ON THE SOLUTIONS OF A FAMILY
OF QUARTIC THUE EQUATIONS

ALAIN TOGBÉ

ABSTRACT. In this paper, we solve a certain family of diophantine equations associated with a family of cyclic quartic number fields. In fact, we prove that for $n \leq 5 \times 10^6$ and $n \geq N = 1.191 \times 10^{19}$, with n, $n+2$, $n^2 + 4$ square-free, the Thue equation

$$\Phi_n(x, y) = x^4 - n^2 x^3 y - (n^3 + 2n^2 + 4n + 2)x^2 y^2 - n^2 x y^3 + y^4 = 1$$

has no integral solution except the trivial ones: $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$.

1. INTRODUCTION

We consider the following Thue equation:

$$(1.1) \qquad \Phi_n(x, y) = x^4 - n^2 x^3 y - (n^3 + 2n^2 + 4n + 2)x^2 y^2 - n^2 x y^3 + y^4 = 1,$$

and we plan to prove that for $n \leq 5 \times 10^6$ and $n \geq N = 1.191 \times 10^{19}$, with n, $n + 2$, $n^2 + 4$ square-free, this equation has only the solutions $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$.

In 1991, L. C. Washington [13] and O. Lecacheux [4] independently studied the units of $\mathbb{Q}(\omega)$, where ω is a root the polynomial $\Phi_n(x, 1)$. Their results were helpful to solve the equation (1.1). For the proof, we use estimates from linear forms in the logarithms of algebraic numbers, techniques from diophantine approximation and new methods of computation.

This paper is based upon the results of our Ph.D. dissertation [12] and is divided into 5 sections. In the second section, we shall give some elementary properties of the above polynomial and recall the result of L. C. Washington and O. Lecacheux about the fundamental system of units of the number field associated with this Thue equation. We shall study the approximative properties of the solutions in the third section. In the last two sections, we solve the equation using respectively the method proposed by M. Mignotte in [6] and the Bilu-Hanrot method (see [2]).

2. ELEMENTARY PROPERTIES OF THE POLYNOMIAL

We have the following properties.

(a) $\Phi_n(\pm 1, 0) = \Phi_n(0, \pm 1) = 1$. So $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$ are solutions of (1.1) and are called *the trivial solutions.*

(b) $\Phi_n(x,y) = \Phi_n(-x,-y) = \Phi_n(y,x) = \Phi_n(-y,-x)$, so if (x,y) is a solution of (1.1), so are $(-x,-y)$, (y,x), $(-y,-x)$. Without loss of generality, we shall consider only the solutions (x,y) with y positive.

Lemma 2.1. *If* $n \neq 0$, *then* $\Phi(x) = \Phi_n(x,1)$ *is irreducible.*

Proof. Let α be a root of $\Phi(x)$ and $\eta = \alpha + \frac{1}{\alpha}$. Then, η is a root of

$$z^2 - n^2 z - (n^3 + 2n^2 + 4n + 4).$$

The discriminant of $z^2 - n^2 z - (n^3 + 2n^2 + 4n + 4)$ is $\Delta = (n+2)^2(n^2+4)$, which is not a square for $n \neq 0$. So the quadratic subfield of $\mathbb{K}$ is $k = \mathbb{Q}(\sqrt{n^2+4})$. Suppose now that $\alpha \in \mathbb{Q}(\eta)$. We know that α is a root of $X^2 - X\eta + 1 = 0$ for which the discriminant is

$$\omega := \eta^2 - 4 = (n^3 + 2n^2 + 4n) + n^2\eta.$$

We have $N_{\mathbb{Q}(\eta)/\mathbb{Q}}(\omega) = n^2(n+2)^2(n^2+4)$. This norm is a square if and only if $n = 0$. So $X^2 - X\eta + 1$ is irreducible over $\mathbb{Q}(\eta)$, i.e., $\alpha \notin \mathbb{Q}(\eta)$ if $n \neq 0$. $\square$

We know from [4] and [13] that the Galois group of $\mathbb{K} = \mathbb{Q}(\alpha)$, α being a root of $\Phi(x) = \Phi_n(x,1)$, is a cyclic group of order 4, generated by σ. Moreover

$$\Phi(x) = (x - \alpha)(x + \beta)(x - 1/\alpha)(x + 1/\beta),$$

where $\alpha = \alpha_1$ is the largest root of $\Phi(x)$, the other roots being $\alpha_2 = \sigma(\alpha_1)$ with

$$\alpha_2 = \left(n + \tfrac{1}{n+2}\right) - \left(n^3 + n^2 + 3n + \tfrac{3}{n+2}\right)\alpha_1 + \left(-n^2 + n - 2 + \tfrac{3}{n+2}\right)\alpha_1^2 + \left(1 - \tfrac{1}{n+2}\right)\alpha_1^3,$$

$$\alpha_3 = \sigma^2(\alpha) = 1/\alpha_1, \qquad \alpha_4 = \sigma^3(\alpha) = 1/\alpha_2.$$

For $n \geq 4$, we have the following inequalities for α_i:

$$\begin{array}{ccccc}
n^2 + n + 1 + \tfrac{2}{n} - \tfrac{3}{n^2} & < & \alpha_1 & < & n^2 + n + 1 + \tfrac{2}{n} - \tfrac{2}{n^2}, \\
-n - 1 - \tfrac{1}{n} + \tfrac{1}{n^3} & < & \alpha_2 & < & -n - 1 - \tfrac{1}{n} + \tfrac{2}{n^2}, \\
\tfrac{1}{n^2} - \tfrac{2}{n^3} & < & \alpha_3 & < & \tfrac{1}{n^2} - \tfrac{1}{n^3}, \\
-\tfrac{1}{n} + \tfrac{1}{n^2} - \tfrac{1}{n^3} & < & \alpha_4 & < & -\tfrac{1}{n} + \tfrac{1}{n^2}.
\end{array}$$

Now let $n \neq -1$ be an odd integer such that n, $n+2$ and n^2+4 are square-free. Then the discriminant of $\mathbb{K}$ is $D = n^2(n+2)^2(n^2+4)^3$ and the fundamental unit of k is

$$\varepsilon = \frac{n + \sqrt{n^2+4}}{2}.$$

We know from [13] that $\{1, \varepsilon, \alpha_1, \alpha_2\}$ is a basis for the ring of integers $\mathcal{O}_\mathbb{K}$ of $\mathbb{K}$; moreover, $\{1, \varepsilon, \alpha_1, \varepsilon\alpha_1\}$ is an integral basis for $\mathbb{K}$, which shows that $\{1, \alpha_1\}$ is a basis for $\mathbb{K}$ over k. We have the following result (see [4] or [13]).

Theorem 2.2 (L. C. Washington, O. Lecacheux). *When* n, $n+2$ *and* n^2+4 *are square-free,* $\{\varepsilon, \alpha_1, \alpha_2\}$ *is a fundamental system of units of* $\mathbb{K}$ *when* $|n| \geq 3$. *When* $n = 1$, *the subgroup* $\langle -1, \varepsilon, \alpha_1, \alpha_2 \rangle$ *is of index* 2 *in the full group of units.*

3. Approximative properties of the solutions of (1.1)

We shall begin this section with the following result.

Proposition 3.1. *Let (x, y) be an integral solution of (1.1); then we have $|y| \geq 2$ or $(x, y) = (1, 0), (-1, 0), (0, 1), (0, -1)$.*

Proof. We know that if $y = 0$, then $x = \pm 1$. Now suppose $y = 1$; then

$$x = 0 \quad \text{or} \quad P(x) = x^3 - n^2 x^2 - (n^3 + 2n^2 + 4n + 2)x - n^2 = 0.$$

So for $n \geq 1$, $P(x)$ has three real roots. Moreover, we have

$$P(-n - 1) = n^2 + 3n + 1 > 0, \quad P(-n - 2) = -n^3 - 3n^2 - 2n - 4 < 0,$$

$$P(-1) = n^3 + 4n + 1 > 0, \quad P(0) = -n^2 < 0,$$

$$P(n^2 + n + 1) = -2n^3 - 4n^2 - 3n - 1 < 0, \quad P(n^2 + n + 2) = n^4 + n^3 + 3n^2 + 2n + 4 > 0.$$

Then these roots are in $]-n - 2, -n - 1[, \quad]-1, 0[, \quad]n^2 + n + 1, n^2 + n + 2[$, respectively. $\qquad\blacksquare$

Now let (x, y) be a non-trivial solution of (1.1). Then

$$(3.1) \qquad \Phi_n(x, y) = (x - \alpha y)(x + \beta y)\left(x - \frac{1}{\alpha}y\right)\left(x + \frac{1}{\beta}y\right) = 1$$

is the norm of the algebraic unit $\gamma_1 := x - \alpha_1 y \in \mathcal{O}_{\mathbb{K}}^{\times}$, so there are unique integers $u_0 \in \{0, 1\}$, u_1, u_2, u_3 such that

$$\gamma_1 = (-1)^{u_0}\, \varepsilon^{u_1} \alpha^{u_2} \beta^{u_3}.$$

We call u_0, u_1, u_2, u_3 *the associated exponents* of the solution (x, y). Moreover, we have

$$\begin{cases} N_{k/\mathbb{Q}}(\varepsilon) = \varepsilon\,\sigma(\varepsilon) = -1, \\ \gamma_{i+1} = \sigma(\gamma_i) = x - \alpha_{i+1}y, \quad 1 \leq i \leq 3. \end{cases}$$

So we have

$$(3.2) \qquad \begin{cases} |\gamma_1| = \varepsilon^{u_1} \alpha^{u_2} \beta^{u_3}, \\ |\gamma_2| = \varepsilon^{-u_1} \beta^{u_2} \alpha^{-u_3}, \\ |\gamma_3| = \varepsilon^{u_1} \alpha^{-u_2} \beta^{-u_3}, \\ |\gamma_4| = \varepsilon^{-u_1} \beta^{-u_2} \alpha^{u_3}. \end{cases}$$

The inequalities for the α_i give

$$(3.3) \qquad -\beta < -\frac{1}{\beta} < \frac{1}{\alpha} < \alpha,$$

so

$$(3.4) \qquad \frac{x}{y} - \alpha < \frac{x}{y} - \frac{1}{\alpha} < \frac{x}{y} + \frac{1}{\beta} < \frac{x}{y} + \beta.$$

There are four possibilities to obtain the minimum of $|\gamma_i|$, $1 \leq i \leq 4$. We shall say that (x, y) is a of type 1, 2, 3 or 4:

- **type 1**, if all γ_i's are positive and $x - \alpha y = \min_{1 \leq i \leq 4}\{x - \alpha_i y\}$;

- **type 2**, if all γ_i's are negative and $-x - y\beta = \min_{1 \leq i \leq 4}\{|x - \alpha_i y|\}$;

- **type 3**, if γ_1, γ_3 are negative, γ_2, γ_4 are positive and $y\frac{1}{\alpha} - x = \min_{1 \leq i \leq 4}\{|x - \alpha_i y|\}$;

- **type 4**, if γ_1, γ_3 are negative, γ_2, γ_4 are positive and $y\frac{1}{\beta} + x = \min_{1 \leq i \leq 4}\{|x - \alpha_i y|\}$.

The above types exhaust all the possibilities.

4. Part I

4.1. **Solutions of type 1.** Suppose that $(x, y) \in \mathbb{Z}^2$ is a non-trivial solution of type 1.

Lemma 4.1. *We have*

$$
\begin{array}{ccc}
0 & < \ \gamma_1 \ < & y \frac{1}{2n^6}, \\
y\left(\alpha + \beta\right) & < \ \gamma_2 \ < & y\left(\alpha + \beta + \frac{1}{2n^6}\right), \\
y\left(\alpha - \frac{1}{\alpha}\right) & < \ \gamma_3 \ < & y\left(\alpha - \frac{1}{\alpha} + \frac{1}{2n^6}\right), \\
y\left(\alpha + \frac{1}{\beta}\right) & < \ \gamma_4 \ < & y\left(\alpha + \frac{1}{\beta} + \frac{1}{2n^6}\right).
\end{array}
$$

Proof. We have $y|\alpha - \alpha_i| \leq |x - \alpha y| + |x - \alpha_i y| \leq 2(x - \alpha_i y)$ for $2 \leq i \leq 4$, and then

$$
\frac{1}{x - \alpha_i y} \leq \frac{2}{y(\alpha - \alpha_i)}, \quad 2 \leq i \leq 4.
$$

So we obtain

$$
x - \alpha y = \prod_{i=2}^{4} \frac{1}{x - \alpha_i y} \leq \prod_{i=2}^{4} \frac{2}{(\alpha - \alpha_i)y} = \frac{8}{\Phi'(\alpha)} y^{-3},
$$

where $\Phi'(\alpha) = (\alpha - \alpha_2)(\alpha - \alpha_3)(\alpha - \alpha_4)$. But $\Phi'(\alpha) > n^6$, for $n \geq 1$; therefore

$$
\gamma_1 = x - \alpha y \leq \frac{8}{\Phi'(\alpha)} y^{-3} < \frac{8}{y^3 n^6} \leq \frac{1}{n^6},
$$

whereupon $0 < \frac{x}{y} - \alpha < \frac{1}{yn^6} \leq \frac{1}{2n^6}$ ($y \geq 2$), from which we conclude that

$$
y(\alpha - \alpha_i) < \gamma_i < y\left(\alpha - \alpha_i + \frac{1}{2n^6}\right), \quad \text{for } 1 \leq i \leq 4.
$$

So we obtain the desired result.   □

Let us consider the Siegel identity

$$
0 < \frac{\gamma_4(\alpha_1 - \alpha_2)}{\gamma_2(\alpha_1 - \alpha_4)} = 1 - \frac{\gamma_1(\alpha_2 - \alpha_4)}{\gamma_2(\alpha_1 - \alpha_4)},
$$

so

$$
\alpha^{2u_3} \beta^{-2u_2} \left(\frac{\alpha + \beta}{\alpha + \frac{1}{\beta}}\right) = 1 + \frac{\gamma_1}{\gamma_2}\left(\frac{\beta^2 - 1}{\alpha\beta + 1}\right).
$$

Put $A_1 = 2u_3$, $B_1 = -2u_2$, $\lambda_1 = \frac{\alpha + \beta}{\alpha + \frac{1}{\beta}}$, $\tau_1 = \frac{\gamma_1}{\gamma_2}\left(\frac{\beta^2 - 1}{\alpha\beta + 1}\right)$, to obtain the following linear form in logarithms:

(4.1) $\Lambda_1 = A_1 \log(\alpha) + B_1 \log(\beta) + \log(\lambda_1) = \log(1 + \tau_1).$

Several results can be deduced from this.

Lemma 4.2. $\Lambda_1 \neq 0$.

Proof. Because $\tau_1 > 0$.   □

Lemma 4.3. *We have*

$$
u_2 < u_3 < 0 \quad \text{and} \quad 0 < |A_1| < B_1.
$$

Proof. From the formula (3.2), we obtain

$$u_1 = -\frac{c_2 + c_4}{2\log(\varepsilon)} = \frac{c_1 + c_3}{2\log(\varepsilon)}, \qquad u_2 = \frac{(c_1 - c_3)\log(\alpha) + (c_2 - c_4)\log(\beta)}{2[\log^2(\alpha) + \log^2(\beta)]},$$

$$u_3 = \frac{(c_1 - c_3)\log(\beta) - (c_2 - c_4)\log(\alpha)}{2[\log^2(\alpha) + \log^2(\beta)]},$$

with $c_i = \log|\gamma_i|$. So $c_1 < c_3 < c_4 < c_2$, $c_1 - c_3 < 0$ and $c_2 - c_4 > 0$, for $n \geq 3$; then $u_3 < 0$ and $A_1 < 0$. Moreover, we have

$$(c_1 - c_3)\log(\alpha) + (c_2 - c_4)\log(\beta)$$

$$< \log(\beta)\log\left(\frac{\alpha + \beta + \frac{1}{2n^6}}{\alpha + \frac{1}{\beta}}\right) - \log(\alpha)\log\left(2n^6\left(\alpha - \frac{1}{\alpha}\right)\right).$$

The right hand side of the inequality is negative for $n \geq 3$. Therefore $u_2 < 0$ and $B_1 > 0$, for $n \geq 3$.

Finally, the expression $\log\left(\frac{\gamma_1}{\gamma_3}\right)(\log(\alpha) - \log(\beta)) + \log\left(\frac{\gamma_2}{\gamma_4}\right)(\log(\alpha) + \log(\beta))$ is less than

$$-\log\left(2n^6\left(\alpha - \frac{1}{\alpha} + \frac{1}{2n^6}\right)\right)(\log(\alpha) - \log(\beta))$$

$$+ \log\left(\frac{\alpha + \beta + \frac{1}{2n^6}}{\alpha + \frac{1}{\beta}}\right)(\log(\alpha) + \log(\beta)),$$

which is negative for $n \geq 3$; so $u_2 - u_3 < 0$, i.e., $u_2 < u_3 < 0$, $0 < -2u_3 < -2u_2$, i.e., $0 < |A_1| < B_1$. □

We use the Baker-Wüstholz theorem, (see [1], page 20) to obtain a lower bound of Λ_1.

Lemma 4.4. *We have*

$$\log(\Lambda_1) > -2^{33} \times 3^7 \times \log(24)\log^3(\alpha\beta)\log(B_1).$$

Proof. We have

$$h'(\alpha) = h'(\beta) = \frac{1}{4}\log(\alpha\beta),$$

$$h'(\lambda_1) = \frac{1}{4}\log\left(\frac{\alpha + \beta}{\alpha + 1/\beta} \cdot \frac{\alpha + 1/\beta}{1/\beta + 1/\alpha}\right) = \frac{1}{4}\log(\alpha\beta),$$

because the conjugates of λ_1 are

$$\lambda_1^{(1)} = \frac{\alpha + \beta}{\alpha + 1/\beta} > 1, \qquad \lambda_1^{(2)} = \frac{\beta + 1/\alpha}{\beta + \alpha} < 1,$$

$$\lambda_1^{(3)} = \frac{1/\alpha + 1/\beta}{1/\alpha + \beta} < 1, \qquad \lambda_1^{(4)} = \frac{1/\beta + \alpha}{1/\beta + 1/\alpha} > 1;$$

and because $B = \max\{-2u_2, -2u_3, 1\} = -2u_2 = B_1$ and $C(3, 4) = 3^7 \times 2^{39}\log(24)$. So we obtain the desired result. □

The following result gives us an upper bound for $\log(\Lambda_1)$.

Lemma 4.5. *For $n \geq 4$, we have*

$$\log(\Lambda_1) < -1.4B_1\log(\alpha).$$

Proof. We know that $\Lambda_1 = \log(1 + \tau_1) \leq \tau_1 = \frac{\gamma_1}{\gamma_2}\left(\frac{\beta^2 - 1}{\alpha\beta + 1}\right)$, so

$$\log(\Lambda_1) \leq \log\left(\frac{\gamma_1}{\gamma_2}\left(\frac{\beta^2 - 1}{\alpha\beta + 1}\right)\right) = -\log\left(\frac{\gamma_2}{\gamma_1}\left(\frac{\alpha\beta + 1}{\beta^2 - 1}\right)\right).$$

We have to show that $1.4B_1 \log(\alpha) < \log\left(\frac{\gamma_2}{\gamma_1}\right) + \log\left(\frac{\alpha\beta+1}{\beta^2-1}\right)$. According to the expression of B_1, it remains to show that

$$\frac{1.4\log^2(\alpha)}{\log^2(\alpha) + \log^2(\beta)}\log\left(\frac{\gamma_3}{\gamma_1}\right) + \frac{1.4\log(\alpha)\log(\beta)}{\log^2(\alpha)\,! + \log^2(\beta)}\log\left(\frac{\gamma_4}{\gamma_2}\right)$$
$$< \log\left(\frac{\gamma_2}{\gamma_1}\right) + \log\left(\frac{\alpha\beta + 1}{\beta^2 - 1}\right),$$

i.e.,

$$\frac{1.4\log(\alpha)\log(\beta)}{\log^2(\alpha) + \log^2(\beta)}\log\left(\frac{\gamma_4}{\gamma_2}\right)$$
$$< \log\left(\frac{\alpha\beta + 1}{\beta^2 - 1}\right) + \log\left(\frac{\gamma_2}{\gamma_3}\right) + \left(1 - \frac{1.4\log^2(\alpha)}{\log^2(\alpha) + \log^2(\beta)}\right)\log\left(\frac{\gamma_3}{\gamma_1}\right).$$

We know that

$$\log\left(\frac{\gamma_4}{\gamma_2}\right) < \log\left(\frac{\alpha + \frac{1}{\beta} + \frac{1}{2n^6}}{\alpha + \beta}\right), \qquad \log\left(\frac{\alpha + \beta}{\alpha - \frac{1}{\alpha} + \frac{1}{2n^6}}\right) < \log\left(\frac{\gamma_2}{\gamma_3}\right),$$

and

$$\log\left(2n^6\left(\alpha - \frac{1}{\alpha}\right)\right) < \log\left(\frac{\gamma_3}{\gamma_1}\right).$$

Then we must prove that

$$\frac{1.4\log(\alpha)\log(\beta)}{\log^2(\alpha) + \log^2(\beta)}\log\left(\frac{\alpha + \frac{1}{\beta} + \frac{1}{2n^6}}{\alpha + \beta}\right)$$

is less than

$$\log\left(\frac{\alpha\beta + 1}{\beta^2 - 1}\right) + \log\left(\frac{\alpha + \beta}{\alpha - \frac{1}{\alpha} + \frac{1}{2n^6}}\right) + \left(1 - \frac{1.4\log^2(\alpha)}{\log^2(\alpha) + \log^2(\beta)}\right)\log\left(2n^6\left(\alpha - \frac{1}{\alpha}\right)\right),$$

but one can check that this is indeed the case. $\qquad\square$

Using the last two results, we obtain

$$1.4B_1 \log(\alpha) - 2^{33} \times 3^7 \times \log(24)\log^3(\alpha\beta)\log(B_1) < 0.$$

Put

$$K_1 = \frac{2^{33} \times 3^7 \times \log(24)\log^3(\alpha\beta)}{1.4\log(\alpha)}, \qquad G_{1,n}(x) = x - K_1\log(x);$$

then we have

$$G_{1,n}(B_1) < 0, \qquad G'_{1,n}(x) = 1 - \frac{K_1}{x}, \qquad G'_{1,n}(K_1) = 0,$$
$$G_{1,n}(K_1) < 0, \qquad G_{1,n}(K_1^{1.106}) > 0,$$

and then $B_1 < K_1^{1.106}$. So $K_1^{1.106}$ is an upper bound for B_1.

Now we use a method considered by Maurice Mignotte to solve the problem (see [6]). We know that, for $|x| < 1$,

$$\log(1+x) = x - \frac{\theta}{2}x^2, \quad \frac{1}{1+x} = 1 - x + \theta_0 x^2, \qquad 0 < \theta, \theta_0 < 1,$$

and we use the inequalities on α_i to obtain respectively

$$\alpha = n^2\left(1 + \frac{1}{n} + \frac{1}{n^2} + \frac{\theta_1}{n^3}\right), \quad \beta = n\left(1 + \frac{1}{n} + \frac{1}{n^2} - \frac{\theta_2}{n^3}\right), \qquad 1 < \theta_1, \theta_2 < 2,$$

$$\lambda_1 = \frac{\alpha + \beta}{\alpha + 1/\beta} = \frac{1 + \frac{2}{n} + \frac{2}{n^2} + \frac{\theta_3}{n^3}}{1 + \frac{1}{n} + \frac{1}{n^2} + \frac{\theta_4}{n^3}}, \qquad 2 < \theta_3, \theta_4 < 3.$$

Then we deduce that

$$\delta = \frac{\log(\beta)}{\log(\alpha)} = \frac{1}{2} + \frac{1}{4n\log(n)} + \frac{\vartheta}{8n^2\log(n)}, \qquad \text{with} \quad |\vartheta| < 3,$$

$$\delta_1 = \frac{\log(\lambda_1)}{\log(\alpha)} = \frac{1}{2n\log(n)} + \frac{\vartheta_1}{4n^2\log(n)}, \qquad \text{with} \quad |\vartheta_1| < 3.$$

Moreover, we have

$$\left| A_1 + B_1 \frac{\log(\beta)}{\log(\alpha)} + \frac{\log(\lambda_1)}{\log(\alpha)} \right| = |A_1 + B_1\delta + \delta_1| < \frac{\tau_1}{\log(\alpha)} < \frac{1}{n^2\log(n)}, \quad \text{for } n \geq 2.$$

Then we obtain

$$(4.2) \qquad \left\| B_1\left(\frac{1}{2} + \frac{1}{4n\log(n)}\right) + \frac{1}{2n\log(n)} \right\| < \frac{B_1\vartheta + 2\vartheta_1}{8n^2\log(n)} < \frac{3B_1 + 6}{8n^2\log(n)},$$

where $\|x\| = |x - [x]| = \mathrm{dist}(x, \mathbb{Z})$. As $B_1 = -2u_2$ is an even integer, we obtain

$$\left\| B_1\left(\frac{1}{2} + \frac{1}{4n\log(n)}\right) + \frac{1}{2n\log(n)} \right\| = \left\| \frac{B_1 + 2}{4n\log(n)} \right\|.$$

Moreover, we have

$$\frac{B_1 + 2}{4n\log(n)} < \frac{63}{256} < \frac{1}{4} \iff B_1 < \frac{63}{64}n\log(n) - 2.$$

We know that B_1 has an upper bound $K_1^{1.106}$, and for $n \geq N_1 = 1.999 \times 10^{18}$, we have

$$B_1 < K_1^{1.106} < \frac{63}{64}n\log(n) - 2.$$

In this case, we obtain

$$\left\| B_1\left(\frac{1}{2} + \frac{1}{4n\log(n)}\right) + \frac{1}{2n\log(n)} \right\| = \left\| \frac{B_1 + 2}{4n\log(n)} \right\| = \frac{B_1 + 2}{4n\log(n)}.$$

But we have

$$\frac{B_1 + 2}{4n\log(n)} > \frac{3B_1 + 6}{8n^2\log(n)},$$

for $n \geq 2$. So this contradicts (4.2), i.e., there does not exist a non-trivial solution of type 1, for $n \geq N_1$.

4.2. **Solutions of type 2.** Suppose now $(x, y) \in \mathbb{Z}^2$ is a non-trivial solution of type 2.

Lemma 4.6. *We have*

$$\begin{cases} y\left(-\beta - \alpha - \frac{1}{2n^4}\right) < \gamma_1 < y\left(-\beta - \alpha\right), \\ -y\frac{1}{2n^4} < \gamma_2 < 0, \\ y\left(-\beta - \frac{1}{\alpha} - \frac{1}{2n^4}\right) < \gamma_3 < y\left(-\beta - \frac{1}{\alpha}\right), \\ y\left(-\beta + \frac{1}{\beta} - \frac{1}{2n^4}\right) < \gamma_4 < y\left(-\beta + \frac{1}{\beta}\right). \end{cases}$$

Proof. The same as the proof of Lemma 4.1. $\qquad\qquad\qquad\qquad\square$

We use the Siegel identity

$$0 < \frac{\gamma_3(\alpha_2 - \alpha_1)}{\gamma_1(\alpha_2 - \alpha_3)} = 1 - \frac{\gamma_2(\alpha_3 - \alpha_1)}{\gamma_1(\alpha_3 - \alpha_2)}$$

to obtain the following linear form in logarithms:

$$\Lambda_2 = A_2 \log(\alpha) + B_2 \log(\beta) + \log(\lambda_2) = \log(1 + \tau_2),$$

where $A_2 = -2u_2$, $B_2 = -2u_3$, $\lambda_2 = \frac{\alpha + \beta}{\beta + \frac{1}{\alpha}}$, $\tau_2 = \frac{\gamma_2}{\gamma_1}\left(\frac{\alpha - 1/\alpha}{1/\alpha + \beta}\right)$.

We obtain the equivalent of Lemmas 4.2, 4.3, 4.4 and 4.5.

Lemma 4.7. $\Lambda_2 \neq 0$.

Lemma 4.8. *We have*

$$u_2 < 0 < u_3, \quad 0 < |u_2| < u_3, \quad and \quad 0 < A_2 < |B_2|.$$

Lemma 4.9. *We have*

$$\log(\Lambda_2) > -2^{33} \times 3^7 \times \log(24) \log^3(\alpha\beta) \log(|B_2|).$$

Lemma 4.10. *For $n \geq 4$, we have*

$$\log(\Lambda_2) < -1.12|B_2| \log(\alpha).$$

So $K_2^{1.104}$ is an upper bound for $|B_2|$, and we can use the approach of type 1 to reach a contradiction for $n \geq N_2 = 2.322 \times 10^{18}$. Hence there is no non-trivial solution for $n \geq N_2$.

4.3. **Solutions of type 3.** Suppose that $(x, y) \in \mathbb{Z}^2$ is a non-trivial solution of type 3.

Lemma 4.11. *We have*

$$\begin{cases} y\left(\frac{1}{\alpha} - \alpha - \frac{1}{2n^2}\right) < \gamma_1 < y\left(\frac{1}{\alpha} - \alpha\right), \\ y\left(\frac{1}{\alpha} + \beta - \frac{1}{2n^2}\right) < \gamma_2 < y\left(\frac{1}{\alpha} + \beta\right), \\ -y\frac{1}{2n^2} < \gamma_3 < 0, \\ y\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{2n^2}\right) < \gamma_4 < y\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \end{cases}$$

Proof. The same as the proof of Lemma 4.1. $\qquad\qquad\qquad\qquad\square$

We use the Siegel identity

$$0 < \frac{\gamma_2(\alpha_3 - \alpha_4)}{\gamma_4(\alpha_3 - \alpha_2)} = 1 - \frac{\gamma_3(\alpha_2 - \alpha_4)}{\gamma_4(\alpha_2 - \alpha_3)},$$

to obtain the following linear form in logarithms:

$$\Lambda_3 = A_3 \log(\alpha) + B_3 \log(\beta) + \log(\lambda_3) = \log(1 + \tau_3),$$

where $A_3 = -2u_3$, $B_3 = 2u_2$, $\lambda_3 = \frac{1/\alpha + 1/\beta}{1/\alpha + \beta}$, $\tau_3 = -\frac{\gamma_3}{\gamma_4}\left(\frac{\beta - 1/\beta}{1/\alpha + \beta}\right)$.

As before, we obtain the following results.

Lemma 4.12. $\Lambda_3 \neq 0$.

Lemma 4.13. *We have*

$$0 < u_3 < u_2 \quad and \quad 0 < |A_3| < B_3.$$

Lemma 4.14. *We have*

$$\log(\Lambda_3) > -2^{33} \times 3^7 \times \log(24) \log^3(\alpha\beta) \log(B_3).$$

Lemma 4.15. *For $n \geq 4$, we have*

$$\log(\Lambda_3) < -0.25 B_3 \log(\alpha).$$

So $K_3^{1.102}$ is an upper bound for B_3, and we can use the approach of type 1 to reach a contradiction for $n \geq N_3 = 1.168 \times 10^{19}$. Hence there is no non-trivial solution for $n \geq N_3$.

4.4. **Solutions of type 4.** Suppose that $(x, y) \in \mathbb{Z}^2$ is a non-trivial solution of type 4.

Lemma 4.16. *We have*

$$\begin{cases} y\left(-\frac{1}{\beta} - \alpha\right) < \gamma_1 < y\left(-\frac{1}{\beta} - \alpha + \frac{1}{2n^2}\right), \\ y\left(-\frac{1}{\beta} + \beta\right) < \gamma_2 < y\left(-\frac{1}{\beta} + \beta + \frac{1}{2n^2}\right), \\ -y\left(-\frac{1}{\beta} - \frac{1}{\alpha}\right) < \gamma_3 < y\left(-\frac{1}{\beta} - \frac{1}{\alpha} + \frac{1}{2n^2}\right), \\ 0 < \gamma_4 < y\frac{1}{2n^2}. \end{cases}$$

Proof. The same as the proof of Lemma 4.1. $\qquad \square$

We use the Siegel identity

$$0 < \frac{\gamma_1(\alpha_4 - \alpha_3)}{\gamma_3(\alpha_4 - \alpha_1)} = 1 - \frac{\gamma_4(\alpha_1 - \alpha_3)}{\gamma_3(\alpha_1 - \alpha_4)},$$

to obtain the following linear form in logarithms:

$$\Lambda_4 = A_4 \log(\alpha) + B_4 \log(\beta) + \log(\lambda_4) = \log(1 + \tau_4).$$

where $A_4 = 2u_2$, $B_4 = 2u_3$, $\lambda_4 = \frac{1/\alpha + 1/\beta}{\alpha + 1/\beta}$, $\tau_4 = -\frac{\gamma_4}{\gamma_3}\left(\frac{\alpha - 1/\alpha}{\alpha + 1/\beta}\right)$.

We obtain the following results.

Lemma 4.17. $\Lambda_4 \neq 0$.

Lemma 4.18. *We have*

$$u_3 < 0 < u_2, \qquad 0 < u_2 < |u_3|, \qquad 0 < A_4 < |B_4|.$$

Lemma 4.19. *We have*

$$\log(\Lambda_4) > -2^{33} \times 3^7 \times \log(24) \log^3(\alpha\beta) \log(|B_4|).$$

Lemma 4.20. *For $n \geq 4$, we have*

$$\log(\Lambda_4) < -0.27|B_4|\log(\alpha).$$

So $K_4^{1.104}$ is an upper bound for B_4, we can use the approach of type 1 to reach a contradiction for $n \geq N_4 = 1.169 \times 10^{19}$. Hence there is no non-trivial solution for $n \geq N_4$.

Now let us put $N = \max\{N_1,\ N_2,\ N_3,\ N_4\}$.

Theorem 4.21. *For $n \geq N = 1.191 \times 10^{19}$, with n, $n+2$, n^2+4 square-free, the equation*

$$\Phi_n(x,y) = x^4 - n^2 x^3 y - (n^3 + 2n^2 + 4n + 2)x^2 y^2 - n^2 xy^3 + y^4 = 1,$$

has only the trivial solutions $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$.

5. Part II: Small values of n

For small values of n, the previous method is not successful, so we use a method of Y. Bilu and G. Hanrot (see [2]). It is a computational method which allows us to determine a Baker's bound for the exponents $b_i = u_i$ (see (3.2)), to reduce this bound, and then to solve completely the equation. We obtain the following result.

Theorem 5.1. *For $n \leq 5 \times 10^6$, with n, $n+2$, n^2+4 square-free, the equation*

$$\Phi_n(x,y) = x^4 - n^2 x^3 y - (n^3 + 2n^2 + 4n + 2)x^2 y^2 - n^2 xy^3 + y^4 = 1$$

has only the trivial solutions $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$.

Here are a few remarks about the computation. In order to reduce Baker's bound, we determined the expressions X_i, c_i, δ_i, which allows to determine a Baker's bound for the exponents b_i, to make the reduction, see [2], and we wrote a program in PARI/GP. We know that $\varepsilon < n + 1$, $\alpha < (n+1)^2$ and $\beta < n + 2$, so we put $n_2 = (n_1 + 2)(|b_1| + 2|b_2| + |b_3| + 1) + 50$ as the computation's precision, where n_1 is the number of digits of $n + 2$.

After almost 3 reductions, we obtain $B_0 \leq 4$ and $X_3 = 1$. The integral solutions (x,y) verifying $|x| \leq X_3$, $|y| \leq X_3$ are: $(-1,0)$, $(0,-1)$, $(0,1)$, $(1,0)$. In fact, we let b_1, b_2, b_3 run between $-B_0$ and B_0, and we observed the exponents associated with the integral solutions. Then we obtain:

for types 1 and 2: $b_1 = b_2 = b_3 = 0$, $x = 1$, $y = 0$, so $(1,0)$ is a solution;

for types 3 and 4: $b_1 = 0$, $b_2 = 1$, $b_3 = 0$, $x = 0$, $y = -1$, so $(0,-1)$ is a solution.

The property (b) of section 2 gave us the other solutions. The computations were done with a SUN PARC ULTRA1, and for each value of n, the time of computation is between 3 and 5 seconds.

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Département de Mathematiques et de Statistique, Université Laval, Québec, Québec, G1K 7P4 Canada

Current address: Department of Mathematics & Computer Science, Greenville College, 315 E. College Avenue, Greenville, IL 62246

E-mail address: `atogbe@greenville.edu`