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CONVERGENCE RATES TO THE DISCRETE TRAVELLING WAVE FOR RELAXATION SCHEMES

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ABSTRACT. This paper is concerned with the asymptotic convergence of numerical solutions toward discrete travelling waves for a class of relaxation numerical schemes, approximating the scalar conservation law. It is shown that if the initial perturbations possess some algebraic decay in space, then the numerical solutions converge to the discrete travelling wave at a corresponding algebraic rate in time, provided the sums of the initial perturbations for the u-component equal zero. A polynomially weighted l^2 norm on the perturbation of the discrete travelling wave and a technical energy method are applied to obtain the asymptotic convergence rate.

1. INTRODUCTION

We shall investigate here the convergence rates to the stationary discrete travelling wave for a class of relaxation numerical schemes of the type introduced by Jin and Xin [7] as well as Aregba-Driollet and Natalini [1] to approximate the scalar conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

when the relaxation time is small.

The relaxation numerical schemes we consider take the form

$$\begin{aligned} u_j^{n+1} - u_j^n + \tfrac{\lambda}{2}(v_{j+1}^n - v_{j-1}^n) - \tfrac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) &= 0, \\ v_j^{n+1} - v_j^n + \tfrac{a\lambda}{2}(u_{j+1}^n - u_{j-1}^n) - \tfrac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) &= -\kappa(v_j^{n+1} - f(u_j^{n+1})). \end{aligned} \tag{1.2}$$

The discrete solution $(u^n, v^n) := (u_j^n, v_j^n)_{j\in\mathbb{Z}}$ is a numerical approximation of the point values $(u, v)(x_j, t_n)$ on the grid given by $x_j = j\Delta x$ and $t_n = n\Delta t$, with $\Delta x = r$ and $\Delta t = h$ being the spatial and the temporal mesh lengths. Further, we assume that the mesh ratio $\lambda = \frac{\Delta t}{\Delta x}$ satisfies the Courant-Friedrichs-Lewy (CFL) condition

$$\mu := \sqrt{a}\lambda < 1. \tag{1.3}$$

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The relaxation scheme (1.2) with $\kappa = \Delta t/\varepsilon > 0$ was introduced in [7] as an approximation to the system

$$(1.4)\qquad \begin{aligned} u_t + v_x &= 0, \quad x \in \mathbb{R}, \quad t > 0,\\ v_t + au_x &= -\tfrac{1}{\varepsilon}(v - f(u)), \end{aligned}$$

which approximates scalar conservation laws (1.1) when the relaxation rate $\varepsilon > 0$ is small. For rigorous justification of such a kind of zero relaxation limit we refer to Chen, Levermore and Liu [2], Liu [14] and Natalini [22], etc., for 2×2 systems. One of the main advantages of the system (1.4) is its form of local relaxation structure and linearity in convection which makes it possible to solve this system quite easily by underresolved stable numerical discretization using neither Riemann solvers spatially nor nonlinear systems of algebraic equations solvers temporally [7].

In (1.4), the constant $a > 0$ is assumed to satisfy the *subcharacteristic condition* introduced by Liu [14]:

$$(1.5)\qquad -\sqrt{a} < f'(u) < \sqrt{a} \quad \text{for all } u \text{ under consideration.}$$

To see that (1.4) is a good approximation of (1.1), application of the *Chapman-Enskog expansion* to (1.4) implies

$$(1.6)\qquad u_t + f(u)_x = \epsilon[(a^2 - f'(u)^2)u_x]_x.$$

The Cauchy problem for (1.6) is well-posed if (1.5) holds.

Throughout this paper it is assumed that the flux function f is smooth and convex, i.e.,

$$(1.7)\qquad f''(u) > 0, \quad \text{for all } u \text{ under consideration.}$$

Our interest here will be on the discrete traveling wave solution of (1.2) propagating at subcharacteristic speed $s = 0$ in the sense that

$$(1.8)\qquad -\sqrt{a} < s < \sqrt{a}.$$

If $(u^n, v^n) = (U_j, V_j)_{j\in\mathbb{Z}}$ is a travelling wave solution to (1.2) connecting constant states $(u_\pm, v_\pm) = (U_{\pm\infty}, V_{\pm\infty})$, we must have

$$v_\pm = f(u_\pm),$$

since the only constant state solutions of (1.2) are equilibrium states which are on the equilibrium curve $v = f(u)$ in the state space (u, v).

Under the CFL condition (1.3) and the subcharacteristic condition (1.5), related to an admissible stationary shock denoted by $(u_-, u_+, 0)$ for the equation (1.1), the scheme (1.2) admits a unique **stationary discrete travelling wave** $(U_j, V_j)_{j\in\mathbb{Z}}$ with U_j taking on a given value $u_* \in]u_+, u_-[$ at $j = 0$, i.e., it satisfies the conditions

$$(1.9)\qquad \begin{aligned} &(V_{j+1} - V_{j-1}) - \sqrt{a}(U_{j+1} - 2U_j + U_{j-1}) = 0,\\ &(U_{j+1} - U_j) - \tfrac{1}{\sqrt{a}}(V_{j+1} - 2V_j + V_{j-1}) = -\tfrac{2\kappa}{a\lambda}(V_j - f(U_j)), \quad j \in \mathbb{Z},\\ &\lim_{j\to\pm\infty}(U_j, V_j) = (u_\pm, f(u_\pm)),\\ &U_j|_{j=0} = u_*. \end{aligned}$$

The existence of such discrete solutions and further properties, see Proposition 2.1, were proved by Liu, Wang and Yang [17]. Moreover, by using the energy method the authors in [17] were able to show that these stationary discrete travelling waves are nonlinearly stable with respect to initial perturbations, provided the total mass of the perturbation is zero.

The main goal of this paper is to improve the nonlinear stability results in [17] by establishing the time convergence rates to a stationary discrete travelling wave $(U_j, V_j)_{j\in\mathbb{Z}}$. To the author's knowledge, this seems the first time-asymptotic convergence rate for a difference scheme applied to a relaxation system of conservation laws. The result is based on an observation in Liu, Woo and Yang [19] that the perturbation of travelling waves that initially decay in space with some algebraic rate yields a corresponding decay rate in time; see also Zingano [27]. This observation suggests that we should use L^2-based weighted norms on the initial perturbations to investigate the algebraic decay. This strategy was initiated in a paper by Kawashima and Matsumura [8] for the scalar viscous conservation law. Using this approach, a time decay result in the context of numerical scheme was obtained by Liu and Wang [13]. In the present paper the algebraic decay in space will be encoded through the use of an algebraic discrete weight analogous to [13]. The result then states that the perturbation will decay algebraically in time.

Now we state the main theorem in this paper.

Theorem 1.1. *Assume that the CFL condition* (1.3)*, the subcharacteristic condition* (1.5)*, and* (1.7) *hold. Let $(U_j, V_j)_{j\in\mathbb{Z}}$ be a stationary discrete travelling wave defined by* (1.9) *connecting $(u_+, f(u_+))$ to $(u_-, f(u_-))$. Assume that*

$$\sum_{j\in\mathbb{Z}}(u_j^0 - U_j) = 0 \tag{1.10}$$

and, for some $\alpha > 0$,

$$\sum_{j\in\mathbb{Z}}\left[(1+j^2)^{\frac{\alpha}{2}+1}|u_j^0 - U_j|^2 + (1+j^2)^{\frac{\alpha}{2}}|v_j^0 - V_j|^2\right] \le \delta \tag{1.11}$$

for some positive constant δ. Then the unique global solution $(u_j^n, v_j^n)_{j\in\mathbb{Z}}$ to the relaxation scheme (1.2) *with the initial data $(u_j^0, v_j^0)_{j\in\mathbb{Z}}$ tends in the maximum norm to the discrete travelling wave $(U_j, V_j)_{j\in\mathbb{Z}}$ at the rate*

$$\sup_j |(u_j^n, v_j^n) - (U_j, V_j)| \le C(1+nh)^{-\alpha/2}\sqrt{\delta}, \quad n \ge 0, \tag{1.12}$$

provided λ is suitably small, $\kappa \in \mathbb{R}^+$.

A few remarks are in order concerning the theorem and its proof.

Remark 1. Although we only present the results for implicit scheme in the theorem and its proof, however, it should be clear from our analysis and the nonlinear stability analysis in [17] that the corresponding result holds for the explicit scheme in the form

$$\begin{aligned} u_j^{n+1} - u_j^n + \tfrac{\lambda}{2}(v_{j+1}^n - v_{j-1}^n) - \tfrac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) &= 0, \\ v_j^{n+1} - v_j^n + \tfrac{a\lambda}{2}(u_{j+1}^n - u_{j-1}^n) - \tfrac{\mu}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) &= -\kappa(v_j^n - f(u_j^n)), \end{aligned}$$

under some technical restriction on κ (for stability result, $0 < \kappa \le 1$ in [17]).

Remark 2. Our result shows that there is a relationship between the spatial decay assumed of the initial perturbation and the rate of decay in time. In this sense the theorem exhibits the transformation of spatial decay into temporal decay.

Remark 3. In the theorem and its proof we just use the fact that $\kappa > 0$. If we take $\kappa = \exp(\Delta t/\epsilon) - 1$ instead, then (1.2) will reduce to a relaxation scheme studied in the paper of Aregba-Driollet and Natalini [1]. They considered a fractional-step scheme, where the homogeneous (linear) part is treated by some monotone scheme and then the source term is solved exactly thanks to its particular structure. Therefore the asymptotic convergence rates presented in Theorem 1.1 still hold true for a first order relaxation scheme in [1].

It is well known that in general the initial value problem of (1.1) develops discontinuities in a finite time which present difficulties for numerical computation of solutions to (1.1). Discrete shock profiles of the numerical schemes for (1.1) epitomize the propagation of solutions and structure properties of shocks in numerical solutions. In recent years existence and stability of discrete shocks has been an interesting subject of study. The existence of a discrete shock was first studied by Jennings [6] for a monotone scheme. For a first order system, Majda and Ralston [21] used a center manifold theory and proved the existence of a discrete shock; see also Michelson [20]. The asymptotic stability for scalar equations was studied by Jennings [6], Tadmor [25], Smyrlis [24], Engquist and Yu [3], Liu and Wang [11], [12], and other authors. For a first order system, Liu and Yu [15] recently showed both the existence and stability of a discrete shock when the relative discrete shock speed is a diophantine number. For a modified Lax-Friedrichs scheme Liu and Xin [9], [10] proved the stability of discrete shocks. For a general initial perturbation, Ying [26] obtained a stability result for the Lax-Friedrichs scheme. For the existence and stability of the discrete travelling wave for some relaxing schemes, see [17], [18]. Existence and stability of discrete shocks are essential for error analysis of a difference scheme approximating (1.1); see [6], [9], [3] and [4]. These and our other references also quote and describe further earlier work.

The rest of the paper is outlined as follows. In Section 2 we recall the existence and stability of the stationary discrete travelling wave, then reformulate the original problem and restate the main theorem. In Section 3 the basic time decay estimates are proved by using a weighted energy analysis. The proofs of some intermediate technical energy estimates summarized in Lemma 3.4 are relegated to Section 4, from which the restriction on λ is clarified. Finally, the main theorem is proved in Section 5. Some computations in grouping of terms for constructing the energy function are carried out in the Appendix. Grouping terms in this way yields a simpler energy function than that in [17]. Actually our proof here reduces to a slightly simplified version of the stability proof in [17] after replacing the weight by 1 in this novel energy expression.

We end this section by presenting the following definitions of discrete norms to be used in subsequent analysis.

First let us define the weighted l^2-norm. Suppose that $\{K_j > 0, j \in \mathbb{Z}\}$ is any discrete weight function. For any infinite dimensional vector $u \equiv (u_j)_{j\in\mathbb{Z}}$, we define

$$|u|_K = \left(\sum_j |u_j|^2 K_j\right)^{\frac{1}{2}}.$$

We denote the corresponding space by

$$l^2_K = \{u \quad |u|_K < \infty\}.$$

When for $r = \Delta x$ specifically $K_j = \langle jr\rangle^\alpha = (1+(jr)^2)^{\frac{\alpha}{2}}$ for some $\alpha \geq 0$, we write

$$l_K^2 = l_\alpha^2 \quad \text{with norm } \ |\cdot|_K = |\cdot|_\alpha.$$

If $\alpha = 0$, $|\cdot|_\alpha$ becomes the regular l^2 norm $\|\cdot\| = |\cdot|_0$. We will denote the difference of a discrete function $(u_j)_{j\in\mathbb{Z}}$ in space by

$$\Delta u := (u_{j+1} - u_j)_{j\in\mathbb{Z}}.$$

2. Discrete travelling wave and main theorems

Let $(U_j, V_j)_{j\in\mathbb{Z}}$ be a stationary discrete travelling wave connecting $(u_\pm, f(u_\pm))$ for the relaxation scheme (1.2). For the existence of $(U_j, V_j)_{j\in\mathbb{Z}}$ a necessary and sufficient condition is Rankine-Hugonoit relation

$$f(u_-) = f(u_+) \tag{2.1}$$

combined with Lax's shock condition

$$f'(u_+) < 0 < f'(u_-) \tag{2.2}$$

when the propagation speed $s = 0$. Due to the convexity of the flux function f, the shock condition (2.2) is equivalent to

$$u_+ < u_-.$$

Further, it was shown in [17] that $(U_j)_{j\in\mathbb{Z}}$, the u-component of the discrete travelling wave $(U_j, V_j)_{j\in\mathbb{Z}}$, is the stationary discrete shock profile of a monotone conservative difference scheme which becomes as $\epsilon \to 0$,

$$u_j^{n+1} = u_j^n - \frac{1}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\mu}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \tag{2.3}$$

The scheme (2.3) is a first order monotone difference scheme for the scalar conservation laws (1.1). This yields the monotonicity of $(U_j)_{j\in\mathbb{Z}}$, which is crucial in our stability analysis.

Proposition 2.1 ([17])**.** *Under the Rankine-Hugonoit condition* (2.1)*, Lax's shock condition* (2.2) *and the subcharacteristic condition* (1.5)*, for each given* $u_* \in]u_+, u_-[$*, there exists a unique stationary discrete travelling wave* $(U_j, V_j)_{j\in\mathbb{Z}}$ *for the scheme* (1.2)*, i.e.,* $(U_j, V_j)_{j\in\mathbb{Z}}$ *satisfy* (1.9)*. Moreover,*

$$U_{j+1} < U_j \qquad \text{for any} \quad j \in \mathbb{Z}.$$

Let $(u_j^n, v_j^n)_{j\in\mathbb{Z}, n\in\mathbb{N}}$ be the numerical solution of (1.2) corresponding to a slight perturbation of the wave profile $(U_j, V_j)_{j\in\mathbb{Z}}$, i.e.,

$$(u_j^0, v_j^0) = (U_j, V_j) + (\tilde{u}_j, \tilde{v}_j)$$

with $(\tilde{u}_{\pm\infty}, \tilde{v}_{\pm\infty}) = (0,0)$ and $(U_{\pm\infty}, V_{\pm\infty}) = (u_\pm, f(u_\pm))$. After assuming

$$\sum_{j\in\mathbb{Z}} (u_j^0 - U_j) = 0, \tag{2.4}$$

we have

$$\sum_{j\in\mathbb{Z}} (u_j^n - U_j) = 0$$

indicated by the conservation form of the first equation in scheme (1.2). Then we can expect to show that

$$(u_j^n, v_j^n) \to (U_j, V_j) \qquad \text{as} \quad n \to \infty. \tag{2.5}$$

Denote

$$\bar{u}_j^0 := \sum_{i=-\infty}^{j} (u_i^0 - U_i), \quad \bar{v}_j^0 := v_j^0 - V_j. \tag{2.6}$$

When $\|\bar{u}^0\| + \|\bar{v}^0\|$ is small, the authors in [17] could derive energy estimates related to (2.5), leading to the following result.

Theorem 2.2 ([17])**.** *Let $(U_j, V_j)_{j\in\mathbb{Z}}$ be a discrete stationary travelling wave of the relaxation scheme (1.2). If $\|\bar{u}^0\| + \|\bar{v}^0\|$ is suitably small, then there exists a unique global solution, $(u_j^n, v_j^n)_{j\in\mathbb{Z}}$, to the scheme (1.2) with initial value $(u_j^0, v_j^0)_{j\in\mathbb{Z}}$ such that*

$$\lim_{n\to\infty} \sum_j (|u_j^n - U_j|^2 + |v_j^n - V_j|^2) = 0$$

provided λ is suitably small, and $\kappa \in R^+$.

Also, experience suggests that, in the case of discrete shock profiles, perturbations might decay quite fast as $n \to \infty$, provided they are sufficiently localized in space. Our main result in Theorem 1.1 shows that this is indeed the case; and Theorem 1.1 can be obtained from the following theorem.

Theorem 2.3 (Convergence Rate)**.** *Let $(u_j^n, v_j^n)_{j\in\mathbb{Z}}$ be a solution obtained in Theorem 2.2, and $(\bar{u}_j^0, \bar{v}_j^0)_{j\in\mathbb{Z}} \in l_\alpha^2$ for some $\alpha > 0$. If $|\bar{u}^0|_\alpha + |\bar{v}^0|_\alpha$ is suitably small, then*

$$\sup_j |(u_j^n - U_j, v_j^n - V_j)| \le C(1+nh)^{-\frac{\alpha}{2}} (|\bar{u}^0|_\alpha + |\bar{v}^0|_\alpha)$$

for all $n \in \mathbb{N}_0$.

Theorem 1.1 is a direct consequence of Theorem 2.3 if we note that the condition (1.11) in Theorem 1.1 implies that $(\bar{u}_j^0, \bar{v}_j^0)_{j\in\mathbb{Z}} \in l_\alpha^2$ (whose proof requires a discrete version of a weighted Poincaré inequality and is omitted here; for details, see [5]).

In order to prove Theorem 2.3, we reformulate the scheme (1.2) by formally introducing

$$\bar{u}_j^n := \sum_{k=-\infty}^{j} (u_k^j - U_k), \qquad \bar{v}_j^n := v_j^n - V_j. \tag{2.7}$$

It will be shown below that the summation always gives a finite value.

Now, both $(u_j^n, v_j^n)_{j\in\mathbb{Z}}$ and $(U_j, V_j)_{j\in\mathbb{Z}}$ satisfy (1.2); by taking the difference of the two systems of the scheme and summing the first equation with respect to j over $(-\infty, j)$, we obtain, after linearizing the resulting system around the wave profile $(U_j, V_j)_{j\in\mathbb{Z}}$,

$$\begin{cases} \bar{u}_j^{n+1} - \bar{u}_j^n + \frac{\lambda}{2}(\bar{v}_{j+1}^n + \bar{v}_j^n) - \frac{\mu}{2}(\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n) = 0, \\ \bar{v}_j^{n+1} - \bar{v}_j^n + \frac{a\lambda}{2}(\bar{u}_{j+1}^n - \bar{u}_j^n - \bar{u}_{j-1}^n + \bar{u}_{j-2}^n) - \frac{\mu}{2}(\bar{v}_{j+1}^n - 2\bar{v}_j^n + \bar{v}_{j-1}^n) \\ \qquad = -\kappa[\bar{v}_j^{n+1} - \Lambda_j(\bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1}) - \theta_j^{n+1}], \end{cases} \tag{2.8}$$

$$\begin{aligned} \theta_j^{n+1} &= f(u_j^{n+1}) - f(U_j) - f'(U_j)(u_j^{n+1} - U_j) \\ &= \frac{1}{2} f''\big(\eta u_j^{n+1} + (1-\eta)U_j)(u_j^{n+1} - U_j\big)^2, \qquad 0 < \eta < 1, \end{aligned} \tag{2.9}$$

$$\Lambda_j = f'(U_j), \tag{2.10}$$

$$\bar{u}_j^n - \bar{u}_{j-1}^n = u_j^n - U_j. \tag{2.11}$$

Set

$$L_j^n := -\frac{\lambda}{2}(\bar{v}_{j+1}^n + \bar{v}_j^n). \tag{2.12}$$

This by the first equation of (2.8) yields

$$L_j^n = \bar{u}_j^{n+1} - \bar{u}_j^n - \frac{\mu}{2}(\bar{u}_{j+1}^n - 2\bar{u}_j^n + \bar{u}_{j-1}^n). \tag{2.13}$$

For simplicity of presentation, we introduce

$$w_j^n = \bar{u}_{j+1}^n - \bar{u}_{j-1}^n, \;\; \bar{w}_j^n = \bar{u}_{j+1}^n - \bar{u}_j^n. \tag{2.14}$$

Further summing both sides of the second equaion of (2.8) with index j and $j+1$, then multiplying by $\frac{-\lambda}{2}$, we get

(2.15)

$$\begin{aligned}\mathcal{L}(\bar{u}_j^n) := \;\; & L_j^{n+1} - L_j^n - \tfrac{\mu}{2(\kappa+1)}(L_{j+1}^n - 2L_j^n + L_{j-1}^n) - \tfrac{\mu^2}{4(\kappa+1)}(w_{j+1}^n - w_{j-1}^n) \\ & + \tfrac{\kappa}{\kappa+1}L_j^n + \tfrac{\kappa\lambda\Lambda_j}{2(\kappa+1)}\bar{w}_{j-1}^{n+1} + \tfrac{\kappa\lambda\Lambda_{j+1}}{2(\kappa+1)}\bar{w}_j^{n+1} \\ = \;\; & \tfrac{\kappa}{(\kappa+1)}e_j^{n+1},\end{aligned}$$

where

$$e_j^{n+1} = -\frac{\lambda}{2}(\theta_j^{n+1} + \theta_{j+1}^{n+1}), \;\; \mu = \sqrt{a}\lambda. \tag{2.16}$$

The corresponding initial data for the reformulated scheme (2.15) are

$$\bar{u}_j^n|_{n=0} = \bar{u}_j^0, \;\; L_j^n|_{n=0} = L_j^0 := -\frac{\lambda}{2}(\bar{v}_{j+1}^0 + \bar{v}_j^0). \tag{2.17}$$

We observe from (2.9) that the right hand side $\frac{\kappa}{(\kappa+1)}e_j^{n+1}$ in (2.15) involves only high powers of terms which we expect to be small and have little effect in the subsequent energy analysis for small perturbations. As shown in [17], the most important properties for the stability of the discrete travelling wave are its compressity, expressed by the inequality

$$\Lambda_j > \Lambda_{j+1}, \quad j \in \mathbb{Z}, \tag{2.18}$$

which is implied by the convexity of f and the monotonicity of U_j in j; as well as the fact that the wave travels at subcharacteristic speed, see (1.8). In fact, under the assumptions in Theorem 2.2, the authors in [17] were able to derive the energy estimate

$$\sup_{0\le i\le n}\sum_j\left\{4(L_j^i)^2+\frac{k^2}{(k+1)^2}(\bar u_j^i)^2+\frac{k\mu}{(k+1)^2}(\bar\omega_j^i)^2+\frac{2\mu^2}{k+1}(\omega_j^i)^2\right\} \tag{2.19}$$
$$+\sum_{i=0}^n\sum_j\left\{\frac{k\mu^2}{(k+1)^2}(\omega_j^i)^2+2(\Lambda_j-\Lambda_{j+1})\frac{k^2\lambda}{(k+1)^2}(u_j^{i+1})^2+\frac{k\mu^2}{(k+1)^2}(\bar\omega_{j+1}^i-\bar\omega_j^i)^2\right.$$
$$+\frac{k^2}{2(k+1)^2}(\bar\omega_j^i)^2+\frac{k^2}{(k+1)^2}(\bar u_j^{i+1}-\bar u_j^i)^2+\frac{4}{(k+1)}(L_{j+1}^i-L_j^i)^2$$
$$\left.+\frac{\mu^3}{(k+1)}(\omega_{j+1}^i-\omega_j^i)^2+\frac{k}{k+1}(L_j^i)^2\right\}\le C(\|\bar u^0\|^2+\|\bar v^0\|^2),$$

provided we take $\|\bar u^0\|+\|\bar v^0\|$ sufficiently small (see [17] for the detailed derivation). Let us point out that (2.19) implies the stability result in Theorem 2.2, but, due to the fact that

$$\Lambda_j-\Lambda_{j+1}\to 0 \qquad \text{as} \quad j\to\pm\infty,$$

no decay rate can be directly inferred from (2.19). This will be done by a different, though related, analysis.

We now restate Theorem 2.3 in terms of $(\bar u_j^n, L_j^n)_{j\in\mathbb Z}$ as follows.

Theorem 2.4. *Under the assumptions of Theorem 2.3, there exists a positive constant ϵ_1 such that if $|\bar u^0|_\alpha+|L^0|_\alpha\le\epsilon_1$, then the Cauchy problem* (2.15), (2.17) *has a unique global solution $(\bar u_j^n, L_j^n)_{j\in\mathbb Z}$ such that*

$$(1+nh)^\alpha\Big[\|\bar u^n\|^2+\|L^n\|^2\Big]+(1+nh)^{-p}\sum_{i<n}(1+ih)^{\alpha+p} \tag{2.20}$$
$$\cdot\Big[\|L^i\|^2+\mu\|\Delta L^i\|^2+\mu\|\Delta\bar u^i\|^2\Big]\le C\Big[|\bar u^0|_\alpha^2+|L^0|_\alpha^2\Big]$$

for any $p>0$.

It is easy to get the unique solution $(\bar u_j^n, L_j^n)_{j\in\mathbb Z}$ from the scheme (2.15) for some $n>0$. Our effort henceforth is concentrated on establishing the basic time decay estimate (2.20) which is carried out in Sections 3-4. The proof of Theorem 2.3 based on Theorem 2.4, is given at the final Section 5.

3. Time decay analysis

In this section, we investigate the time decay estimates for $(L_j^n,\bar u_j^n)_{j\in\mathbb Z}$ generated by the reformulated scheme (2.15) with initial data (2.17). First we present the basic reasoning behind the argument. Let us rewrite the scheme (2.15) as

$$\mathcal L_1(\bar u_j^n)+\mathcal L_2(\bar u_j^n)=\frac{\kappa}{\kappa+1}e_j^{n+1}, \tag{3.1}$$

where

$$\mathcal L_1(\bar u_j^n):=L_j^{n+1}-L_j^n-\frac{\mu}{2(\kappa+1)}(L_{j+1}^n-2L_j^n+L_{j-1}^n)-\frac{\mu^2}{4(\kappa+1)}(w_{j+1}^n-w_{j-1}^n),$$
$$\mathcal L_2(\bar u_j^n):=\frac{\kappa}{\kappa+1}L_j^n+\frac{\kappa\lambda}{2(\kappa+1)}[\Lambda_j\bar w_{j-1}^{n+1}+\Lambda_{j+1}\bar w_j^{n+1}].$$

Because of the subcharacteristic speed of the wave profile, the dynamics for the perturbations is expected to be mainly governed by the first order approximation scheme

$$\mathcal{L}_2(\bar{u}_j^n) = 0$$

with propagation speed $(\Lambda_j)_{j\in\mathbb{Z}}$. Since a discrete shock profile $(U_j)_{j\in\mathbb{Z}}$ is strictly decreasing in $j \in \mathbb{Z}$, see Proposition 2.1, and $f(u)$ is convex, there exists a unique $j_0 \in \mathbb{Z}$ such that $U_{j_0} \le \bar{u} < U_{j_0-1}$, with $\bar{u} \in \,]u_+, u_-[$ uniquely determined by $f'(\bar{u}) = \frac{f(u_+)-f(u_-)}{u_+-u_-} = 0$. Again by the convexity of f and $\Lambda_j = f'(U_j)$, we have

$$\Lambda_{j_0} \le 0 < \Lambda_{j_0-1}, \;\; \Lambda_j < \Lambda_{j-1}, \;\; j \in \mathbb{Z}. \tag{3.2}$$

Experience suggests that at large times most of the information for solutions of (3.1) come from points j away from j_0 on the initial line. Thus we can consider a decay factor $n^\gamma |j-j_0|^\beta$ in deriving our time decay estimates. Without loss of generality, we may assume $j_0 = 0$. We introduce for $r = \Delta x$ and $h = \Delta t$ the abbreviations

$$P_j := \langle jr\rangle^\beta \quad \text{and} \quad H_j := (1+nh)^\gamma,$$

where $\beta \in \,]0,\alpha]$ and γ are positive constants at our disposal. To avoid the singularities we choose a time-dependent discrete weight function of the form

$$K_j^n = H^n P_j, \qquad j \in \mathbb{Z},$$

which will be used to characterize the decay rate.

In fact, the above choice of j_0 and the convexity of f give us a lower bound for

$$A_j = \lambda(\Lambda_j P_j - \Lambda_{j+1} P_{j+1})$$

with $\Lambda_j = f'(U_j)$ satisfying (3.2) and $P_j = \langle jr\rangle^\beta$, $\beta \in [0,\alpha]$. This lower bound on A_j plays a crucial role in our later argument and is summarized in the following lemma.

Lemma 3.1. *For any $\beta \in [0,\alpha]$, there exists a positive constant c_0 independent of β such that*

$$A_j \ge c_0 \beta \langle jr\rangle^{\beta-1} h \tag{3.3}$$

for any $j \in \mathbb{Z}$, provided λ is suitably small.

Proof. The proof can be done by an analysis similar to [13]. We omit the details. □

To handle the weighted terms, we further state some basic estimates on the weights $P_j = \langle jr\rangle^\beta = (1+(jr)^2)^{\beta/2}$ and $H^n = (1+nh)^\gamma$.

Lemma 3.2. *(i) For any $j \in \mathbb{Z}$ and $\beta \in [0,\alpha]$, there exist constants $\theta \in \,]0,1[$ and $c_r > 0$, $C_r > 0$ such that*

$$\theta^{-1} P_j \ge P_{j+1} \ge \theta P_j,$$
$$c_r \beta r \langle jr\rangle^{\beta-1} \le |P_{j+1} - P_j| \le C_r \beta r \langle jr\rangle^{\beta-1}.$$

(ii) For any $n \in \mathbb{N}_0$ and $\gamma > 0$,

$$H^n < H^{n+1} \le (1+h)^\gamma H^n,$$
$$H^{n+1} - H^n \le \gamma(1+h)^\gamma (1+nh)^{\gamma-1} h.$$

Proof. The proof of (i) can be found in [16]; and (ii) can be easily verified by using the Taylor expression. □

Armed with Lemmas 3.1 and 3.2, we turn to establish the basic time decay estimate. Set

$$N(n,\alpha) := \sup_{0\le i\le n} [|\bar u^i|_\alpha^2 + |L^i|_\alpha^2], \quad N(n) := N(n,0), \tag{3.4}$$

where $|\cdot|_\alpha$ denotes the norm in the weighted l^2_α space.

In what follows, we always assume that $N(n_1)$ is small for any given $n_1 > 0$. This assumption will be verified by an a priori estimate in subsequent sections, if the initial perturbation $N(0,\alpha)$ is sufficiently small. To derive such an a priori estimate, we need the following inequalities:

$$\begin{aligned} \sup_{0\le i\le n}\sup_j |\bar u_j^i| &\le \sqrt{N(n)}, \\ \sup_{0\le i\le n}\sup_j |L_j^i| &\le \sqrt{N(n)}, \\ \sup_{0\le i\le n}\sup_j |(\bar w_j^i, w_j^i)| &\le 2\sqrt{N(n)}. \end{aligned} \tag{3.5}$$

In order to shorten notation, we introduce

$$G(i,\beta) := |\bar u^i|_\beta^2 + |L^i|_\beta^2, \quad i\in\mathbb{N}_0, \tag{3.6}$$

which satisfies $G(0,\beta) = N(0,\beta)$ and

$$G(i,\beta) \le N(n,\beta), \text{ for } i\le n. \tag{3.7}$$

We will solve the Cauchy problem (2.15), (2.17) in $0 < n \le n_1$ for a given $n_1 > 0$. The most important step of the whole analysis is to establish the following estimate.

Lemma 3.3. *Let $(\bar u_j^n, L_j^n)_{j\in\mathbb{Z}}$ be a solution of (2.15) for $n\le n_1$. Assume that $N(n_1)$ and λ are suitably small. Then for any $\beta\in[0,\alpha]$ there exists a positive constant C independent of n_1 such that for all $n\le n_1$ and*

$$|\Gamma^i|_\beta^2 = |L^i|_\beta^2 + \kappa|\bar u^{i+1} - \bar u^i|_\beta^2 + \mu|\Delta \bar u^i|_\beta^2 + \mu|\Delta L^i|_\beta^2, \quad i\in\mathbb{N}_0,$$

the following estimate holds:

$$\begin{aligned} &(1+nh)^\gamma G(n,\beta) + \beta\sum_{i<n}(1+ih)^\gamma G(i,\beta-1)h + \sum_{i<n}(1+ih)^\gamma |\Gamma^i|_\beta^2 \\ &\quad\le C\left\{ G(0,\beta) + \gamma\sum_{i<n}(1+ih)^{\gamma-1}G(i,\beta)h \right. \\ &\qquad\qquad \left. +\beta\sum_{i<n}(1+ih)^\gamma\left[\|\Delta\bar u^i\|^2 + \|\Delta L^i\|^2\right]h \right\}. \end{aligned} \tag{3.8}$$

Proof. The proof consists of three steps: grouping of terms, energy estimates, and concluding the proof.

Step 1. *Grouping of terms.* Multiplying (3.1) by $2L_j^nK_j^n$ and summing the result over $j \in \mathbb{Z}$ for $0 < n \le n_1$ gives

$$\begin{aligned}
&\sum_j 2L_j^nK_j^n(L_j^{n+1}-L_j^n) - \frac{\mu}{\kappa+1}\sum_j L_j^nK_j^n(L_{j+1}^n - 2L_j^n + L_{j-1}^n)\\
&\quad -\frac{\mu^2}{2(\kappa+1)}\sum_j L_j^nK_j^n(w_{j+1}^n - w_{j-1}^n) + \frac{2\kappa}{\kappa+1}\sum_j K_j^n(L_j^n)^2\\
&\quad +\frac{\kappa\lambda}{\kappa+1}\sum_j K_j^nL_j^n(\Lambda_j\bar{w}_{j-1}^{n+1} + \Lambda_{j+1}\bar{w}_j^{n+1}) = \sum_j \frac{2\kappa}{\kappa+1}L_j^nK_j^ne_j^{n+1}.
\end{aligned} \tag{3.9}$$

After a few summations by parts, we obtain

$$\begin{aligned}
&\sum_j\Big\{(L_j^{n+1})^2K_j^{n+1} - (L_j^n)^2K_j^n - (K_j^{n+1}-K_j^n)(L_j^{n+1})^2 - (L_j^{n+1}-L_j^n)^2K_j^n\\
&\quad + \frac{\mu}{\kappa+1}\left[K_j^n(L_{j+1}^n - L_j^n)^2 + (K_{j+1}^n - K_j^n)L_{j+1}^n(L_{j+1}^n - L_j^n)\right]\\
&\quad + \frac{\mu^2}{2(\kappa+1)}\left[(K_{j+1}^n - K_{j-1}^n)L_{j+1}^nw_j^n + K_{j-1}^nw_j^n(L_{j+1}^n - L_{j-1}^n)\right]\\
&\quad + \frac{\kappa}{\kappa+1}\Big[2K_j^n(L_j^n)^2 + A_j^n(L_j^n)^2 + \lambda\Lambda_{j+1}K_{j+1}^n(L_{j+1}^n + L_j^n)\\
&\quad \cdot\Big((1-\mu)\bar{w}_j^n + \frac{\mu}{2}(\bar{w}_{j+1}^n + \bar{w}_{j-1}^n)\Big) - \lambda(K_{j+1}^n - K_j^n)\Lambda_{j+1}L_j^n\bar{w}_j^{n+1}\Big]\Big\}\\
&= \frac{2\kappa}{\kappa+1}\sum_j K_j^nL_j^ne_j^{n+1},
\end{aligned} \tag{3.10}$$

where

$$A_j^n = \lambda(K_j^n\Lambda_j - K_{j+1}^n\Lambda_{j+1}) = A_jH^n.$$

In fact, two typical terms involved in leading to (3.10) can be given as follows. Using the indentity

$$(L_j^{n+1})^2 - (L_j^n)^2 = 2L_j^n(L_j^{n+1} - L_j^n) + (L_j^{n+1} - L_j^n)^2$$

in $2\sum_j K_j^nL_j^n(L_j^{n+1} - L_j^n)$, we arrive at

$$\begin{aligned}
&\sum_j K_j^n\left[(L_j^{n+1})^2 - (L_j^{n+1} - L_j^n)^2 - (L_j^n)^2\right]\\
&\quad = \sum_j (L_j^{n+1})^2K_j^{n+1} - \sum_j (L_j^n)^2K_j^n - \sum_j (L_j^{n+1} - L_j^n)^2K_j^n\\
&\qquad - \sum_j (L_j^{n+1})^2(K_j^{n+1} - K_j^n).
\end{aligned}$$

Using the relation

$$\bar{w}_j^{n+1} = L_{j+1}^n - L_j^n + \bar{w}_j^n + \frac{\mu}{2}(\bar{w}_{j+1}^n - 2\bar{w}_j^n + \bar{w}_{j-1}^n)$$

in the last term of the right side of (3.9), one obtains

$$
\begin{aligned}
&\frac{\kappa\lambda}{\kappa+1}\sum_j \Lambda_{j+1}(L^n_{j+1}K^n_{j+1}+L^n_jK^n_j)\bar{w}^{n+1}_j \\
&\quad = \frac{\kappa\lambda}{\kappa+1}\sum_j \Lambda_{j+1}\left[(L^n_{j+1}+L^n_j)K^n_{j+1}-L^n_j(K^n_{j+1}-K^n_j)\right] \\
&\quad\quad \cdot\left[L^n_{j+1}-L^n_j+\bar{w}^n_j+\frac{\mu}{2}(\bar{w}^n_{j+1}-2\bar{w}^n_j+\bar{w}^n_{j-1})\right] \\
&\quad = \frac{\kappa}{\kappa+1}\sum_j A^n_j(L^n_j)^2+\frac{\kappa\lambda}{\kappa+1}\sum_j \Lambda_{j+1}K^n_{j+1}(L^n_{j+1}+L^n_j) \\
&\quad\quad \cdot\left[\bar{w}^n_j+\frac{\mu}{2}(\bar{w}^n_{j+1}-2\bar{w}^n_j+\bar{w}^n_{j-1})\right]-\frac{\kappa\lambda}{\kappa+1}\sum_j \Lambda_{j+1}(K^n_{j+1}-K^n_j)L^n_j\bar{w}^{n+1}_j.
\end{aligned}
$$

To make (3.10) useful in constructing the weighted energy function $G(n,\beta)$, we have to combine it with some additional terms. To this end, we multiply (3.1) by $2K^n_j\bar{u}^{n+1}_j$ and sum the result over $j\in\mathbb{Z}$ for $0\le n\le n_1$, which gives, after a few computations (carried out in the Appendix),

(3.11)

$$
\begin{aligned}
&\sum_j\Big[2(K^{n+1}_jL^{n+1}_j\bar{u}^{n+1}_j-K^n_jL^n_j\bar{u}^n_j)-2(K^{n+1}_j-K^n_j)L^{n+1}_j\bar{u}^{n+1}_j \\
&\qquad\qquad -2K^n_j(\bar{u}^{n+1}_j-\bar{u}^n_j)L^n_j\Big] \\
&\quad+\frac{\mu}{\kappa+1}\sum_j\Big[K^n_j(L^n_{j+1}-L^n_j)^2+(K^n_{j+1}-K^n_j)(L^n_{j+1}-L^n_j)(L^n_{j+1}+\bar{u}^n_{j+1}) \\
&\qquad\qquad +K^n_j\bar{w}^n_j(L^n_{j+1}-L^n_j)\Big] \\
&\quad-\frac{\mu^2}{2(\kappa+1)}\sum_j(\bar{w}^n_j-\bar{w}^n_{j-1})(L^n_{j+1}-2L^n_j+L^n_j) \\
&\quad+\frac{\mu^2}{2(\kappa+1)}\sum_j\Big[K^n_{j-1}w^n_j(L^n_{j+1}-L^n_{j-1})+(K^n_{j+1}-K^n_{j-1})w^n_j(L^n_{j+1}+\bar{u}^n_{j+1}) \\
&\qquad\qquad +K^n_j(w^n_{j+1})^2-\frac{\mu}{2}K^n_j(\bar{w}^n_j-\bar{w}^n_{j-1})(w^n_{j+1}-w^n_{j-1})\Big] \\
&\quad+\frac{\kappa}{\kappa+1}\sum_j\Big[K^{n+1}_j(\bar{u}^{n+1}_j)^2-K^n_j(\bar{u}^n_j)^2-(K^{n+1}_j-K^n_j)(\bar{u}^{n+1}_j)^2 \\
&\qquad\qquad +K^n_j(\bar{u}^{n+1}_j-\bar{u}^n_j)^2\Big] \\
&\quad+\frac{\kappa\mu}{\kappa+1}\sum_j\Big[K^n_j(\bar{w}^n_j)^2-K^n_j(\bar{u}^{n+1}_j-\bar{u}^n_j)(\bar{w}^n_j-\bar{w}^n_{j-1})+(K^n_{j+1}-K^n_j)\bar{u}^n_{j+1}\bar{w}^n_j\Big] \\
&\quad+\frac{\kappa}{\kappa+1}\sum_j\Big[A^n_j(\bar{u}^{n+1}_j)^2-\lambda\Lambda_{j+1}(K^n_{j+1}-K^n_j)\bar{u}^{n+1}_j\bar{w}^{n+1}_j\Big] \\
&=\sum_j\frac{2\kappa}{\kappa+1}K^n_j\bar{u}^{n+1}_je^{n+1}_j
\end{aligned}
$$

with $A_j^n = (1+nh)^\gamma A_j$. We perform (3.10) + (3.11) $\times$ τ with a positive number τ (determined later on) and suitably group the terms in the result to obtain the following inequality:

$$H^{n+1}E(n+1) - H^n E(n) + L_1(n) + L_2(n) \leq \sum_{i=1}^{3} R_i(n). \tag{3.12}$$

The individual expressions are, using $K_j^n = (1+nh)^\gamma \langle jr\rangle^\beta = H^n P_j$,

$$\begin{aligned}
E(n) &:= \sum_j \big[(L_j^n)^2 + 2\tau \bar{u}_j^n L_j^n + \frac{\kappa\tau}{\kappa+1}(\bar{u}_j^n)^2\big] P_j,\\
L_1(n) &:= \frac{\kappa}{\kappa+1}(1+nh)^\gamma \sum_j A_j \Big[(L_j^n)^2 + \tau(\bar{u}_j^{n+1})^2\Big],\\
L_2(n) &: = \frac{\kappa}{\kappa+1}(1+nh)^\gamma \left[2|L^n|_\beta^2 + \tau|\bar{u}^{n+1} - \bar{u}^n|_\beta^2 + \mu\tau|\Delta\bar{u}^n|_\beta^2\right]\\
&\quad + \frac{\mu}{\kappa+1}(1+nh)^\gamma \left[(1+\tau)|\Delta L^n|_\beta^2 + \frac{\tau\mu}{2}\sum_j P_j(\bar{w}_{j+1}^n)^2\right],\\
R_1(n) &:= [H^{n+1} - H^n]E(n+1),\\
R_2(n) &:= H^n \sum_j P_j \Big[(L_j^{n+1} - L_j^n)^2 + 2\tau(\bar{u}_j^{n+1} - \bar{u}_j^n)L_j^n\\
&\qquad\qquad + \frac{\kappa\mu\tau}{\kappa+1}(\bar{u}_j^{n+1} - \bar{u}_j^n)(\bar{w}_j^n - \bar{w}_{j-1}^n) - \frac{\mu\tau}{\kappa+1}\bar{w}_j^n(L_{j+1}^n - L_j^n)\\
&\qquad\qquad + \frac{\kappa\mu}{\kappa+1}\frac{\Lambda_j}{\sqrt{a}}(L_j^n + L_{j-1}^n)\Big[(1-\mu)w_{j-1}^n + \frac{\mu}{2}(\bar{w}_j^n + \bar{w}_{j-2}^n)\Big]\\
&\qquad\qquad - \frac{(1+\tau)\mu^2}{2(\kappa+1)}w_{j+1}^n(L_{j+2}^n - L_j^n)\\
&\qquad\qquad + \frac{\tau\mu^2}{2(\kappa+1)}(\bar{w}_j^n - \bar{w}_{j-1}^n)(L_{j+1}^n - 2L_j^n + L_{j-1}^n)\\
&\qquad\qquad + \frac{\tau\mu^3}{4(\kappa+1)}(\bar{w}_j^n - \bar{w}_{j-1}^n)(w_{j+1}^n - w_{j-1}^n)\Big]\\
&\quad + \frac{2\kappa}{\kappa+1}H^n \sum_j P_j(L_j^n + \tau\bar{u}_j^{n+1})e_j^{n+1},\\
R_3(n) &:= \frac{\mu}{\kappa+1}H^n \sum_j |P_{j+1} - P_j| \Big\{|L_{j+1}^n - L_j^n|\Big[(1+\tau)|L_{j+1}^n| + \tau|\bar{u}_{j+1}^n|\Big]\\
&\quad + \frac{(1+\tau)}{2}|\mu(L_{j+1}^n w_j^n + L_{j+2}^n w_{j+1}^n)| + \frac{\tau\mu}{2}|(\bar{u}_{j+1}^n w_j^n + \bar{u}_{j+2}^n w_{j+1}^n)|\Big\}\\
&\quad + \frac{\kappa\mu}{\kappa+1}H^n \sum_j |P_{j+1} - P_j| \Big[\tau|\bar{u}_{j+1}^n \bar{w}_j^n| + |\frac{\Lambda_{j+1}}{\sqrt{a}}(L_j^n + \tau\bar{u}_j^{n+1})\bar{w}_j^{n+1}|\Big].
\end{aligned}$$

Step 2. *Energy estimates.* In order to get the desired estimate, one has to bound the above grouped terms $L_i(n)$, $i = 1, 2$, and $R_i(n)$, $i = 1, 2, 3$, respectively. This is done in Lemma 3.4, which will be proved in Section 4.

Lemma 3.4. *Assume that λ and $N(n_1)$ are suitably small. Then for any $0 < n \leq n_1$ and $\beta \in [0, \alpha]$, there exist positive constants C, c_1 and $0 < \sigma < 1$, independent of n, such that*

(i) $L_1(n) \geq c_1\beta(1+nh)^\gamma G(n, \beta-1)h$,

(ii) $R_1(n) \leq C\gamma(1+nh)^{\gamma-1}G(n,\beta)h$,

(iii) $R_2(n) < L_2(n)$,

(iv) $R_2(n) + R_3(n)$
$$\leq \sigma L_2(n) + (1+nh)^\gamma \left\{C\beta[\|\Delta\bar{u}^n\|^2 + \|\Delta L^n\|^2]h + \tfrac{c_1}{2}\beta G(n,\beta-1)h\right\}.$$

Based on these estimates, we continue the proof of Lemma 3.3.

Step 3. *Concluding the proof.* From (3.12) and Lemma 3.4, we find that

$$\begin{aligned} &H^{n+1}E(n+1) - H^n E(n) + \frac{c_1}{2}\beta H^n G(n,\beta-1)h + (1-\sigma)L_2(n) \\ &\quad \leq C\left[\beta(1+nh)^\gamma\Big(\|\Delta\bar{u}^n\|^2 + \|\Delta L^n\|^2\Big)h + \gamma h(1+nh)^{\gamma-1}G(n,\beta)\right]. \end{aligned} \tag{3.13}$$

Taking $\tau = \frac{\kappa}{4(\kappa+1)}$, we have
$$E(n) \sim G(n,\beta)$$
and
$$L_2(n) \geq \frac{\tau}{\kappa+1}\min\{\kappa,1\}(1+nh)^\gamma|\Gamma^n|_\beta^2,$$
with
$$|\Gamma^n|_\beta^2 = |L^n|_\beta^2 + \kappa|\bar{u}^{n+1} - \bar{u}^n|_\beta^2 + \mu|\Delta\bar{u}^n|_\beta^2 + \mu|\Delta L^n|_\beta^2.$$
Noting the above facts and summing (3.13) in n from 0 to $n-1$, we at once obtain (3.8). This proves the desired result in Lemma 3.3. □

Equipped with the basic estimate in Lemma 3.3, we are in a good position to proceed. First, taking $\beta = \gamma = 0$ in (3.8), we immediately get

Lemma 3.5. *There exists a positive constant C, independent of n_1, such that for $n \in [0, n_1]$*
$$G(n,0) + \sum_{i<n}\|\Gamma^i\|^2 \leq CG(0,0) \tag{3.14}$$
provided $N(n_1)$ and λ are suitably small.

Applying induction to (3.8) in β and γ, we have

Lemma 3.6. *Let $\gamma \in [0,\alpha]$ be an integer. Then, for any $n \leq n_1$,*

(3.15)
$$\begin{aligned} &(1+nh)^\gamma G(n,\alpha-\gamma) + (\alpha-\gamma)\sum_{i<n}(1+ih)^\gamma G(i,\alpha-\gamma-1)h + \sum_{i<n}(1+ih)^\gamma|\Gamma^i|_{\alpha-\gamma}^2 \\ &\quad \leq CG(0,\alpha). \end{aligned}$$

Consequently, if $\alpha = [\alpha]$, then for any $\gamma \leq \alpha$ we have
$$(1+nh)^\gamma G(n,0) + \sum_{i<n}(1+nh)^\gamma\|\Gamma^i\|^2 \leq CG(0,\alpha) \tag{3.16}$$
for any $n \leq n_1$.

Proof. Step 1. We take $0 \le \alpha < 1$. Letting $\beta = \alpha$ and $\gamma = 0$ in (3.8), we have

$$
\begin{aligned}
&G(n,\alpha) + \alpha \sum_{i<n} G(i,\alpha-1)h + \sum_{i<n} |\Gamma^i|_\alpha^2 \\
&\qquad \le C\Big[G(0,\alpha) + \alpha \sum_{i<n} [\|\Delta \bar{u}^i\|^2 + \|\Delta L^i\|^2]h\Big].
\end{aligned} \tag{3.17}
$$

The second term on the right hand side is bounded by $C\sum_{i<n} \|\Gamma^i\|^2$, which together with Lemma 3.5 leads to

$$G(n,\alpha) + \alpha \sum_{i<n} G(i,\alpha-1)h + \sum_{i<n} |\Gamma^i|_\alpha^2 \le C[G(0,\alpha) + G(0,0)].$$

Combining this with $G(0,0) \le G(0,\alpha)$ gives (3.15) with $\gamma = 0$. Therefore Lemma 3.6 is proved for $0 \le \alpha < 1$.

Step 2. We take $1 \le \alpha < 2$. First, letting $\beta = 0$, $\gamma = 1$ in (3.8), we have

$$(1+nh)G(n,0) + \sum_{i<n} (1+ih)|\Gamma^i|_0^2 \le C\{G(0,0) + \sum_{i<n} G(i,0)h\},$$

and by combining this with (3.15) ($\gamma = 0$) we obtain (3.16) with $\gamma = 1$, where we have used the inequality $\sum_{i<n} G(i,0)h \le \sum_{i<n} \|\Gamma^i\|^2 \le CG(0,0)$. Then, letting $\beta = \alpha - 1$ and $\gamma = 1$ in (3.8), we have

$$
\begin{aligned}
&(1+nh)G(n,\alpha-1) + \sum_{i<n} (1+ih)\Big[(\alpha-1)G(i,\alpha-1-1)h + |\Gamma^i|_{\alpha-1}^2\Big] \\
&\le C\Big\{G(0,\alpha-1) + \sum_{i<n} G(i,\alpha-1)h + (\alpha-1)\sum_{i<n} (1+ih)\Big[\|\Delta \bar{u}^i\|^2 + \|\Delta L^i\|^2\Big]\Big\},
\end{aligned}
$$

which, together with (3.16) with $\gamma = 1$ and (3.15) ($\gamma = 0$), yields (3.15) with $\gamma = 1$, where we have used the fact that $\|\Delta \bar{u}^i\|^2 + \|\Delta L^i\|^2$ is bounded by $C\|\Gamma^i\|^2$. Therefore the proof is completed for $\alpha < 2$.

Proceeding in this way, i.e., taking successively in (3.8) $\beta = \alpha - m$, $\gamma = m$, and then $\beta = 0$ $\gamma = m + 1$, for $m = 0, 1, \cdots, [\alpha] - 1$, we can get the desired estimate (3.15) for any $\alpha \ge 0$. This completes the proof of Lemma 3.6. □

From Lemma 3.6, if α is an integer, then

$$(1+nh)^\alpha G(n,0) + \sum_{i<n} (1+ih)^\alpha \|\Gamma^i\|^2 \le CG(0,\alpha), \tag{3.18}$$

which obviously implies (2.20).

We show a sharper estimate when α in not an integer. Taking $\beta = 0$ in (3.8) gives

$$(1+nh)^\gamma G(n,0) + \sum_{i<n} (1+ih)^\gamma \|\Gamma^i\|^2 \le C\{G(0,0) + \gamma \sum_{i<n} (1+ih)^{\gamma-1} G(i,0)h\}. \tag{3.19}$$

Taking also $\gamma = [\alpha]$ in (3.15), i.e.,

$$
\begin{aligned}
&(1+nh)^{[\alpha]} G(n,\alpha-[\alpha]) + (\alpha-[\alpha]) \sum_{i<n} (1+ih)^{[\alpha]} G(i,\alpha-[\alpha]-1)h \\
&\qquad\qquad + \sum_{i<n} (1+ih)^{[\alpha]} |\Gamma^i|_{\alpha-[\alpha]}^2 \le CG(0,\alpha),
\end{aligned} \tag{3.20}
$$

we get an estimate for the final term in (3.19).

Introducing the notation $|g^i|_\alpha^2 := G(i,\alpha)$ for simplicity of presentation, we have

$$\begin{aligned}
&\sum_{i<n}(1+ih)^{\gamma-1}G(i,0)h = \sum_{i<n}(1+ih)^{\gamma-1}|g^i|_0^2 h\\
&= \sum_{i<n}(1+ih)^{\gamma-1}\sum_{j\in\mathbb{Z}}\langle jr\rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha)-(\alpha-[\alpha])([\alpha]+1-\alpha)}(|g_j^i|^2)^{[\alpha]+1-\alpha+(\alpha-[\alpha])}h\\
&\le \sum_{i<n}(1+ih)^{\gamma-1}\left(\sum_{j\in\mathbb{Z}}\langle jr\rangle^{\alpha-[\alpha]}|g_j^i|^2\right)^{[\alpha]+1-\alpha}\left(\sum_{j\in\mathbb{Z}}\langle jr\rangle^{-([\alpha]+1-\alpha)}|g_j^i|^2\right)^{\alpha-[\alpha]}h\\
&= \sum_{i<n}(1+ih)^{-([\alpha]+1-\gamma)}\Big((1+ih)^{[\alpha]}G(i,\alpha-[\alpha])\Big)^{[\alpha]+1-\alpha}\\
&\quad\cdot\Big((1+ih)^{[\alpha]}G(i,\alpha-[\alpha]-1)\Big)^{\alpha-[\alpha]},
\end{aligned}$$

where we have used the Hölder inequality

$$\sum ab \le \left(\sum a^p\right)^{1/p}\left(\sum b^{p'}\right)^{1/p'},\quad \frac{1}{p}+\frac{1}{p'}=1.$$

Here $p=\frac{1}{[\alpha]+1-\alpha}$ and $p'=\frac{1}{\alpha-[\alpha]}$ and $G(i,\alpha-[\alpha])=|g^i|^2_{\alpha-[\alpha]}$.

Further, again using the Hölder inequality and (3.20), one obtains

(3.21)
$$\begin{aligned}
&\sum_{i<n}(1+ih)^{\gamma-1}G(i,0)h\\
&\quad\le CG(0,\alpha)^{[\alpha]+1-\alpha}\sum_{i<n}(1+ih)^{-([\alpha]+1-\gamma)}\Big((1+ih)^{[\alpha]}G(i,\alpha-[\alpha]-1)\Big)^{\alpha-[\alpha]}h\\
&\quad\le CG(0,\alpha)^{[\alpha]+1-\alpha}\left(\sum_{i<n}(1+ih)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}}h\right)^{[\alpha]+1-\alpha}\\
&\qquad\cdot\left(\sum_{i<n}(1+ih)^{[\alpha]}G(i,\alpha-[\alpha]-1)h\right)^{\alpha-[\alpha]}\\
&\quad\le CG(0,\alpha)\left(\sum_{i<n}(1+ih)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}}h\right)^{[\alpha]+1-\alpha}.
\end{aligned}$$

Now take $\gamma=\alpha+p$ for any $p>0$, as was done in [23] for scalar viscous conservation law. Then it is easy to find that

$$\sum_{i<n}(1+ih)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}}h \le C(1+nh)^{\frac{p}{[\alpha]+1-\alpha}}. \tag{3.22}$$

Applying (3.21), (3.22) into (3.19) leads to the following estimate.

Lemma 3.7. *For any $p>0$ and $n\le n_1$,*

$$(1+nh)^{\alpha+p}G(n,0)+\sum_{i<n}(1+ih)^{\alpha+p}\|\Gamma^i\|^2 \le CG(0,\alpha)(1+nh)^p. \tag{3.23}$$

Combining the above estimates yields the following uniform time decay estimate.

Proposition 3.8. *If λ and $N(n_1)$ are suitably small, then*

$$(1+nh)^{\alpha}N(n,0)+(1+nh)^{-p}\sum_{i<n}(1+ih)^{\alpha+p}\|\Gamma^i\|^2\le CN(0,\alpha) \tag{3.24}$$

for any $n\le n_1$ and $p>0$.

4. Energy estimate

Now we justify the estimates in Lemma 3.4. Hereafter we fix

$$\tau=\frac{\kappa}{4(\kappa+1)}$$

and denote by C a generic positive constant, which arises from using the Young inequality and Lemma 3.2 (i) in various circumstances, and also depends on a and $f'(u)$ for the u under consideration.

To prove (i), we claim that there exists a positive constant $\tilde{c}$ such that

$$|L^n|_\beta^2+|\bar{u}^{n+1}|_\beta^2\ge \tilde{c}G(n,\beta),\ \text{ for any } \beta\in[0,\alpha]. \tag{4.1}$$

This estimate together with the estimate on A_j in (3.3) yields

$$\begin{aligned} L_1(n)\ge\ & \tfrac{\kappa}{\kappa+1}\tau H^n\sum_j A_j(|L_j^n|^2+|\bar{u}_j^{n+1}|^2)\\ \ge\ & \tfrac{c_0\kappa\tau}{\kappa+1}\beta H^n(|L^n|_{\beta-1}^2+|\bar{u}^{n+1}|_{\beta-1}^2)h\\ \ge\ & \tfrac{\tilde{c}c_0\kappa\tau}{\kappa+1}\beta H^nG(n,\beta-1)h, \end{aligned}$$

which proves the desired estimate (i) with $c_1=\frac{\tilde{c}c_0\kappa\tau}{\kappa+1}$.

Finally, in order to conclude the proof of (i), we need to prove (4.1). Using the identity

$$\bar{u}_j^{n+1}=L_j^n+(1-\mu)\bar{u}_j^n+\frac{\mu}{2}(\bar{u}_{j+1}^n+\bar{u}_{j-1}^n), \tag{4.2}$$

and the Young inequality, one obtains

$$\begin{aligned} (L_j^n)^2+(\bar{u}_j^{n+1})^2 &\ge (2-\delta_1-\delta_2)(L_j^n)^2+(1-\mu)^2(1-\tfrac{1}{\delta_1}-\tfrac{1}{\delta_3})(\bar{u}_j^n)^2\\ &\quad -\tfrac{\mu^2}{4}(\delta_3+\tfrac{1}{\delta_2})(\bar{u}_{j+1}^n+\bar{u}_{j+1}^n)^2\\ &\ge \tfrac14(L_j^n)^2+\tfrac16(1-\mu)^2(\bar{u}_j^n)^2-5\mu^2[(\bar{u}_{j+1}^n)^2+(\bar{u}_{j+1}^n)^2], \end{aligned}$$

where we have chosen $\delta_1=3/2$, $\delta_2=1/4$ and $\delta_3=6$ for definiteness.

Using the smallness of λ and Lemma 3.2, we have

$$|L^n|_\beta^2+|\bar{u}^{n+1}|_\beta^2\ge\tilde{c}G(n,\beta)$$

with $\tilde{c}<\frac16(1-\mu)^2$. This concludes the proof of (4.1).

Concerning (ii), the factor $\gamma(1+nh)^{\gamma-1}h$ comes from the estimate of $H^{n+1}-H^n$ in Lemma 3.2 (ii). It remains to show that

$$E(n+1)\le CG(n,\beta),\quad \beta\in[0,\alpha]. \tag{4.3}$$

Using Lemma 3.2 (i) and (4.2), one gets

$$|\bar{u}^{n+1}|_\beta^2\le C\Big[|L^n|_\beta^2+|\bar{u}^n|_\beta^2\Big]. \tag{4.4}$$

By (2.15) one may write

$$L_j^{n+1} = \frac{(1-\mu)}{\kappa+1} L_j^n + \frac{\mu}{2(\kappa+1)}(L_{j+1}^n + L_{j-1}^n) + \frac{\mu^2}{4(\kappa+1)}(w_{j+1}^n - w_{j-1}^n) - \frac{\kappa\mu}{2(\kappa+1)}\left[\frac{\Lambda_j}{\sqrt{a}}\bar{w}_{j-1}^{n+1} + \frac{\Lambda_{j+1}}{\sqrt{a}}\bar{w}_j^{n+1}\right] + \frac{\kappa}{\kappa+1}e_j^{n+1}. \tag{4.5}$$

Note that (2.16) with (2.9) and (3.5) yields

$$|e_j^{n+1}| \le C\sqrt{N(n+1)}\mu\left[|\bar{w}_j^{n+1}| + |\bar{w}_{j-1}^{n+1}|\right]. \tag{4.6}$$

Thus, using $|\Delta \bar{u}^{n+1}|_\beta^2 \le C|\bar{u}^{n+1}|_\beta^2$ and (4.4), we have

$$\begin{aligned} |L^{n+1}|_\beta^2 &= \sum_j \langle jr\rangle^\beta |L_j^{n+1}|^2 \\ &\le C\left[|L^n|_\beta^2 + |\Delta\bar{u}^{n+1}|_\beta^2 + |\Delta\bar{u}^n|_\beta^2\right] \\ &\le C\Big[|L^n|_\beta^2 + |\bar{u}^n|_\beta^2\Big]. \end{aligned} \tag{4.7}$$

Combining (4.4), (4.7) and the expression for $E(n+1)$ gives (4.3), which proves (ii).

Next we estimate $R_2(n) := \sum_{i=1}^7 R_{2i}(n)\cdot H^n$.

We will frequently use the Young inequality and the definitions of w_j^n and $\bar{w}_j^n$ in (2.14). Thus, using (4.5), (4.6) and Lemma 3.2 (i), one gets

$$\begin{aligned} R_{21} &= |L^{n+1} - L^n|_\beta^2 \\ &= \sum_j P_j\Bigg[\frac{\mu^2}{4(\kappa+1)}(w_{j+1}^n - w_{j-1}^n) + \frac{\kappa\mu}{2(\kappa+1)}(L_{j+1}^n - 2L_j^n + L_{j-1}^n) \\ &\qquad - \frac{\kappa}{\kappa+1}L_j^n - \frac{\kappa\lambda\Lambda_j}{2(\kappa+1)}\bar{w}_{j-1}^{n+1} - \frac{\kappa\lambda\Lambda_{j+1}}{2(\kappa+1)}\bar{w}_j^{n+1} + \frac{\kappa}{\kappa+1}e_j^{n+1}\Bigg]^2 \\ &\le \frac{5\kappa}{4(\kappa+1)}\sum_j P_j|L_j^n|^2 \\ &\quad + \frac{C\kappa}{\kappa+1}\left[\mu^2\sum_j P_j(L_{j+1}^n - L_j^n)^2 + \mu^2\sum_j P_j(\bar{w}_j^{n+1})^2 + \mu^2\sum_j P_j(\bar{w}_j^{n+1})^4\right] \\ &\quad + \frac{\mu^4}{(\kappa+1)^2}\sum_j P_j(\bar{w}_j^n)^2. \end{aligned}$$

Noting that $\sup_j |\bar{u}_j^n|^2 \le N(n)$, $\sup_j |\bar{w}_j^n|^2 \le 2N(n)$, and

$$\bar{w}_j^{n+1} = L_{j+1}^n - L_j^n + \bar{w}_j^n + \frac{\mu}{2}(\bar{w}_{j+1}^n - 2\bar{w}_j^n + \bar{w}_{j-1}^n),$$

we may find a constant C such that

$$R_{21} \le \frac{5\kappa}{4(\kappa+1)}|L^n|_\beta^2 + \frac{C\kappa}{\kappa+1}\mu^2\left[|\Delta L^n|_\beta^2 + |\Delta\bar{u}^n|_\beta^2\right] + \frac{C\mu^4}{\kappa+1}\sum_j P_j(\bar{w}_{j+1}^n)^2. \tag{4.8}$$

Here and after in the simplifying process the terms in higher orders of μ (<1) are, if necessary, absorbed into the terms in lower orders.

Using

$$\bar{u}_j^{n+1} - \bar{u}_j^n = L_j^n + \frac{\mu}{2}(\bar{w}_j^n - \bar{w}_{j-1}^n),$$

one bounds R_{22}:

$$\begin{aligned}
|R_{22}| &= \left| 2\tau \sum_j P_j L_j^n (\bar{u}_j^{n+1} - \bar{u}_j^n) \right| \\
&= \left| 2\tau \sum_j P_j (L_j^n)^2 + \tau\mu \sum_j P_j L_j^n (\bar{w}_j^n - \bar{w}_{j-1}^n) \right| \\
&\leq \tau(2 + C\mu)|L^n|_\beta^2 + \frac{\kappa\mu\tau}{4(\kappa+1)}|\Delta\bar{u}^n|_\beta^2.
\end{aligned}$$

Using the Young inequality, we estimate R_{23} as follows:

$$\begin{aligned}
|R_{23}| &\leq \left| \frac{\kappa\mu\tau}{\kappa+1} \sum_j P_j (\bar{w}_j^n - \bar{w}_{j-1}^n)(\bar{u}_j^{n+1} - \bar{u}_j^n) \right| + \left| -\frac{\mu\tau}{\kappa+1} \sum_j P_j \bar{w}_j^n (L_{j+1}^n - L_j^n) \right| \\
&\leq \frac{\kappa}{\kappa+1} \left[\frac{\tau}{2}|\bar{u}^{n+1} - \bar{u}^n|_\beta^2 + C\tau\mu^2|\Delta\bar{u}^n|_\beta^2 \right] + \frac{\mu}{\kappa+1} \left[\frac{1}{2}|\Delta L^n|_\beta^2 + \frac{\tau^2}{2}|\Delta\bar{u}^n|_\beta^2 \right].
\end{aligned}$$

The subcharacteristic condition (1.5) implies $|\Lambda_j/\sqrt{a}| \leq 1$, so

$$\begin{aligned}
|R_{24}| &= \left| \frac{\kappa\mu}{\kappa+1} \sum_j P_j \frac{\Lambda_j}{\sqrt{a}} (L_j^n + L_{j-1}^n) \left[(1-\mu)\bar{w}_{j-1}^n + \mu \frac{\bar{w}_j^n + \bar{w}_{j-2}^n}{2} \right] \right| \\
&\leq \frac{\kappa}{\kappa+1} \left[\mu \sum_j P_j |L_j^n + L_{j-1}^n| \left| (1-\mu)\bar{w}_{j-1}^n + \frac{\mu}{2}(\bar{w}_j^n + \bar{w}_{j-2}^n) \right| \right] \\
&\leq \frac{C\kappa}{\kappa+1} \left[\mu^{1/2}|L^n|_\beta^2 + \mu^{3/2}|\Delta\bar{u}^n|_\beta^2 \right],
\end{aligned}$$

where C may depend on θ, and we use Lemma 3.2 (i) for combining terms of the same weighted order.

Similarly, we have

$$\begin{aligned}
|R_{25}| &= \frac{\mu^2}{2(\kappa+1)} \left| \sum_j P_j \left[-(1+\tau) w_{j+1}^n (L_{j+2}^n - L_j^n) \right. \right. \\
&\qquad \left. \left. + \tau|(\bar{w}_j^n - \bar{w}_{j-1}^n)(L_{j+1}^n - 2L_j^n + L_{j-1}^n) \right] \right| \\
&\leq \frac{C\mu}{\kappa+1} \left(\mu^{1/2}|\Delta L^n|_\beta^2 + \mu^{3/2} \sum_j P_j (w_{j+1}^n)^2 \right)
\end{aligned}$$

and

$$|R_{26}| = \frac{\tau\mu^3}{4(\kappa+1)} \sum_j P_j |\bar{w}_j^n - \bar{w}_{j-1}^n||w_{j+1}^n - w_{j-1}^n| \leq \frac{C\tau\mu^3}{\kappa+1} \sum_j P_j (\bar{w}_{j+1}^n)^2.$$

Finally, using (2.16) and $\sup_j |\bar{u}_j^{n+1}|^2 \le N(n+1)$, $\sup_j |\bar{L}_j^n|^2 \le N(n)$ one obtains

$$\begin{aligned}
|R_{27}| &= \left| 2\frac{\kappa}{\kappa+1} \sum_j (L_j^n + \tau \bar{u}_j^{n+1}) e_j^{n+1} \right| \\
&\le \frac{C\kappa\mu}{\kappa+1} \sum_j P_j(|L_j^n| + \tau|\bar{u}_j^{n+1}|) \left(|\bar{w}_j^{n+1}|^2 + |\bar{w}_{j-1}^{n+1}|^2\right) \\
&\le \frac{C\kappa\mu}{\kappa+1} \left[(\sqrt{N(n)} + \sqrt{N(n+1)}) |\bar{w}^{n+1}|_\beta^2 \right] \\
&\le \frac{C\kappa\mu}{\kappa+1} \sqrt{N(n+1)} \left[|\Delta L^n|_\beta^2 + |\Delta \bar{u}^n|_\beta^2 \right].
\end{aligned}$$

Combining the above estimates, we derive that, for any $n < n_1$,

$$\begin{aligned}
|R_2(n)| \le \frac{\kappa}{\kappa+1} H^n &\left[\left(\frac{5}{4} + \frac{2 + C\mu}{4} + C\mu^{1/2} \right) |L^n|_\beta^2 + \frac{\tau}{2} |\bar{u}^{n+1} - \bar{u}^n|_\beta^2 \right] \\
&+ \frac{\kappa}{\kappa+1} H^n \left(c\mu^2 + \frac{\mu\tau}{4} + c\tau\mu^2 + \frac{\tau\mu}{8(\kappa+1)} + C\mu\sqrt{N(n+1)} \right) |\Delta \bar{u}^n|_\beta^2 \\
&+ \frac{\mu}{\kappa+1} \left[\frac{1}{2} + C\kappa\mu + C\mu^{1/2} + C\kappa\sqrt{N(n+1)} \right] |\Delta L^n|_\beta^2 \\
&+ \frac{\mu}{\kappa+1} H^n \left(C\mu^3 + C\mu^{3/2} + C\tau\mu^2 \right) \sum_j P_j |\bar{w}_{j+1}^n|^2,
\end{aligned}$$

which, for a suitably small μ, allows us to get the desired estimate (iii). Moreover, we may find a suitable constant $\sigma \in]0,1[$ such that

$$R_2(n) + \frac{2\kappa}{5(\kappa+1)} \mu^2 \tau |\Delta \bar{u}^n|_\beta^2 + \frac{2\kappa}{5(\kappa+1)} \mu^2 (1+\tau) |\Delta L^n|_\beta^2 \le \sigma L_2(n). \tag{4.9}$$

Now we turn to (iv). Here we have to carefully separate suitable terms which will be absorbed by $L_1(n)$ or $L_2(n)$. Set

$$R_3(n) := H^n \cdot \sum_{i=1}^{3} R_{3i}(n).$$

Using the estimates in Lemma 3.2(i) and taking

$$\bar{\epsilon} = \frac{(\kappa+1)c_1}{6\sqrt{a}C_r} \min_{j\in\mathbb{Z}} \left\{ \frac{\langle (j+1)r \rangle^{\beta-1}}{\langle jr \rangle^{\beta-1}} \right\}, \quad \beta \in [0,\alpha],$$

we get

$$\begin{aligned}
R_{31}(n) &= \frac{\mu}{\kappa+1} \sum_j |P_{j+1} - P_j| \left[|L_{j+1}^n - L_j^n| \left((1+\tau)|L_{j+1}^n| + \tau|\bar{u}_{j+1}^n| \right) \right] \\
&\le \frac{\mu}{\kappa+1} C_r \beta r \sum_j \langle jr \rangle^{\beta-1} \left[\bar{\epsilon}\left(|L_{j+1}^n|^2 + |\bar{u}_{j+1}^n|^2 \right) + \frac{(1+\tau)^2 + \tau^2}{4\bar{\epsilon}} |L_{j+1}^n - L_j^n|^2 \right] \\
&\le \frac{1}{6} c_1 \beta h \left(|L^n|_{\beta-1}^2 + |\bar{u}^n|_{\beta-1}^2 \right) + C_{\bar{\epsilon}} \beta h \sum_j \langle jr \rangle^{\beta-1} |L_{j+1}^n - L_j^n|^2,
\end{aligned}$$

Furthermore,

$$
\begin{aligned}
C_{\bar\epsilon}\beta h\sum_j\langle jr\rangle^{\beta-1}|L^n_{j+1}-L^n_j|^2
&= \sum_{|j|\ge J_1}\frac{C_{\bar\epsilon}\beta h}{\langle jr\rangle}\langle jr\rangle^{\beta}|L^n_{j+1}-L^n_j|^2+C_{\bar\epsilon}\beta h\sum_{|j|\le J_1}\langle jr\rangle^{\beta-1}|L^n_{j+1}-L^n_j|^2\\
&\le \frac{\kappa\mu^2(1+\tau)}{5(\kappa+1)}|\Delta L^n|^2_\beta+\beta C_{J_1}\|\Delta L^n\|^2h
\end{aligned}
$$

for some large $J_1>0$.

With a similar argument we arrive at

$$
\begin{aligned}
R_{32}(n) &= \frac{\mu^2(1+\tau)}{2(\kappa+1)}\sum_j|P_{j+1}-P_j||(L^n_{j+1}w^n_j+L^n_{j+2}w^n_{j+1})|\\
&\quad+\frac{\tau\kappa\mu}{\kappa+1}\sum_j|P_{j+1}-P_j\|\bar u^n_{j+1}\bar w^n_j|\\
&\quad+\frac{\mu^2\tau}{2(\kappa+1)}\sum_j|P_{j+1}-P_j||(\bar u^n_{j+1}w^n_j+\bar u^n_{j+2}w^n_{j+1})|\\
&\le \frac{1}{6}c_1\beta h\big(|L^n|^2_{\beta-1}+|\bar u^n|^2_{\beta-1}\big)+\frac{\kappa\mu\tau}{5(\kappa+1)}\mu|\Delta\bar u^n|^2_\beta+\beta C_{J_2}\|\Delta\bar u^n\|^2h
\end{aligned}
$$

for some $J_2>0$.

Finally we treat the term $R_{33}(n)$. In fact, if we use the two indentities

$$\bar u^{n+1}_j=L^n_j+\bar u^n_j+\frac{\mu}{2}(\bar w^n_j-\bar w^n_{j-1})$$

and

$$\bar w^{n+1}_j=L^n_{j+1}-L^n_j+\bar w^n_j+\frac{\mu}{2}(\bar w^n_{j+1}-2\bar w^n_j+\bar w^n_{j-1}),$$

$R_{33}(n)$ can be estimated by using the estimates of $R_{31}(n)$ and $R_{32}(n)$ given above. Thus we have

$$
\begin{aligned}
R_{33}(n) &\le \frac{\kappa\mu}{\kappa+1}\sum_j|P_{j+1}-P_j|\big(|L^n_j+\tau\bar u^{n+1}_j\|\bar w^{n+1}_j|\big)\\
&\le \frac{1}{6}c_1\beta h\big(|L^n|^2_{\beta-1}+|\bar u^n|^2_{\beta-1}\big)+\frac{\kappa\mu\tau}{5(\kappa+1)}\mu|\Delta\bar u^n|^2_\beta\\
&\quad+\frac{\kappa\mu^2(1+\tau)}{5(\kappa+1)}|\Delta L^n|^2_\beta+\beta C_{J_3}\big(\|\Delta\bar u^n\|^2+\|\Delta L^n\|^2\big)h
\end{aligned}
$$

with some large $J_3>0$. Taking $J=\max\{J_1,J_2,J_3\}$ and $C_J=(C_{J_1}+C_{J_2}+C_{J_3})$, we have

$$
\begin{aligned}
R_3(n) &\le \frac{1}{2}c_1\beta h\big(|L^n|^2_{\beta-1}+|\bar u^n|^2_{\beta-1}\big)+\frac{2\kappa}{5(\kappa+1)}\mu^2\tau|\Delta\bar u^n|^2_\beta\\
&\quad+\frac{2\kappa}{5(\kappa+1)}\mu^2(1+\tau)|\Delta L^n|^2_\beta+\beta C_J\big(\|\Delta\bar u^n\|^2+\|\Delta L^n\|^2\big)h.
\end{aligned}
\tag{4.10}
$$

Combining (4.9), (4.10) and the expression for $L_2(n)$, we immediately obtain (iv). This completes the proof of Lemma 3.4.

5. Convergence rates

Now we are in a position to prove our main results.

Proof of Theorem 2.4. In fact, if the initial weighted norm $|\bar{u}_j^0|_\alpha + |L_j^0|_\alpha$ is small enough, i.e., $N(0,\alpha)$ is small enough, then (3.24) is true a little longer and hence forever. This is a form of continuous induction which is given fully in many places (we omit it here).

The smallness of $N(n,0)$ used in proving Lemma 3.4 can be ensured by (3.24). Noting the definition of $G(n,\alpha)$, we now immediately get Theorem 2.4. $\square$

Proof of Theorem 2.3. It remains to establish the convergence rates in Theorem 2.3. From (2.20) we have

$$\sum_{i<n}(1+ih)^{\alpha+p}\|\Gamma^i\|^2 \le C(1+nh)^p N(0,\alpha), \quad \text{for any } \ n\in\mathbb{N}.$$

This implies that

$$\sup_{i\le n}\|\Gamma^i\| \le C(1+nh)^{-\frac{\alpha}{2}}\sqrt{N(0,\alpha)}, \tag{5.1}$$

where

$$\|\Gamma^i\|^2 = \|L^i\|^2 + \kappa\|\bar{u}^{i+1}-\bar{u}^i\|^2 + \mu\|\Delta\bar{u}^i\|^2 + \mu\|\Delta L^i\|^2.$$

Note that by (2.11)

$$|u_j^n - U_j| = |\bar{u}_j^n - \bar{u}_{j-1}^n| \le \|\Delta\bar{u}^n\|,$$

which together with (5.1) yields

$$\sup_j |u_j^n - U_j| \le C\|\Gamma^n\| \le C(1+nh)^{-\frac{\alpha}{2}}\Big[|\bar{u}_j^0|_\alpha + |L_j^0|_\alpha\Big]. \tag{5.2}$$

Now it remains to estimate the maximum norm of $\bar{v}_j^n = v_j^n - V_j$. It follows from the second equation of (2.8) and (2.9)-(2.12) that

$$\begin{aligned}
&(1+\kappa)\bar{v}_j^{n+1}\\
&\quad= (1-2\mu)\bar{v}_j^n + \frac{\mu}{2}(\bar{v}_{j+1}^n + 2\bar{v}_j^n + \bar{v}_{j-1}^n) - \frac{a\lambda}{2}(\bar{w}_j^n - \bar{w}_{j-2}^n) + \kappa\Lambda_j\bar{w}_{j-1}^{n+1} + \kappa\theta_j^{n+1}\\
&\quad= (1-2\mu)\bar{v}_j^n - \sqrt{a}(L_j^n + L_{j-1}^n) - \frac{a\lambda}{2}(\bar{w}_j^n - \bar{w}_{j-2}^n) + \kappa\Lambda_j\bar{w}_{j-1}^{n+1} + C\kappa(\bar{w}_{j-1}^{n+1})^2,
\end{aligned}$$

that is,

$$\begin{aligned}
\bar{v}_j^{n+1} = {}& \frac{1-2\mu}{1+\kappa}\bar{v}_j^n - \frac{\sqrt{a}}{1+\kappa}(L_j^n + L_{j-1}^n) - \frac{a\lambda}{2(\kappa+1)}(\bar{w}_j^n - \bar{w}_{j-2}^n)\\
&+ \frac{\kappa\Lambda_j}{\kappa+1}\bar{w}_{j-1}^{n+1} + \frac{C\kappa}{\kappa+1}(\bar{w}_{j-1}^{n+1})^2,
\end{aligned}$$

where $0 < \frac{1-2\mu}{1+\kappa} < 1$ for $0<\mu<\frac{1}{2}$. Using the Young inequality, we have

$$\begin{aligned}
\sum_j(\bar{v}_j^{n+1})^2 &\le \nu\sum_j(\bar{v}_j^n)^2 + C\sum_j\left[(L_j^n)^2 + (\bar{w}_j^n)^2 + |\bar{u}_j^{n+1} - \bar{u}_j^n|^2\right]\\
&\le \nu\|\bar{v}^n\|^2 + C\|\Gamma^n\|^2,
\end{aligned} \tag{5.3}$$

where

$$\nu = (\frac{1-2\mu}{1+\kappa})^2 + (\frac{1-2\mu}{1+\kappa})^M < 1$$

for M suitably large, and C is a positive constant of order $(\frac{1-2\mu}{1+\kappa})^{-M}$ and depends on $N(0,\alpha)$.

Define

$$T(n) = (1+nh)^{\alpha+p}\|\bar{v}^n\|^2.$$

It follows from (5.3) that

$$T(n+1) \le \nu_1 T(n) + C(1+nh)^{\alpha+p}\|\Gamma^n\|^2, \tag{5.4}$$

where

$$\nu_1 = \sup_{n>0} \nu \left(\frac{1+(n+1)h}{1+nh}\right)^{\alpha+p} < 1$$

for suitably small h, and $\mu < 1$.

Summing both sides of the inequality (5.4) over n from 0 to $n-1$ and using the estimate (2.20) yields

$$\begin{aligned} T(n) &\le T(0) - (1-\nu_1)\sum_{i<n} T(i) + C\sum_{i<n}(1+ih)^{\alpha+p}\|\Gamma^i\|^2 \\ &\le \|\bar{v}^0\|^2 + C(1+nh)^p N(0,\alpha). \end{aligned}$$

Noting that $\|\bar{v}^0\|^2 \le |\bar{v}^0|_\alpha^2$ and $N(0,\alpha) \le C[|\bar{u}^0|_\alpha^2 + |\bar{v}^0|_\alpha^2]$, one has

$$(1+nh)^{\alpha+p}\|\bar{v}^n\|^2 \le C(1+nh)^p[|\bar{u}^0|_\alpha + |\bar{v}^0|_\alpha]^2.$$

This estimate implies that

$$\sup_j |\bar{v}_j^n| \le \|\bar{v}^n\| \le C(1+nh)^{-\frac{\alpha}{2}}[|\bar{u}^0|_\alpha + |\bar{v}^0|_\alpha]. \tag{5.5}$$

Combining (5.2) and (5.5) completes the proof of Theorem 2.3. □

Appendix

We want here to compute the terms in $\sum_j 2K_j^n \bar{u}_j^{n+1}\mathcal{L}(\bar{u}_j^n)$ to obtain (3.11). By the definition of $\mathcal{L}$,

(A.1)

$$\begin{aligned} &\sum_j 2K_j^n \bar{u}_j^{n+1}\mathcal{L}(\bar{u}_j^n) = \sum_{i=1}^{6} J_i \\ &\quad = \sum_j 2K_j^n \bar{u}_j^{n+1}(L_j^{n+1} - L_j^n) - \frac{\mu}{\kappa+1}\sum_j K_j^n \bar{u}_j^{n+1}(L_{j+1}^n - 2L_j^n + L_{j-1}^n) \\ &\quad - \frac{\mu^2}{2(\kappa+1)}\sum_j K_j^n \bar{u}_j^{n+1}(w_{j+1}^n - w_{j-1}^n) + \sum_j 2\frac{\kappa}{\kappa+1}\bar{u}_j^{n+1}K_j^n(\bar{u}_j^{n+1} - \bar{u}_j^n) \\ &\quad - \mu\sum_j \frac{\kappa}{\kappa+1}\bar{u}_j^{n+1}K_j^n(\bar{w}_j^n - \bar{w}_{j-1}^n) + \lambda\sum_j \frac{\kappa}{\kappa+1}\bar{u}_j^{n+1}K_j^n(\Lambda_j\bar{w}_{j-1}^{n+1} + \Lambda_{j+1}\bar{w}_j^{n+1}). \end{aligned}$$

We now estimate each term J_i for $i = 1, \cdots, 6$ in (A.1). We rewrite the first term as

$$\begin{aligned} J_1 = &\sum_j 2(K_j^{n+1}\bar{u}_j^{n+1}L_j^{n+1} - K_j^n\bar{u}_j^nL_j^n) \\ &- 2\sum_j (K_j^{n+1} - K_j^n)\bar{u}_j^{n+1}L_j^{n+1} - 2\sum_j K_j^n(\bar{u}_j^{n+1} - \bar{u}_j^n)L_j^n. \end{aligned}$$

Using the identity $\bar{u}_j^{n+1} = L_j^n + \bar{u}_j^n + \frac{\mu}{2}(\bar{w}_j^n - \bar{w}_{j-1}^n)$ for the second term, we get

$$\begin{aligned} J_2 &= -\frac{\mu}{\kappa+1}\sum_j K_j^n\left[L_j^n + \bar{u}_j^n + \frac{\mu}{2}(\bar{w}_j^n - \bar{w}_{j-1}^n)\right](L_{j+1}^n - 2L_j^n + L_{j-1}^n) \\ &= \frac{\mu}{\kappa+1}\sum_j \left[K_j^n(L_{j+1}^n - L_j^n)^2 + (K_{j+1}^n - K_j^n)(L_{j+1}^n - L_j^n)L_{j+1}^n\right] \\ &\quad + \frac{\mu}{\kappa+1}\sum_j (K_{j+1}^n - K_j^n)\bar{u}_{j+1}^n(L_{j+1}^n - L_j^n) \\ &\quad + \frac{\mu}{\kappa+1}\sum_j K_j^n(\bar{u}_{j+1}^n - \bar{u}_j^n)(L_{j+1}^n - L_j^n) \\ &\quad - \frac{\mu^2}{2(\kappa+1)}\sum_j K_j^n(\bar{w}_j^n - \bar{w}_{j-1}^n)(L_{j+1}^n - 2L_j^n + L_{j-1}^n). \end{aligned}$$

Similarly, for the third term we have

$$\begin{aligned} J_3 &= -\frac{\mu^2}{2(\kappa+1)}\sum_j K_j^n\left[L_j^n + \bar{u}_j^n + \frac{\mu}{2}(\bar{w}_j^n - \bar{w}_{j-1}^n)\right](w_{j+1}^n - w_{j-1}^n) \\ &= \frac{\mu^2}{2(\kappa+1)}\sum_j \left[K_{j-1}^n w_j^n(L_{j+1}^n - L_{j-1}^n) + (K_{j+1}^n - K_{j-1}^n)w_j^nL_{j+1}^n\right] \\ &\quad + \frac{\mu^2}{2(\kappa+1)}\sum_j \left[(K_{j+1}^n - K_{j-1}^n)\bar{u}_{j+1}^n w_j^n + K_j^n(w_{j+1}^n)^2\right] \\ &\quad - \frac{\mu^3}{4(\kappa+1)}\sum_j (\bar{w}_j^n - \bar{w}_{j-1}^n)(w_{j+1}^n - w_{j-1}^n)K_j^n \end{aligned}$$

and the fourth is rewritten as

$$\begin{aligned} J_4 = \frac{\kappa}{\kappa+1}\sum_j [&K_j^{n+1}(\bar{u}_j^{n+1})^2 - K_j^n(\bar{u}_j^n)^2 \\ &- (K_j^{n+1} - K_j^n)(\bar{u}_j^{n+1})^2 + K_j^n(\bar{u}_j^{n+1} - \bar{u}_j^n)^2]. \end{aligned}$$

Using summation by parts for the last two terms, we obtain

$$\begin{aligned}
J_5 =& -\frac{\kappa\mu}{\kappa+1}\sum_j \bar{u}_j^{n+1}K_j^n(\bar{w}_j^n - \bar{w}_{j-1}^n)\\
=& -\frac{\kappa\mu}{\kappa+1}\sum_j (\bar{u}_j^{n+1} - u_j^n)K_j^n(\bar{w}_j^n - \bar{w}_{j-1}^n) + \frac{\kappa\mu}{\kappa+1}\sum_j (K_{j+1}^n\bar{u}_{j+1}^n - K_j^n\bar{u}_j^n)\bar{w}_j^n\\
=& -\frac{\kappa\mu}{\kappa+1}\sum_j K_j^n(\bar{u}_j^{n+1} - u_j^n)(\bar{w}_j^n - \bar{w}_{j-1}^n) + \frac{\kappa\mu}{\kappa+1}\sum_j K_j^n(\bar{w}_j^n)^2\\
&+ \frac{\kappa\mu}{\kappa+1}\sum_j (K_{j+1}^n - K_j^n)\bar{u}_{j+1}^n\bar{w}_j^n
\end{aligned}$$

and

$$\begin{aligned}
J_6 &= \frac{\kappa\lambda}{\kappa+1}\sum_j K_j^n\bar{u}_j^{n+1}(\Lambda_j\bar{w}_{j-1}^{n+1} + \Lambda_{j+1}\bar{w}_j^{n+1})\\
&= \frac{\kappa\lambda}{\kappa+1}\sum_j \Lambda_{j+1}(K_{j+1}^n\bar{u}_{j+1}^{n+1} + K_j^n\bar{u}_j^{n+1})\bar{w}_j^{n+1}\\
&= \frac{\kappa}{\kappa+1}\sum_j A_j^n(\bar{u}_j^{n+1})^2 - \frac{\kappa\lambda}{\kappa+1}\sum_j \Lambda_{j+1}(K_{j+1}^n - K_j^n)\bar{u}_j^{n+1}\bar{w}_j^{n+1}
\end{aligned}$$

with

$$A_j^n = \lambda(\Lambda_j K_j^n - \Lambda_{j+1}K_j^n) = (1+nh)^\gamma A_j.$$

Inserting the new terms for J_i with $i = 1, \cdots, 6$ into (A.1), we immediately get (3.11).

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References

1. D. Aregba-Driollet and R. Natalini, *Convergence of relaxation schemes for conservation laws*, Appl. Anal. **61** (1996), 163-190. CMP 98:13
2. G.-Q. Chen, C.D. Levermore and T.P. Liu, *Hyperbolic conservation laws with stiff relaxation terms and entropy*, Comm. Pure Appl. Math. **47** (1993), 787-830. MR **95h:**35133
3. B. Engquist and Shih-Hsien Yu, *Convergence of finite difference schemes for piecewise smooth solutions with shocks*, preprint (1996).
4. H. Fan, *Existence and Uniqueness of travelling waves and error estimates for Godunov schemes of Conservation Laws*, Math. Comp. **67** (1998), 87-109. MR **98h:**65040
5. G. H. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, London, New York, 1934. MR **13:**727e (2nd ed.)
6. G. Jennings, *Discrete shocks*, Comm. Pure Appl. Math. **27** (1974), 25-37. MR **49:**3358
7. S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math. **48** (1995), 235–276. MR **96c:**65134
8. S. Kawashima and A. Matsumura, *Asymptotic stability of travelling wave solutions of system for one-dimensional gas motion*, Comm. Math. Phys. **101**(1985), 97-127. MR **87h:**35035
9. J.G. Liu and Z. Xin, *Nonlinear stability of discrete shocks for systems of conservation laws*, Arch. Rational Mech. Anal. **125** (1993), 217-256.
10. G. Liu and Z. Xin, L^1 *stability of stationary discrete shocks*, Math. Comp. **60** (1993), 233-244. MR **93d:**35097
11. H.L. Liu and J. Wang, *Nonlinear stability of stationary discrete shocks for non-convex scalar conservation laws*, Math. Comp. **65** (1996), 1137-1153. MR **96j:**35155

12. H.L. Liu and J. Wang, *Asymptotic stability of stationary discrete shocks of Lax-Friedrichs scheme for non-convex conservation laws*, Japan J. Indust. Appl. Math. **15** (1998), 147-162. CMP 98:09
13. H.L. Liu and J. Wang, *Decay rate for perturbations of stationary discrete shocks for convex scalar conservation laws*, Math. Comp. **66** (1997), 69-84. MR **97k:**39004
14. T.-P. Liu, *Hyperbolic conservation laws with relaxation*, Commun. Math. Phys. **108** (1987), 153-175. MR **88f:**35092
15. T.-P. Liu and S.-H. Yu, *Continuum shock profiles for discrete conservation laws, I. Construction*, Comm. Pure Appl. Math. **52** (1999), 85–127.
16. H.L. Liu, J. Wang and G. Warnecke, *Convergence rates to discrete shocks for nonconvex conservation laws*, Preprint No.14 (1998), Otto-von-Guericke-University Magdeburg.
17. H.L. Liu, J. Wang and T. Yang, *Nonlinear stability and existence of stationary discrete travelling waves for the relaxing schemes*, to appear in Japan J. Indust. Appl. Math.
18. H.L. Liu, J. Wang and T. Yang, *Existence of the Discrete Travelling Waves for a Relaxing Scheme*, Appl. Math. Lett. **10** (1997), 117-122. CMP 97:15
19. H.L. Liu, C. W. Woo and T. Yang, *Decay rate for travelling waves of a relaxation model*, J. Diff. Equ. **134** (1997), 343-367. MR **97j:**35091
20. D. Michelson, *Discrete shocks for difference approximations to systems of conservation laws*, Adv. in Appl. Math. **5** (1984), 433-469. MR **86f:**65159
21. A. Majda and J. Ralston, *Discrete shock profiles for systems of conservation laws*, Comm. Pure Appl. Math. **32** (1979), 445-483. MR **81i:**35108
22. R. Natalini, *Convergence to equilibrium for the relaxation approximations of conservation laws*, Comm. Pure Appl. Math. **49** (1996), 795-823. MR **97c:**35131
23. M. Nishikawa, *Convergence rate to the travelling wave for viscous conservation laws*, Funkcial. Ekvac. **41** (1998), 107–132. CMP 98:14
24. Y.S. Smyrlis, *Existence and stability of stationary profiles of the LW scheme*. Comm. Pure Appl. Math., **43** (1990), 509-545. MR **91d:**65143
25. E. Tadmor, *The large-time behavior of the scalar genuinely nonlinear Lax-Friedrichs scheme*, Math. Comp. **43** (1984), 353-368. MR **86g:**65162
26. L.-A. Ying, *Asymptotic stability of discrete shock waves for the Lax-Friedrichs scheme to hyperbolic systems of conservations*, Japan J. Indust. Appl. Math. **14** (1997), 437-468. MR **99a:**65703
27. P. R. Zingano, *Nonlinear stability with decay rate for travelling wave solutions of a hyperbolic system with relaxation*, J. Diff. Equ. **130** (1996), 36-58. MR **97j:**35090

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