# Restricted Routing and Wide Diameter of the Cycle Prefix Network 

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#### Abstract

The cycle prefix network is a Cayley coset digraph based on sequences over an alphabet which has been proposed as a vertex symmetric communication network. This network has been shown to have many remarkable communication properties such as a large number of vertices for a given degree and diameter, simple shortest path routing, Hamiltonicity, optimal connectivity, and others. These considerations for designing symmetric and directed interconnection networks are well justified in practice and have been widely recognized in the research community. Among the important properties of a good network, efficient routing is probably one of the most important. In this paper, we further study routing schemes in the cycle prefix network. We confirm an observation first made from computer experiments regarding the diameter change when certain links are removed in the original network, and we completely determine the wide diameter of the network. The wide diameter of a network is now perceived to be even more important than the diameter. We show by construction that the wide diameter of the cycle prefix network is very close to the ordinary diameter. This means that routing in parallel in this network costs little extra time compared to ordinary single path routing.


Suggested Running Title:
Cycle Prefix Network

## 1 Introduction

The cycle prefix network is a vertex symmetric, directed graph which has recently been proposed for use as a communication network [ [5]. It has been shown that the cycle prefix network has many remarkable communication properties such as a large number of vertices for a given degree and diameter, simple shortest path routing, Hamiltonicity, optimal connectivity, and others [5],3, 9]. These considerations for designing symmetric and directed networks are well justified in practice and have been widely recognized in the research community. In the search for highly efficient network models and in the study of communication algorithms on these networks, Cayley graph techniques have been used successfully in discovering new models and in analyzing network efficiencies. Another interesting idea in network design which is used for the construction of many networks is to represent nodes as certain sequences over an alphabet with links represented by suitable operations on sequences. The hypercube, de Bruijn, Kautz, star and pancake networks can all be constructed in this fashion. In the case of cycle prefix digraphs, both the idea of Cayley graphs (Cayley coset digraphs, to be precise) and that of sequences over an alphabet can be used as the underlying representation, and each has its own advantage. The former idea was the point of departure for the discovery of the cycle prefix network, motivated by the fundamental theorem of Sabidussi [1] which shows that all vertex symmetric digraphs are Cayley coset digraphs. The sequence representation of a cycle prefix network is more useful for studying its properties and for implementing it in practice. For this reason, we shall utilize the sequence representation of the cycle prefix digraph throughout the paper.

In the performance evaluation of networks, the diameter and routing efficiency are among the most critical concerns. In this paper we shall further study these issues for the cycle prefix network. The paper has two objectives. The first objective is to confirm an observation first made on the basis of computer experiments concerning the diameter change when certain links are removed. By proving a reachability theorem we show that in the case of cycle prefix digraphs, the resulting networks often possess better degree-diameter properties than the original one. Using the method of Conway and Guy [目, one may construct large symmetric networks with small degree and diameter. In a recent paper [6], Comellas and Fiol describe a routing scheme with implies an upper bound on the diameter of the link deleted cycle prefix digraphs. However, they left open the question of whether the bound is exact. We settle this question by exhibiting vertices that achieve the diameter bound.

The second objective of this paper is concerned with the recently introduced notion of the wide diameters of networks. Mathematically, the notion of the wide diameter of a graph naturally stems from the classical theorem of Menger relating connectivity to disjoint paths. However, such a notion has not been studied in graph theory until very recently when it became relevant from an engineering point of view. D. F. Hsu gives a vivid account on the
background of wide diameters [8]:
"The concept of the wide diameter of a graph $G$ arises naturally from the study of routing, reliability, randomized routing, fault tolerance, and other communication protocols (such as the byzantine algorithm) in parallel architecture and distributed computer networks. By considering both the width and the length of a container, we are able to give a global and systematic treatment on the interconnection network for various distributed systems."
"Although the concept of a container and the notion of wide diameter have been discussed and used in practical applications, the graph theory questions suggested have not, at least until recently, been studied as extensively as the questions in the hardware and software design, development, and implementation of distributed computing systems."

In the case of the cycle prefix networks, we completely determine its wide diameter through an explicit construction. It turns out that the wide diameter is very close to the ordinary diameter of the network. In other words, routing in parallel in the cycle prefix network costs little extra time compared to ordinary single path routing. This property undoubtedly increases the usefulness of the cycle network.

## 2 The Cycle Prefix Digraphs $\Gamma_{\Delta}(D)$ and $\Gamma_{\Delta}(D,-r)$

The cycle prefix digraph $\Gamma_{\Delta}(D)(\Delta \geq D)$ is defined as a digraph whose vertex set consists of sequences $x_{1} x_{2} \cdots x_{D}$ over an alphabet $\{1,2, \ldots, \Delta+1\}$, where $x_{1}, x_{2}, \ldots, x_{D}$ are distinct. Such sequences are called partial permutations or $D$-permutations. The adjacency relations for a vertex $x_{1} x_{2} \cdots x_{D}$ are described as follows:

$$
x_{1} x_{2} \cdots x_{D} \Rightarrow \begin{cases}x_{k} x_{1} \cdots x_{k-1} x_{k+1} \cdots x_{d}, & \text { for } 2 \leq k \leq D \\ y x_{1} x_{2} \cdots x_{D-1}, & \text { for } y \neq x_{1}, x_{2}, \ldots, x_{D}\end{cases}
$$

We say that the vertex $x_{k} x_{1} \cdots x_{k-1} x_{k+1} \cdots x_{D}$ is obtained from $x_{1} x_{2} \cdots x_{D}$ via a rotation on the prefix $x_{1} x_{2} \cdots x_{k}$, denoted,

$$
x_{k} x_{1} \cdots x_{k-1} x_{k+1} \cdots x_{D}=R_{k}\left(x_{1} x_{2} \cdots x_{D}\right) .
$$

In particular, the operation $R_{D}$ is called a full rotation, and $x_{D} x_{1} x_{2} \cdots x_{D-1}$ is called a full rotation of $x_{1} x_{2} \cdots x_{D}$. The rotations $R_{k}$ for $k<D$ are called partial rotations. Similarly, we say that the sequence $y x_{1} x_{2} \cdots x_{D-1}$ is obtained from $x_{1} x_{2} \cdots x_{D}$ via a shift, denoted

$$
y x_{1} x_{2} \cdots x_{D-1}=S_{y}\left(x_{1} x_{2} \cdots x_{D}\right) .
$$

Since the term "adjacent" is somewhat ambiguous for a directed graph, when $(u, v)$ is an arc in a digraph we shall say that $u$ is adjacent to $v$ while $v$ is next to to $u$. The term "adjacent from" is used by some authors to distinguish it from "adjacent to". From the above sequence
definition of $\Gamma_{\Delta}(D)$, it is easily seen to be vertex symmetric, a fact that immediately follows from the Cayley coset digraph definition $\Gamma_{\Delta}(D)$ [5] .

We notice that some authors prefer the sequence shift in the direction from right to left, like the shift for de Bruijn digraphs: $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \Rightarrow\left(x_{2}, \ldots, x_{n}, z\right)$. For the sake of consistency, we shall follow the notation in [5] , and continue the left-to-right shift, which reflects the rotations on the prefixes. A left-handed notation is however adopted by Comellas and Fiol [6].

In the original study of cycle prefix digraphs $\Gamma_{\Delta}(D)$, Faber and Moore observed from computational experiments that if one rules out the double arcs in $\Gamma_{\Delta}(D)$, then the diameter of the resulting digraph, denoted, $\Gamma_{\Delta}(D,-1)$, increases only by one. They then considered a new construction based on the cycle prefix digraph $\Gamma_{\Delta}(D)$. Suppose $\Gamma_{\Delta}(D,-r)$ is the digraph obtained from $\Gamma_{\Delta}(D)$ by deleting the arcs represented by the partial rotations $R_{2}, R_{3}, \ldots, R_{r+1}$. Formally speaking, $\Gamma_{\Delta}(D,-r)$ has the same vertex set as $\Gamma_{\Delta}(D)$, and the adjacency relations for a vertex $x_{1} x_{2} \cdots x_{D}$ are described by

$$
x_{1} x_{2} \cdots x_{D} \Rightarrow \begin{cases}x_{k} x_{1} \cdots x_{k-1} x_{k+1} \cdots x_{d}, & \text { for } r+2 \leq k \leq D, \\ y x_{1} x_{2} \cdots x_{D-1}, & \text { for } y \neq x_{1}, x_{2}, \ldots, x_{D}\end{cases}
$$

Note that the degree of $\Gamma_{\Delta}(D,-r)$ decreases by $r$ compared with $\Gamma_{\Delta}(D)$. There is an intuitive reason to surmise that the diameter of $\Gamma_{\Delta}(D,-r)$ would increase by the same scale. Recently, Comellas and Fiol [6] have shown that in most cases the diameter increase of $\Gamma_{\Delta}(D,-r)$ is in fact bounded by $r$. However, they did not solve the problem of whether this bound is exact or not. We will fill this gap by proving the exactness of the diameter bound. Moreover, we shall study the reachability property of the digraph $\Gamma_{\Delta}(D,-r)$ (this property for $\Gamma_{\Delta}(D)$ has been studied in [6] $]$. A vertex symmetric digraph with the reachability property is of great use in constructing new classes of vertex symmetric digraphs with small degree and diameter, as proposed by Conway and Guy [4]. The details are presented in the next section.

## 3 Restricted Routing for $\Gamma_{\Delta}(D,-r)$

For the sake of easier presentation, we shall start with the reachability of the digraph $\Gamma_{\Delta}(D,-r)$. It is easy to see that $\Gamma_{\Delta}(D,-r)$ is vertex symmetric and that one may choose the standard origin $12 \cdots D$ while considering routing for any two vertices. For simplicity the vertex $X$ is always referred to as $x_{1} x_{2} \cdots x_{D}$.

Definition 3.1 ( $k$-Reachable digraphs) A digraph is said to be $k$-reachable if for any two vertices $u$ and $v$, which are not necessarily distinct, there exists a path (with repeated vertices and arcs allowed) from $u$ to $v$ of length $k$.

Comellas and Fiol [6] have shown that the digraph $\Gamma_{\Delta}(D)$ is $D$-reachable for $\Delta \geq D \geq 3$. Here we will present a stronger result for $\Gamma_{\Delta}(D,-r)$.

Theorem 3.2 Suppose $r \geq 0$ and $\Delta \geq D \geq 2 r+3$. Then the vertex symmetric digraph $\Gamma_{\Delta}(D,-r)$ is $(D+r)$-reachable.

The major reason for the above theorem lies in the following observation about dead angles. Given a vertex $X=x_{1} x_{2} \cdots x_{D}$, the prefix $x_{1} x_{2} \cdots x_{r+1}$ is called the dead angle of $X$ in $\Gamma_{\Delta}(D,-r)$. We say that a letter $z$ is in the dead angle of $X$ if $z=x_{i}$ for some $1 \leq i \leq r+1$.

Lemma 3.3 (Dead Angle Principle) Let $X$ be a vertex in $\Gamma_{\Delta}(D,-r)$. Then there exists a vertex $Y$ next to $X$ such that $Y$ begins with a letter $z$, if and only if $z$ is not in the dead angle of $P$.

The proof of the above lemma is straightforward. It is based on a property of $\Gamma_{\Delta}(D)$ regarding how the rotation and shift operation work together to complement each other: Suppose $z$ is not in the dead angle of $X$. If $z$ is indeed in $X$, then a rotation operation on $X$ may put $z$ back to the beginning of the sequence; otherwise, a shift operation can achieve the same goal with ease. For the above reason, one sees that the two operations are coherent with each other, although the look rather unrelated. Moreover, suppose $Y$ is next to $X$ in $\Gamma_{\Delta}(D,-r)$, then $Y$ is determined by its first element. We now give the proof of Theorem 3.2.

Proof. Let $I=12 \cdots D$ be the standard origin and $X=x_{1} x_{2} \cdots x_{D}$ be the destination. It suffices to show that there is a directed path of length $D+r$ from $I$ to $X$. We first consider the case when $x_{D} \neq 1$. Let $A$ be the set $\left\{x_{1}, x_{2}, \ldots, x_{D-r-1}\right\}$. Since there are $r+1$ elements in the dead angle of $X$, and $D-(r+1)>r+1$, there exists an element $y_{1}$ in $A$ that is not in the dead angle of $I$. By the dead angle principle, $I$ is adjacent to a vertex $P_{1}$ with prefix $y_{1} 1 \cdots(r+1)$. Let $A=A \backslash\left\{y_{1}\right\}$. Considering the dead angle of $P_{1}$, the same condition on $D$ and $r$ ensures that that there exists an element $y_{2}$ in $A_{1}$ that is not in the dead angle of $P_{1}$ (implying that $y_{1}$ and $y_{2}$ are distinct). Hence $P_{1}$ is adjacent to a vertex $P_{2}$ with prefix $y_{2} y_{1} 1 \cdots r$. Repeating the above procedure, one may reach a vertex $P_{r}$ such that $P_{r}$ has prefix $y_{r} \cdots y_{2} y_{1} 1$ and $y_{1}, y_{2}, \ldots, y_{r}$ come from $A$. It has already taken $r$ steps to get to $P_{r}$ from $I$. Since $x_{D} \neq 1$, we may construct a path of length $D$ from $P_{r}$ to $X$, and display it by showing the prefixes:

$$
\begin{gathered}
y_{r} \cdots y_{2} y_{1} 1 \Rightarrow x_{D} y_{r} \cdots y_{2} y_{1} \quad \Rightarrow \quad x_{D-1} x_{D} y_{r} \cdots y_{2} y_{1} \quad \Rightarrow \quad \cdots \\
\Rightarrow \quad x_{D-r} \cdots x_{D-1} x_{D} y_{r} \cdots y_{2} y_{1} \quad \Rightarrow \quad x_{D-r-1} x_{D-r} \cdots x_{D-1} x_{D} \quad \Rightarrow \quad \cdots \\
\Rightarrow \quad x_{1} x_{2} \cdots x_{D} .
\end{gathered}
$$

We next consider the case when $x_{D}=1$. Let $A=\left\{x_{1}, x_{2}, \ldots, x_{D-r-2}\right\}$ and $B=$ $\{2,3, \ldots, r+1\}$. Since $D-r-2>r$, there exists an element $y_{1}$ such that $y_{1} \in A$ but $y_{1} \notin B$. Note that $1 \notin A, B$. Hence by the dead angle principle, $I$ is adjacent to a vertex $P_{1}$ with prefix $y_{1} 12 \cdots(r+1)$. Let $A_{1}=A \backslash\left\{y_{1}\right\}, B_{1}=B \backslash\{r+1\}$. The same condition on $D$ and $r$ ensures that there exists $y_{2} \in A_{1}$, but $y_{2} \notin B_{1}$. It follows that $P_{1}$ is adjacent to a vertex $P_{2}$ with prefix $y_{2} y_{1} 12 \cdots r$. Repeating this procedure, one ends up with a vertex $P_{r}$ having prefix $y_{r} y_{r-1} \cdots y_{1} 1$, where $y_{i} \in A$. We continue with the following path of length $D-r$ starting from $P_{r}$ (with only prefixes shown):

$$
\begin{aligned}
y_{r} y_{r-1} \cdots y_{1} 1 & \Rightarrow x_{D-r-1} y_{r} y_{r-1} \cdots y_{1} 1 \\
& \Rightarrow x_{D-1} x_{D-r-1} y_{r} y_{r-1} \cdots y_{1} 1 \\
& \Rightarrow x_{D-2} x_{D-1} x_{D-r-1} y_{r} y_{r-1} \cdots y_{1} 1 \\
& \cdots \\
& \Rightarrow x_{D-r} \cdots x_{D-1} x_{D-r-1} y_{r} y_{r-1} \cdots y_{1} 1 \\
& \Rightarrow x_{D-r-1} x_{D-r} \cdots x_{D-1} y_{r} y_{r-1} \cdots y_{1} 1
\end{aligned}
$$

The last vertex is labeled by $P_{2 r+2}$ according to the length. Note that $y_{i} \in A$, we claim that there is a path from $P_{2 r+2}$ to $X$ of the following form:

$$
\begin{gathered}
P_{2 r+2} \Rightarrow x_{D-r-2} x_{D-r-1} \cdots x_{D-1} \cdots 1 \quad \Rightarrow \quad x_{D-r-3} \cdots x_{D-1} \cdots 1 \Rightarrow \cdots \\
\Rightarrow \quad x_{1} x_{2} \cdots x_{D-1} 1=X,
\end{gathered}
$$

because $y_{i} \in A$ at each step it is impossible to bump 1 out of the sequence so that the last vertex has to be $X$. Summing up all the segment, we get a path of length $D+r$.

Specializing the above theorem for $r=0$, it follows the reachability of $\Gamma_{\Delta}(D)$ first observed in [6]. Moreover, using the method of Conway and Guy [7], one may construct large symmetric digraphs with small degree and diameter based on $\Gamma_{\Delta}(D,-r)$. Since the digraph $\Gamma_{\Delta}(D,-r)$ in some cases has more vertices than $\Gamma_{\Delta-r}(D+r)$, one may use the above theorem in constructing new symmetric digraphs. However, we will not discuss this aspect here.

The rest of this section is concerned with the diameter of $\Gamma_{\Delta}(D,-r)$. It is shown in [母] that the diameter of $\Gamma_{\Delta}(D,-r)$ does not exceed $D+r$ for $\Delta \geq D \geq 2 r+2$. This upper bound is achieved by the following construction that is a much simpler version than the construction for the reachability of $\Gamma_{\Delta}(D,-r)$. Note that the reachability result requires a slightly stronger condition on the parameters of $\Gamma_{\Delta}(D,-r)$. Let's give an outline. Let $I=12 \cdots D$ be the standard origin, and $X=x_{1} x_{2} \cdots x_{D}$ be any vertex in $\Gamma_{\Delta}(D,-r)$. For the case $x_{D} \neq 1$, one may first try to reach from $I$ a vertex with prefix $y_{r} y_{r-1} \cdots y_{1} 1$, where $y_{i} \neq x_{D-1}, x_{D-2}, \ldots, x_{D-r}$. Then one may continue with vertices having prefixes $x_{D}, x_{D-1}$, etc. For the case $x_{D}=1$, one may get to a vertex with prefix $y_{r} y_{r-1} \cdots y_{1} 1$ such
that $y_{i} \in\left\{x_{1}, x_{2}, \ldots, x_{D-r-2}\right\}$. Then one may get to $X$ via vertices with prefixes $x_{D-1}$, $x_{D-2} x_{D-1}$, etc. The last element $x_{D}=1$ will eventually takes care of itself for the reason given in the proof of the reachability theorem.

It is harder to show that the above diameter bound is exact. For this purpose, we find a class of vertices that achieve the bound. A vertex $X$ in $\Gamma_{\Delta}(D,-r)$ is called a remote vertex if $x_{D-1}=1$ and $x_{D}=D$, and there exists $x_{i}>D$ for some $1 \leq i \leq D-2$. We shall use the common notation $d(X, Y)$ to denote the distance from $X$ to $Y$ in a digraph. Then we have the following theorem:

Theorem 3.4 Let $\Delta \geq D \geq 2 r+2$, and $X$ be a remote vertex in $\Gamma_{\Delta}(D,-r)$, then the distance from the standard origin $I=12 \cdots D$ to $X$ equals $D+r$.

Proof. The diameter upper bound is already established, so it suffices to show that $d(I, X) \geq D+r$. Since $X=x_{1} x_{2} \cdots x_{D-2} 1 D$ and there exists $x_{i}>D$ for some $i$, to reach $X$ from $I$ requires at least one shift operation. Thus, element $D$ in $I$ cannot remain in the last position during the process to reach $X$ from $I$. Since $D$ is in the destination vertex $X$, it is either moved back to the beginning position at some point, or is removed out of the sequence by a shift operation and then put back to the beginning by another shift operation. Let $I \Rightarrow P_{1} \Rightarrow P_{2} \Rightarrow \cdots \Rightarrow P_{m}=X$ be a shortest path from $I$ to $X$. Since either a shift or rotation operation on a vertex, say $Y=y_{1} y_{2} \cdots y_{D}$, moves the elements in the dead angle to the positions on the right hand side, the vertex $P_{r+1}$ must have the prefix $z_{r+1} \cdots z_{2} z_{1} 1$. If $D$ does not appear in $z_{r+1} \cdots z_{2} z_{1}$, then it will take at least $D$ steps to reach $X$ from $P_{r+1}$ as far as the last element of $X$ is concerned. This contradicts the upper bound $D+r$ on the diameter of $\Gamma_{\Delta}(D,-r)$. We now assume that $z_{i}=D$. It follows that $P_{r+i+1}$ has a prefix of the form $w_{i} \cdots w_{2} w_{1} z_{r+1} \cdots z_{i+1} D$. Consider the following two cases:

Case 1. The element 1 appears in $w_{i} \cdots w_{2} w_{1}$. If $D$ is shifted out of a vertex after the vertex $P_{r+i+1}$, then it will take at least $D$ steps to put $D$ to the last position of $P_{m}$, a contradiction. Thus, $D$ has to remain in the vertices $P_{r+i+1}, P_{r+i+2}, \ldots, P_{m}$. Moreover, $D$ will never be put back to the beginning of a vertex by a rotation because after that rotation one needs at least $D-1$ steps to move $D$ to the last position of $P_{m}$, which is also impossible. Hence, in the path from $P_{r+i+1}$ to $P_{m}, 1$ has to remain in all the vertices on this path segment, and 1 is always to the left of $D$. We define $\delta(Y)$ to be the number of elements between $D$ and 1. Let $f$ be the number of operations used to reach $P_{m}$ from $P_{r+i+1}$ that move the element $D$ to its right, and $g$ be other operations used in the same path. For a rotation or a shift operation $T$ on $P_{j}(r+i+1 \leq j \leq m-1)$, if $T$ moves $D$ to its right, then $T$ leaves the value of $\delta\left(P_{j}\right)$ unchanged; otherwise $T$ may reduce the value of $\delta\left(P_{j}\right)$ at most by one. It follows that

$$
m-r-i-1=f+g \geq(D-r-2)+(r-i+1)=D-i-1
$$

Hence $m \geq D+r$.
Case 2. The element 1 does not appear in $w_{i} \cdots w_{2} w_{1}$. As in Case 1 , the element $D$ has to remain in the vertices on the path from $P_{r+i+1}$ to $P_{m}$, and $D$ is never moved back to the beginning of any vertex on the path. It is clear that at some step, 1 has to be put back to the beginning of a vertex on the aforementioned path either by a rotation or a shift operation. Suppose this happens to $P_{j}=1 \cdots(j \geq r+i+2)$. Since 1 is never moved back to the beginning of a vertex, in $P_{j}$ there are at least $r+1$ elements between 1 and $D$. Thus, we need at least $r+1$ steps to move 1 next to $D$. It follows that

$$
m \geq(r+i+1)+(D-r-2)+(r+1)=D+r+i \geq D+r .
$$

This completes the proof.
We remark that when $r \geq 1$ the digraph $\Gamma_{\Delta}(D,-r)$ does not have the unique shortest path property like $\Gamma_{\Delta}(D)$. For example, in $\Gamma_{4}(4,-1)$ there are two shortest paths from 1234 to 5214, shown below:

$$
\begin{aligned}
& 1234 \Rightarrow 4123 \Rightarrow 5412 \Rightarrow 1542 \Rightarrow 2154 \Rightarrow 5214, \\
& 1234 \Rightarrow 5123 \Rightarrow 4512 \Rightarrow 1452 \Rightarrow 2145 \Rightarrow 5214 .
\end{aligned}
$$

## 4 The Wide Diameter of $\Gamma_{\Delta}(D)$

Connectivity considerations of a network was primarily motivated by its fault tolerance capabilities, while the diameter is a measurement of routing efficiency along a single path. Interestingly, the recent notion of wide diameter is a kind of unification of both the diameter and the connectivity due to the classical theorem of Menger. This notion also has a strong practical background. Let $G$ be a digraph of connectivity $k$ and diameter $D$. By Menger's theorem, between any two distinct vertices $x$ and $y$ in $G$ there are $k$ vertex-disjoint paths. Such a set of disjoint paths, denoted by $C(x, y)$, is called a container, and its length is defined as maximum length among the paths in the container. The wide distance from $x$ to $y$ is then defined to be the minimum length of the containers from $x$ to $y$, and the wide diameter is the maximum wide distance among all the pairs of distinct vertices. As we have mentioned before, the consideration of the wide diameter of a network has solid practical background which we will not get into the discussions. Clearly, the wide diameter of a digraph is at least as large as the ordinary diameter. However, it is rather remarkable that for most of the popular interconnection networks the wide diameters are not significantly bigger than (actually, a small constant bigger than) the ordinary diameter, like the hypercube, the de Bruijn, the Kautz, and the star networks. The main result of this section is to show that such a remarkable phenomenon also occurs in the cycle prefix network. For a vertex $X$ in $\Gamma_{\Delta}(D)$, we shall use $N(X)$ to denoted the set of vertices next to $X$, and $M(X)$ the set of
vertices adjacent to $X$. We shall use the notation $i \circ X$ to denote the vertex adjacent to $X$ that is obtained by rotating the element $i$ to the beginning position if $i$ is $X$, or by shifting $i$ into $X$ and bumping the last element out of $X$, namely $S_{i}(x)$ by the previous notation. If $i=x_{1}$, let $i \circ X=X$. Note that $N(X)$ consists of vertices $X_{i}=i \circ X$ for $i \neq x_{1}$, and $M(Y)$ consists of vertices

$$
Y_{i}= \begin{cases}23 \cdots i 1(i+1) \cdots D, & \text { if } 2 \leq i \leq D \\ 23 \cdots D i & \text { if } D<i \leq \Delta+1 \\ Y & \text { if } i=1\end{cases}
$$

Theorem 4.1 The wide diameter of $\Gamma_{\Delta}(D)$ is at most $D+2$. It is exactly $D+2$ for $D \geq$ 4. Specifically, if $X$ and $Y$ are distinct vertices in $\Gamma_{\Delta}(D)$, then there is a bijection $\theta$ of $N(X) \backslash\{Y\}$ to $M(Y) \backslash\{X\}$ such that the shortest paths from $Z$ to $\theta(Z)$ are vertex disjoint and do not contain either $X$ or $Y$.

Since the proof of the above theorem heavily depends on the unique shortest path property, we here give a brief review of the shortest path routing in $\Gamma_{\Delta}(D)$. Given two vertices $X$ and $Y$ in $\Gamma_{\Delta}(D)$, a tail of $Y$ with respect to $X$ (as the origin) is a suffix $y_{k+1} \cdots y_{D}$ such that it forms a subsequence of $X$, say $x_{i_{1}} x_{i_{2}} \cdots x_{i_{D-k}}$, and all the elements $x_{1}, x_{2}, \ldots, x_{i_{D-k}}$ occur in $Y$. Note that a tail can be the empty sequence. A header of $Y$ with respect to $X$ is a prefix $y_{1} \cdots y_{k}$ such that the complement suffix $y_{k+1} \cdots y_{D}$ is a tail. It is proved in [5] that the distance from $X$ to $Y$ is the length of the shortest header of $Y$ with respect to $X$. Suppose $y_{1} y_{2} \cdots y_{k}$ is the shortest header of $Y$ with respect to $X$, then the shortest path from $X$ to $Y$ is determined by the following prefixes:

$$
X \quad \Rightarrow \quad y_{k} * * \quad \Rightarrow \quad y_{k-1} y_{k} * * * \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad y_{1} \cdots y_{k} * * * *=Y
$$

where $* * *$ is the usual wild-card notation to mean "some sequence" in order to fill the gap in the notation of a sequence. Without loss of generality $Y$ can be assumed to be the standard origin. For clarity we list the following conditions which together are equivalent to $d(X, Y)=k$ for $k<D:$
(a). $y_{D}$ appears in $X$, say $x_{j}=y_{D}$.
(b). $x_{1}, x_{2}, \ldots, x_{j}$ are all in $Y$ (but they do not necessarily form a subsequence).
(c). $y_{k+1} \cdots y_{D}$ is a subsequence of $X$, but $y_{k} y_{k+1}$ is not.

When $d(X, Y)=D$, it is equivalent to the following statement:
(a). either $y_{D}$ is not in $X$,
(b). or $y_{D}$ is in $X$, say $x_{j}=y_{D}$, but there exists $x_{r}$ with $r<j$ that is not in $Y$.

For example, let $X=47285136$ and $Y=82164753$, the shortest header of $Y$ with respect to $X$ is illustrated by $8216 \mid 4753$ The following property is helpful in understanding the routing scheme in $\Gamma_{\Delta}(D)$ and it implies the uniqueness of the shortest path.

Proposition 4.2 Given $X=x_{1} x_{2} \cdots x_{D}$ and $Y=y_{1} y_{2} \cdots y_{D}$ in $\Gamma_{\Delta}(D)$, suppose $k=$ $d(X, Y)$. Then $d(i \circ X, Y) \geq d(X, Y)$ unless $i=y_{k}$, in which case $d(i \circ X, Y)=d(X, Y)-1$.

In order to reach the conclusion in the above theorem concerning the wide diameter of $\Gamma_{\Delta}(D)$, we shall start with the easiest case $x_{1}=1$. We give a complete treatment of this case. For the other cases, we only give an outline of the proof. The details are similar to the case $x_{1}=1$ but more tedious. In this regard, we hope that a simpler construction will be found with a better understanding of the the wide diameter of $\Gamma_{\Delta}(D)$. There is no doubt that the construction given in this paper is $a d$ hoc, although it does give the best bound.

For the case $x_{1}=1$, the mapping $\theta$ is defined by

$$
\theta\left(X_{i}\right)=Y_{i}, \quad(2 \leq i \leq \Delta+1)
$$

The following is an example for $\Delta=5, D=4$ and $X=1325$.

$$
\begin{aligned}
& X_{2}=2135 \rightarrow \underline{4} 213 \rightarrow \underline{34} 21 \rightarrow \underline{134} 2 \rightarrow \underline{2134}=Y_{2} \\
& X_{3}=3125 \rightarrow \underline{4} 312 \rightarrow \underline{14} 32 \rightarrow \underline{314} 2 \rightarrow \underline{2314}=Y_{3} \\
& X_{4}=4132 \rightarrow \underline{3412} \rightarrow \\
& X_{5}=5132 \rightarrow \underline{4} 513 \rightarrow \underline{3451} \rightarrow \\
& X_{6}=6132 \rightarrow \underline{4} 613 \rightarrow \underline{34} 61 \rightarrow
\end{aligned}
$$

The following lemma gives the distance from $X_{i}$ to $Y_{i}$, from which the shortest path routing is determined in terms of the shortest header.

Lemma 4.3 Let $X=x_{1} x_{2} \cdots x_{D}$ be a vertex in $\Gamma_{\Delta}(D)$ such that $x_{1}=1$. Suppose the distance from $X$ to $Y=12 \cdots D$ is $k$, then the distance from $X_{i}$ to $Y_{i}$ is given by

$$
d\left(X_{i}, Y_{i}\right)= \begin{cases}k, & \text { if } 1<i<k \\ k-2, & \text { if } i=k, \\ i-2, & \text { if } k<i \leq D \\ D-1, & \text { if } i>D\end{cases}
$$

Proof. We first consider the case when $k<D$. Note that $d(X, Y)=k$ is equivalent to the above conditions (a), (b) and (c) altogether. We need to check the same conditions for the corresponding distances in various cases.

For $1<i<k$, the verification for $d\left(i \circ X, Y_{i}\right)=k$ is divided into the following three steps:
(a). $Y_{D}=D$ appears in $X$ : Suppose $i \circ X$ does not contain $D$. Since $D$ is in $X$, it must be the last element in $X$ and $i \circ X$ is obtained from $X$ via a shift operation. By the condition (b) for $d(X, Y)=k<D, x_{1}, x_{2}, \ldots, x_{D}$ are all in $Y$, implying that $X$ is a permutation on $1,2, \ldots, D$. Thus, $i \circ X$ is obtained from $X$ via a rotation, which is a contradiction.
(b). Suppose $x_{j}=D$. If $i \circ X$ is next to $X$ via a rotation, then every element prior to $D$ in $i \circ X$ is in $Y_{i}$ since $Y_{i}$ is a permutation of $Y$. If $i \circ X$ is next to $X$ via a shift operation, the above argument for (a) shows that $D$ cannot be the last element of $X$. It also follows that every element prior to $D$ in $i \circ X$ is still in $Y_{i}$.
(c). Since $(k+1, k+2, \ldots, D)$ is a subsequence of $X$, it follows that it is also a subsequence of $i \circ X$ because $i<k$ and $D$ stays in $i \circ X$. Clearly, $(k, k+1)$ cannot be a subsequence of $i \circ X$ because it is not a subsequence of $X$.

Combining (a), (b) and (c) one sees that $d\left(X_{i}, Y_{i}\right)=k$, and the shortest header of $Y_{i}$ with respect to $X_{i}$ is illustrated as follows:

$$
23 \cdots i 1(i+1) \cdots k \mid(k+1) \cdots D
$$

For the case $i=k, d\left(X_{k}, Y_{k}\right)=k-2$ : The verification of conditions (a) and (b) are the same as for the previous case. The only catch for condition (c) is that $(k, 1, k+1, \ldots D)$ is a subsequence of $k \circ X$. Since $k$ is the first element of $k \circ X$, it follows that $d\left(X_{k}, Y_{k}\right)=k-2$ and the shortest header of $Y_{k}$ is illustrated below:

$$
23 \cdots(k-1) \mid k 1(k+1) \cdots D .
$$

For the case $k<i \leq D, d\left(X_{i}, Y_{i}\right)=i-2$ : The argument for conditions (a) and (b) remain the same. Noticing that the first two elements of $i \circ X$ is $i 1$ and that $(i, 1, i+1, \cdots, D)$ is a subsequence of $i \circ X$, it follows that the shortest header of $Y_{i}$ is illustrated below:

$$
23 \cdots, k(k+1) \mid i 1(i+1) \cdots D .
$$

It now comes the last subcase: $D<i \leq \Delta+1$. The tail containing the single element $i$ of $Y_{i}$ is clearly the longest tail with respect to $X_{i}$, and it is illustrated below:

$$
23 \cdots D \mid i
$$

We finally finish up the main case $k=D$, for which $Y$ is not a closed vertex with respect to $X$. For $1<i<D$, if $D$ is not in $X$, then $D$ is not in $i \circ X$ either. Suppose $x_{j}=D$ and
there exists $x_{r}$ with $r<j$ that is not in $Y$. If $i \circ X$ does not contain $D$, then we are done. If $i \circ X$ contains $D$, then $x_{r}$ stays in $i \circ X$, but it is not in $Y_{i}$. Hence we still have $d\left(X_{i}, Y_{i}\right)=D$.

For $i=D$, we have $D \circ X=D 1 * * *$ and $Y_{D}=23 \cdots D 1$. Clearly $d\left(X_{D}, Y_{D}\right)=D-2$. For $i>D$, this is an easy matter, and the same as for the case $k<D$. This completes all the cases.

The shortest path routing from $X_{i}$ to $Y_{i}$ easily follows from the above lemma. Our next goal is to show that all the shortest paths from $X_{i}$ to $Y_{i}$ are vertex-disjoint. To this end, we need to define two statistics on a vertex so that they can be used to distinguish the vertices along the shortest paths from $X_{i}$ to $Y_{i}$. Given $X=x_{1} x_{2} \cdots x_{D}$, suppose $d(X, Y)=k$ where $Y=12 \cdots D$. Define $\alpha(X)$ to be the first element $x_{i}$ such that $x_{i} \notin\{k+1, k+2, \ldots, D+1\}$. Let $\beta(X, i)=j+1$, where $j$ is obtained as follows: Let $Y^{\prime}$ be the second to the last vertex on the shortest path from $X$ to $i \circ Y$. Let $j$ be the element immediately preceding $i$ in $Y^{\prime}$ or, if $j$ does not occur in $Y^{\prime}$, the last element of $Y^{\prime}$. Equivalently, $\beta(X, i)$ is the minimum of $D+2$ and the smallest $x>k$ such that $x$ is to the right of $i$ or not in $X$. Let $\beta(X)=\beta(X, \alpha(X))$. For example, suppose $X=531624$ and $Y=123456$. Then $d(X, Y)=4, \alpha(X)=3, \beta(X)=6$, $\beta(X, 2)=7$. We call $(\alpha(x), \beta(x))$ the characteristic pair of $X$.

It turns out that the characteristic pair of a vertex on the shortest path from $X_{i}$ to $Y_{i}$ can be easily determined along with the routing. The following table illustrates the shortest path $P_{i}$ from $X_{i}$ to $Y_{i}$ in various cases, together with the characteristic pairs from which one sees that all the vertices are indeed distinct. The notation $j \rightarrow$ means the operation of getting $j \circ Z$ from $Z=z_{1} z_{2} \cdots z_{D}$ for $j \neq z_{1}$.

The cases other than $x_{1}=1$ are more tedious. The critical part is to construct the mapping from $N(x)$ to $M(Y)$. Recall that $d(X, Y)=k$.

For $x_{1}=k+1$, we have

$$
\theta(i \circ X)= \begin{cases}Y_{k}, & \text { if } \quad i=1 \\ Y_{i}, & \text { if } 1<i<k \\ Y_{\beta(X, 1)-1} & \text { if } i=k \\ Y_{i-1} & \text { if } k+1 \leq i<\beta(X, 1) \\ Y_{i} & \text { if } \beta(X, 1) \leq i \leq \Delta+1\end{cases}
$$

Table 1: Case 1: $x_{1}=1$.

| $i \quad P_{i}$ | $\alpha(v)$ | $\beta(v)$ | Notes |
| :---: | :---: | :---: | :---: |
| (a) $1<i<k$ : $\left.\begin{array}{l}i \rightarrow \\ k \rightarrow \\ \vdots \\ i+1 \rightarrow \\ 1 \rightarrow \\ i \rightarrow \\ \vdots \\ 2 \rightarrow\end{array}\right\}$ | 1 | $k+1$ $i+1$ | $i \neq 1$ $i+1 \neq k+1$ |
| $\left.\begin{array}{c} \text { (b) } i=k: \\ k \rightarrow \\ \vdots \\ 2 \rightarrow \end{array}\right\}$ | 1 | $k+1$ |  |
| $\left.\begin{array}{rl} (\mathrm{c}) k+1 & \leq i \leq D+1: \\ i & \rightarrow \\ \vdots \\ 2 & \rightarrow \end{array}\right\}$ | 1 | $i+1$ |  |
| $\begin{gathered} \hline \text { (d) } D+1<i \leq \Delta+1: \\ i \rightarrow \\ D \rightarrow \\ \vdots \\ 2 \rightarrow \end{gathered}$ | $i$ | $D+1$ |  |

It is straightforward to see that $\theta$ is a bijection. The distance from $i \circ X$ to $\theta(i \circ X)$ are given below:

$$
d(i \circ X, \theta(i \circ X))= \begin{cases}k-1, & \text { if } \quad i=1, \\ k-1, & \text { if } \quad 1<i<k \\ k-1, & \text { if } \quad i=k \\ i, & \text { if } \quad k+1 \leq i<\beta(X, 1) \\ D-1 & \text { if } \quad \beta(X, 1) \leq i \leq \Delta+1\end{cases}
$$

We omit the detailed verification of the above distances. Based on these distances, we have the following table which illustrates the shortest path routing from $i \circ X$ to $\theta(i \circ X)$. In addition to the characteristic pairs, we need one more characteristic to distinguish the vertices. For the sake of easy description, we assume that the elements in $X$ that are greater than $D$ occur in increasing order starting $D+1, D+2, \ldots$, because a permutation on the set $\{D+1, D+2, \ldots, \Delta+1\}$ can map the vertex $X$ into this form without affecting the destination vertex or the other elements in $X$ that do not exceed $D$.

For $x_{1} \neq 1$ or $k+1$, we have

$$
\theta(i \circ X)= \begin{cases}Y_{\beta(X, 1)-1}, & \text { if } \quad i=1, \\ Y_{i}, & \text { if } \quad 1<i<k \\ Y_{\alpha(X)} & \text { if } \quad i=k, \\ Y_{i-1} & \text { if } \quad k+1 \leq i<\beta(X, 1), \\ Y_{i} & \text { if } \quad \beta(X, 1) \leq i \leq \Delta+1\end{cases}
$$

In this case, the distances are given below:

$$
d(i \circ X, \theta(i \circ X))= \begin{cases}\beta(X, 1)-1, & \text { if } i=1, \\ k-1, & \text { if } 1<i<k, i \neq x_{1}, \\ k-2, & \text { if } i=k, \\ i, & \text { if } k+1 \leq i<\beta(X, 1), \\ i-2, & \text { if } \beta(X, 1) \leq i \leq D \\ D-1 & \text { if } D<i \leq \Delta+1, i \neq k+1\end{cases}
$$

Note that the assumption on the elements in $X$ that are greater than $D$ implies that $x_{1}<k$. The detailed information on the shortest path routing from $i \circ X$ to $\theta(i \circ X)$ is given in the table below, which also includes the statistic $\beta(V, 1)$ to distinguish the vertices.

After completing the above case by case analysis, we arrive at the conclusion that the wide diameter of $\Gamma_{\Delta}(D)$ does not exceed $D+2$. To determine when the wide diameter is exactly $D+2$, let $X=(\Delta+1, \Delta, \ldots, \Delta-D+2)$ and $Y=12 \cdots D$. The distance from $i \circ Y$

Table 2: Case 2: $x_{1}=k+1$.


Table 3: $\quad$ Case 3.

| $P_{i}$ | $\alpha(v)$ | $\beta(v)$ | $\beta(v, 1)$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| (a) $i=1$ : |  |  |  |  |
| $1 \rightarrow \quad\}$ | 1 | $k+1$ | $k+1$ |  |
| $\beta(x, 1)-1 \rightarrow$ |  |  |  |  |
|  | 1 | $\beta(x, 1)$ | $\beta(x, 1)$ |  |
|  |  |  |  |  |
| (b) $1<i<k, i \neq x_{1}$ : |  |  |  |  |
| $i \rightarrow$ ) |  |  |  |  |
| $k \rightarrow \quad\}$ | $i$ | $k+1$ | $\beta(x, 1)$ | $i \neq x_{1}$ |
| $i+1 \rightarrow$ |  |  |  |  |
| $1 \rightarrow$ ) |  |  |  |  |
| $i \rightarrow \quad$ | 1 |  |  | $i+1 \neq k+1$ |
|  |  |  |  | $i+1 \neq x_{1}+1$ |
|  |  |  |  |  |
| (c) $i=k$ : |  |  |  |  |
| $k \rightarrow \quad$ |  |  |  |  |
| \} | $x_{1}$ | $k+1$ | $\beta(x, 1)$ | $d(v, Y)<k$ |
| $x_{1}+1 \rightarrow \quad$ |  |  |  |  |
| $1 \rightarrow$ ) |  |  |  |  |
| $x_{1} \rightarrow$ |  |  |  |  |
| \% | 1 | $x_{1}+1$ | $x_{1}+1$ | $x_{1}<k$ |
| $2 \rightarrow \quad$ |  |  |  |  |
| (d) $k+1 \leq i<\beta(x, 1):$ |  |  |  |  |
| $i \rightarrow \quad\}$ | $x_{1}$ | $i+1$ | $\beta(x, 1)$ | $i+1 \neq k+1$ |
| $1 \rightarrow \quad$ |  |  |  |  |
| $i-1 \rightarrow$ | 1 |  |  |  |
| $\vdots$ |  |  |  |  |
| $2 \rightarrow$, |  |  |  |  |
| (e) $\beta(x, 1) \leq i \leq D+1$ : |  |  |  |  |
|  |  |  |  |  |
| \} | $x_{1}$ | $i+1$ | $i+1$ |  |
| $2 \rightarrow$ 仡 |  |  |  |  |
| (f) $D+1<i \leq \Delta+1, i \neq k+1$ : |  |  |  |  |
| $i \rightarrow$ ) |  |  |  |  |
| $D \rightarrow$ |  |  |  |  |
|  | $i$ | $D+1$ | $D+1$ |  |
| $2 \rightarrow \quad \int$ |  |  |  |  |

is $D$ except for $i=D$. If $1<i<D$, the unique shortest path from $i \circ X$ to $Y$ goes through $23 \cdots D(\Delta+1)$. Provided that $D-2 \geq 2$, this implies that, of any $\Delta$ disjoint path from $X$ to $Y$, at least one must have length at least $D+2$.

We have left open the problem of finding the wide diameter of $\Gamma_{\Delta}(D,-r)$, which is of great interest if it is determined.

Acknowledgments. This work was performed under the auspices of the U. S. Department of Energy. We thank D. F. Hsu for helpful discussions.

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