FEM. Make a circle $\Gamma_0=\{(x,y)|x^2+y^2=R^2\}$ for an appropriate radius R such that the boundary Γ is surrounded by Γ_0, and let Γ_0 divide the domain Ω into a bounded part $\Omega_1=\{(x,y)|x^2+y^2<R^2,(x,y)\in\Omega\}$ and an unbounded part $\Omega_2=\{(x,y)|x^2+y^2>R^2\}$. From the last section we see that the function u satisfying equation (1.1a) in Ω_2 and the radiation condition (1.3) satisfies

$$\frac{\partial u}{\partial n}=K(\gamma_0 u)\qquad \text{on } \Gamma_0$$

with the trace operator γ_0 defined by $\gamma_0 u=u|_{\Gamma_0}$. Thus the exterior Dirichlet problem (1.1),(1.3) and the exterior Neumann problem (1.2),(1.3) are transformed into

$$\begin{cases}\triangle u+k^2u=0 & \text{in } \Omega_1,\\ u=u_0 & \text{on } \Gamma,\\ \dfrac{\partial u}{\partial n}=K(\gamma_0 u) & \text{on } \Gamma_0,\end{cases}\tag{3.1}$$

and

$$\begin{cases}\triangle u+k^2u=0 & \text{in } \Omega_1,\\ \dfrac{\partial u}{\partial n}=u_n & \text{on } \Gamma,\\ \dfrac{\partial u}{\partial n}=K(\gamma_0 u) & \text{on } \Gamma_0,\end{cases}\tag{3.2}$$

respectively.

Now we can apply FEM to the nonlocal boundary value problems (3.1) and (3.2) in the bounded domain Ω_1. The variational problem corresponding to (3.2) is: Find $u\in H^1(\Omega_1)$ such that

$$A(u,v)=\langle u_n,v\rangle\qquad \forall v\in H^1(\Omega_1)\tag{3.3}$$

where

$$\begin{aligned}&A(u,v)=a_1(u,v)+b(\gamma_0u,\gamma_0v),\\ &a_1(u,v)=\int_{\Omega_1}(\nabla u\cdot\nabla v-k^2uv)d\Omega,\quad b(\gamma_0u,\gamma_0v)=\int_{\Gamma_0}(\gamma_0v)K(\gamma_0u)ds,\\ &\langle u_n,v\rangle=\int_\Gamma u_nvds,\quad \nabla u\cdot\nabla v=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}.\end{aligned}\tag{3.4}$$

Obviously, $a_1(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $A(\cdot,\cdot)$ are symmetric bilinear forms.

It must be pointed out that the unknown $\gamma_0(\frac{\partial u}{\partial n})$ is not involved in (3.3). This is why we use the integral equation (2.13) on Γ_0, not (2.17), and select a circle as the common boundary, not another closed curve. This coupled procedure is superior to that based upon the direct BEM.

For the following discussion, we introduce the function spaces

$$
\begin{aligned}
&H^1_{loc}(\Omega) = \{u \bigm| u \in H^1(\tilde{\Omega}) \quad \text{for any compact set } \tilde{\Omega} \subset \Omega\},\\
&H^1_r(\Omega) = \{u \bigm| u \in H^1_{loc}(\Omega),\ u \text{ satisfying the radiation condition (1.3)}\},\\
&H^1_L(\Omega) = \{u \bigm| u \in H^1_r(\Omega),\ \triangle u + k^2 u = 0 \text{ in } \Omega\},\\
&H^1_c(\Omega) = \{u \bigm| u \in H^1_{loc}(\Omega),\ u \text{ having compact support in } \Omega\},
\end{aligned}
$$

and define

$$
a_2(u,v) = \int_{\Omega_2} (\nabla u \cdot \nabla v - k^2 uv) d\Omega, \quad a(u,v) = a_1(u,v) + a_2(u,v). \tag{3.5}
$$

Lemma 3.1.

$$
b(\gamma_0 u, \gamma_0 v) = a_2(u,v) \qquad \forall u \in H^1_L(\Omega_2), v \in H^1_c(\Omega_2).
$$

Proof. It follows from Green's formula. □

Theorem 3.1. *Let $u_n \in H^{-1/2}(\Gamma)$ be given. Thus the variational problem (3.3) in Ω_1 is equivalent to the variational problem corresponding to (1.2), (1.3) in Ω: Find $u \in H^1_r(\Omega)$ such that*

$$
a(u,v) = \langle u_n, v\rangle \qquad \forall v \in H^1_c(\Omega). \tag{3.6}
$$

In other words, if $u_1 \in H^1(\Omega_1)$ is the solution of (3.3), then

$$
u = \begin{cases} u_1 & \text{in } \overline{\Omega}_1 \\ u_2 \equiv P(\gamma_0 u_1) & \text{in } \Omega_2 \end{cases}
$$

is the solution of (3.6); conversely, if $u \in H^1_r(\Omega)$ is the solution of (3.6), then u in Ω_1 is the solution of (3.3).

Proof. Let $u_1 \in H^1(\Omega_1)$ satisfy the variational formula (3.3). So $u_2 \equiv P(\gamma_0 u_1) \in H^1_L(\Omega_2), \gamma_0 u_2 = \gamma_0 u_1 \in H^{1/2}(\Gamma_0)$ and $u \in H^1_r(\Omega)$. From Lemma 3.1, it follows that

$$
a(u,v) = a_1(u,v) + b(\gamma_0 u, \gamma_0 v) = A(u_1, v) = \langle u_n, v\rangle \qquad \forall v \in H^1_c(\Omega).
$$

Hence u is the solution of (3.6).

Conversely, assume $u \in H^1_r(\Omega)$ satisfies the variational formula (3.6); then u is the generalized solution of (1.2),(1.3). For any $v_1 \in H^1(\Omega_1)$, it is known that $\gamma_0 v_1 \in H^{1/2}(\Gamma_0)$, and there exists $v_2 \in H^1_c(\Omega_2)$ such that $\gamma_0 v_2 = \gamma_0 v_1$ from the trace theorem. Writing

$$
v = \begin{cases} v_1 & \text{in } \overline{\Omega}_1 \\ v_2 & \text{in } \Omega_2 \end{cases},
$$

we see that $v \in H^1_c(\Omega)$, and

$$
A(u, v_1) = a_1(u, v_1) + a_2(u, v_2) = a(u,v) = \langle u_n, v\rangle = \langle u_n, v_1\rangle.
$$

This shows that u in Ω_1 is the solution of (3.3). □

From the solvability and uniqueness of (3.6), now we can say that for any given $u_n \in H^{-1/2}(\Gamma)$ there exists exactly one variational solution to (3.3).

In the following discussion, $\|\cdot\|_{s,D}$ denotes the norm in the Sobolev space $H^s(D)$ and $\|\cdot\|_{0,D}$ is $L^2(D)$-norm for $s=0$.

Lemma 3.2. *The linear operator $K : H^s(\Gamma_0) \to H^{s-1}(\Gamma_0)$ is continuous, i.e.*

$$\|Kf\|_{s-1,\Gamma_0} \le c\|f\|_{s,\Gamma_0} \qquad \forall f \in H^s(\Gamma_0) \tag{3.7}$$

where s is any real number and c is a positive constant independent of f.

Proof. Let $f(\theta) \in H^s(\Gamma_0)$ and expand it in the Fourier series

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

with the Fourier coefficients

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta.$$

The norm in $H^s(\Gamma_0)$ can be defined using the Fourier coefficiets as

$$\|f\|_{s,\Gamma_0} = \left[2\pi R \sum_{n=-\infty}^{\infty}(n^2+1)^s|a_n|^2\right]^{1/2}.$$

Thus from (2.14) we have

$$Kf = k\sum_{-\infty}^{\infty} a_n H_{|n|}(kR)e^{in\theta},$$

$$\|Kf\|^2_{s-1,\Gamma_0} = 2\pi|k|^2 R\sum_{-\infty}^{\infty}(n^2+1)^{s-1}|a_n H_{|n|}(kR)|^2.$$

By (2.16), there exists a positive integer N such that $\frac{1}{2} < |\frac{kR}{n}H_n(kR)| < \frac{3}{2}$ when $n > N$, and we obtain (3.7) with

$$c = \max\left\{\frac{3}{2R}, \max_{0\le n\le N}\left\{|kH_n(kR)|\Big/\sqrt{n^2+1}\right\}\right\}.$$

$\square$

Theorem 3.2. *The symmetric bilinear form $b(\cdot,\cdot)$ on the space $H^{1/2}(\Gamma_0)\times H^{1/2}(\Gamma_0)$ has the following properties:*

1. $|b(f,g)| \le c_1\|f\|_{1/2,\Gamma_0}\|g\|_{1/2,\Gamma_0}$ *(continuity).*
2. $\mathrm{Im}(-\overline{k}b(f,\overline{f})) \ge c_2\|f\|^2_{1/2,\Gamma_0}$*($H^{1/2}(\Gamma_0)$-coercivity).*

Here $f, g \in H^{1/2}(\Gamma_0)$ are arbitrary functions and $c_1,\ c_2 > 0$ are constants independent of f and g.

Proof. (1) It comes from Lemma 3.2 that

$$|b(f,g)| = \left|\int_{\Gamma_0} gKf ds\right| \le \|g\|_{1/2,\Gamma_0}\|Kf\|_{-1/2,\Gamma_0} \le c_1\|f\|_{1/2,\Gamma_0}\|g\|_{1/2,\Gamma_0}.$$

(2) Let

$$u(r,\theta) = Pf(\theta).$$

Thus $u \in H^1_L(\Omega_2), \gamma_0 u = f$. Defining $\Omega_d = \{(x,y) \mid R^2 < x^2+y^2 < d^2\}$ and $\Gamma_d = \{(x,y) \mid x^2+y^2 = d^2\}$ for $d > R$ and using the radiation condition and Green's formula, we have

$$0 = \lim_{d\to\infty} \int_{\Gamma_d} |\frac{\partial u}{\partial r} - iku|^2 ds = \lim_{d\to\infty} \int_{\Gamma_d} \left[|\frac{\partial u}{\partial r}|^2 + |ku|^2 - 2\mathrm{Im}(\overline{k}\overline{u}\frac{\partial u}{\partial r}) \right] ds,$$

$$\int_{\Gamma_d} \overline{u}\frac{\partial u}{\partial r} ds = -\int_{\Omega_d} (k^2|u|^2 - |\nabla u|^2) d\Omega - \int_{\Gamma_0} \overline{u}\frac{\partial u}{\partial n} ds.$$

Combine them to give

(3.8)

$$\begin{aligned}\mathrm{Im}(-\overline{k}b(f,\overline{f})) &= \mathrm{Im}\Big(-\overline{k}\int_{\Gamma_0} \overline{u}\frac{\partial u}{\partial n} ds \Big) \\ &= \lim_{d\to\infty} \left[\frac{1}{2}\int_{\Gamma_d} (|\frac{\partial u}{\partial r}|^2 + |ku|^2) ds + \mathrm{Im}(k)\int_{\Omega_d} (|\nabla u|^2 + |ku|^2) d\Omega \right].\end{aligned}$$

If $\mathrm{Im}(k) > 0$, from the trace theorem we get

$$\mathrm{Im}(-\overline{k}b(f,\overline{f})) \geq \mathrm{Im}(k)\min\{1,|k|^2\}\|u\|^2_{1,\Omega_2} \geq c_2\|f\|^2_{1/2,\Gamma_0}.$$

□

Theorem 3.2 can also be proved like Lemma 3.2, and from Theorem 3.2 we obtain the following theorem.

Theorem 3.3. *The symmetric bilinear form $A(\cdot,\cdot)$ on the space $H^1(\Omega_1)\times H^1(\Omega_1)$ has the following properties:*

1. $|A(u,v)| \leq c_1\|u\|_{1,\Omega_1}\|v\|_{1,\Omega_1}$ *(continuity).*
2. $\mathrm{Im}(-\overline{k}A(u,\overline{u})) \geq c_2\|u\|^2_{1,\Omega_1}$ *($H^1(\Omega_1)$-coercivity).*

Here $u, v \in H^1(\Omega_1)$ are arbitrary functions and $c_1, c_2 > 0$ are constants independent of u and v.

Theorem 3.3 also shows that the variational problem (3.3) has only one solution. By (3.8) and Theorem 3.2, there exists a constant $c > 0$ such that

$$\|Pf\|_{1,\Omega_2} \leq c\|f\|_{1/2,\Gamma_0} \qquad \forall f \in H^{1/2}(\Gamma_0). \tag{3.9}$$

The above discussion can be applied to the boundary value problem (3.1), in which the boundary condition on Γ is essential.

4. Convergence analysis and error estimate

In the previous section a boundary value problem in an unbounded domain has been transformed into the corresponding variational problem in a bounded one. Now FEM can be applied in Ω_1. Let $\psi_1(x,y),\ldots,\psi_M(x,y)$ be the local base functions of interpolation in FEM, and $u^h(x,y)$ be the interpolation function for $u(x,y)$ on $\overline{\Omega}_1$ defined by

$$u^h(x,y) = \sum_{j=1}^{M} u_j\psi_j(x,y) \qquad \text{in} \quad \overline{\Omega}_1. \tag{4.1}$$

So the discrete variational problem corresponding to (3.3) is: Find $u^h \in S^h(\Omega_1)$ such that

$$A(u^h,v^h) = \langle u_n, v^h\rangle \qquad \forall v^h \in S^h(\Omega_1), \tag{4.2}$$

where $S^h(\Omega_1)$ is the finite element space defined by

$$S^h(\Omega_1) = \mathrm{Span}\{\psi_1, \ldots, \psi_M\} \subset H^1(\Omega_1).$$

In the following, u and u^h denote the solutions for the variational problem (3.3) and the discrete variational problem (4.2), respectively, h is the maximal length of the diameters of elements, $\Pi : H^1(\Omega_1) \to S^h(\Omega_1)$ is the interpolation operator, and c denotes a positive constant independent of u and u^h; it can have different values at different places.

Theorem 4.1. *If $u \in H^{j+1}(\Omega_1)$ and*

$$\|v - \Pi v\|_{1,\Omega_1} \le ch^j \|v\|_{j+1,\Omega_1} \qquad \forall v \in H^{j+1}(\Omega_1),$$

then

$$\|u - u^h\|_{1,\Omega_1} \le ch^j \|u\|_{j+1,\Omega_1}.$$

Proof. By Theorem 3.3, we use Céa's lemma to obtain

$$\|u - u^h\|_{1,\Omega_1} \le c\|u - \Pi u\|_{1,\Omega_1} \le ch^j \|u\|_{j+1,\Omega_1}.$$

□

Theorem 4.2.

$$|P(\gamma_0 u) - P(\gamma_0 u^h)| \le c_r \|u - u^h\|_{0,\Gamma_0} \le c_r \|u - u^h\|_{1,\Omega_1}, \quad r > R, \tag{4.3}$$

$$\|P(\gamma_0 u) - P(\gamma_0 u^h)\|_{1,\Omega_2} \le c\|u - u^h\|_{1/2,\Gamma_0} \le c\|u - u^h\|_{1,\Omega_1}, \tag{4.4}$$

where c_r depends on r.

Proof. By (2.11),

$$\begin{aligned} |P(\gamma_0 u) - P(\gamma_0 u^h)| &= \frac{1}{2\pi}\Big|\int_0^{2\pi} \big[G_0 + 2\sum_{n=1}^{\infty} G_n \cos n(\theta - \theta')\big](\gamma_0 u - \gamma_0 u^h) d\theta'\Big| \\ &\le \frac{1}{2\pi}\Big[2\pi\Big(|G_0|^2 + 2\sum_{n=1}^{\infty}|G_n|^2\Big)\int_0^{2\pi} |\gamma_0 u - \gamma_0 u^h|^2 d\theta'\Big]^{1/2} \\ &= \Big[\frac{1}{2\pi R}\Big(|G_0|^2 + 2\sum_{n=1}^{\infty}|G_n|^2\Big)\Big]^{1/2} \|u - u^h\|_{0,\Gamma_0}, \end{aligned}$$

and the series is convergent for $r > R$ by (2.10), and (4.3) is proved by the trace theorem. We use (3.9) and the trace theorem to arrive at (4.4). □

The last theorem shows that the errors in Ω_2 can be controlled by the errors on Γ_0 and in Ω_1.

For time-harmonic acoustic wave propagation in a homogeneous isotropic medium, $k^2 = \omega(\omega + i\gamma)/c^2$, where c is the speed of sound, ω is the frequency of the acoustic wave and γ is the damping coefficient. If $|k|$ is large, the Helmholtz equation has a rapidly oscillating solution and the quality of a finite element solution depends significantly on the wavenumber k as well as the stepwidth h of meshes. Ihlenburg and Babuška [9] applied the FEM with piecewise linear approximation to a one-dimensional Helmholtz equation, and their results show that the relative error of the FE-solution in the H^1-seminorm is controlled by a term of order kh. Bayliss, et al., in [10] dealt with a two-dimentional Helmholtz equation with a real wavenumber $k > 0$ in a bounded domain, and stated a convergence theorem which shows that the relative error bound is $O(k(kh)^{j+1})$ in the L^2-norm and $O(k(kh)^j)$ in the H^1-norm, where j is the order of polynomial approximation in FEM. In this paper, $\mathrm{Im}(k) > 0$ in an unbounded domain is assumed. For a large $|k|$, the constants c_1 and c_2 in Theorem 3.3 are $|k|^2$ and $\mathrm{Im}(k)$, respectively, so the constant in Céa's lemma is of order $|k|^3/\mathrm{Im}(k)$. For a linear interpolation, the error of the FE-solution is

$$||u - u^h||_{1,\Omega_1} \leq c\frac{|k|^3}{\mathrm{Im}(k)}h||u||_{2,\Omega_1},$$

with a constant c independent of k and h. In addition, the constant c in (4.4) is $\sqrt{|k|/\mathrm{Im}(k)}$.

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