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FINITE DIFFERENCES OF THE LOGARITHM OF THE PARTITION FUNCTION

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Abstract. Let $p(n)$ denote the partition function. DeSalvo and Pak proved that $\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}$ for $n \geq 2$. Moreover, they conjectured that a sharper inequality $\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}$ holds for $n \geq 45$. In this paper, we prove the conjecture of Desalvo and Pak by giving an upper bound for $-\Delta^2 \log p(n-1)$, where Δ is the difference operator with respect to n. We also show that for given $r \geq 1$ and sufficiently large n, $(-1)^{r-1}\Delta^r \log p(n) > 0$. This is analogous to the positivity of finite differences of the partition function. It was conjectured by Good and proved by Gupta that for given $r \geq 1$, $\Delta^r p(n) > 0$ for sufficiently large n.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^r \lambda_i = n$. Let $p(n)$ denote the number of partitions of n. In particular, we set $p(0) = 1$. The Hardy-Ramanujan-Rademacher formula for $p(n)$ states that

$$p(n) =\frac{\sqrt{12}}{24n-1}\sum_{k=1}^{N}\frac{A_k(n)}{\sqrt{k}}\left[\left(1-\frac{k}{\mu(n)}\right)e^{\mu(n)/k}+\left(1+\frac{k}{\mu(n)}\right)e^{-\mu(n)/k}\right] + R_2(n,N),$$

where $A_k(n)$ is an arithmetic function, $R_2(n, N)$ is the remainder term and

$$\mu(n) = \frac{\pi}{6}\sqrt{24n-1}; \tag{1.1}$$

see, for example, Hardy and Ramanujan [11], Rademacher [18]. Note that $A_1(n) = 1$ and $A_2(n) = (-1)^n$ for $n \geq 1$. Lehmer [14, 15] gave the error bound

$$|R_2(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}}\left[\left(\frac{N}{\mu(n)}\right)^3 \sinh\frac{\mu(n)}{N}+\frac{1}{6}-\left(\frac{N}{\mu(n)}\right)^2\right],$$

which is valid for all positive integers n and N.

Employing Rademacher's convergent series and Lehmer's error bound, DeSalvo and Pak [8] proved the following inequality conjectured by Chen [6].

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Theorem 1.1. *For $n \geq 2$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}. \tag{1.2}$$

The above relation has been improved by DeSalvo and Pak [8].

Theorem 1.2. *For $n \geq 7$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.3}$$

They also proposed the following conjecture.

Conjecture 1.3. *For $n \geq 45$, we have*

$$\frac{p(n-1)}{p(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}. \tag{1.4}$$

It should be mentioned that by using Lehmer's error bound for the remainder term of $p(n)$, Bessenrodt and Ono [5] proved the following inequality.

Theorem 1.4. *For any integers a, b satisfying $a, b > 1$ and $a + b > 9$, we have*

$$p(a)p(b) > p(a+b).$$

In this paper, we shall prove Conjecture 1.3 by giving an upper bound for $-\Delta^2 \log p(n-1)$ for $n \geq 5000$. Moreover, for any given r, we give an upper bound for $(-1)^{r-1}\Delta^r \log p(n)$.

In 1977, Good [9] conjectured that $\Delta^r p(n)$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive. Using the Hardy-Rademacher series [19] for $p(n)$, Gupta [10] proved that for any given r, $\Delta^r p(n) > 0$ for sufficiently large n. In 1988, Odlyzko [16] proved the conjecture of Good and obtained the following asymptotic formula for $n(r)$:

$$n(r) \sim \frac{6}{\pi^2} r^2 \log^2 r \quad \text{as } r \to \infty.$$

Knessl and Keller [12, 13] obtained an approximation $n(r)'$ for $n(r)$ for which $|n(r)' - n(r)| \leq 2$ up to $r = 75$. Almkvist [2, 3] proved that $n(r)$ satisfies certain equations.

By using the bounds of the modified Bessel function of the first kind, we shall prove that for any given $r \geq 1$, there exists a positive integer $n(r)$ such that $(-1)^{r-1}\Delta^r \log p(n) > 0$ for $n \geq n(r)$.

2. Proof of Conjecture 1.3

In this section, we give a proof of Conjecture 1.3 by using an inequality of DeSalvo and Pak [8]. Letting

$$p_2(n) = 2\log p(n) - \log p(n-1) - \log p(n+1),$$

DeSalvo and Pak have shown that for $n \geq 50$,

$$\begin{aligned} p_2(n) < {} & \frac{24\pi}{(24(n-1)-1)^{3/2}} + \frac{288\pi(-3+\pi\sqrt{24(n-1)-1})}{(24(n-1)-1)^{3/2}(-6+\pi\sqrt{24(n-1)-1})^2} \\ & - \frac{864}{(24(n+1)-1)^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}. \end{aligned} \tag{2.1}$$

We shall give an estimate of the right hand side of (2.1), leading to a proof of the conjecture.

Proof of Conjecture 1.3. The conjecture can be restated as follows:

$$p_2(n) < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right), \tag{2.2}$$

where $n \geq 45$. We proceed to give an estimate of each term of the right hand side of (2.1).

We begin with the first term. We claim that for $n \geq 50$,

$$\frac{24\pi}{(24(n-1)-1)^{3/2}} < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2 + \frac{3}{2n^{5/2}}. \tag{2.3}$$

For $0 < x \leq \frac{1}{48}$, it can be easily checked that

$$\frac{1}{(1-x)^{3/2}} < 1 + \frac{3}{2}x + \frac{3}{8}x^{3/2}. \tag{2.4}$$

For $n \geq 50$, we have $\frac{25}{24n} \leq \frac{1}{48}$, and hence we can apply (2.4) to deduce that

$$\begin{aligned}\frac{24\pi}{(24(n-1)-1)^{3/2}} &= \frac{24\pi}{(24n)^{3/2}\left(1-\frac{25}{24n}\right)^{3/2}} \\ &< \frac{24\pi}{(24n)^{3/2}}\left(1 + \frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{3/2}\right).\end{aligned} \tag{2.5}$$

For $n \geq 50$, we have

$$\frac{3}{8}\left(\frac{25}{24n}\right)^{3/2} < \frac{3}{8}\left(\frac{25}{24}\right)^{3/2}\frac{1}{50^{1/2}n},$$

$$\frac{24\pi}{(24n)^{3/2}} < \frac{24\pi}{(24)^{3/2}50^{1/2}n}.$$

It follows that

$$\begin{aligned}&\frac{24\pi}{(24n)^{3/2}}\left(\frac{75}{48n} + \frac{3}{8}\left(\frac{25}{24n}\right)^{3/2} + \frac{24\pi}{(24n)^{3/2}}\right) \\ &\leq \frac{24\pi}{(24n)^{3/2}n}\left(\frac{25}{16} + \frac{3}{8}\left(\frac{25}{24}\right)^{3/2}\frac{1}{50^{1/2}} + \frac{24\pi}{(24)^{3/2}50^{1/2}}\right) \\ &< \frac{3}{2n^{5/2}}.\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain (2.3).

As for the second term of the right hand side of (2.1), it can be shown that for $n \geq 50$,

$$\frac{288\pi(-3+\pi\sqrt{24(n-1)-1})}{(24(n-1)-1)^{3/2}(-6+\pi\sqrt{24(n-1)-1})^2} < \frac{1}{2n^2} + \frac{1}{n^{5/2}}. \tag{2.7}$$

To this end, we need the following inequality for $\alpha \geq \frac{1}{2}$ and $0 < x \leq c < 1$:

$$\frac{1}{(1-x)^{\alpha}} \leq 1 + \left(\frac{1}{1-c}\right)^{\alpha+1} \alpha x. \tag{2.8}$$

Let

$$f(x) = \frac{1}{(1-x)^{\alpha}} - 1 - \left(\frac{1}{1-c}\right)^{\alpha+1} \alpha x.$$

For $\alpha \geq \frac{1}{2}$ and $0 \leq x \leq c < 1$, we see that

$$f'(x) = \frac{\alpha}{(1-x)^{\alpha+1}} - \left(\frac{1}{1-c}\right)^{\alpha+1} \alpha \leq 0.$$

Since $f(0) = 0$, we obtain that $f(x) \leq 0$ under the above assumption. This yields that $f(x) < 0$ for $0 < x \leq c < 1$ and $\alpha \geq \frac{1}{2}$, and hence (2.8) is proved.

The left hand side of (2.7) can be rewritten as

$$\frac{144\pi^2\sqrt{24n-25}}{(24n-25)^{3/2}(-6+\pi\sqrt{24n-25})^2} + \frac{288\pi(-3+\frac{\pi}{2}\sqrt{24n-25})}{(24n-25)^{3/2}(-6+\pi\sqrt{24n-25})^2},$$

which can be simplified to

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} + \frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)}. \tag{2.9}$$

Setting $x = \frac{25}{24n}$, $\alpha = 2$ and $c = \frac{1}{48}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we find that for $n \geq 50$,

$$\frac{1}{\left(1-\frac{25}{24n}\right)^2} \leq 1 + \left(\frac{48}{47}\right)^3 \frac{25}{12n}. \tag{2.10}$$

Setting $x = \frac{6}{\pi\sqrt{24n-25}}$, $\alpha = 2$ and $c = \frac{1}{15}$, for $n \geq 50$, we also have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. Again, using (2.8), we see that for $n \geq 50$,

$$\frac{1}{\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} < 1 + \left(\frac{15}{14}\right)^3 \frac{6}{\pi\sqrt{24n-25}} < 1 + \frac{24}{\pi\sqrt{24n-25}}. \tag{2.11}$$

Combining (2.10) and (2.11), we deduce that for $n \geq 50$,

$$\begin{aligned} &\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} \\ &\leq \frac{1}{4n^2}\left(1+\left(\frac{48}{47}\right)^3\frac{25}{12n}\right)\left(1+\frac{24}{\pi\sqrt{24n-25}}\right). \end{aligned} \tag{2.12}$$

It is easily seen that

$$\frac{24}{\pi\sqrt{24n-25}} = \frac{24}{\pi(24n)^{1/2}} \frac{1}{\left(1-\frac{25}{24n}\right)^{1/2}}. \tag{2.13}$$

Setting $x = \frac{25}{24n}$, $\alpha = \frac{1}{2}$ and $c = \frac{1}{48}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), for $n \geq 50$, we get

$$\frac{1}{\left(1-\frac{25}{24n}\right)^{1/2}} < 1 + \left(\frac{48}{47}\right)^{3/2} \frac{25}{48n}. \tag{2.14}$$

Combining (2.12), (2.13) and (2.14), we find that for $n \geq 50$,

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2}$$
$$\leq \frac{1}{4n^2}\left(1+\left(\frac{48}{47}\right)^3\frac{25}{12n}\right)\left(1+\frac{24}{\pi(24n)^{1/2}}\left(1+\left(\frac{48}{47}\right)^{3/2}\frac{25}{48n}\right)\right). \tag{2.15}$$

The right hand side of (2.15) can be expanded as follows:

$$\frac{1}{4n^2}+\frac{\sqrt{6}}{2\pi n^{5/2}}+\frac{25}{48n^3}\left(\frac{48}{47}\right)^3+\frac{25\sqrt{6}}{96\pi n^{7/2}}\left(\frac{48}{47}\right)^{3/2}$$
$$+\frac{25\sqrt{6}}{24\pi n^{7/2}}\left(\frac{48}{47}\right)^3+\frac{25^2\sqrt{24}}{48^2\pi n^{9/2}}\left(\frac{48}{47}\right)^{9/2}. \tag{2.16}$$

Clearly, for $\alpha > \frac{5}{2}$ and $n \geq 50$,

$$\frac{1}{n^\alpha} \leq \frac{1}{50^{\alpha-5/2}n^{5/2}},$$

which implies that for $n \geq 50$,

$$\frac{1}{n^3} \leq \frac{1}{50^{1/2}n^{5/2}}, \tag{2.17}$$

$$\frac{1}{n^{7/2}} \leq \frac{1}{50n^{5/2}}, \tag{2.18}$$

$$\frac{1}{n^{9/2}} \leq \frac{1}{50^2n^{5/2}}. \tag{2.19}$$

Applying (2.17), (2.18) and (2.19) to the last four terms of (2.16), we obtain that for $n \geq 50$,

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)^2} < \frac{1}{4n^2}+\frac{1}{2n^{5/2}}. \tag{2.20}$$

Setting $x = \frac{6}{\pi\sqrt{24n-25}}$, $\alpha = 1$ and $c = \frac{1}{15}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we see that for $n \geq 50$,

$$\frac{1}{1-\frac{6}{\pi\sqrt{24n-25}}} < 1+\left(\frac{15}{14}\right)^2\frac{6}{\pi\sqrt{24n-25}} < 1+\frac{12}{\pi\sqrt{24n-25}}. \tag{2.21}$$

Using (2.21) and the same argument as in the derivation of (2.20), it can be shown that for $n \geq 50$,

$$\frac{1}{4n^2\left(1-\frac{25}{24n}\right)^2\left(1-\frac{6}{\pi\sqrt{24n-25}}\right)} < \frac{1}{4n^2}+\frac{1}{2n^{5/2}}. \tag{2.22}$$

In view of (2.20) and (2.22), we arrive at (2.7).

To estimate the third term of the right hand side of (2.1), we aim to show that for $n \geq 50$,

$$-\frac{864}{(24(n+1)-1)^2} < \frac{1}{2n^{5/2}}-\frac{3}{2n^2}. \tag{2.23}$$

It is easily verified that for $\alpha \geq 1/2$ and $0 \leq x \leq 1$,

$$1 \geq \frac{1}{(1+x)^{\alpha}} \geq 1 - \alpha x. \tag{2.24}$$

So for $n \geq 50$, we have

$$\frac{1}{\left(1+\frac{23}{24n}\right)^2} \geq 1 - \frac{23}{12n}.$$

Consequently, for $n \geq 50$,

$$-\frac{864}{(24(n+1)-1)^2} = -\frac{3}{2n^2\left(1+\frac{23}{24n}\right)^2} \leq \frac{23}{8n^3} - \frac{3}{2n^2} \leq \frac{1}{2n^{5/2}} - \frac{3}{2n^2}.$$

Utilizing the above upper bounds (2.3), (2.7) and (2.23) for the three terms of the right hand side of (2.1), we conclude that for $n \geq 50$,

$$p_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2 - \frac{1}{n^2} + \frac{3}{n^{5/2}} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}}.$$

Next we show that for $n \geq 5000$,

$$p_2(n) < \frac{24\pi}{(24n)^{3/2}} - \left(\frac{24\pi}{(24n)^{3/2}}\right)^2. \tag{2.25}$$

Clearly, for $n \geq 100$,

$$-\frac{1}{n^2} + \frac{3}{n^{5/2}} < -\frac{2}{3n^2}.$$

To prove that for $n \geq 5000$,

$$-\frac{2}{3n^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2n}{3}}} < 0, \tag{2.26}$$

let

$$g(x) = -\frac{2}{3x^2} + 2e^{-\frac{\pi}{10}\sqrt{\frac{2x}{3}}}.$$

The equation $g(x) = 0$ has two solutions:

$$x_1 = \frac{2400}{\pi^2}\left(W_0\left(-\frac{\pi\sqrt{2}}{40\cdot 3^{3/4}}\right)\right)^2,$$

$$x_2 = \frac{2400}{\pi^2}\left(W_{-1}\left(-\frac{\pi\sqrt{2}}{40\cdot 3^{3/4}}\right)\right)^2,$$

where $W_0(z)$ and $W_{-1}(z)$ are two branches of Lambert W function $W(z)$; see Corless, Gonnet, Hare, Jeffrey and Knuth [7]. More explicitly, we have $x_1 \approx 0.64$ and $x_2 \approx 4996.47$. It can be checked that $g(5000) < 0$. Thus for $x \geq 5000$,

$$g(x) < 0.$$

This proves (2.26). Hence (2.25) holds.

Using (2.25), we shall show that inequality (2.2) holds for $n \geq 5000$. It is easily verified that for $x > 0$,

$$x(1-x) < \log(1+x). \tag{2.27}$$

Let

$$h(x) = \log(1+x) - x + x^2.$$

For $x \geq 0$, we see that

$$h'(x) = \frac{x+2x^2}{1+x} \geq 0.$$

Since $h(0) = 0$, we have $h(x) > 0$ for $x > 0$. Combining (2.25) and (2.27), we deduce that for $n \geq 5000$,

$$p_2(n) < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right).$$

Since DeSalvo and Pak [8] have verified the above relation for $45 \leq n \leq 8000$, we reach the conclusion that inequality (2.2) holds for $n \geq 45$, and hence the proof is complete. □

3. An upper bound for $(-1)^{r-1}\Delta^r \log p(n)$

The conjecture of DeSalvo and Pak can be formulated as an upper bound for $2\log p(n) - \log p(n-1) - \log p(n+1)$; namely, for $n \geq 45$,

$$-\Delta^2 \log p(n-1) < \log\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right), \tag{3.1}$$

where Δ is the difference operator as given by $\Delta f(n) = f(n+1) - f(n)$.

In this section, we give an upper bound for $(-1)^{r-1}\Delta^r \log p(n)$. When $r = 2$, this upper bound reduces to the above relation (3.1). In the following theorem, we adopt the notation $(a)_k$ for the rising factorial, namely, $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

Theorem 3.1. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) < \log\left(1 + \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

In the proof of the above theorem, we shall use the Hardy-Ramanujan-Rademacher series for $n \geq 1$,

$$p(n) = 2\pi\left(\frac{\pi}{12}\right)^{3/2}\sum_{k=1}^{\infty} A_k(n)k^{-5/2}L_{3/2}\left(\frac{\pi^2}{6k^2}\left(n - \frac{1}{24}\right)\right), \tag{3.2}$$

and the estimate for $A_k(n)$,

$$|A_k(n)| \leq 2k^{3/4}; \tag{3.3}$$

see Rademacher [19]. Note that $A_k(n) = 1$ in (3.2) are the same as the Hardy-Ramanujan-Rademacher formula in the previous section. The function $L_\nu(x)$ in (3.2) is defined by

$$L_\nu(x) = \sum_{m=0}^{\infty}\frac{x^m}{m!\Gamma(m+\nu+1)}, \tag{3.4}$$

where $\Gamma(m+\nu+1)$ is the Gamma function.

With the notation of $\mu(n)$ as in (1.1), we have

$$\frac{\pi^2}{6}\left(n - \frac{1}{24}\right) = \frac{\mu^2(n)}{4},$$

and so (3.2) can be rewritten as

$$p(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} \sum_{k=1}^{\infty} A_k(n) k^{-5/2} L_{3/2}\left(\frac{\mu^2(n)}{4k^2}\right). \tag{3.5}$$

Denote the kth summand in (3.5) by $f_k(n)$, namely,

$$f_k(n) = 2\pi \left(\frac{\pi}{12}\right)^{3/2} A_k(n) k^{-5/2} L_{3/2}\left(\frac{\mu^2(n)}{4k^2}\right). \tag{3.6}$$

Then (3.5) can be restated as

$$p(n) = f_1(n)\left(1 + \frac{f_2(n)}{f_1(n)}\right)\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n) + f_2(n)}\right). \tag{3.7}$$

It is known that

$$L_{3/2}(x) = \frac{1}{\sqrt{\pi}} \frac{d}{dx}\left(\frac{\sinh 2\sqrt{x}}{\sqrt{x}}\right);$$

see Abramowitz and Stegun [1] or Almkvist [2]. Since $A_1(n) = 1$, $f_1(n)$ can be expressed as

$$f_1(n) = \frac{\sqrt{12}}{24n-1}\left[\left(1 - \frac{1}{\mu(n)}\right)e^{\mu(n)} + \left(1 + \frac{1}{\mu(n)}\right)e^{-\mu(n)}\right]. \tag{3.8}$$

Recalling $A_2(n) = (-1)^n$, by (3.4) and (3.6) we obtain that for $n \geq 1$,

$$f_1(n) - |f_2(n)| = 2\pi\left(\frac{\pi}{12}\right)^{3/2} \sum_{m=0}^{\infty}\left(\frac{1}{4^m} - \frac{1}{2^{5/2}16^m}\right)\frac{\mu^{2m}(n)}{m!\Gamma(m+5/2)}.$$

Clearly, $\frac{1}{4^m} - \frac{1}{2^{5/2}16^m} > 0$ for $m \geq 0$. Hence for $n \geq 1$,

$$f_1(n) - |f_2(n)| > 0, \tag{3.9}$$

which implies that for $n \geq 1$, $f_1(n)$ is positive and

$$f_1(n) + f_2(n) > 0.$$

It is also clear that for $n \geq 1$, both $\mu(n) - 1$ and $1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}$ are positive. Applying (3.8) to (3.7), we obtain that for $n \geq 1$,

$$\begin{aligned}\log p(n) = &\log\frac{\pi^2}{6\sqrt{3}} - 3\log\mu(n) + \log(\mu(n)-1) + \mu(n)\\ &+ \log\left(1 + \frac{\mu(n)+1}{\mu(n)-1}e^{-2\mu(n)}\right) + \log\left(1 + \frac{f_2(n)}{f_1(n)}\right)\\ &+ \log\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}\right).\end{aligned}$$

Hence

$$(-1)^{r-1}\Delta^r \log p(n) = H_r + F_1 + F_2 + F_3, \tag{3.10}$$

where

$$H_r = (-1)^{r-1}\Delta^r \left(-3\log\mu(n) + \log(\mu(n)-1) + \mu(n)\right),$$

$$F_1 = (-1)^{r-1}\Delta^r \log\left(1 + \frac{\mu(n)+1}{\mu(n)-1}e^{-2\mu(n)}\right),$$

$$F_2 = (-1)^{r-1}\Delta^r \log\left(1 + \frac{f_2(n)}{f_1(n)}\right),$$

$$F_3 = (-1)^{r-1}\Delta^r \log\left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}\right).$$

Let

$$G_r = F_1 + F_2 + F_3. \tag{3.11}$$

To estimate $(-1)^{r-1}\Delta^r \log p(n)$, we shall give upper bounds for H_r and G_r. We first consider G_r.

Theorem 3.2. *For $n \geq 50$, we have*

$$|G_r| < 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{3.12}$$

To prove Theorem 3.2, we recall a monotone property of the ratio of two power series; see Ponnusamy and Vuorinen [17]. We also need a lower bound and an upper bound on the ratio of $L_\nu(x)$ and $L_\nu(y)$, which can be deduced from known bounds on the ratio of two modified Bessel functions of the first kind.

Proposition 3.3. *Suppose that the power series*

$$f(x) = \sum_{m=0}^{\infty} \alpha_m x^m \quad \text{and} \quad g(x) = \sum_{m=0}^{\infty} \beta_m x^m$$

both converge for $|x| < \infty$ and $\beta_m > 0$ for all $m > 0$. Then the function $\frac{f(x)}{g(x)}$ is strictly decreasing for $x > 0$ if the sequence $\{\alpha_m/\beta_m\}_{m=0}^{\infty}$ is strictly decreasing.

Let $I_\nu(x)$ be the modified Bessel function of the first kind as given by

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^m}{m!\Gamma(m+\nu+1)};$$

see Watson [20]. It is known that for $\nu \geq 1/2$ and $0 < x < y$, $I_\nu(x)$ increases with x and

$$e^{x-y}\left(\frac{x}{y}\right)^\nu < \frac{I_\nu(x)}{I_\nu(y)} < e^{x-y}\left(\frac{y}{x}\right)^\nu;$$

see Baricz [4, inequalities 2.2 and 2.4]. For $x > 0$, from (3.4) we see that $L_\nu(x)$ can be expressed by $I_\nu(x)$:

$$L_\nu(x) = x^{-\nu/2} I_\nu(2\sqrt{x}).$$

Thus the above properties of $I_\nu(x)$ can be restated in terms of $L_\nu(x)$.

Proposition 3.4. *For $\nu \geq 1/2$ and $0 < x < y$, we have*

$$e^{2\sqrt{x}-2\sqrt{y}} < \frac{L_\nu(x)}{L_\nu(y)} < e^{2\sqrt{x}-2\sqrt{y}}\left(\frac{y}{x}\right)^\nu.$$

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Since $|G_r| \leq |F_1| + |F_2| + |F_3|$, in order to estimate G_r, we shall estimate $|F_1|$, $|F_2|$ and $|F_3|$. By the definition of $f_k(n)$, we have

$$|f_k(n)| = 2\pi \left(\frac{\pi}{12}\right)^{3/2} |A_k(n)| k^{-5/2} L_{3/2}\left(\frac{\mu(n)^2}{4k^2}\right).$$

It follows from (3.3) that for $n \geq 1$,

$$|f_k(n)| \leq 4\pi \left(\frac{\pi}{12}\right)^{3/2} k^{-7/4} L_{3/2}\left(\frac{\mu(n)^2}{4k^2}\right),$$

which yields that

$$\sum_{k=3}^{\infty} |f_k(n)| \leq 4\pi \left(\frac{\pi}{12}\right)^{3/2} \zeta(7/4) L_{3/2}\left(\frac{\mu(n)^2}{36}\right), \tag{3.13}$$

where $\zeta(x)$ is the Riemann zeta function. For convenience, we denote by $g(n)$ the right hand side of the above inequality, so that (3.13) becomes

$$\sum_{k=3}^{\infty} |f_k(n)| \leq g(n). \tag{3.14}$$

To estimate F_1, F_2 and F_3, we shall make use of the monotonicity of $\frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}$, $\frac{|f_2(n)|}{f_1(n)}$ and $\frac{g(n)}{f_1(n)-|f_2(n)|}$. It is easily seen that $\frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}$ decreases with n for $n \geq 1$, since $\frac{y+1}{y-1} e^{-2y}$ decreases with y for $y > 0$ and $\mu(n)$ increases with n. By (3.6), we have

$$\frac{|f_2(n)|}{f_1(n)} = \frac{L_{3/2}(\mu^2(n)/16)}{2^{5/2} L_{3/2}(\mu^2(n)/4)}.$$

The ratio of the coefficients of x^m in $L_{3/2}(\mu^2(n)/16)$ and $L_{3/2}(\mu^2(n)/4)$ is $\frac{4^m}{16^m}$. By Proposition 3.3, we see that $\frac{L_{3/2}(y/16)}{L_{3/2}(y/4)}$ decreases with y for $y > 0$. Notice that $\mu^2(x)$ increases with x for $x \geq 1$. So $\frac{L_{3/2}(\mu^2(x)/16)}{L_{3/2}(\mu^2(x)/4)}$ decreases with x for $x \geq 1$. This implies that $\frac{|f_2(n)|}{f_1(n)}$ decreases with n.

Next we prove the monotonicity of $\frac{g(n)}{f_1(n)-|f_2(n)|}$. Recall that

$$\frac{g(n)}{f_1(n) - |f_2(n)|} = \frac{2\zeta(7/4) L_{3/2}(\mu^2(n)/36)}{L_{3/2}(\mu^2(n)/4) - 2^{-5/2} L_{3/2}(\mu^2(n)/16)}.$$

The ratio of the coefficients of x^m in $L_{3/2}(y/36)$ and $L_{3/2}(y/4) - 2^{-5/2} L_{3/2}(y/16)$ equals

$$\frac{\frac{1}{36^m}}{\frac{1}{4^m} - \frac{1}{2^{5/2} 16^m}},$$

which decreases with m for $m \geq 0$. By Proposition 3.3, we deduce that for $y > 0$,

$$\frac{L_{3/2}(y/36)}{L_{3/2}(y/4) - 2^{-5/2} L_{3/2}(y/16)}$$

decreases with y. Hence $\frac{g(n)}{f_1(n)-|f_2(n)|}$ decreases with n for $n \geq 1$.

Using the above monotone properties, we proceed to derive upper bounds for $|F_1|$, $|F_2|$ and $|F_3|$. It is known that for $0 < x < 1$,

$$\log(1-x) \geq \frac{-x}{1-x}, \tag{3.15}$$

$$|\log(1 \pm x)| \leq -\log(1-x); \tag{3.16}$$

see DeSalvo and Pak [8].

We first estimate F_1. Since

$$\Delta^r f(n) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(n+k),$$

we have

$$F_1 = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \log\left(1 + \frac{\mu(n+k)+1}{\mu(n+k)-1} e^{-2\mu(n+k)}\right).$$

It follows that

$$|F_1| \leq \sum_{k=0}^{r} \binom{r}{k} \log\left(1 + \frac{\mu(n+k)+1}{\mu(n+k)-1} e^{-2\mu(n+k)}\right). \tag{3.17}$$

By the monotonicity of $\frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}$, we see that for $n \geq 1$ and $0 \leq k \leq r$,

$$\log\left(1 + \frac{\mu(n+k)+1}{\mu(n+k)-1} e^{-2\mu(n+k)}\right) \leq \log\left(1 + \frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}\right). \tag{3.18}$$

Applying (3.18) to (3.17), we find that for $n \geq 1$,

$$|F_1| \leq 2^r \log\left(1 + \frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}\right).$$

Since $\log(1+x) \leq x$ for $x \geq 0$, we see that for $n \geq 1$,

$$|F_1| \leq 2^r \frac{\mu(n)+1}{\mu(n)-1} e^{-2\mu(n)}. \tag{3.19}$$

To estimate F_2, we begin with the following expression:

$$F_2 = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \log\left(1 + \frac{f_2(n+k)}{f_1(n+k)}\right). \tag{3.20}$$

It follows from (3.9) that

$$0 < 1 - \frac{|f_2(n)|}{f_1(n)} < 1.$$

Using (3.16), we find that for $n \geq 1$,

$$\left|\log\left(1 + \frac{f_2(n+k)}{f_1(n+k)}\right)\right| \leq -\log\left(1 - \frac{|f_2(n+k)|}{f_1(n+k)}\right). \tag{3.21}$$

Combining (3.20) and (3.21), we obtain that for $n \geq 1$,

$$|F_2| \leq -\sum_{k=0}^{r} \binom{r}{k} \log\left(1 - \frac{|f_2(n+k)|}{f_1(n+k)}\right).$$

In view of the monotonicity of $\frac{|f_2(n)|}{f_1(n)}$, we see that for $n \geq 1$,

$$|F_2| \leq -2^r \log\left(1 - \frac{|f_2(n)|}{f_1(n)}\right).$$

Hence, by (3.15), we obtain that for $n \geq 1$,

$$|F_2| \leq 2^r \frac{|f_2(n)|}{f_1(n) - |f_2(n)|}. \tag{3.22}$$

To estimate F_3, we use the following expression:

$$F_3 = \sum_{k=0}^{r} (-1)^{k+1} \binom{r}{k} \log \left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n+k)}{f_1(n+k) + f_2(n+k)}\right). \tag{3.23}$$

By Proposition 3.4, we find that for $n \geq 1$,

$$2^{-\frac{5}{2}} e^{-\frac{\mu(n)}{2}} < \frac{|f_2(n)|}{f_1(n)} < \sqrt{2} e^{-\frac{\mu(n)}{2}} \tag{3.24}$$

and

$$2\zeta(7/4) e^{-\frac{2\mu(n)}{3}} < \frac{g(n)}{f_1(n)} < 54\zeta(7/4) e^{-\frac{2\mu(n)}{3}}. \tag{3.25}$$

Consequently, for $n \geq 1$,

$$\frac{|f_2(n)|}{f_1(n)} + \frac{g(n)}{f_1(n)} < \sqrt{2} e^{-\frac{\mu(n)}{2}} + 54\zeta(7/4) e^{-\frac{2\mu(n)}{3}}. \tag{3.26}$$

For $n \geq 50$, it can be checked that

$$\sqrt{2} e^{-\frac{\mu(n)}{2}} + 54\zeta(7/4) e^{-\frac{2\mu(n)}{3}} < 1. \tag{3.27}$$

Combining (3.26) and (3.27), we obtain that for $n \geq 50$,

$$\frac{|f_2(n)|}{f_1(n)} + \frac{g(n)}{f_1(n)} < 1,$$

or equivalently,

$$f_1(n) - |f_2(n)| - g(n) > 0. \tag{3.28}$$

Combining (3.14) and (3.28), we see that for $n \geq 50$,

$$f_1(n) - |f_2(n)| - |\sum_{k\geq 3}^{\infty} f_k(n)| > 0,$$

which can be rewritten as

$$1 \geq 1 - \frac{\left|\sum_{k\geq 3}^{\infty} f_k(n)\right|}{f_1(n) - |f_2(n)|} > 0.$$

Thus, we can use (3.16) to deduce that for $n \geq 50$,

$$\left|\log \left(1 + \frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n) + f_2(n)}\right)\right| \leq -\log \left(1 - \frac{|\sum_{k\geq 3}^{\infty} f_k(n)|}{f_1(n) - |f_2(n)|}\right). \tag{3.29}$$

Since $-\log(1-x)$ is increasing for $x > -1$, according to (3.14) and (3.29), we deduce that for $n \geq 50$,

$$-\log \left(1 - \frac{|\sum_{k\geq 3}^{\infty} f_k(n)|}{f_1(n) - |f_2(n)|}\right) < -\log \left(1 - \frac{g(n)}{f_1(n) - |f_2(n)|}\right). \tag{3.30}$$

Combining (3.29) and (3.30), we see that for $n \geq 50$,

$$\left|\log\left(1+\frac{\sum_{k\geq 3}^{\infty} f_k(n)}{f_1(n)+f_2(n)}\right)\right| \leq -\log\left(1-\frac{g(n)}{f_1(n)-|f_2(n)|}\right). \tag{3.31}$$

It follows from (3.23) and (3.31) that for $n \geq 50$,

$$|F_3| \leq -\sum_{k=0}^{r}\binom{r}{k}\log\left(1-\frac{g(n+k)}{f_1(n+k)-|f_2(n+k)|}\right).$$

Based on the monotonicity of $\frac{g(n)}{f_1(n)-|f_2(n)|}$, we find that for $n \geq 50$,

$$|F_3| \leq -2^r\log\left(1-\frac{g(n)}{f_1(n)-|f_2(n)|}\right).$$

Hence, by (3.15), we obtain that for $n \geq 50$,

$$|F_3| \leq 2^r\frac{g(n)}{f_1(n)-|f_2(n)|-g(n)}. \tag{3.32}$$

By Proposition 3.4, we see that for $n \geq 1$,

$$2^{\frac{7}{2}}\zeta(7/4)e^{-\frac{\mu(n)}{6}} < \frac{g(n)}{|f_2(n)|} < 27\sqrt{2}\zeta(7/4)e^{-\frac{\mu(n)}{6}}. \tag{3.33}$$

In view of (3.19) and (3.24), we deduce that for $n \geq 50$,

$$\frac{|F_1|}{F_4} < 2^{\frac{5}{2}}\frac{\mu(n)+1}{\mu(n)-1}e^{-\frac{3}{2}\mu(n)}, \tag{3.34}$$

where F_4 is defined by

$$F_4 = 2^r\frac{|f_2(n)|}{f_1(n)}.$$

As a consequence of (3.22) and (3.24), it can be checked that for $n \geq 50$,

$$\frac{|F_2|}{F_4} < \frac{1}{1-\sqrt{2}e^{-\frac{\mu(n)}{2}}}. \tag{3.35}$$

Applying (3.24), (3.25) and (3.33) to (3.32), we obtain that for $n \geq 50$,

$$\frac{|F_3|}{F_4} < \frac{27\sqrt{2}\zeta(7/4)}{e^{\frac{\mu(n)}{6}}-\sqrt{2}e^{-\frac{\mu(n)}{3}}-54\zeta(7/4)e^{-\frac{\mu(n)}{2}}}. \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), we conclude that for $n \geq 50$,

$$|F_1|+|F_2|+|F_3| < 5F_4. \tag{3.37}$$

It follows from (3.24) that for $n \geq 1$,

$$F_4 < 2^{r+\frac{1}{2}}e^{-\frac{\mu(n)}{2}}. \tag{3.38}$$

Thus (3.37) and (3.38) lead to an upper bound for $|F_1|+|F_2|+|F_3|$. This completes the proof. □

To prove Theorem 3.1, we still need to estimate H_r and we shall use two inequalities due to Odlyzko [16] on the relations between the higher order differences and derivatives.

Proposition 3.5. *Let r be a positive integer. Suppose that $f(x)$ is a function with infinite continuous derivatives for $x \geq 1$, and $(-1)^{k-1} f^{(k)}(x) > 0$ for $k \geq 1$. Then for $r > 1$,*

$$(-1)^{r-1} f^{(r)}(x+r) \leq (-1)^{r-1} \Delta^r f(x) \leq (-1)^{r-1} f^{(r)}(x).$$

Proof of Theorem 3.1. First, we treat the case $r = 1$, which states that for $n \geq 12$,

$$\Delta \log p(n) < \log \left(1 + \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}}\right). \tag{3.39}$$

Since we have estimated $|G_r|$, we only need to estimate H_r for $r = 1$. By Proposition 3.5, we have

$$H_1 \leq \frac{2\pi}{\sqrt{24n-1}} - \frac{36}{24(n+1)-1} + \frac{12}{(24n-1)(1-\frac{6}{\pi\sqrt{24n-1}})}. \tag{3.40}$$

We claim that for $n \geq 50$,

$$H_1 < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{1}{n+1} + \frac{5}{4\,(n+1)^{3/2}}. \tag{3.41}$$

We proceed to estimate each term of the right hand side of (3.40). For the first term, we need to show that for $n \geq 50$,

$$\frac{2\pi}{\sqrt{24n-1}} < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{3}{2(n+1)}. \tag{3.42}$$

Setting $x = \frac{25}{24(n+1)}$, $\alpha = 1/2$ and $c = \frac{1}{48}$, for $n \geq 50$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. It follows from (2.8) that for $n \geq 50$,

$$\begin{aligned}\frac{2\pi}{\sqrt{24n-1}} &= \frac{2\pi}{\sqrt{24}\,(n+1)^{1/2}\left(1-\frac{25}{24(n+1)}\right)^{1/2}}\\ &\leq \frac{2\pi}{\sqrt{24}\,(n+1)^{1/2}}\left(1+\left(\frac{48}{47}\right)^{3/2}\frac{25}{48(n+1)}\right).\end{aligned}$$

This proves (3.42).

For the second term of the right hand side of (3.40), for $n \geq 50$, we have

$$-\frac{36}{24(n+1)-1} < -\frac{3}{2(n+1)}. \tag{3.43}$$

For the last term of the right hand side of (3.40), using the same argument as in the proof of (2.20), we obtain that for $n \geq 50$,

$$\frac{12}{(24n-1)(1-\frac{6}{\pi\sqrt{24n-1}})} < \frac{1}{2(n+1)} + \frac{1}{2\,(n+1)^{3/2}}. \tag{3.44}$$

Combining (3.42), (3.43) and (3.44), we arrive at (3.41).

By the estimate of H_1 in (3.41) and the estimate of G_1 in (3.12), we find that for $n \geq 50$,

$$\Delta \log p(n) < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{1}{n+1} + \frac{5}{4\,(n+1)^{3/2}} + 10\sqrt{2}e^{-\frac{\pi}{12}\sqrt{(24n-1)}}.$$

Notice that for $n \geq 200$,

$$\frac{5}{4\,(n+1)^{3/2}} < \frac{12-\pi^2}{24(n+1)},$$

and for $n \geq 50$,

$$10\sqrt{2}e^{-\frac{\pi}{12}\sqrt{(24n-1)}} < \frac{12-\pi^2}{24(n+1)}.$$

Hence, for $n \geq 200$,

$$\Delta \log p(n) < \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{\pi^2}{12(n+1)}. \tag{3.45}$$

Moreover, it can be easily checked that for $x > 0$,

$$x\left(1 - \frac{x}{2}\right) < \log(1+x).$$

Thus, for $n \geq 1$,

$$\frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}} - \frac{\pi^2}{12(n+1)} < \log\left(1 + \frac{\sqrt{6}\pi}{6\,(n+1)^{1/2}}\right).$$

Combining the above relation and (3.45), we reach (3.39) for $n \geq 200$.

It can be checked that (3.39) is valid for $12 \leq n \leq 200$, and so Theorem 3.1 holds for $r = 1$.

We now turn to the case $r \geq 2$. We proceed to show that there exists an integer $n(r)$ such that for $n \geq n(r)$,

$$(-1)^{r-1}\Delta^r \log p(n) < U_r, \tag{3.46}$$

where

$$U_r = \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\left(1 - \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

Since $x(1-x) < \log(1+x)$ for $x > 0$, we have that for $n \geq 1$,

$$U_r < \log\left(1 + \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{(n+1)^{r-\frac{1}{2}}}\right).$$

Thus (3.46) implies Theorem 3.1 for $r \geq 2$.

By (3.10), we see that for $n \geq 1$,

$$(-1)^{r-1}\Delta^r \log p(n) \leq H_r + |G_r|.$$

To prove (3.46), it suffices to show that for $n \geq n(r)$,

$$H_r + |G_r| < U_r. \tag{3.47}$$

Since Theorem 3.2 gives an upper bound for $|G_r|$, we need an upper bound for H_r. Recall that for $n \geq 1$,

$$H_r = (-1)^{r-1}\Delta^r\left(-3\log\mu(n) + \log(\mu(n)-1) + \mu(n)\right). \tag{3.48}$$

For $x \geq 1$, write

$$\log(\mu(x)-1) = \log\mu(x) - \sum_{k=1}^{\infty}\frac{1}{k\mu(x)^k}.$$

By exchanging the order of summations, it can be seen that for $x \geq 1$,

$$\Delta^r \log(\mu(x) - 1) = \Delta \log \mu(n) - \sum_{k=1}^{\infty} \Delta^r \left(\frac{1}{k\mu(n)^k} \right).$$

Hence (3.48) implies that for $n \geq 1$,

$$H_r = (-1)^{r-1} \Delta^r \left(\mu(n) - 2 \log \mu(n) \right) - \sum_{k=1}^{\infty} (-1)^{r-1} \Delta^r \left(\frac{1}{k\mu(n)^k} \right).$$

The rth derivatives of $\mu(x) = \frac{\pi}{6}\sqrt{24x-1}, \log \mu(x)$ and $\mu(x)^{-k}$ are given as follows:

$$\mu^{(r)}(x) = \frac{(-1)^{r-1}(\frac{1}{2})_{r-1} 24^r \pi}{12(24x-1)^{r-\frac{1}{2}}},$$

$$\log^{(r)}(\mu(x)) = \frac{(-1)^{r-1}(r-1)! 24^r}{(24x-1)^r},$$

$$\left(\frac{1}{\mu^k} \right)^{(r)} = \left(\frac{k}{2} \right)_r \frac{(-144)^r}{\pi^k (24x-1)^{\frac{k}{2}+r}}.$$

Therefore, the functions $\mu(x) = \frac{\pi}{6}\sqrt{24x-1}, \log \mu(x)$ and $-\mu(x)^{-k}$ satisfy the conditions of Proposition 3.5 for $r \geq 1$ and $k \geq 1$. Hence,

$$H_r \leq \frac{(\frac{1}{2})_{r-1} 24^r \pi}{12(24n-1)^{r-\frac{1}{2}}} - \frac{(r-1)! 24^r}{(24(n+r)-1)^r}$$

$$+ \sum_{k=1}^{\infty} \left(\frac{k}{2} \right)_r \frac{144^r}{k\pi^k (24n-1)^{\frac{k}{2}+r}}. \tag{3.49}$$

To bound the first term of (3.49), we note that

$$\frac{(\frac{1}{2})_{r-1} 24^r \pi}{12(24n-1)^{r-\frac{1}{2}}} = \frac{(\sqrt{6}\pi \frac{1}{2})_{r-1}}{(n+1)^{r-\frac{1}{2}} \left(1 - \frac{25}{24(n+1)}\right)^{r-\frac{1}{2}}}.$$

We claim that for $n \geq 48r - 3$,

$$\frac{\sqrt{6}\pi (\frac{1}{2})_{r-1}}{6(n+1)^{r-\frac{1}{2}} \left(1 - \frac{25}{24(n+1)}\right)^{r-\frac{1}{2}}} \leq U_r + \frac{a_1}{(n+1)^{r+\frac{1}{2}}}, \tag{3.50}$$

where

$$a_1 = \left(\frac{1}{2} \right)_{r-1} \left(\frac{48}{47} \right)^{r+\frac{1}{2}} (2r-1) \frac{25\pi}{24^{\frac{3}{2}}} + \frac{\pi^2}{6} \left(\left(\frac{1}{2} \right)_{r-1} \right)^2 \frac{1}{(48r-2)^{r-\frac{3}{2}}}.$$

Setting $x = \frac{25}{24(n+1)}$, $\alpha = r - 1/2$ and $c = \frac{1}{48}$, for $n \geq 48r-3$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. Invoking (2.8), we find that for $n \geq 48r - 3$,

$$\frac{1}{\left(1 - \frac{25}{24(n+1)}\right)^{r-1/2}} \leq 1 + \left(\frac{48}{47} \right)^{r+1/2} \frac{25(2r-1)}{48(n+1)}.$$

It follows that for $n \geq 48r-3$,

$$\frac{\sqrt{6}\pi(\frac{1}{2})_{r-1}}{6(n+1)^{r-\frac{1}{2}}\left(1-\frac{25}{24(n+1)}\right)^{r-\frac{1}{2}}}$$

$$\leq U_r + \frac{\pi^2\left(\left(\frac{1}{2}\right)_{r-1}\right)^2}{6(n+1)^{2r-1}} + \frac{25\pi(2r-1)\left(\frac{1}{2}\right)_{r-1}\left(\frac{48}{47}\right)^{r+\frac{1}{2}}}{24^{3/2}\,(n+1)^{r+1/2}}.$$

It is easily seen that for $n \geq 48r-3$,

$$\frac{1}{(n+1)^{2r-1}} \leq \frac{1}{(n+1)^{r+1/2}\,(48r-2)^{r-3/2}}.$$

So we arrive at (3.50).

As for the second term of (3.49), notice that

$$\frac{(r-1)!24^r}{(24(n+r)-1)^r} = \frac{(r-1)!}{(n+1)^r\left(1-\frac{24r-25}{24(n+1)}\right)^r},$$

and for $n \geq 48r-3$,

$$0 < \frac{24r-25}{24(n+1)} < 1.$$

Consequently, for $n \geq 48r-3$,

$$\frac{(r-1)!24^r}{(24(n+r)-1)^r} \geq \frac{(r-1)!}{(n+1)^r}. \tag{3.51}$$

Next we estimate the last term of (3.49). It can be checked that

$$\sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_r \frac{144^r}{k\pi^k(24n-1)^{\frac{k}{2}+r}} = \sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_r \frac{6^r}{k\pi^k 24^{\frac{k}{2}}\,(n+1)^{\frac{k}{2}+r}\left(1-\frac{25}{24(n+1)}\right)^{\frac{k}{2}+r}}.$$

We aim to show that for $n \geq 48r-3$,

$$\sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_r \frac{6^r}{k\pi^k 24^{\frac{k}{2}}\,(n+1)^{\frac{k}{2}+r}\left(1-\frac{25}{24(n+1)}\right)^{\frac{k}{2}+r}} \leq \frac{a_2+a_3}{(n+1)^{r+\frac{1}{2}}}, \tag{3.52}$$

where

$$a_2 = \sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_r\left(\frac{1}{48r-2}\right)^{\frac{k-1}{2}}\frac{6^k}{k\pi^k 24^{\frac{k}{2}}},$$

$$a_3 = \sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_{r+1}\left(\frac{1}{48r-2}\right)^{\frac{k+1}{2}}\left(\frac{48}{47}\right)^{\frac{k}{2}+r+1}\frac{25\cdot 6^k(r+\frac{k}{2})}{k\pi^k 24^{\frac{k}{2}+1}}.$$

Note that for any given r, it can be shown that a_2+a_3 are convergent. Setting $x=\frac{25}{24(n+1)}$, $\alpha=k/2+r$ and $c=\frac{1}{48}$, for $n \geq 48r-3$, we have $0<x<c<1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we find that for $n \geq 48r-3$,

$$\frac{1}{\left(1-\frac{25}{24(n+1)}\right)^{r-1/2}} \leq 1 + \left(\frac{48}{47}\right)^{k/2+r+1}\frac{25(2r+k)}{48(n+1)}. \tag{3.53}$$

Clearly, for $n \geq 48r-3$ and $k \geq 1$,

$$\frac{1}{(n+1)^{k/2+r}} \leq \frac{1}{(n+1)^{r+1/2}(48r-2)^{\frac{k-1}{2}}}, \tag{3.54}$$

$$\frac{1}{(n+1)^{k/2+r+1}} \leq \frac{1}{(n+1)^{r+1/2}(48r-2)^{\frac{k+1}{2}}}. \tag{3.55}$$

Thus, (3.52) follows from (3.53), (3.54) and (3.55).

Combining (3.50), (3.51) and (3.52), we obtain that for $n \geq 48r-3$,

$$H_r(n) < U_r - \frac{(r-1)!}{(n+1)^r} + \frac{a_1+a_2+a_3}{(n+1)^{r+\frac{1}{2}}}.$$

Let

$$u_1 = \frac{4(a_1+a_2+a_3)^2}{((r-1)!)^2}.$$

Notice that for given r, $a_1+a_2+a_3$ is finite. It can be verified that for $n \geq u_1+1$,

$$\frac{a_1+a_2+a_3}{(n+1)^{r+\frac{1}{2}}} < \frac{(r-1)!}{2(n+1)^r}.$$

Thus, for $n \geq \max\{48r-3, u_1+1\}$,

$$H_r(n) < U_r - \frac{(r-1)!}{2(n+1)^r}.$$

Employing the above inequality and (3.12), we deduce that for $n \geq \max\{50, 48r-3, u_1+1\}$,

$$H_r + |G_r| < U_r - \frac{(r-1)!}{2(n+1)^r} + 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}.$$

Observe that for $n \geq 1$,

$$\frac{1}{(n+1)^r} \geq \frac{\left(\frac{23}{48}\right)^r}{\left(n-\frac{1}{24}\right)^r}.$$

It follows that for $n \geq \max\{50, 48r-3, u_1+1\}$,

$$H_r + |G_r| < U_r - \frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(n-\frac{1}{24}\right)^r} + 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{3.56}$$

To deduce (3.47) from (3.56), we consider the equation

$$\frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(x-\frac{1}{24}\right)^r} = 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}}. \tag{3.57}$$

Keep in mind that $\mu(x)$ is defined for $x \geq 1/24$. We claim that equation (3.57) has two real roots. Recall that the Lambert W function $W(z)$ is defined to be a function satisfying

$$W(z)e^{W(z)} = z, \tag{3.58}$$

for any complex number z; see Corless, Gonnet, Hare, Jeffrey and Knuth [7]. So a solution of (3.57) has the form

$$x = \frac{1}{24} + \frac{6r^2}{\pi^2}\left(W\left(-\frac{\sqrt{46}\pi}{48r}\left(\frac{(r-1)!}{10\sqrt{2}}\right)^{\frac{1}{2r}}\right)\right)^2.$$

It is known that $W(z)$ is a multi-valued function. In particular, $W(z)$ has two real values, $W_0(z)$ and $W_{-1}(z)$, for $-\frac{1}{e} < z < 0$. Using the inequality (see Abramowitz and Stegun [1])

$$m! < \sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m+\frac{1}{12m}}, \tag{3.59}$$

we see that for $r \geq 2$,

$$\frac{\sqrt{46}\pi}{48r} \left(\frac{(r-1)!}{10\sqrt{2}} \right)^{\frac{1}{2r}} < \frac{1}{e}.$$

Hence (3.57) has two real roots. Let u_2 be the larger real root. Clearly, for sufficiently large x,

$$5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}} < \frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(x - \frac{1}{24}\right)^r}.$$

It follows that for $n \geq u_2 + 1$,

$$5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}} < \frac{\left(\frac{23}{48}\right)^r (r-1)!}{2\left(n - \frac{1}{24}\right)^r}. \tag{3.60}$$

Combining (3.56) and (3.60), we conclude that (3.47) holds for $n \geq n(r)$, where

$$n(r) = \max\{50, 48r - 3, u_1 + 1, u_2 + 1\}.$$

This completes the proof for the case $r \geq 2$. □

4. The positivity of $(-1)^{r-1}\Delta^r \log p(n)$

In this section, we prove the positivity of $(-1)^{r-1}\Delta^r \log p(n)$ for $r \geq 1$ and sufficiently large n. This is analogous to the positivity of the differences of the partition function conjectured by Good [9] and proved by Gupta [10]. The proof relies on the estimates of H_r and G_r in the previous section.

Theorem 4.1. *For each $r \geq 1$, there exists a positive integer $n(r)$ such that for $n \geq n(r)$,*

$$(-1)^{r-1}\Delta^r \log p(n) > 0. \tag{4.1}$$

Proof. The case $r = 1$ is obvious since $p(n+1) > p(n)$ for $n \geq 1$. For $r = 2$, DeSalvo and Pak [8] have shown that the sequence $p(n)$ is log-concave for $n > 25$, or equivalently, for $n \geq 25$,

$$-\Delta^2 \log p(n) > 0.$$

We now consider the case $r \geq 3$. Recall that

$$(-1)^{r-1}\Delta^r \log p(n) = H_r + G_r,$$

where H_r and G_r are given in (3.10) and (3.11). Hence, we see that for $r \geq 1$,

$$(-1)^{r-1}\Delta^r \log p(n) \geq H_r - |G_r|. \tag{4.2}$$

An upper bound for $|G_r|$ has been given in Theorem 3.2, so we only need a suitable lower bound for H_r. By the definition of H_r, we find that

$$H_r = (-1)^{r-1}\Delta^r \left(\mu(n) - 2\log\mu(n) - \sum_{k=1}^{\infty} \frac{1}{k\mu(n)^k} \right). \tag{4.3}$$

Applying Proposition 3.5 to the right hand side of the above equation, we get

$$H_r \geq \frac{(\frac{1}{2})_{r-1}24^r\pi}{12(24(n+r)-1)^{r-\frac{1}{2}}} - \frac{(r-1)!24^r}{(24n-1)^r}$$

$$+\sum_{k=1}^{\infty}\left(\frac{k}{2}\right)_r \frac{144^r}{k\pi^k(24(n+r)-1)^{\frac{k}{2}+r}}. \tag{4.4}$$

The first term of the right hand side of (4.4) has the following lower bound for $n \geq 48r-2$:

$$\frac{(\frac{1}{2})_{r-1}24^r\pi}{12(24(n+r)-1)^{r-\frac{1}{2}}} \geq \frac{b_1}{n^{r-\frac{1}{2}}} - \frac{b_2}{n^r}, \tag{4.5}$$

where

$$b_1 = \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1},$$

$$b_2 = \frac{\pi\sqrt{48r-2}}{24^{\frac{3}{2}}}\left(\frac{1}{2}\right)_r.$$

Setting $x = \frac{24r-1}{24n}$ and $\alpha = r-1/2$, for $n \geq 48r-2$, we have $0 < x < 1$ and $\alpha \geq \frac{1}{2}$. It follows from (2.24) that for $n \geq 48r-2$,

$$\frac{1}{\left(1+\frac{24r-1}{24n}\right)^{r-\frac{1}{2}}} \geq 1 - \frac{24r-1}{24n}\left(r-\frac{1}{2}\right),$$

or equivalently,

$$\frac{(\frac{1}{2})_{r-1}24^r\pi}{12(24(n+r)-1)^{r-\frac{1}{2}}} \geq \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_{r-1}\frac{1}{n^{r-\frac{1}{2}}} - \frac{\sqrt{6}\pi}{6}\left(\frac{1}{2}\right)_r\frac{24r-1}{24n^{r+\frac{1}{2}}}.$$

Observing that for $n \geq 48r-2$,

$$\frac{1}{n^{r+\frac{1}{2}}} \leq \frac{1}{\sqrt{48r-2}\,n^r},$$

we obtain (4.5) for $n \geq 48r-2$.

For the second term of the right hand side of (4.4), we claim that for $n \geq 48r-2$,

$$\frac{(r-1)!24^r}{(24n-1)^r} \leq \frac{b_3}{n^r}, \tag{4.6}$$

where

$$b_3 = (r-1)!\left(1+\frac{r}{24}\left(\frac{1}{48r-2}\right)\left(\frac{48}{47}\right)^{r+1}\right).$$

Setting $x = \frac{1}{24n}$, $\alpha = r$ and $c = \frac{1}{48}$, for $n \geq 48r-2$, we have $0 < x < c < 1$ and $\alpha \geq \frac{1}{2}$. By (2.8), we see that for $n \geq 48r-2$,

$$\frac{1}{\left(1-\frac{1}{24n}\right)^r} \leq 1 + \left(\frac{48}{47}\right)^{r+1}\frac{r}{24n}.$$

So we obtain (4.6) for $n \geq 48r-2$.

Since the last term of the right hand side of (4.4) is positive, combining (4.5) and (4.6), we deduce that for $n \geq 48r-2$,

$$H_r \geq \frac{b_1}{n^{r-\frac{1}{2}}} - \frac{b_2+b_3}{n^r}. \tag{4.7}$$

To derive a simpler expression for a lower bound of H_r, let

$$m_1 = \frac{4(b_2+b_3)^2}{b_1^2}.$$

Thus, for $n \geq m_1+1$, it can be checked that

$$\frac{b_2+b_3}{n^r} < \frac{b_1}{2n^{r-\frac{1}{2}}}.$$

It follows that for $n \geq \max\{48r-2, m_1+1\}$,

$$H_r(n) > \frac{b_1}{2n^{r-\frac{1}{2}}}. \tag{4.8}$$

Combining (4.2) and (4.8), we find that for $n \geq \max\{50, 48r-2, m_1+1\}$,

$$(-1)^{r-1}\Delta^r \log p(n) > \frac{b_1}{2n^{r-\frac{1}{2}}} - 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{4.9}$$

Notice that for $r \geq 1$ and $n \geq 1$,

$$\frac{1}{n^{r-\frac{1}{2}}} \geq \frac{\left(\frac{23}{24}\right)^{r-\frac{1}{2}}}{\left(n-\frac{1}{24}\right)^{r-\frac{1}{2}}}.$$

Thus, for $n \geq \max\{50, 48r-2, m_1+1\}$,

$$(-1)^{r-1}\Delta^r \log p(n) > \left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2n^{r-\frac{1}{2}}} - 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}}. \tag{4.10}$$

To prove that the right hand side of (4.10) is positive for sufficiently large n, consider the following equation:

$$\left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2x^{r-\frac{1}{2}}} = 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}}. \tag{4.11}$$

The solution of (4.11) can be expressed in terms of the Lambert W function, namely,

$$x = \frac{1}{24} + \frac{6\,(2r-1)^2}{\pi^2} W\left(-\frac{\sqrt{46}\pi}{24(2r-1)}\left(\frac{\pi\left(\frac{1}{2}\right)_{r-1}}{20\sqrt{6}}\right)^{\frac{1}{2r-1}}\right)^2. \tag{4.12}$$

For $r \geq 1$, we have $\left(\frac{1}{2}\right)_r < r!$. Using the estimate of $r!$ as given by (3.59), we obtain that for $r \geq 3$,

$$-\frac{1}{e} < -\frac{\sqrt{46}\pi}{24(2r-1)}\left(\frac{\pi\left(\frac{1}{2}\right)_{r-1}}{20\sqrt{6}}\right)^{\frac{1}{2r-1}} < 0.$$

Thus (4.11) has two real roots. Let m_2 be the larger real root of equation (4.11). Clearly, for sufficiently large x,

$$\left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2x^{r-\frac{1}{2}}} - 5\cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(x)}{2}} > 0. \tag{4.13}$$

It follows that for $n \geq m_2 + 1$,

$$\left(\frac{23}{24}\right)^{r-\frac{1}{2}} \frac{b_1}{2n^{r-\frac{1}{2}}} - 5 \cdot 2^{r+\frac{1}{2}} e^{-\frac{\mu(n)}{2}} > 0. \tag{4.14}$$

Let

$$n(r) = \max\{50, 48r - 2, m_1 + 1, m_2 + 1\}.$$

Combining (4.9) and (4.14), we conclude that for $n \geq n(r)$,

$$(-1)^{r-1} \Delta^r \log p(n) > 0. \tag{4.15}$$

This completes the proof. $\square$

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