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SOME EXAMPLES RELATED TO THE *abc*–CONJECTURE FOR ALGEBRAIC NUMBER FIELDS

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Abstract. We present a numerical method for finding extreme examples of identities related to the uniform *abc*–conjecture for algebraic number fields.

1. Introduction

Let K be an algebraic number field and let V_K denote the set of primes on K, that is, $v \in V_K$ is an equivalence class of non–trivial norms on K (finite or infinite). Let $\|x\|_v = N_{K/\mathbb{Q}}(\mathfrak{p})^{-v_\mathfrak{p}(x)}$ if v is a prime defined by a prime ideal $\mathfrak{p}$ of the ring of integers $\mathfrak{O}_K$ in K and $v_\mathfrak{p}$ is the corresponding valuation. Let $\|x\|_v = |\varphi(x)|^e$ for all distinct (non–conjugate) embeddings $\varphi : K \to \mathbb{C}$, with $e = 1$ if $\varphi(K) \subset \mathbb{R}$ and $e = 2$ otherwise. We define the *height* of $(a, b, c) \in (K^*)^3$ to be

$$H_K(a, b, c) = \prod_{v \in V_K} \max\left(\|a\|_v, \|b\|_v, \|c\|_v\right),$$

and the *conductor* of (a, b, c) to be

$$N_K(a, b, c) = \prod_{\mathfrak{p} \in I_K(a,b,c)} N_{K/\mathbb{Q}}(\mathfrak{p}),$$

where $I_K(a, b, c)$ is the set of all prime ideals $\mathfrak{p}$ of $\mathfrak{O}_K$ for which $\|a\|_\mathfrak{p}$, $\|b\|_\mathfrak{p}$, $\|c\|_\mathfrak{p}$ are not all equal. Let $\Delta_{K/\mathbb{Q}}$ denote the discriminant of K.

The uniform *abc*–conjecture. For every $\varepsilon > 0$ there exists a constant C_ε, depending only on ε, such that

$$H_K(a, b, c) < C_\varepsilon^{[K:\mathbb{Q}]} \left(|\Delta_{K/\mathbb{Q}}|\, N_K(a, b, c)\right)^{1+\varepsilon},$$

for all a, b, $c \in K^*$ satisfying $a + b + c = 0$.

Remark. In [4], $\Delta_{K/\mathbb{Q}}^{1+\varepsilon}$ is replaced by $\Delta_{K/\mathbb{Q}}^{A}$ for some constant A. The choice $A = 1 + \varepsilon$ is suggested by a theorem in [1].

We define a real valued function on $K \setminus \{0, 1\}$ by

$$l_K(x) = \frac{\log H_K(x, 1 - x, 1)}{\log|\Delta_{K/\mathbb{Q}}| + \log N_K(x, 1 - x, 1)}.$$

The uniform *abc*–conjecture is equivalent to the statement that $l_K(x)$ is bounded and its biggest limit point equals 1. Examples of $x \in K \setminus \{0, 1\}$ for which $l_K(x)$ is big may therefore be of interest. The definition of $l_K(x)$ suggests defining a function

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on the algebraic numbers (excluding 0 and 1) by $l\,(x) = l_{\mathbb{Q}\,(x)}(x)$. It is not hard to show that the conjecture implies that $l\,(x)$ is bounded, and one could expect that the biggest limit point of $l\,(x)$ also equals 1.

2. Examples

We are looking for algebraic numbers x for which $l_K(x)$ is large, that is, numbers x for which $H_K(x, 1-x, 1)$ is relatively large and $N_K(x, 1-x, 1)$ relatively small. One method is to approximate a number $\sqrt[n]{k}$, $k \in K$, by an element y in K and then hope that $l\,(k/y^n)$ is large. We will try to do so in a few norm–Euclidean quadratic fields.

Let $K = \mathbb{Q}\,(\sqrt{d})$, for a square free integer d. An integral basis for K over $\mathbb{Q}$ is $\{1, \alpha\}$, where

$$\alpha = \begin{cases} (1+\sqrt{d})/2 & \text{if } d \equiv 1 \pmod 4, \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

Consider $\varphi : K \to \mathbb{R}^2$, where $\varphi\,(x + y\,\alpha) = (x, y)$, and define multiplication on $\mathbb{R}^2$ by

$$(x_1, y_1)(x_2, y_2) = \begin{cases} (x_1\,x_2 + y_1\,y_2\,d, x_1\,y_2 + y_1\,x_2) & \text{if } \alpha = \sqrt{d}, \\ (x_1\,x_2 + y_1\,y_2\,\frac{d-1}{4}, x_1\,y_2 + y_1\,x_2 + y_1\,y_2) & \text{if } \alpha = \frac{1+\sqrt{d}}{2}. \end{cases}$$

Then $\varphi\,(x\,y) = \varphi\,(x)\,\varphi\,(y)$ for all $x, y \in K$ and $\varphi\,(\mathfrak{O}_K) = \mathbb{Z}^2$. We extend the norm from the image of K to $\mathbb{R}^2$,

$$N : \mathbb{R}^2 \to \mathbb{R}, \quad N\,(a, b) = |(a + b\,\alpha)\,(a - b\,\alpha)|,$$

so $N\,(a, b) = |N_{K/\mathbb{Q}}(a + b\,\alpha)|$ for all a, $b \in \mathbb{Q}$. For any $x \in \mathbb{R}^2$ we define the subset T_x of $\mathfrak{O}_K$ to be $\{r \in \mathfrak{O}_K : N\,(x - \varphi\,(r)) < 1\}$. Note that if T_x is non–empty for all $x \in \mathbb{R}^2$, then K is a norm–Euclidean domain. The following theorem from [3] gives a non–empty subset of T_x in some special cases:

Theorem. *Let d be 2, 3, 6, or 7. For any $(x, y) \in \mathbb{R}^2$ let*

$$S_{(x,y)} = \{[x] + [y]\,\alpha + a + b\,\alpha : a = -1, 0, 1, 2, b = 0, 1\} \subset \mathfrak{O}_K,$$

where $[a]$ denotes the largest integer less than a. Then $T_{(x,y)} \cap S_{(x,y)} \neq \emptyset$ for all $(x, y) \in \mathbb{R}^2$. For $d = -11$, -7, -3, -2, -1, the statement is true with

$$S_{(x,y)} = \{[x] + [y]\,\alpha + a + b\,\alpha : a = 0, 1, b = 0, 1\}.$$

Now select an element $a \in \varphi\,(\mathfrak{O}_K)$ such that the equation $x^n - a = 0$ has a solution $x \in \mathbb{R}^2 \setminus \varphi\,(K)$, where n is a positive integer. We want to expand x in a continued fraction p_i/q_i, where p_i, $q_i \in \varphi\,(\mathfrak{O}_K)$. Set $x_0 = x$ and construct the sequences $\{x_i\} \subset \mathbb{R}^2$ and $\{a_i\} \subset \varphi\,(\mathfrak{O}_K)$ by

$$x_{i+1} = \frac{(1, 0)}{x_i - a_i}, \qquad \text{where} \quad a_i \in \varphi\,(T_{x_i}),$$

i.e. x_{i+1} is the inverse of $x_i - a_i$ with respect to the multiplication in $\mathbb{R}^2$ defined above. To get uniqueness, one needs a rule for selecting a particular $a_i \in \varphi\,(T_{x_i})$. To do this, choose an ordering of the S_{x_i} of the theorem and let a_i be the first element in S_{x_i} satisfying $N(x_i - a_i) \leq N(x_i - a)$ for all $a \in S_{x_i}$. Let p_i/q_i be the continued fraction given by

$$p_i = a_i\, p_{i-1} + p_{i-2}, \qquad p_{-1} = (1,0), \qquad p_0 = a_0,$$
$$q_i = a_i\, q_{i-1} + q_{i-2}, \qquad q_{-1} = (0,0), \qquad q_0 = (1,0).$$

Then one can check that

$$x\, q_i - p_i = -\frac{x\, q_{i-1} - p_{i-1}}{x_{i+1}} = \frac{(-1,0)^i}{x_1\, x_2 \cdots x_{i+1}},$$

and, if we take the norm on both sides,

$$N(x\, q_i - p_i) = \frac{1}{N(x_1) \cdots N(x_{i+1})} = N(x_0 - a_0) \cdots N(x_i - a_i) < 1.$$

Note that $N(x - p_i/q_i) \to 0$ does not have to imply $|\varphi^{-1}(x) - \varphi^{-1}(p_i/q_i)| \to 0$, where $\varphi^{-1} : \mathbb{R}^2 \to \mathbb{C} : (x,y) \mapsto x + y\,\alpha$ and $|\ \ |$ is the usual absolute-value on $\mathbb{C}$.

Now for some examples of identities $a + b = c$ for which $l(a/c)$ are large. The examples are computed using the method described above, for $n = 2, 3, 4$ in the real cases and $n = 2, 3, 4, 5$ in the complex cases. We only searched among equations $x^n - a = 0$ with $N(a) \leq 10000$. The rational examples are well known and are included here for completeness. There is a table of extremal (rational) abc–examples to be found at URL: `http://www.math.chalmers.se/~jub/abc`.

l	identity	author
1.768124	$(\sqrt{2})^{17} + (1-\sqrt{2})^5\ (3-\sqrt{2}) = (1+\sqrt{2})^5(3+\sqrt{2})$	N.B.
1.707221	$(1 - 2\frac{1+\sqrt{-7}}{2}) + (1 - \frac{1+\sqrt{-7}}{2})^{13} = (\frac{1+\sqrt{-7}}{2})^{13}$	N.B.
1.629912	$2 + 3^{10}\ 109 = 23^5$	E.R.
1.625991	$11^2 + 3^2\ 5^6\ 7^3 = 2^{21}\ 23$	B.W.
1.623490	$19\ 1307 + 7\ 29^2\ 31^8 = 2^8\ 3^{22}\ 5^4$	J.B.-J.B.
1.580756	$283 + 5^{11}\ 13^2 = 2^8\ 3^8\ 17^3$	J.B.-J.B., A.N.
1.567887	$1 + 2\ 3^7 = 5^4\ 7$	B. W.
1.561437	$(1+\sqrt{2})^{14} + 1 = (1+\sqrt{2})^7\ (\sqrt{2})^3\ 13^2$	N.B.
1.547075	$7^3 + 3^{10} = 2^{11}\ 29$	B.W.
1.544434	$7^2\ 41^2\ 311^3 + 11^{16}\ 13^2\ 79 = 2\ 3^3\ 5^{23}\ 953$	A.N.
1.536714	$5^3 + 2^9\ 3^{17}\ 13^2 = 11^5\ 17\ 31^3\ 137$	H.R.-P.M.
1.528940	$(8-3\sqrt{7})^2\ (5-2\sqrt{7}) + (8-3\sqrt{7})^7\ (3-\sqrt{7})^3\ (5+2\sqrt{7})^{12} = (4-3\sqrt{7})^4$	N.B.
1.526999	$13\ 19^6 + 2^{30}\ 5 = 3^{13}\ 11^2\ 31$	A.N.
1.522160	$3^{18}\ 23\ 2269 + 17^3\ 29\ 31^8 = 2^{10}\ 5^2\ 7^{15}$	A.N.
1.518102	$(5+2\sqrt{6})^9\ (2-\sqrt{6})^9\ (3-\sqrt{6})\ (1+\sqrt{6})\ (1-\sqrt{6})\ 7^2 + 1 = (5+2\sqrt{6})^8$	N.B.
1.502839	$239 + 5^8\ 17^3 = 2^{10}\ 37^4$	J.B.-J.B., A.N.

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