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AN hp-SPECTRAL COLLOCATION METHOD FOR NONLINEAR VOLTERRA INTEGRAL EQUATIONS WITH VANISHING VARIABLE DELAYS

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ABSTRACT. In this paper, we propose a multistep Legendre-Gauss spectral collocation method for nonlinear second-kind Volterra integral equations (VIEs) with vanishing variable delays. This method is easy to implement and possesses high-order accuracy. We also provide a rigorous convergence analysis of the hp-version of the multistep spectral collocation method under L^2-norm. Numerical results confirm the theoretical predictions.

1. INTRODUCTION

This paper is concerned with the numerical solutions of nonlinear second-kind VIEs with vanishing variable delays:

(1.1)
$$y(t) = f(t) + \int_0^t K_1(t,s)G_1(s,y(s))ds + \int_0^{\theta(t)} K_2(t,s)G_2(s,y(s))ds,\ t \in I := [0,T],$$

where the delay function $\theta(t)$ is of the form $\theta(t) := t - \tau(t)$, and $\theta(t)$ satisfies the following conditions (cf. [9]):

(C1) $\tau(0) = 0,\ \tau(t) > 0$ for $t \in (0,T]$ (vanishing delay);
(C2) $\theta(t) \le q_1 t$ on I for some $q_1 \in (0,1)$, and $\theta'(t) \ge q_0 > 0$;
(C3) $\theta(t) \in C^1(I)$.

Moreover, $K_i \in C(D_i)$ with $i = 1,2,\ D_1 := \{(t,s) :\ 0 \le s \le t \le T\},\ D_2 := \{(t,s) :\ 0 \le s \le \theta(t) \le \theta(T)\}$, $f \in C(I)$ and G_i are continuous functions. Equation (1.1) includes the special case where $\theta(t) = qt$ is the proportional delay with $0 < q < 1$.

The analysis of second-kind VIEs with proportional delays have been studied; see [7, 15, 23]. Some numerical methods for VIEs with delays have also been proposed. The recent works particularly include the piecewise polynomial collocation methods for Volterra integral and differential equations with vanishing delays [4, 7, 9–12, 30, 31]; and the piecewise polynomial collocation methods and Runge-Kutta methods

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for Volterra integral and differential equations with nonvanishing delays [7–9, 28]. The interested readers may also refer to [21, 29] for other related works.

As we know, the spectral method often provides exceedingly accurate numerical results with relatively fewer degrees of freedom, and has been widely used for scientific computations; see, e.g., [5, 6, 13, 14, 17, 19, 25, 26]. Since the spectral method is fully capable of solving problems with a history of dependence, they have become very helpful for numerical VIEs with delays. In recent years, some authors proposed the spectral collocation methods for Volterra integral and related differential equations with proportional or nonvanishing delays [1–3, 18, 32]. However, these algorithms are basically one-step methods. For an effective implementation, it is more reasonable to use multistep methods due to the following considerations:

- The resulting system for the expansion coefficients can be solved more efficiently for a modest number of unknowns.
- To ensure the convergence of the numerical scheme, the length of T is limited sometimes.
- The multistep methods provide sufficient flexibility, e.g., we are able to place more points in the subintervals that are needed.

Recently, Conte and Paternoster [16] constructed a class of multistep collocation methods for nonlinear VIEs, by using the Lagrange interpolation in each sub-step. Li, Tang and Xu [22] introduced a time parareal method with spectral-subdomain enhancement for VIEs. Moreover, Sheng, Wang and Guo [27] proposed a multistep spectral collocation method for nonlinear VIEs, and derived the convergence of the *hp*-version. To the best of our knowledge, there are few works about the multistep spectral collocation method to VIEs with delays.

The aim of this paper is to propose and analyze an efficient multistep spectral collocation method for equation (1.1). We highlight the main differences between our new strategy and the existing ones as follows:

(i) First, we consider the multistep spectral collocation method for VIEs with nonlinear vanishing delays, the existing works for spectral methods studied the one-step methods for VIEs with proportional or nonvanishing delays.

(ii) Second, we use the Legendre expansions in each sub-step (much more stable than the usual Lagrange approach [26]), which lead to quite a neat implementation through manipulating the expansion coefficients of the consecutive steps (see equation (2.34) of this paper).

(iii) Finally, we fully analyze and characterize the *hp*-convergence of the suggested collocation method. The interplay between h and p can significantly enhance numerical accuracy. More precisely, as shown in Theorem 4.2 of this paper, for a given numerical error tolerance $\varepsilon > 0$ and a given solution in the Sobolev space with regularity index m, we roughly have $h^m M^{-m} \sim \varepsilon$ (where M is the number of Legendre modes). Accordingly, we may refine the mesh and/or increase the degree of the polynomial to achieve higher accuracy. In other words, this new process possesses the same fascinating merits as the *hp*-version of the finite element method and the spectral element method for PDEs.

Our numerical results demonstrate that the suggested algorithm possesses the following advantages:

(a) It provides flexibility with respect to variable time steps. Particularly, it enables us to cope with VIEs with delays involving oscillations or steep gradients in the solutions;
(b) It oftentimes works well even for long time calculations;
(c) It also works for nonsmooth solutions.

This paper is organized as follows. In Section 2, we design the multistep Legendre-Gauss spectral collocation method for nonlinear VIEs with vanishing variable delays (1.1). Some lemmas useful for the convergence analysis are provided in Section 3. The convergence analysis for the proposed new method is given in Section 4. Numerical experiments are carried out in Section 5, which confirm the theoretical results. The final section is devoted to some concluding remarks.

2. The Multistep Legendre-Gauss collocation method

In this section, we propose a multistep Legendre-Gauss collocation method for second-kind VIEs with delays (1.1).

2.1. Preliminaries.

2.1.1. *Mesh design.* This subsection is devoted to the mesh design for the multistep Legendre-Gauss collocation method of VIEs with delays (1.1). To this end, we consider the coarse/fine partitions:

(i) The coarse grid $\{\xi_\mu\}_{\mu=0}^{N^c}$ on the interval I is created by

$$\xi_0 = 0, \quad \xi_{N^c} = T \quad \text{and} \quad \xi_{\mu-1} := \theta(\xi_\mu)\ (2 \le \mu \le N^c),$$

where N^c is a positive integer. It is clear that for any $\Lambda_\mu = [\xi_{\mu-1}, \xi_\mu]$, it holds that $\theta(\Lambda_2) \subseteq \Lambda_1$ and $\Lambda_{\mu-1} = \theta(\Lambda_\mu)$ with $3 \le \mu \le N^c$.

(ii) The fine grid in each sub-interval $[\xi_{\mu-1}, \xi_\mu]$ is denoted by

$$I_h^{(\mu)} := \{t_j^{(\mu)} : \ \xi_{\mu-1} = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_{N_\mu^f}^{(\mu)} = \xi_\mu\},$$

satisfying $I_h^{(\mu-1)} \subseteq \theta(I_h^{(\mu)})$, where N_μ^f is a positive integer.

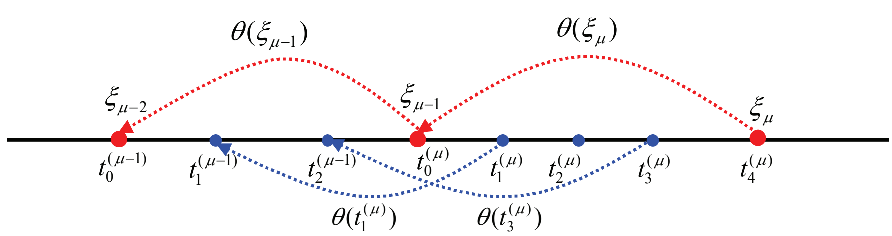

Figure 2.1. A simple mesh.

For clarity, we plot a simple mesh in Figure 2.1 with $I_h^{(\mu-1)} := \{\xi_{\mu-2}, t_1^{(\mu-1)}, t_2^{(\mu-1)}, \xi_{\mu-1}\}$ and $I_h^{(\mu)} := \{\xi_{\mu-1}, t_1^{(\mu)}, t_2^{(\mu)}, t_3^{(\mu)}, \xi_\mu\}$. Since

$$t_0^{(\mu-1)} = \xi_{\mu-2} = \theta(\xi_{\mu-1}), \quad t_0^{(\mu)} = \xi_{\mu-1} = \theta(\xi_\mu), \quad t_1^{(\mu-1)} = \theta(t_1^{(\mu)}), \quad t_2^{(\mu-1)} = \theta(t_3^{(\mu)}),$$

it holds that $I_h^{(\mu-1)} \subseteq \theta(I_h^{(\mu)})$. It is pointed out that Bellen, Brunner, Maset and Torelli [4], and Brunner and Hu [9] also designed a mesh with $I_h^{(\mu-1)} \equiv \theta(I_h^{(\mu)})$. However, the numerical examples of this paper show that our mesh design provides a relatively convenient way for variable step size.

For convenience, we rename the previous global mesh as

$$I_h := \{t_n : \ 0 = t_0 < t_1 < \cdots < t_N = T\}.$$

We also denote $h_n = t_n - t_{n-1}$, $h_{\max} = \max_{1 \le n \le N} h_n$, $I_n = (t_{n-1}, t_n]$ and $y^n(t)$ the solution of equation (1.1) on the n-th element, namely,

$$y^n(t) = y(t), \qquad \forall t \in I_n, \qquad 1 \le n \le N.$$

From equation (1.1) we have that for any $t \in I_n$,

$$\begin{aligned} y(t) = f(t) &+ \int_0^{t_{n-1}} K_1(t,\xi)G_1(\xi, y(\xi))d\xi + \int_{t_{n-1}}^t K_1(t,s)G_1(s,y(s))ds \\ &+ \int_0^{\theta(t_{n-1})} K_2(t,\varsigma)G_2(\varsigma, y(\varsigma))d\varsigma + \int_{\theta(t_{n-1})}^{\theta(t)} K_2(t,\eta)G_2(\eta, y(\eta))d\eta. \end{aligned} \tag{2.1}$$

Clearly, for any given interval $\widetilde{I}_k := (\theta(t_{k-1}), \theta(t_k)]$ with $k > 1$, there exists a unique interval I_j with $1 \le j < k$, such that $\widetilde{I}_k \subseteq I_j$. Correspondingly, we denote

$$y^{\widehat{k}}(t) := y^j(t), \quad t \in \widetilde{I}_k. \tag{2.2}$$

Then, equation (2.1) is equivalent to

$$\begin{aligned} y^n(t) = f(t) &+ \sum_{k=1}^{n-1} \int_{I_k} K_1(t,\xi)G_1(\xi, y^k(\xi))d\xi + \int_{t_{n-1}}^t K_1(t,s)G_1(s,y^n(s))ds \\ &+ \sum_{k=1}^{n-1} \int_{\widetilde{I}_k} K_2(t,\varsigma)G_2(\varsigma, y^{\widehat{k}}(\varsigma))d\varsigma + \int_{\theta(t_{n-1})}^{\theta(t)} K_2(t,\eta)G_2(\eta, y^{\widehat{n}}(\eta))d\eta. \end{aligned} \tag{2.3}$$

In order to transfer the integral intervals $(t_{n-1}, t]$ to I_n and $(\theta(t_{n-1}), \theta(t)]$ to $\widetilde{I}_n$, we make two linear transformations:

$$\begin{aligned} s &= \sigma(t,\lambda) := t_{n-1} + \frac{(\lambda - t_{n-1})(t - t_{n-1})}{h_n}, & \lambda \in I_n, \\ \eta &= \rho(t,\nu) := \theta(t_{n-1}) + \frac{(\nu - \theta(t_{n-1}))(\theta(t) - \theta(t_{n-1}))}{\theta(t_n) - \theta(t_{n-1})}, & \nu \in \widetilde{I}_n. \end{aligned} \tag{2.4}$$

Then, equation (2.3) becomes

$$
\begin{aligned}
y^n(t) = f(t) &+ \sum_{k=1}^{n-1} \int_{I_k} K_1(t,\xi)G_1(\xi, y^k(\xi))d\xi \\
&+ \frac{t-t_{n-1}}{h_n} \int_{I_n} K_1(t,\sigma(t,\lambda))G_1(\sigma(t,\lambda), y^n(\sigma(t,\lambda)))d\lambda \\
&+ \sum_{k=1}^{n-1} \int_{\widetilde{I}_k} K_2(t,\varsigma)G_2(\varsigma, y^{\widehat{k}}(\varsigma))d\varsigma \\
&+ \frac{\theta(t)-\theta(t_{n-1})}{\theta(t_n)-\theta(t_{n-1})} \int_{\widetilde{I}_n} K_2(t,\rho(t,\nu))G_2(\rho(t,\nu), y^{\widehat{n}}(\rho(t,\nu)))d\nu.
\end{aligned} \tag{2.5}
$$

2.1.2. *The shifted Legendre-Gauss interpolation on I_n.* Let $L_l(x)$, $x \in (-1,1)$ be the standard Legendre polynomial of degree l. Clearly, we have (cf. [26])

$$(l+1)L_{l+1}(x) - (2l+1)xL_l(x) + lL_{l-1}(x) = 0, \qquad l \geq 1, \tag{2.6}$$

$$\frac{d}{dx}L_{l+1}(x) - \frac{d}{dx}L_{l-1}(x) = (2l+1)L_l(x), \qquad l \geq 1. \tag{2.7}$$

The shifted Legendre polynomial $L_{n,l}(t)$, $t \in I_n$ is defined by

$$L_{n,l}(t) = L_l(\frac{2t-t_{n-1}-t_n}{h_n}), \qquad l = 0,1,2,\cdots.$$

According to (2.6) and (2.7), it holds that

$$(l+1)L_{n,l+1}(t) - (2l+1)(\frac{2t-t_{n-1}-t_n}{h_n})L_{n,l}(t) + lL_{n,l-1}(t) = 0, \qquad l \geq 1, \tag{2.8}$$

$$\frac{d}{dt}L_{n,l+1}(t) - \frac{d}{dt}L_{n,l-1}(t) = \frac{4l+2}{h_n}L_{n,l}(t), \qquad l \geq 1. \tag{2.9}$$

In particular,

$$
\begin{aligned}
L_{n,0}(t) &= 1, \qquad L_{n,1}(t) = \frac{2t-t_{n-1}-t_n}{h_n}, \\
L_{n,2}(t) &= \frac{6t^2 - 6(t_{n-1}+t_n)t + 4t_{n-1}t_n + t_{n-1}^2 + t_n^2}{h_n^2}.
\end{aligned}
$$

The set of $L_{n,l}(t)$ is a complete $L^2(I_n)$-orthogonal system, namely,

$$\int_{I_n} L_{n,l}(t)L_{n,m}(t)dt = \frac{h_n}{2l+1}\delta_{l,m}, \tag{2.10}$$

where $\delta_{l,m}$ is the Kronecker symbol. Thus for any $v \in L^2(I_n)$, we can write

$$v(t) = \sum_{l=0}^{\infty} v_{n,l}L_{n,l}(t), \qquad v_{n,l} = \frac{2l+1}{h_n}\int_{I_n} v(t)L_{n,l}(t)dt. \tag{2.11}$$

We now turn to the Legendre-Gauss interpolation. For any integer $M_n > 0$, we denote by $\{x_{n,j}, \omega_{n,j}\}_{j=0}^{M_n}$ the nodes and the corresponding Christoffel numbers of the standard Legendre-Gauss interpolation on the interval $(-1,1)$. Let $\mathcal{P}_{M_n}(I_n)$ be the set of polynomials of degree at most M_n on the interval I_n, and let $t_{n,j}$ be the Legendre-Gauss quadrature nodes on the interval I_n given by

$$t_{n,j} = \frac{h_n x_{n,j} + t_{n-1} + t_n}{2} \in I_n, \qquad 1 \leq n \leq N, \qquad 0 \leq j \leq M_n. \tag{2.12}$$

Due to the property of the standard Legendre-Gauss quadrature, it follows that for any $\phi \in \mathcal{P}_{2M_n+1}(I_n)$,

$$\begin{aligned}\int_{I_n} \phi(t)dt &= \frac{h_n}{2}\int_{-1}^{1} \phi(\frac{h_n x + t_{n-1} + t_n}{2})dx \\ &= \frac{h_n}{2}\sum_{j=0}^{M_n} \omega_{n,j}\phi(\frac{h_n x_{n,j} + t_{n-1} + t_n}{2}) = \frac{h_n}{2}\sum_{j=0}^{M_n}\omega_{n,j}\phi(t_{n,j}).\end{aligned} \tag{2.13}$$

Next, let $(u, v)_{I_n}$ and $\|v\|_{I_n}$ be the inner product and the norm of space $L^2(I_n)$, respectively. We also introduce the following discrete inner product and norm on the interval I_n,

$$\langle u, v\rangle_{I_n} = \frac{h_n}{2}\sum_{j=0}^{M_n} u(t_{n,j})v(t_{n,j})\omega_{n,j}, \qquad \|v\|_{I_n,M_n} = \langle v, v\rangle_{I_n}^{\frac{1}{2}}. \tag{2.14}$$

Thanks to the identity (2.13), for any $\phi\psi \in \mathcal{P}_{2M_n+1}(I_n)$ and $\varphi \in \mathcal{P}_{M_n}(I_n)$, we have

$$(\phi, \psi)_{I_n} = \langle \phi, \psi\rangle_{I_n}, \quad \|\varphi\|_{I_n} = \|\varphi\|_{I_n,M_n}. \tag{2.15}$$

Denote by $\mathcal{I}^t_{M_n} : \; C(t_{n-1}, t_n) \to \mathcal{P}_{M_n}(t_{n-1}, t_n)$ the shifted Legendre-Gauss interpolation operator in the t-direction such that

$$\mathcal{I}^t_{M_n} v(t_{n,j}) = v(t_{n,j}), \qquad 0 \le j \le M_n.$$

Because of equation (2.15), we have that for any $\phi \in \mathcal{P}_{M_n+1}(I_n)$,

$$(\mathcal{I}^t_{M_n} v, \phi)_{I_n} = \langle \mathcal{I}^t_{M_n} v, \phi\rangle_{I_n} = \langle v, \phi\rangle_{I_n}. \tag{2.16}$$

We can expand $\mathcal{I}^t_{M_n} v(t)$ as

$$\mathcal{I}^t_{M_n} v(t) = \sum_{l=0}^{M_n} \widehat{v}_{n,l} L_{n,l}(t). \tag{2.17}$$

With the aid of (2.10) and (2.16), we obtain from (2.17) that

$$\widehat{v}_{n,l} = \frac{2l+1}{h_n}(\mathcal{I}^t_{M_n} v, L_{n,l})_{I_n} = \frac{2l+1}{h_n}\langle v, L_{n,l}\rangle_{I_n}, \quad 0 \le l \le M_n. \tag{2.18}$$

2.1.3. *The shifted Legendre-Gauss interpolation on* $\widetilde{I}_n$. For the purpose of convergence analysis, we also need another shifted Legendre polynomial $\widetilde{L}_{n,l}(t)$, defined by

$$\widetilde{L}_{n,l}(t) = L_l(\frac{2t - \theta(t_{n-1}) - \theta(t_n)}{\theta(t_n) - \theta(t_{n-1})}), \quad t \in \widetilde{I}_n.$$

Obviously, the set of $\widetilde{L}_{n,l}(t)$ is a complete $L^2(\widetilde{I}_n)$-orthogonal system, namely,

$$\int_{\widetilde{I}_n} \widetilde{L}_{n,l}(t)\widetilde{L}_{n,m}(t)dt = \frac{\theta(t_n) - \theta(t_{n-1})}{2l+1}\delta_{l,m}. \tag{2.19}$$

Let $\widetilde{t}_{n,j}$ be the Legendre-Gauss quadrature nodes on the interval $\widetilde{I}_n$,

$$\widetilde{t}_{n,j} := \frac{(\theta(t_n) - \theta(t_{n-1}))x_{n,j} + \theta(t_{n-1}) + \theta(t_n)}{2} \in \widetilde{I}_n, \qquad 0 \le j \le M_{\widehat{n}}, \tag{2.20}$$

where $M_{\widehat{n}} = M_i$ with $\widetilde{I}_n \subseteq I_i$ (cf. equation (2.2)). For any $\phi \in \mathcal{P}_{2M_{\widehat{n}}+1}(\widetilde{I}_n)$, we have

(2.21)
$$\begin{aligned}\int_{\widetilde{I}_n} \phi(t)dt &= \frac{\theta(t_n)-\theta(t_{n-1})}{2}\int_{-1}^{1}\phi\Big(\frac{(\theta(t_n)-\theta(t_{n-1}))x+\theta(t_{n-1})+\theta(t_n)}{2}\Big)dx\\ &= \frac{\theta(t_n)-\theta(t_{n-1})}{2}\sum_{j=0}^{M_{\widehat{n}}}\omega_{\widehat{n},j}\phi\Big(\frac{(\theta(t_n)-\theta(t_{n-1}))x_{\widehat{n},j}+\theta(t_{n-1})+\theta(t_n)}{2}\Big)\\ &= \frac{\theta(t_n)-\theta(t_{n-1})}{2}\sum_{j=0}^{M_{\widehat{n}}}\omega_{\widehat{n},j}\phi(\widetilde{t}_{n,j}),\end{aligned}$$

where $\omega_{\widehat{n},j}$ are the corresponding Christoffel numbers. Let $(u,v)_{\widetilde{I}_n}$ and $\|v\|_{\widetilde{I}_n}$ be the inner product and the norm of space $L^2(\widetilde{I}_n)$, respectively. We also introduce the following discrete inner product and norm on the interval $\widetilde{I}_n$,

(2.22)
$$\langle u,v\rangle_{\widetilde{I}_n} = \frac{\theta(t_n)-\theta(t_{n-1})}{2}\sum_{j=0}^{M_{\widehat{n}}}u(\widetilde{t}_{n,j})v(\widetilde{t}_{n,j})\omega_{\widehat{n},j}, \qquad \|v\|_{\widetilde{I}_n,M_{\widehat{n}}} = \langle v,v\rangle^{\frac{1}{2}}_{\widetilde{I}_n}.$$

Thanks to the identity (2.21), for any $\phi\psi \in \mathcal{P}_{2M_{\widehat{n}}+1}(\widetilde{I}_n)$ and $\varphi \in \mathcal{P}_{M_{\widehat{n}}}(\widetilde{I}_n)$, we have

$$(\phi,\psi)_{\widetilde{I}_n} = \langle\phi,\psi\rangle_{\widetilde{I}_n}, \qquad \|\varphi\|_{\widetilde{I}_n} = \|\varphi\|_{\widetilde{I}_n,M_{\widehat{n}}}. \tag{2.23}$$

Denote by $\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}} : C(\theta(t_{n-1}),\theta(t_n)) \to \mathcal{P}_{M_{\widehat{n}}}(\theta(t_{n-1}),\theta(t_n))$ the shifted Legendre-Gauss interpolation operator in the t-direction such that

$$\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}}v(\widetilde{t}_{n,j}) = v(\widetilde{t}_{n,j}), \qquad 0 \le j \le M_{\widehat{n}}.$$

Because of equation (2.23), for any $\phi \in \mathcal{P}_{M_{\widehat{n}}+1}(\widetilde{I}_n)$,

$$(\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}}v,\phi)_{\widetilde{I}_n} = \langle\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}}v,\phi\rangle_{\widetilde{I}_n} = \langle v,\phi\rangle_{\widetilde{I}_n}. \tag{2.24}$$

We can expand $\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}}v(t)$ as

$$\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}}v(t) = \sum_{l=0}^{M_{\widehat{n}}}\widetilde{v}_{n,l}\widetilde{L}_{n,l}(t). \tag{2.25}$$

With the aid of (2.19) and (2.24), we obtain from (2.25) that

(2.26)
$$\widetilde{v}_{n,l} = \frac{2l+1}{\theta(t_n)-\theta(t_{n-1})}(\widetilde{\mathcal{I}}^t_{M_{\widehat{n}}}v,\widetilde{L}_{n,l})_{\widetilde{I}_n} = \frac{2l+1}{\theta(t_n)-\theta(t_{n-1})}\langle v,\widetilde{L}_{n,l}\rangle_{\widetilde{I}_n}, \qquad 0 \le l \le M_{\widehat{n}}.$$

2.2. **The multistep collocation scheme of** (2.5)**.** The multistep collocation scheme for solving (2.5) is to seek $Y^n(t) \in \mathcal{P}_{M_n}(I_n)$, such that

$$
\begin{aligned}
Y^n(t) = \mathcal{I}^t_{M_n}\Big[f(t) &+ \sum_{k=1}^{n-1}\int_{I_k}\mathcal{I}^{\xi}_{M_k}\Big(K_1(t,\xi)G_1(\xi,Y^k(\xi))\Big)d\xi \\
&+ \frac{t-t_{n-1}}{h_n}\int_{I_n}\mathcal{I}^{\lambda}_{M_n}\Big(K_1(t,\sigma(t,\lambda))G_1(\sigma(t,\lambda),Y^n(\sigma(t,\lambda)))\Big)d\lambda \\
&+ \sum_{k=1}^{n-1}\int_{\widetilde{I}_k}\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}}\Big(K_2(t,\varsigma)G_2(\varsigma,Y^{\widehat{k}}(\varsigma))\Big)d\varsigma \\
&+ \frac{\theta(t)-\theta(t_{n-1})}{\theta(t_n)-\theta(t_{n-1})}\int_{\widetilde{I}_n}\widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}}\Big(K_2(t,\rho(t,\nu))G_2(\rho(t,\nu),Y^{\widehat{n}}(\rho(t,\nu)))\Big)d\nu\Big], \qquad t \in I_n,
\end{aligned}
\tag{2.27}
$$

where Y^k is the numerical solution of y^k on the interval I_k, and $Y^{\widehat{k}} \in \mathcal{P}_{M_{\widehat{k}}}$ is the corresponding numerical solution on the interval $\widetilde{I}_k$.

We now describe the numerical implementations and present an algorithm for scheme (2.27). To this end, we set

$$
\begin{aligned}
&Y^n(t) = \sum_{p=0}^{M_n}\widetilde{y}^n_p L_{n,p}(t), \qquad \mathcal{I}^t_{M_n}f(t) = \sum_{p=0}^{M_n}\widetilde{f}^n_p L_{n,p}(t), \\
&\mathcal{I}^t_{M_n}\mathcal{I}^{\xi}_{M_k}\Big(K_1(t,\xi)G_1(\xi,Y^k(\xi))\Big) \\
&\qquad = \sum_{p=0}^{M_n}\sum_{p'=0}^{M_k} a^k_{pp'}L_{n,p}(t)L_{k,p'}(\xi), \quad 1 \le k \le n-1, \\
&\mathcal{I}^t_{M_n}\mathcal{I}^{\lambda}_{M_n}\Big((t-t_{n-1})K_1(t,\sigma(t,\lambda))G_1(\sigma(t,\lambda),Y^n(\sigma(t,\lambda)))\Big) \\
&\qquad = \sum_{p,p'=0}^{M_n} b^n_{pp'}L_{n,p}(t)L_{n,p'}(\lambda), \\
&\mathcal{I}^t_{M_n}\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}}\Big(K_2(t,\varsigma)G_2(\varsigma,Y^{\widehat{k}}(\varsigma))\Big) = \sum_{p=0}^{M_n}\sum_{p'=0}^{M_{\widehat{k}}} c^k_{pp'}L_{n,p}(t)\widetilde{L}_{k,p'}(\varsigma), \quad 1 \le k \le n-1, \\
&\mathcal{I}^t_{M_n}\widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}}\Big((\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\nu))G_2(\rho(t,\nu),Y^{\widehat{n}}(\rho(t,\nu)))\Big) \\
&\qquad = \sum_{p=0}^{M_n}\sum_{p'=0}^{M_{\widehat{n}}} d^n_{pp'}L_{n,p}(t)\widetilde{L}_{n,p'}(\nu).
\end{aligned}
\tag{2.28}
$$

Then, we have

$$
\begin{aligned}
\int_{I_k}\mathcal{I}^t_{M_n}\mathcal{I}^{\xi}_{M_k}\Big(K_1(t,\xi)G_1(\xi,Y^k(\xi))\Big)d\xi
&= \sum_{p=0}^{M_n}\sum_{p'=0}^{M_k} a^k_{pp'}L_{n,p}(t)\int_{I_k}L_{k,p'}(\xi)d\xi \\
&= h_k\sum_{p=0}^{M_n} a^k_{p0}L_{n,p}(t).
\end{aligned}
\tag{2.29}
$$

Similarly, we can obtain

(2.30)
$$\int_{I_n} \mathcal{I}_{M_n}^t \mathcal{I}_{M_n}^\lambda \Big((t-t_{n-1}) K_1(t,\sigma(t,\lambda)) G_1(\sigma(t,\lambda), Y^n(\sigma(t,\lambda))) \Big) d\lambda = h_n \sum_{p=0}^{M_n} b_{p0}^n L_{n,p}(t),$$
$$\int_{\widetilde{I}_k} \mathcal{I}_{M_n}^t \widetilde{\mathcal{I}}_{M_{\widehat{k}}}^\varsigma \Big(K_2(t,\varsigma) G_2(\varsigma, Y^{\widehat{k}}(\varsigma)) \Big) d\varsigma = (\theta(t_k) - \theta(t_{k-1})) \sum_{p=0}^{M_n} c_{p0}^k L_{n,p}(t),$$
$$\int_{\widetilde{I}_n} \mathcal{I}_{M_n}^t \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^\nu \Big((\theta(t) - \theta(t_{n-1})) K_2(t,\rho(t,\nu)) G_2(\rho(t,\nu), Y^{\widehat{n}}(\rho(t,\nu))) \Big) d\nu$$
$$= (\theta(t_n) - \theta(t_{n-1})) \sum_{p=0}^{M_n} d_{p0}^n L_{n,p}(t).$$

Next, it can be verified from (2.28)-(2.30), (2.18), (2.14), (2.26) and (2.22) that

$$\widetilde{y}_p^n = \frac{2p+1}{2} \sum_{i=0}^{M_n} Y^n(t_{n,i}) L_{n,p}(t_{n,i}) \omega_{n,i},$$
$$\widetilde{f}_p^n = \frac{2p+1}{2} \sum_{i=0}^{M_n} f(t_{n,i}) L_{n,p}(t_{n,i}) \omega_{n,i},$$
$$a_{p0}^k = \frac{2p+1}{4} \sum_{i=0}^{M_n} \sum_{j=0}^{M_k} K_1(t_{n,i}, t_{k,j}) G_1(t_{k,j}, Y^k(t_{k,j})) L_{n,p}(t_{n,i}) \omega_{n,i} \omega_{k,j}, \quad 1 \le k \le n-1,$$
$$b_{p0}^n = \frac{2p+1}{4} \sum_{i,j=0}^{M_n} (t_{n,i} - t_{n-1}) K_1(t_{n,i}, \sigma(t_{n,i}, t_{n,j})) \cdot G_1(\sigma(t_{n,i}, t_{n,j}), Y^n(\sigma(t_{n,i}, t_{n,j}))) L_{n,p}(t_{n,i}) \omega_{n,i} \omega_{n,j}, \tag{2.31}$$
$$c_{p0}^k = \frac{2p+1}{4} \sum_{i=0}^{M_n} \sum_{j=0}^{M_{\widehat{k}}} K_2(t_{n,i}, \widetilde{t}_{k,j}) G_2(\widetilde{t}_{k,j}, Y^{\widehat{k}}(\widetilde{t}_{k,j})) L_{n,p}(t_{n,i}) \omega_{n,i} \omega_{\widehat{k},j}, \quad 1 \le k \le n-1,$$
$$d_{p0}^n = \frac{2p+1}{4} \sum_{i=0}^{M_n} \sum_{j=0}^{M_{\widehat{n}}} (\theta(t_{n,i}) - \theta(t_{n-1})) K_2(t_{n,i}, \rho(t_{n,i}, \widetilde{t}_{n,j})) \cdot G_2(\rho(t_{n,i}, \widetilde{t}_{n,j}), Y^{\widehat{n}}(\rho(t_{n,i}, \widetilde{t}_{n,j}))) L_{n,p}(t_{n,i}) \omega_{n,i} \omega_{\widehat{n},j}.$$

Moreover, a combination of (2.27)-(2.30) leads to

(2.32)
$$\sum_{p=0}^{M_n} \widetilde{y}_p^n L_{n,p}(t) = \sum_{p=0}^{M_n} \widetilde{f}_p^n L_{n,p}(t) + \sum_{p=0}^{M_n} \widetilde{a}_p^n L_{n,p}(t) + \sum_{p=0}^{M_n} b_{p0}^n L_{n,p}(t) + \sum_{p=0}^{M_n} \widetilde{c}_p^n L_{n,p}(t) + \sum_{p=0}^{M_n} d_{p0}^n L_{n,p}(t),$$

where

$$\widetilde{a}_p^n = \sum_{k=1}^{n-1} h_k a_{p0}^k, \qquad \widetilde{c}_p^n = \sum_{k=1}^{n-1} (\theta(t_k) - \theta(t_{k-1})) c_{p0}^k. \tag{2.33}$$

Hence, we compare the expansion coefficients of equation (2.32) to obtain that

$$\widetilde{y}_p^n = \widetilde{f}_p^n + \widetilde{a}_p^n + b_{p0}^n + \widetilde{c}_p^n + d_{p0}^n, \quad 0 \le p \le M_n. \tag{2.34}$$

This is an implicit scheme. In actual computation, an iterative process can be employed to evaluate the expansion coefficients $\{\widetilde{y}_p^n\}_{p=0}^{M_n}$. In this paper, we present a simple iterative algorithm (also called the successive substitution method). Briefly, we obtain the successive values of Y^n in terms of previously computed quantities $\{Y^k\}_{k=1}^{n-1}$, as stated in Table 2.1 below.

Table 2.1: A simple iterative algorithm

For $n = 1, \cdots, N$,
- Compute $\{\widetilde{f}_p^n\}_{p=0}^{M_n}$ by the second formula of (2.31);
- If $n = 1$, then $\{\widetilde{a}_p^n\}_{p=0}^{M_n} = 0$ and $\{\widetilde{c}_p^n\}_{p=0}^{M_n} = 0$;
- Else
  - For $k = 1, \cdots, n-1$,
    - Compute $\{Y^{\widehat{k}}(\widetilde{t}_{k,j})\}_{j=0}^{M_{\widehat{k}}}$ by the first formula of (2.28);
  - End
  - Compute $Y^{\widehat{n}}(\rho(t_{n,i}, \widetilde{t}_{n,j}))$, $0 \le i \le M_n,\ 0 \le j \le M_{\widehat{n}}$ by the first formula of (2.28);
  - Compute $\{\widetilde{a}_p^n\}_{p=0}^{M_n}$, $\{\widetilde{c}_p^n\}_{p=0}^{M_n}$ and $\{d_{p0}^n\}_{p=0}^{M_n}$ by (2.31) and (2.33);
- End
- Provide the initial guess of $\{Y^n(t_{n,j})\}_{j=0}^{M_n}$;
- Compute $\{\widetilde{y}_p^n\}_{p=0}^{M_n}$ by the first formula of (2.31);
- While the maximum absolute difference between two consecutive coefficients of $\{\widetilde{y}_p^n\}_{p=0}^{M_n}$ is bigger than the desired tolerance,
  - If $n = 1$,
    - Compute $\{Y^1(\sigma(t_{1,i}, t_{1,j}))\}_{i,j=0}^{M_1}$ and $\{Y^1(\rho(t_{1,i}, \widetilde{t}_{1,j}))\}_{i,j=0}^{M_1}$ by (2.28);
    - Compute $\{b_{p0}^1\}_{p=0}^{M_1}$ and $\{d_{p0}^1\}_{p=0}^{M_1}$ by (2.31);
  - Else
    - Compute $\{Y^n(\sigma(t_{n,i}, t_{n,j}))\}_{i,j=0}^{M_n}$ by (2.28);
    - Compute $\{b_{p0}^n\}_{p=0}^{M_1}$ by (2.31);
  - End
  - Update the coefficients $\{\widetilde{y}_p^n\}_{p=0}^{M_n}$ by (2.34);
- End
- Compute the values $\{Y^n(t_{n,j})\}_{j=0}^{M_n}$ by the first formula of (2.28);

End

Remark 2.1. For the linear VIEs with delays ($G_j(t, y(t)) = y(t)$, $j = 1, 2$), equation (2.34) is equivalent to the following linear systems:

$$\begin{cases} (\mathbb{I}_1 - \mathbb{E}_1)\mathbf{y}^1 = \mathbf{f}^1, & \text{for} \quad n = 1, \\ (\mathbb{I}_n - \mathbb{E}_n)\mathbf{y}^n = \mathbf{f}^n + \mathbf{a}^n + \mathbf{c}^n + \mathbf{d}^n, & \text{for} \quad n > 1, \end{cases} \tag{2.35}$$

where $\mathbb{I}_n$ is an identity matrix of order $(M_n + 1) \times (M_n + 1)$, and $\mathbb{E}_1 = (e^1_{pp'})$ and $\mathbb{E}_n = (e^n_{pp'})$, $0 \le p, p' \le M_n$ are two matrices with the entries

$$\begin{aligned} e^1_{pp'} = &\frac{2p+1}{4} \sum_{i,j=0}^{M_1} (t_{1,i} - t_0) K_1(t_{1,i}, \sigma(t_{1,i}, t_{1,j})) L_{1,p'}(\sigma(t_{1,i}, t_{1,j})) L_{1,p}(t_{1,i}) \omega_{1,i}\omega_{1,j} \\ &+ \frac{2p+1}{4} \sum_{i,j=0}^{M_1} (\theta(t_{1,i}) - \theta(t_0)) K_2(t_{1,i}, \rho(t_{1,i}, \widetilde{t}_{1,j})) L_{1,p'}(\rho(t_{1,i}, \widetilde{t}_{1,j})) L_{1,p}(t_{1,i}) \omega_{1,i}\omega_{1,j}, \\ e^n_{pp'} = &\frac{2p+1}{4} \sum_{i,j=0}^{M_n} (t_{n,i} - t_{n-1}) K_1(t_{n,i}, \sigma(t_{n,i}, t_{n,j})) L_{n,p'}(\sigma(t_{n,i}, t_{n,j})) L_{n,p}(t_{n,i}) \omega_{n,i}\omega_{n,j} \end{aligned}$$

and

$$\begin{aligned} &\mathbf{y}^n = (\widetilde{y}^n_0, \cdots, \widetilde{y}^n_{M_n})^T, \quad \mathbf{f}^n = (\widetilde{f}^n_0, \cdots, \widetilde{f}^n_{M_n})^T, \quad \mathbf{a}^n = (\widetilde{a}^n_0, \cdots, \widetilde{a}^n_{M_n})^T, \\ &\mathbf{c}^n = (\widetilde{c}^n_0, \cdots, \widetilde{c}^n_{M_n})^T, \quad \mathbf{d}^n = (d^n_{00}, \cdots, d^n_{M_n 0})^T. \end{aligned}$$

The system (2.35) can be solved directly, based on matrix factorizations such as LU decomposition.

3. Some useful lemmas

In this section, we present some lemmas useful for establishing the convergence results.

Let $\omega^m(x) = (1 - x^2)^m$. Denote by c a generic positive constant independent of T, h_k, M_k and the solutions of (1.1) and (2.27). For any integer $m \ge 0$, we introduce the weighted Sobolev space on $(-1, 1)$,

$$H^m_{\omega,A}(-1,1) = \{v : \|v\|_{H^m_{\omega,A}(-1,1)} < \infty\},$$

with the norm

$$\|v\|_{H^m_{\omega,A}(-1,1)} = \Big(\sum_{k=0}^{m} \|\partial^k_x v\|^2_{L^2_{\omega^k}(-1,1)}\Big)^{\frac{1}{2}}.$$

According to Theorem 4.2 of [20], we have

Lemma 3.1. *For any $u \in H^m_{\omega,A}(-1,1)$ with integer $1 \le m \le M_n + 1$,*

$$\|u - \pi_{M_n} u\|_{L^2(-1,1)} \le c M_n^{-m} \|\partial^m_x u\|_{L^2_{\omega^m}(-1,1)}. \tag{3.1}$$

where $\pi_{M_n} : C(-1,1) \to \mathcal{P}_{M_n}(-1,1)$ is the standard Legendre-Gauss interpolation operator with $\pi_{M_n} u(x_{n,j}) = u(x_{n,j})$, $\quad 0 \le j \le M_n$.

With the aid of the above lemma, we can derive the following approximation result.

Lemma 3.2. *For any $v \in H^m(I_n)$ with integer $1 \le m \le M_n + 1$,*

$$\|v - \mathcal{I}^t_{M_n} v\|_{I_n} \le cM_n^{-m} \|\partial_t^m v\|_{L^2_{\chi_n^m}(I_n)} \le ch_n^m M_n^{-m} \|\partial_t^m v\|_{I_n}, \tag{3.2}$$

where $H^m(I_n)$ is the usual Sobolev space and $\chi_n^m(t) = (t_n - t)^m (t - t_{n-1})^m$.

Proof. Let $u(x) = v(t)\Big|_{t=\frac{h_n x + t_{n-1} + t_n}{2}}$. Then

$$\mathcal{I}^t_{M_n} v(t_{n,j}) = v(t_{n,j}) = u(x_{n,j}) = \pi_{M_n} u(x_{n,j}), \qquad 0 \le j \le M_n.$$

Since $\mathcal{I}^t_{M_n} v(t)\Big|_{t=\frac{h_n x + t_{n-1} + t_n}{2}}$ and $\pi_{M_n} u(x)$ belong to $\mathcal{P}_{M_n}(-1,1)$ in the variable x, we get

$$\mathcal{I}^t_{M_n} v(t)\Big|_{t=\frac{h_n x + t_{n-1} + t_n}{2}} = \pi_{M_n} u(x). \tag{3.3}$$

This, along with equation (3.1), yields

$$\begin{aligned}
\|v - \mathcal{I}^t_{M_n} v\|^2_{I_n} &= \frac{h_n}{2} \int_{-1}^{1} (u(x) - \pi_{M_n} u(x))^2 dx \\
&\le ch_n M_n^{-2m} \int_{-1}^{1} (\partial_x^m u(x))^2 (1 - x^2)^m dx \\
&\le cM_n^{-2m} \int_{I_n} (\partial_t^m v(t))^2 (t_n - t)^m (t - t_{n-1})^m dt \\
&\le ch_n^{2m} M_n^{-2m} \int_{I_n} (\partial_t^m v(t))^2 dt.
\end{aligned} \tag{3.4}$$

Thus, the desired result follows. □

The following discrete Gronwall Lemma can be found in [27].

Lemma 3.3. *Assume that $\{k_j\}$ and $\{\rho_j\}$ $(j \ge 0)$ are given nonnegative sequences, and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le \rho_0$ and*

$$\varepsilon_n \le \rho_n + \sum_{j=0}^{n-1} q_j + \sum_{j=0}^{n-1} k_j \varepsilon_j, \quad n \ge 1,$$

with $q_j \ge 0$ $(j \ge 0)$. Then

$$\varepsilon_n \le \rho_n + \sum_{j=0}^{n-1} (q_j + k_j \rho_j) \exp(\sum_{j=0}^{n-1} k_j), \quad n \ge 1.$$

4. Error analysis

In this section, we shall analyze and characterize the *hp*-convergence of scheme (2.27) under reasonable assumptions on the nonlinearity. Because of the nonlinear history-dependent effects, a theoretical convergence analysis for the multistep method becomes much more difficult, compared with the linear and/or one-step cases. Hereafter, we denote $M_{\min} = \min_{1 \le n \le N} M_n$.

Lemma 4.1. *Let y^n be the solution of* (2.5) *and Y^n the solution of* (2.27)*. Then it holds that*

$$y^n(t) - Y^n(t) = \sum_{j=1}^{5} B_j(t), \tag{4.1}$$

where

$$\begin{aligned}
B_1(t) &= y^n(t) - \mathcal{I}_{M_n}^t y^n(t),\\
B_2(t) &= \frac{1}{h_n}\mathcal{I}_{M_n}^t \big((t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), y^n(\sigma(t,\cdot)))\big)_{I_n}\\
&\quad - \frac{1}{h_n}\mathcal{I}_{M_n}^t \langle (t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), Y^n(\sigma(t,\cdot)))\rangle_{I_n},\\
B_3(t) &= \frac{1}{\theta(t_n)-\theta(t_{n-1})}\mathcal{I}_{M_n}^t \big((\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), y^{\widehat{n}}(\rho(t,\cdot)))\big)_{\widetilde{I}_n}\\
&\quad - \frac{1}{\theta(t_n)-\theta(t_{n-1})}\mathcal{I}_{M_n}^t \langle (\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), Y^{\widehat{n}}(\rho(t,\cdot)))\rangle_{\widetilde{I}_n},\\
B_4(t) &= \sum_{k=1}^{n-1}\mathcal{I}_{M_n}^t \big(K_1(t,\cdot), G_1(\cdot, y^k(\cdot))\big)_{I_k} - \sum_{k=1}^{n-1}\mathcal{I}_{M_n}^t \langle K_1(t,\cdot), G_1(\cdot, Y^k(\cdot))\rangle_{I_k},\\
B_5(t) &= \sum_{k=1}^{n-1}\mathcal{I}_{M_n}^t \big(K_2(t,\cdot), G_2(\cdot, y^{\widehat{k}}(\cdot))\big)_{\widetilde{I}_k} - \sum_{k=1}^{n-1}\mathcal{I}_{M_n}^t \langle K_2(t,\cdot), G_2(\cdot, Y^{\widehat{k}}(\cdot))\rangle_{\widetilde{I}_k}.
\end{aligned}$$

Proof. By virtue of (2.5) and (2.27), we deduce

$$\begin{aligned}
y^n(t) =& f(t) + \sum_{k=1}^{n-1}\big(K_1(t,\cdot), G_1(\cdot, y^k(\cdot))\big)_{I_k}\\
&+ \frac{1}{h_n}\big((t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), y^n(\sigma(t,\cdot)))\big)_{I_n}\\
&+ \sum_{k=1}^{n-1}\big(K_2(t,\cdot), G_2(\cdot, y^{\widehat{k}}(\cdot))\big)_{\widetilde{I}_k}\\
&+ \frac{1}{\theta(t_n)-\theta(t_{n-1})}\big((\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), y^{\widehat{n}}(\rho(t,\cdot)))\big)_{\widetilde{I}_n}
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
Y^n(t) &= \mathcal{I}_{M_n}^t f(t) + \sum_{k=1}^{n-1}\mathcal{I}_{M_n}^t \langle K_1(t,\cdot), G_1(\cdot, Y^k(\cdot))\rangle_{I_k}\\
&\quad + \frac{1}{h_n}\mathcal{I}_{M_n}^t \langle (t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), Y^n(\sigma(t,\cdot)))\rangle_{I_n}\\
&\quad + \sum_{k=1}^{n-1}\mathcal{I}_{M_n}^t \langle K_2(t,\cdot), G_2(\cdot, Y^{\widehat{k}}(\cdot))\rangle_{\widetilde{I}_k}\\
&\quad + \frac{1}{\theta(t_n)-\theta(t_{n-1})}\mathcal{I}_{M_n}^t \langle (\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), Y^{\widehat{n}}(\rho(t,\cdot)))\rangle_{\widetilde{I}_n}.
\end{aligned} \tag{4.3}$$

By subtracting (4.3) from (4.2), we obtain

(4.4)

$$\begin{aligned} y^n(t) - Y^n(t) &= f(t) - \mathcal{I}^t_{M_n} f(t) + \sum_{k=1}^{n-1} \big(K_1(t,\cdot), G_1(\cdot, y^k(\cdot))\big)_{I_k} \\ &- \sum_{k=1}^{n-1} \mathcal{I}^t_{M_n} \langle K_1(t,\cdot), G_1(\cdot, Y^k(\cdot))\rangle_{I_k} \\ &+ \frac{1}{h_n}\big((t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), y^n(\sigma(t,\cdot)))\big)_{I_n} \\ &- \frac{1}{h_n}\mathcal{I}^t_{M_n}\langle (t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), Y^n(\sigma(t,\cdot)))\rangle_{I_n} \\ &+ \sum_{k=1}^{n-1} \big(K_2(t,\cdot), G_2(\cdot, y^{\widehat{k}}(\cdot))\big)_{\widetilde{I}_k} - \sum_{k=1}^{n-1} \mathcal{I}^t_{M_n}\langle K_2(t,\cdot), G_2(\cdot, Y^{\widehat{k}}(\cdot))\rangle_{\widetilde{I}_k} \\ &+ \frac{1}{\theta(t_n)-\theta(t_{n-1})}\big((\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), y^{\widehat{n}}(\rho(t,\cdot)))\big)_{\widetilde{I}_n} \\ &- \frac{1}{\theta(t_n)-\theta(t_{n-1})}\mathcal{I}^t_{M_n}\langle (\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), Y^{\widehat{n}}(\rho(t,\cdot)))\rangle_{\widetilde{I}_n}. \end{aligned}$$

Moreover, by (4.2) we have

$$\begin{aligned} f(t) - \mathcal{I}^t_{M_n} f(t) &= y^n(t) - \mathcal{I}^t_{M_n} y^n(t) + \sum_{k=1}^{n-1} \big(\mathcal{I}^t_{M_n} K_1(t,\cdot) - K_1(t,\cdot), G_1(\cdot, y^k(\cdot))\big)_{I_k} \\ &+ \frac{1}{h_n}(\mathcal{I}^t_{M_n} - \mathcal{I})\big((t-t_{n-1})K_1(t,\sigma(t,\cdot)), G_1(\sigma(t,\cdot), y^n(\sigma(t,\cdot)))\big)_{I_n} \\ &+ \sum_{k=1}^{n-1} \big(\mathcal{I}^t_{M_n} K_2(t,\cdot) - K_2(t,\cdot), G_2(\cdot, y^{\widehat{k}}(\cdot))\big)_{\widetilde{I}_k} \\ &+ \frac{1}{\theta(t_n)-\theta(t_{n-1})}(\mathcal{I}^t_{M_n} - \mathcal{I})\big((\theta(t)-\theta(t_{n-1}))K_2(t,\rho(t,\cdot)), G_2(\rho(t,\cdot), y^{\widehat{n}}(\rho(t,\cdot)))\big)_{\widetilde{I}_n}, \end{aligned}$$

with $\mathcal{I}$ the identity operator. A combination of the previous two equations leads to (4.1). □

Next, we define the Nemytskii operators $\mathbb{G}_i(y)(t) := G_i(t, y(t))$, $i = 1, 2$. Then we have

Lemma 4.2. *Assume that $K_1(t,s) \in C^m(D_1)$, $y|_{t\in I_n} \in H^m(I_n)$, $\mathbb{G}_1 : H^m(I_n) \to H^m(I_n)$ with integer $1 \le m \le M_{\min} + 1$, and G_1 fulfills the following Lipschitz condition:*

$$|G_1(s, y_1) - G_1(s, y_2)| \le \gamma_1 |y_1 - y_2|, \quad \gamma_1 \ge 0. \tag{4.5}$$

Then, it holds that

$$\|B_2\|^2_{I_n} \le c h_n^2 \|y^n - Y^n\|^2_{I_n} + c h_n^{2m+2} M_n^{-2m}\Big(\|\partial_t^m y\|^2_{I_n} + \|G_1(\cdot, y(\cdot))\|^2_{H^m(I_n)}\Big). \tag{4.6}$$

Proof. In order to estimate $\|B_2\|_{I_n}$, we first make some necessary preparations. Let $\lambda \in I_n$ and $\mathcal{I}^\lambda_{M_n} : C(I_n) \to \mathcal{P}_{M_n}(I_n)$ be the shifted Legendre-Gauss interpolation operator in the λ-direction. As in equation (2.4), we set

$$s = \sigma(t, \lambda) := t_{n-1} + \frac{(\lambda - t_{n-1})(t - t_{n-1})}{h_n}.$$

It is clear that $s \in (t_{n-1}, t]$. Denote by $\{\lambda_{n,j}\}_{j=0}^{M_n}$ the $M_n + 1$ Legendre-Gauss quadrature nodes on the interval I_n (actually, $\lambda_{n,j} = t_{n,j}$, see equation (2.12)) and $s_{n,j} := s_{n,j}(t) = \sigma(t, \lambda_{n,j})$. We define a new shifted Legendre-Gauss interpolation operator $\mathcal{I}_{M_n}^{s,t} : C(t_{n-1}, t) \to \mathcal{P}_{M_n}(t_{n-1}, t)$ as follows:

$$\mathcal{I}_{M_n}^{s,t} v(s_{n,j}) = v(s_{n,j}), \qquad 0 \le j \le M_n.$$

Obviously,

$$\mathcal{I}_{M_n}^{s,t} v(s_{n,j}) = v(s_{n,j}) = v(\sigma(t, \lambda_{n,j})) = \mathcal{I}_{M_n}^{\lambda} v(\sigma(t, \lambda_{n,j})), \qquad 0 \le j \le M_n.$$

Moreover, $\mathcal{I}_{M_n}^{s,t} v(s)$ and $\mathcal{I}_{M_n}^{\lambda} v(\sigma(t, \lambda))\Big|_{\lambda = t_{n-1} + \frac{h_n(s - t_{n-1})}{t - t_{n-1}}}$ belong to $\mathcal{P}_{M_n}(t_{n-1}, t)$ in the variable s. Hence,

$$\mathcal{I}_{M_n}^{s,t} v(s) = \mathcal{I}_{M_n}^{\lambda} v(\sigma(t, \lambda))\Big|_{\lambda = t_{n-1} + \frac{h_n(s - t_{n-1})}{t - t_{n-1}}}. \tag{4.7}$$

Thus, by (4.7), (2.16) and (2.14), we obtain

$$\begin{aligned} \int_{t_{n-1}}^{t} \mathcal{I}_{M_n}^{s,t} v(s) ds &= \frac{t - t_{n-1}}{h_n} \int_{I_n} \mathcal{I}_{M_n}^{\lambda} v(\sigma(t, \lambda)) d\lambda \\ &= \frac{t - t_{n-1}}{2} \sum_{j=0}^{M_n} v(\sigma(t, \lambda_{n,j})) \omega_{n,j} = \frac{t - t_{n-1}}{2} \sum_{j=0}^{M_n} v(s_{n,j}) \omega_{n,j}. \end{aligned} \tag{4.8}$$

Similarly,

$$\int_{t_{n-1}}^{t} (\mathcal{I}_{M_n}^{s,t} v(s))^2 ds = \frac{t - t_{n-1}}{2} \sum_{j=0}^{M_n} v^2(s_{n,j}) \omega_{n,j}. \tag{4.9}$$

Furthermore, according to (4.7) and (3.2), we get that for integer $1 \le m \le M_n + 1$ and $t \in I_n$,

$$\begin{aligned} \int_{t_{n-1}}^{t} (v(s) - \mathcal{I}_{M_n}^{s,t} v(s))^2 ds &= \frac{t - t_{n-1}}{h_n} \int_{I_n} \Big(v(\sigma(t, \lambda)) - \mathcal{I}_{M_n}^{\lambda} v(\sigma(t, \lambda)) \Big)^2 d\lambda \\ &\le c h_n^{2m-1} M_n^{-2m} (t - t_{n-1}) \int_{I_n} (\partial_{\lambda}^m v(\sigma(t, \lambda)))^2 d\lambda \\ &\le c h_n^{2m} M_n^{-2m} \int_{t_{n-1}}^{t} (\partial_s^m v(s))^2 ds. \end{aligned} \tag{4.10}$$

We now estimate the term $\|B_2\|_{I_n}$. By Lemma 4.1, equation (2.16) and the first formula of (4.8), we get

$$
\begin{aligned}
\|B_2\|^2_{I_n} &= \frac{1}{h_n^2}\Big\|\mathcal{I}^t_{M_n}\int_{I_n}(t-t_{n-1})K_1(t,\sigma(t,\lambda))G_1(\sigma(t,\lambda),y^n(\sigma(t,\lambda)))d\lambda \\
&\quad -\mathcal{I}^t_{M_n}\int_{I_n}\mathcal{I}^\lambda_{M_n}\Big((t-t_{n-1})K_1(t,\sigma(t,\lambda))G_1(\sigma(t,\lambda),Y^n(\sigma(t,\lambda)))\Big)d\lambda\Big\|^2_{I_n} \\
&= \Big\|\mathcal{I}^t_{M_n}\int_{t_{n-1}}^t K_1(t,s)G_1(s,y^n(s))ds \\
&\quad -\mathcal{I}^t_{M_n}\int_{t_{n-1}}^t \mathcal{I}^{s,t}_{M_n}\Big(K_1(t,s)G_1(s,Y^n(s))\Big)ds\Big\|^2_{I_n} \\
&\le 2(\|D_1\|^2_{I_n}+\|D_2\|^2_{I_n}),
\end{aligned}
\tag{4.11}
$$

where

$$
\begin{aligned}
D_1 &= \mathcal{I}^t_{M_n}\int_{t_{n-1}}^t(\mathcal{I}-\mathcal{I}^{s,t}_{M_n})\Big(K_1(t,s)G_1(s,y^n(s))\Big)ds, \\
D_2 &= \mathcal{I}^t_{M_n}\int_{t_{n-1}}^t\mathcal{I}^{s,t}_{M_n}\Big(K_1(t,s)\big(G_1(s,y^n(s))-G_1(s,Y^n(s))\big)\Big)ds.
\end{aligned}
$$

Next, by (2.16), (2.14), (4.10) and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|D_1\|^2_{I_n} &= \Big\|\mathcal{I}^t_{M_n}\int_{t_{n-1}}^t(\mathcal{I}-\mathcal{I}^{s,t}_{M_n})\Big(K_1(t,s)G_1(s,y^n(s))\Big)ds\Big\|^2_{I_n} \\
&= \frac{h_n}{2}\sum_{j=0}^{M_n}\omega_{n,j}\Big[\int_{t_{n-1}}^{t_{n,j}}(\mathcal{I}-\mathcal{I}^{s,t_{n,j}}_{M_n})\Big(K_1(t_{n,j},s)G_1(s,y^n(s))\Big)ds\Big]^2 \\
&\le ch_n^2\sum_{j=0}^{M_n}\omega_{n,j}\int_{t_{n-1}}^{t_{n,j}}\Big[(\mathcal{I}-\mathcal{I}^{s,t_{n,j}}_{M_n})\Big(K_1(t_{n,j},s)G_1(s,y^n(s))\Big)\Big]^2ds \\
&\le ch_n^{2m+2}M_n^{-2m}\sum_{j=0}^{M_n}\omega_{n,j}\int_{t_{n-1}}^{t_{n,j}}\Big(\partial_s^m\big(K_1(t_{n,j},s)G_1(s,y^n(s))\big)\Big)^2ds \\
&\le ch_n^{2m+2}M_n^{-2m}\sum_{i=0}^{m}\int_{I_n}\big(\partial_s^iG_1(s,y(s))\big)^2ds.
\end{aligned}
\tag{4.12}
$$

Thanks to (4.8), (2.16) and (2.14), we get

$$
\begin{aligned}
\|D_2\|^2_{I_n} &= \frac{h_n}{2}\sum_{i=0}^{M_n}\Big[\frac{t_{n,i}-t_{n-1}}{2}\sum_{j=0}^{M_n}K_1(t_{n,i},s_{n,j})\big(G_1(s_{n,j},y^n(s_{n,j})) \\
&\quad -G_1(s_{n,j},Y^n(s_{n,j}))\big)\omega_{n,j}\Big]^2\omega_{n,i},
\end{aligned}
$$

where $s_{n,j} = s_{n,j}(t_{n,i})$ (slightly different from the previous definition $s_{n,j}$). Further, by (4.5), (4.9), (4.10) and Hölder's inequality, we deduce that

(4.13)
$$
\begin{aligned}
\|D_2\|^2_{I_n} &\leq ch_n^2 \sum_{i=0}^{M_n} \Big[\omega_{n,i} \frac{t_{n,i} - t_{n-1}}{2} \sum_{j=0}^{M_n} \big(y^n(s_{n,j}) - Y^n(s_{n,j})\big)^2 \omega_{n,j}\Big] \sum_{j=0}^{M_n} \omega_{n,j} \\
&\leq ch_n^2 \sum_{i=0}^{M_n} \omega_{n,i} \int_{t_{n-1}}^{t_{n,i}} (\mathcal{I}_{M_n}^{s,t_{n,i}} y^n(s) - Y^n(s))^2 ds \\
&\leq ch_n^2 \sum_{i=0}^{M_n} \omega_{n,i} \int_{t_{n-1}}^{t_{n,i}} \Big((\mathcal{I}_{M_n}^{s,t_{n,i}} y^n(s) - y^n(s))^2 + (y^n(s) - Y^n(s))^2\Big) ds \\
&\leq ch_n^2 \|y^n - Y^n\|^2_{I_n} + ch_n^{2m+2} M_n^{-2m} \int_{I_n} (\partial_s^m y(s))^2 ds.
\end{aligned}
$$

Finally, a combination of (4.11)-(4.13) leads to the desired result. □

Lemma 4.3. *Assume that $K_1(t,s) \in C^m(D_1)$, $y|_{t\in I_n} \in H^m(I_n)$, $\mathbb{G}_1 :\ H^m(I_n) \to H^m(I_n)$ with $1 \leq n \leq N$ and integer $1 \leq m \leq M_{\min}+1$, and G_1 fulfills the Lipschitz condition* (4.5)*. Then, it holds that*

(4.14)
$$
\|B_4\|^2_{I_n} \leq cTh_n \sum_{k=1}^{n-1} \Big(\|y^k - Y^k\|^2_{I_k} + h_k^{2m} M_k^{-2m} \big(\|\partial_t^m y\|^2_{I_k} + \|G_1(\cdot, y(\cdot))\|^2_{H^m(I_k)}\big)\Big).
$$

Proof. According to Lemma 4.1 and the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\|B_4\|^2_{I_n} &= \Big\| \sum_{k=1}^{n-1} \mathcal{I}_{M_n}^t \big(K_1(t,\cdot), G_1(\cdot, y^k(\cdot))\big)_{I_k} \\
&\qquad - \sum_{k=1}^{n-1} \mathcal{I}_{M_n}^t \langle K_1(t,\cdot), G_1(\cdot, Y^k(\cdot))\rangle_{I_k} \Big\|^2_{I_n} \\
&\leq \Big(\sum_{k=1}^{n-1} \Big\| \mathcal{I}_{M_n}^t \int_{I_k} K_1(t,\xi) G_1(\xi, y^k(\xi)) d\xi \\
&\qquad - \mathcal{I}_{M_n}^t \int_{I_k} \mathcal{I}_{M_k}^\xi \Big(K_1(t,\xi) G_1(\xi, Y^k(\xi))\Big) d\xi \Big\|_{I_n} \Big)^2 \\
&\leq \sum_{j=1}^{n-1} h_j \sum_{k=1}^{n-1} \frac{1}{h_k} \Big\| \mathcal{I}_{M_n}^t \int_{I_k} K_1(t,\xi) G_1(\xi, y^k(\xi)) d\xi \\
&\qquad - \mathcal{I}_{M_n}^t \int_{I_k} \mathcal{I}_{M_k}^\xi \Big(K_1(t,\xi) G_1(\xi, Y^k(\xi))\Big) d\xi \Big\|^2_{I_n} \qquad (4.15) \\
&\leq T \sum_{k=1}^{n-1} \frac{1}{h_k} \Big\| \mathcal{I}_{M_n}^t \int_{I_k} K_1(t,\xi) G_1(\xi, y^k(\xi)) d\xi \\
&\qquad - \mathcal{I}_{M_n}^t \int_{I_k} \mathcal{I}_{M_k}^\xi \Big(K_1(t,\xi) G_1(\xi, Y^k(\xi))\Big) d\xi \Big\|^2_{I_n} \\
&\leq 2T \sum_{k=1}^{n-1} \frac{1}{h_k} (\|B_{4,1}\|^2_{I_n} + \|B_{4,2}\|^2_{I_n}),
\end{aligned}
$$

where

$$B_{4,1} = \mathcal{I}^t_{M_n} \int_{I_k} (\mathcal{I} - \mathcal{I}^\xi_{M_k}) \Big(K_1(t,\xi) G_1(\xi, y^k(\xi)) \Big) d\xi,$$
$$B_{4,2} = \mathcal{I}^t_{M_n} \int_{I_k} \mathcal{I}^\xi_{M_k} \Big(K_1(t,\xi) \big(G_1(\xi, y^k(\xi)) - G_1(\xi, Y^k(\xi)) \big) \Big) d\xi.$$

Next, by (2.16), (2.14), (3.2) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|B_{4,1}\|^2_{I_n} &= \Big\| \mathcal{I}^t_{M_n} \int_{I_k} (\mathcal{I} - \mathcal{I}^\xi_{M_k}) \Big(K_1(t,\xi) G_1(\xi, y^k(\xi)) \Big) d\xi \Big\|^2_{I_n} \\ &= \frac{h_n}{2} \sum_{j=0}^{M_n} \omega_{n,j} \Big[\int_{I_k} (\mathcal{I} - \mathcal{I}^\xi_{M_k}) \Big(K_1(t_{n,j},\xi) G_1(\xi, y^k(\xi)) \Big) d\xi \Big]^2 \\ &\le c h_k h_n \sum_{j=0}^{M_n} \omega_{n,j} \int_{I_k} \Big[(\mathcal{I} - \mathcal{I}^\xi_{M_k}) \Big(K_1(t_{n,j},\xi) G_1(\xi, y^k(\xi)) \Big) \Big]^2 d\xi \\ &\le c h_n h_k^{2m+1} M_k^{-2m} \sum_{j=0}^{M_n} \omega_{n,j} \int_{I_k} \Big[\partial^m_\xi \big(K_1(t_{n,j},\xi) G_1(\xi, y^k(\xi)) \big) \Big]^2 d\xi \\ &\le c h_n h_k^{2m+1} M_k^{-2m} \sum_{i=0}^{m} \int_{I_k} (\partial^i_\xi G_1(\xi, y(\xi)))^2 d\xi. \end{aligned} \tag{4.16}$$

Thanks to (2.16) and (2.14), we get

$$\begin{aligned} \|B_{4,2}\|^2_{I_n} = \frac{h_n}{2} \sum_{i=0}^{M_n} \Big[\frac{h_k}{2} \sum_{j=0}^{M_k} K_1(t_{n,i}, t_{k,j}) \big(G_1(t_{k,j}, y^k(t_{k,j})) \\ - G_1(t_{k,j}, Y^k(t_{k,j})) \big) \omega_{k,j} \Big]^2 \omega_{n,i}. \end{aligned}$$

Moreover, by (4.5), (2.16), (2.14), (3.2) and Hölder's inequality, we deduce readily that

$$\begin{aligned} \|B_{4,2}\|^2_{I_n} &\le c h_k^2 h_n \sum_{j=0}^{M_k} \big(y^k(t_{k,j}) - Y^k(t_{k,j}) \big)^2 \omega_{k,j} \sum_{j=0}^{M_k} \omega_{k,j} \\ &\le c h_k h_n \int_{I_k} \big(\mathcal{I}^\xi_{M_k} y^k(\xi) - Y^k(\xi) \big)^2 d\xi \\ &\le c h_k h_n \int_{I_k} \Big[(\mathcal{I}^\xi_{M_k} y^k(\xi) - y^k(\xi))^2 + (y^k(\xi) - Y^k(\xi))^2 \Big] d\xi \\ &\le c h_k h_n \|y^k - Y^k\|^2_{I_k} + c h_n h_k^{2m+1} M_k^{-2m} \int_{I_k} \big(\partial^m_\xi y(\xi) \big)^2 d\xi. \end{aligned} \tag{4.17}$$

Therefore, a combination of (4.15)-(4.17) leads to (4.14). □

Theorem 4.1. *Let y^n be the solution of (2.5) and Y^n be the solution of (2.27). Assume that the conditions (C1)-(C3) hold, $K_i(t,s) \in C^m(D_i)$, $y|_{t\in I_n} \in H^m(I_n)$, $\mathbb{G}_i : H^m(I_n) \to H^m(I_n)$ with $1 \le n \le N$ and integer $1 \le m \le M_{\min} + 1$, and G_i fulfill the following Lipschitz conditions:*

$$|G_i(s, y_1) - G_i(s, y_2)| \le \gamma_i |y_1 - y_2|, \quad \gamma_i \ge 0, \quad i = 1, 2. \tag{4.18}$$

Then, for any $1 \le n \le N$ and $h_{\max}$ sufficiently small (cf. equation (4.24) *below),*

$$
\begin{aligned}
\|y_n - Y^n\|_{I_n}^2 \le c\exp(cT^2)\Big[& h_n^{2m} M_n^{-2m}\big(\|\partial_t^m y\|_{I_n}^2 + h_n^2 \|G_1(\cdot, y(\cdot))\|_{H^m(I_n)}^2\big) \\
& + h_n^{2m+2} M_{\widehat{n}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_n}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_n)}^2\big) \\
& + h_n \sum_{k=1}^{n-1}\Big(h_k^{2m} M_k^{-2m}\big(\|\partial_t^m y\|_{I_k}^2 + \|G_1(\cdot, y(\cdot))\|_{H^m(I_k)}^2\big) \\
& + h_k^{2m} M_{\widehat{k}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_k}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_k)}^2\big)\Big)\Big].
\end{aligned} \tag{4.19}
$$

Proof. Clearly, by (3.2) we get that for integer $1 \le m \le M_n + 1$,

$$
\|B_1\|_{I_n}^2 = \|y^n - \mathcal{I}_{M_n}^t y^n\|_{I_n}^2 \le c h_n^{2m} M_n^{-2m} \|\partial_t^m y\|_{I_n}^2. \tag{4.20}
$$

Using similar arguments as in Lemmas 4.2 and 4.3, we derive that (see Appendices A and B)

$$
\|B_3\|_{I_n}^2 \le c h_n^2 \|y^{\widehat{n}} - Y^{\widehat{n}}\|_{\widetilde{I}_n}^2 + c h_n^{2m+2} M_{\widehat{n}}^{-2m}\Big(\|\partial_t^m y\|_{\widetilde{I}_n}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_n)}^2\Big) \tag{4.21}
$$

and

$$
\|B_5\|_{I_n}^2 \le cTh_n \sum_{k=1}^{n-1}\Big(\|y^{\widehat{k}} - Y^{\widehat{k}}\|_{\widetilde{I}_k}^2 + h_k^{2m} M_{\widehat{k}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_k}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_k)}^2\big)\Big). \tag{4.22}
$$

Moreover, by (2.2) we have

$$
\sum_{k=1}^{n} \|y^{\widehat{k}} - Y^{\widehat{k}}\|_{\widetilde{I}_k}^2 \le \sum_{k=1}^{n-1} \|y^k - Y^k\|_{I_k}^2.
$$

Hence, by (4.1), (4.20), (4.6), (4.21), (4.14) and (4.22), we deduce that

$$
\begin{aligned}
(1 - ch_n^2)\|y^n - Y^n\|_{I_n}^2 \le\ & cTh_n \sum_{k=1}^{n-1} \|y^k - Y^k\|_{I_k}^2 \\
& + c h_n^{2m} M_n^{-2m}\big(\|\partial_t^m y\|_{I_n}^2 + h_n^2 \|G_1(\cdot, y(\cdot))\|_{H^m(I_n)}^2\big) \\
& + c h_n^{2m+2} M_{\widehat{n}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_n}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_n)}^2\big) \\
& + cTh_n \sum_{k=1}^{n-1}\Big(h_k^{2m} M_k^{-2m}\big(\|\partial_t^m y\|_{I_k}^2 + \|G_1(\cdot, y(\cdot))\|_{H^m(I_k)}^2\big) \\
& + h_k^{2m} M_{\widehat{k}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_k}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_k)}^2\big)\Big).
\end{aligned} \tag{4.23}
$$

Next, let

$$
\varepsilon_k = h_k^{-1} \|y^k - Y^k\|_{I_k}^2, \qquad 1 \le k \le n,
$$

and assume that $h_{\max}$ is sufficiently small such that

$$
c h_{\max}^2 \le \beta < 1. \tag{4.24}
$$

Then, we use equation (4.23) and Lemma 3.3 to obtain

$$
\begin{aligned}
\varepsilon_n \le c\exp(cT^2)\Big[& h_n^{2m-1} M_n^{-2m}\big(\|\partial_t^m y\|_{I_n}^2 + h_n^2\|G_1(\cdot,y(\cdot))\|_{H^m(I_n)}^2\big) \\
& + h_n^{2m+1} M_{\widehat{n}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_n}^2 + \|G_2(\cdot,y(\cdot))\|_{H^m(\widetilde{I}_n)}^2\big) \\
& + \sum_{k=1}^{n-1}\Big(h_k^{2m} M_k^{-2m}\big(\|\partial_t^m y\|_{I_k}^2 + \|G_1(\cdot,y(\cdot))\|_{H^m(I_k)}^2\big) \\
& + h_k^{2m} M_{\widehat{k}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_k}^2 + \|G_2(\cdot,y(\cdot))\|_{H^m(\widetilde{I}_k)}^2\big)\Big)\Big].
\end{aligned} \tag{4.25}
$$

This implies (4.19). □

Further, let $Y(t)$ be the global numerical solution of equation (1.1), which is given by

$$Y(t) := Y^n(t), \quad t \in I_n, \quad 1 \le n \le N.$$

Then, according to Theorem 4.1, we obtain

Theorem 4.2. *Let $y(t)$ be the solution of equation* (1.1) *and let $Y(t)$ be the global numerical solution of equation* (1.1)*. Assume that the conditions (C1)-(C3) hold, $K_i(t,s) \in C^m(D_i)$, $y|_{t\in I_n} \in H^m(I_n)$, $\mathbb{G}_i : H^m(I_n) \to H^m(I_n)$ with $1 \le n \le N$ and integer $1 \le m \le M_{\min}+1$ and G_i, $i = 1,2$ fulfill the Lipschitz conditions* (4.18)*. Then for $h_{\max}$ sufficiently small (cf.* (4.24)*), it holds that*

$$
\begin{aligned}
\|y - Y\|_{L^2(I)}^2 \le c\exp(cT^2)\sum_{n=1}^{N}\Big[& h_n^{2m} M_n^{-2m}\big(\|\partial_t^m y\|_{I_n}^2 + h_n^2\|G_1(\cdot,y(\cdot))\|_{H^m(I_n)}^2\big) \\
& + h_n^{2m+2} M_{\widehat{n}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_n}^2 + \|G_2(\cdot,y(\cdot))\|_{H^m(\widetilde{I}_n)}^2\big) \\
& + h_n\sum_{k=1}^{n-1}\Big(h_k^{2m} M_k^{-2m}\big(\|\partial_t^m y\|_{I_k}^2 + \|G_1(\cdot,y(\cdot))\|_{H^m(I_k)}^2\big) \\
& + h_k^{2m} M_{\widehat{k}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_k}^2 + \|G_2(\cdot,y(\cdot))\|_{H^m(\widetilde{I}_k)}^2\big)\Big)\Big].
\end{aligned} \tag{4.26}
$$

The above result can be simply written as

$$
\begin{aligned}
\|y - Y\|_{L^2(I)} \le c\exp(cT^2) h_{\max}^m M_{\min}^{-m}(& \|\partial_t^m y\|_{L^2(I)} + \|G_1(\cdot,y(\cdot))\|_{H^m(I)} \\
& + \|G_2(\cdot,y(\cdot))\|_{H^m(0,\theta(T))}),
\end{aligned} \tag{4.27}
$$

provided that $y \in H^m(I)$ and $\mathbb{G}_i : H^m(I) \to H^m(I)$.

Proof. The first bound follows from Theorem 4.1. We next estimate the result (4.27). Obviously,

$$
\begin{aligned}
& \sum_{n=1}^{N}\Big(h_n^{2m} M_n^{-2m}\big(\|\partial_t^m y\|_{I_n}^2 + h_n^2\|G_1(\cdot,y(\cdot))\|_{H^m(I_n)}^2\big) \\
& \quad + h_n^{2m+2} M_{\widehat{n}}^{-2m}\big(\|\partial_t^m y\|_{\widetilde{I}_n}^2 + \|G_2(\cdot,y(\cdot))\|_{H^m(\widetilde{I}_n)}^2\big)\Big) \\
& \le h_{\max}^{2m} M_{\min}^{-2m}\big(\|\partial_t^m y\|_{L^2(I)}^2 + \|G_1(\cdot,y(\cdot))\|_{H^m(I)}^2 + \|G_2(\cdot,y(\cdot))\|_{H^m(0,\theta(T))}^2\big).
\end{aligned}
$$

Similarly,

$$\sum_{n=1}^{N} h_n \sum_{k=1}^{n-1} \Big(h_k^{2m} M_k^{-2m} \big(\|\partial_t^m y\|_{I_k}^2 + \|G_1(\cdot, y(\cdot))\|_{H^m(I_k)}^2 \big)$$
$$+ h_k^{2m} M_{\widehat{k}}^{-2m} \big(\|\partial_t^m y\|_{\widetilde{I}_k}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(\widetilde{I}_k)}^2 \big) \Big)$$
$$\leq T h_{\max}^{2m} M_{\min}^{-2m} \big(\|\partial_t^m y\|_{L^2(I)}^2 + \|G_1(\cdot, y(\cdot))\|_{H^m(I)}^2 + \|G_2(\cdot, y(\cdot))\|_{H^m(0,\theta(T))}^2 \big).$$

Hence, a combination of the previous inequalities leads to (4.27). □

5. Numerical results

In this section, we present some numerical results to illustrate the efficiency of the multistep Legendre-Gauss spectral collocation method. We denote by $\{\xi_\mu\}_{\mu=0}^{N^c}$ the coarse grid, and by $\{t_j^{(\mu)}\}_{j=0}^{N_\mu^f}$ the fine grid in each sub-interval $[\xi_{\mu-1}, \xi_\mu]$ (cf. (i) and (ii) of Subsection 2.1.1). Let $E_1(T)$ and $E_2(T)$ be the maximum error at the mesh points and the discrete L^2-error:

$$E_1(T) = \max_{1 \leq k \leq N} |y(t_k) - Y(t_k)|,$$
$$E_2(T) = \Big(\sum_{k=1}^{N} \frac{h_k}{2} \sum_{j=0}^{M_k} (y^k(t_{k,j}) - Y^k(t_{k,j}))^2 \omega_{k,j} \Big)^{\frac{1}{2}} \approx \Big(\int_0^T (y(t) - Y(t))^2 dt \Big)^{\frac{1}{2}}.$$

5.1. **Linear problem.** Consider the linear VIE with delays (cf. [11]):

$$y(t) = \frac{1}{2}(1 + e^{-qt}) - \int_0^t y(s)ds + \frac{1}{2} \int_0^{qt} y(s)ds, \qquad t \in [0, T], \tag{5.1}$$

with the exact solution $y(t) = e^{-t}$.

We use the multistep Legendre-Gauss collocation scheme (2.35) to resolve equation (5.1) numerically. In Figures 5.1-5.6, we list the maximum errors and the discrete L^2-errors of equation (5.1), with $T = 10$, $q = 0.1,\ 0.5,\ 0.99$, various N^c (the number of coarse grid) and N_μ^f (the number of fine grid in each sub-interval $[\xi_{\mu-1}, \xi_\mu]$), and the uniform mode $M_k \equiv M$. They indicate that the numerical errors decay exponentially as M increases and/or $h_{\max}$ decreases. This means that we may refine the mesh and/or increase the degree of the polynomials to achieve higher accuracy. In fact, this is the main advantage of the *hp*-version.

5.2. **Nonlinear problem.** Consider the nonlinear VIE with delays:

$$y(t) = f(t) + \int_0^t e^{s-t}(y(s) + e^{-y(s)})ds + \int_0^{\theta(t)} e^{s-t}(y(s) + e^{-y(s)})ds, \qquad t \in [0, T], \tag{5.2}$$

with the exact solution $y(t) = \ln(t + e)$ and $f(t) = 2e^{-t} - e^{\theta(t)-t} \ln(\theta(t) + e)$.

We use the multistep algorithm in Table 2.1 to resolve equation (5.2) numerically. In Figures 5.7 and 5.8, we list the maximum errors and the discrete L^2-errors of equation (5.2), with $T = 1$, $\theta(t) = \frac{4}{5}\sin(t)$, various N^c and N_μ^f, and the uniform mode $M_k \equiv M$. In Figures 5.9 and 5.10, we also list the maximum errors and the discrete L^2-errors of equation (5.2), with $T = 100$, $\theta(t) = 1/2\arctan(t)$, $N^c = 5$, the uniform mode $M_k \equiv M$ and various N_μ^f. More precisely, we take

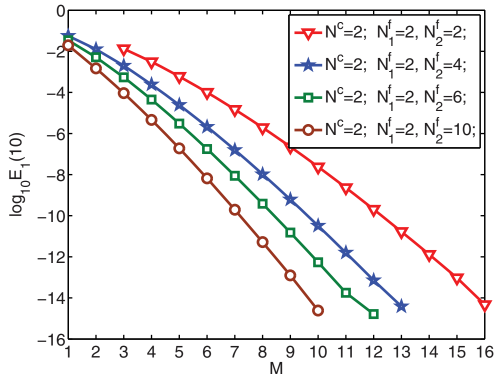

FIGURE 5.1. The maximum errors of equation (5.1) with $q=0.1$.

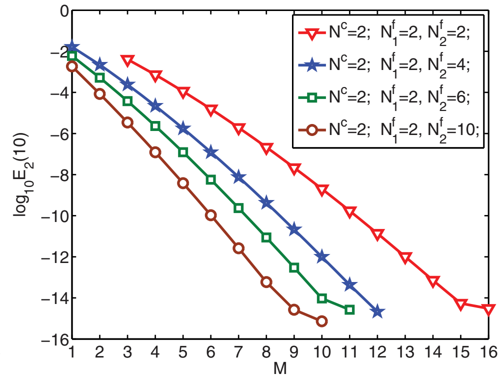

FIGURE 5.2. The discrete L^2-errors of equation (5.1) with $q=0.1$.

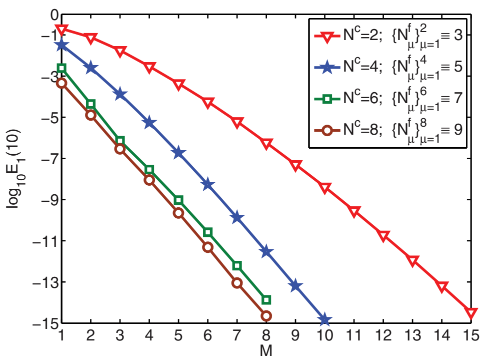

FIGURE 5.3. The maximum errors of equation (5.1) with $q=0.5$.

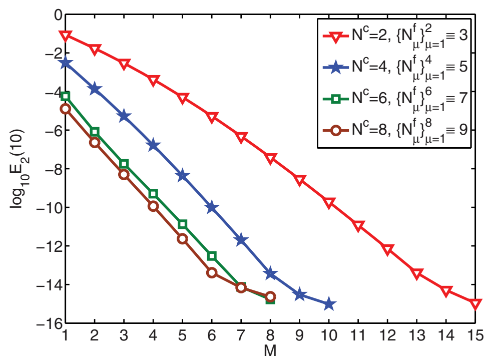

FIGURE 5.4. The discrete L^2-errors of equation (5.1) with $q=0.5$.

①. $N_5^f = 20$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 5$, the total interval number $N = 40$;
②. $N_5^f = 40$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 9$, the total interval number $N = 76$;
③. $N_5^f = 80$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 17$, the total interval number $N = 148$.

We find that the suggested algorithm not only has high-order accuracy for long time numerical simulations, but also provides flexibility with respect to variable time steps.

5.3. **Oscillating solution.** Consider the nonlinear VIE with delays:

$$y(t) = f(t) + \int_0^t t^2 s y^2(s)ds + \int_0^{\theta(t)} y^2(s)ds, \qquad t \in [0,T], \tag{5.3}$$

with $f(t) = -\frac{1}{4}t^4 + \frac{t^2}{4\lambda^2}\sin^2(\lambda t) - \frac{t^3}{4\lambda}\sin(2\lambda t) - \frac{1}{2}\theta(t) + \frac{1}{4\lambda}\sin(2\lambda\theta(t)) + \cos(\lambda t)$ and the exact solution $y(t) = \cos(\lambda t)$.

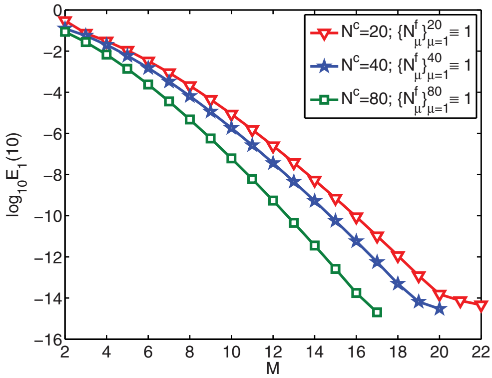

FIGURE 5.5. The maximum errors of equation (5.1) with $q = 0.99$.

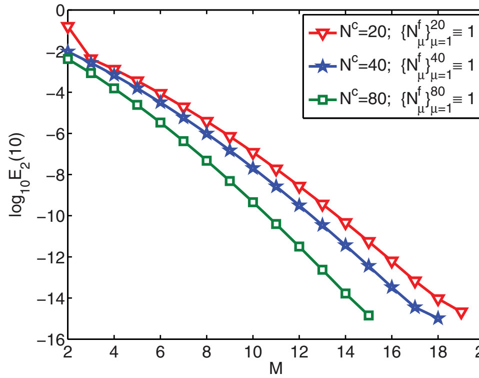

FIGURE 5.6. The discrete L^2-errors of equation (5.1) with $q = 0.99$.

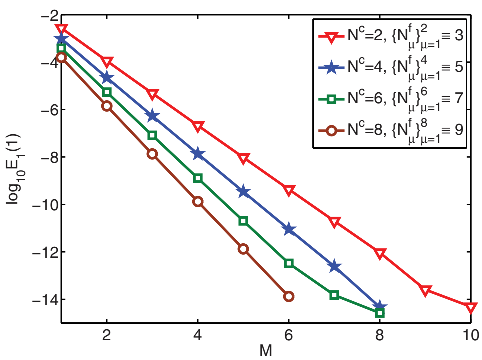

FIGURE 5.7. The maximum errors of equation (5.2).

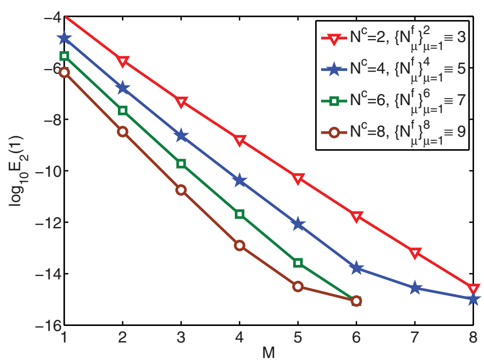

FIGURE 5.8. The discrete L^2-errors of equation (5.2).

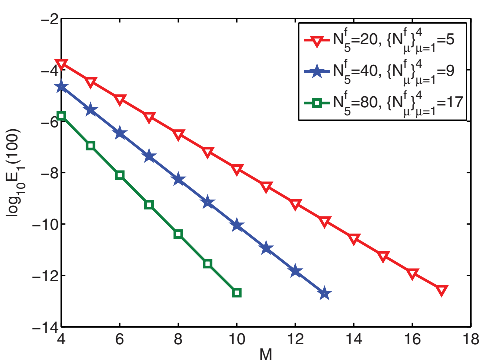

FIGURE 5.9. The maximum errors of equation (5.2).

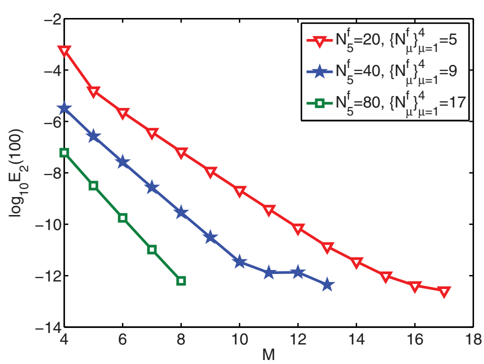

FIGURE 5.10. The discrete L^2-errors of equation (5.2).

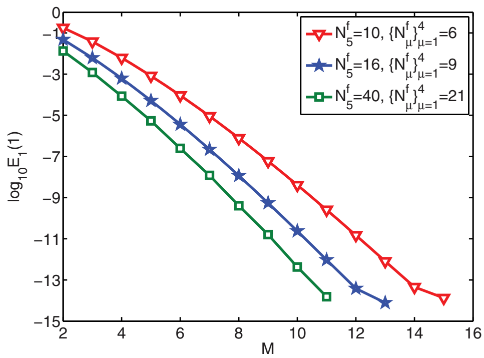

FIGURE 5.11. The maximum errors of equation (5.3).

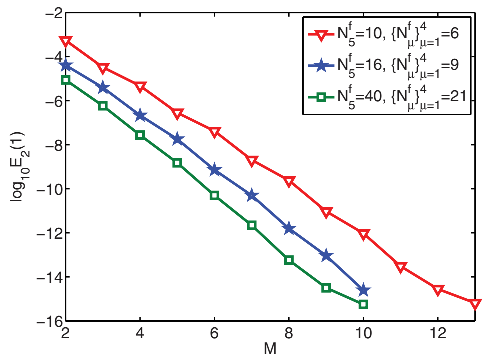

FIGURE 5.12. The discrete L^2-errors of equation (5.3).

We use the multistep algorithm in Table 2.1 to resolve equation (5.3) numerically. In Figures 5.11 and 5.12, we list the maximum errors and the discrete L^2-errors of equation (5.3), with $T = 1$, $\theta(t) = \frac{1}{2}\sin(t)$, $\lambda = 100$, $N^c = 5$, the uniform mode $M_k \equiv M$ and

①. $N_5^f = 10$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 6$, the total interval number $N = 34$;
②. $N_5^f = 16$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 9$, the total interval number $N = 52$;
③. $N_5^f = 40$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 21$, the total interval number $N = 124$.

We observe that the numerical errors decay exponentially as M increases and/or $h_{\max}$ decreases. In particular, they indicate that our algorithm is very effective for highly oscillating solutions. Indeed, this is also one of the main advantages of the hp-version.

5.4. **Steep gradient solution.** Consider the nonlinear VIE with delays:

$$y(t) = f(t) + \int_0^t y^2(s)ds + \int_0^{\theta(t)} y^2(s)ds, \qquad t \in [0, T], \tag{5.4}$$

where

$$f(t) = \frac{\sqrt{\pi}}{2}ca^2\big(\mathrm{erf}(\frac{b-t}{2}) - \mathrm{erf}(\frac{b}{c})\big) - \frac{\sqrt{\pi}}{2}ca^2\mathrm{erf}(\frac{\theta(t)-b}{2}) + a\exp(-\frac{(t-b)^2}{2c^2}).$$

The exact solution $y(t) = a\exp(-\frac{(t-b)^2}{2c^2})$ is a Gaussian function, which has extremely steep gradients near $t = b$.

We use the multistep algorithm in Table 2.1 to resolve equation (5.4) numerically. In Figures 5.13 and 5.14, we list the maximum errors and the discrete L^2-errors of equation (5.4), with $T = 10$, $\theta(t) = \frac{2}{3}\arctan(t)$, $a = 1$, $b = 5$, $c = 0.1$, $N^c = 5$, the uniform mode $M_k \equiv M$ and

①. $N_5^f = 10$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 5$, the total interval number $N = 30$;
②. $N_5^f = 20$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 5$, the total interval number $N = 40$;
③. $N_5^f = 40$, $\{N_\mu^f\}_{\mu=1}^4 \equiv 5$, the total interval number $N = 60$.

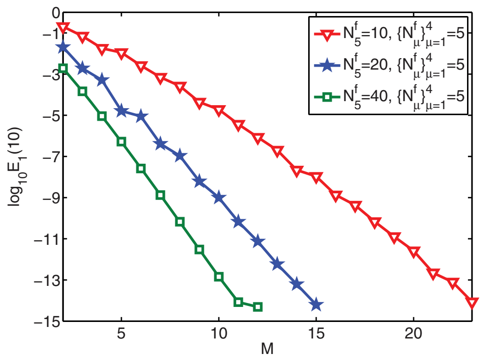

FIGURE 5.13. The maximum errors of equation (5.4).

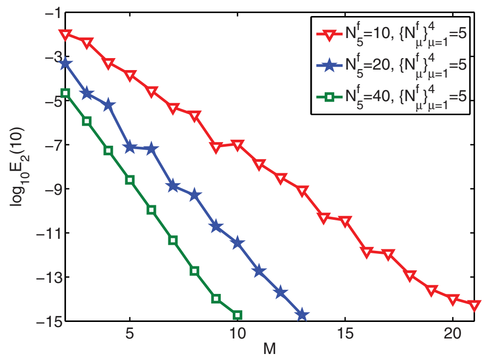

FIGURE 5.14. The discrete L^2-errors of equation (5.4).

They indicate that our algorithm achieves high-order accuracy for the solutions of VIEs with steep gradients by using the locally refined meshes near $t = b$.

5.5. **Nonsmooth solutions.**

5.5.1. *Discontinuous solution.* Consider the linear delay VIE with a discontinuous solution:

$$y(t) = f(t) + \int_0^t y(s)ds + \int_0^{\frac{3}{5}\arctan(t)} y(s)ds, \qquad t \in [0, 5], \tag{5.5}$$

where

$$f(t) = \begin{cases} t - \frac{t^2}{2} - \frac{9}{50}\arctan^2(t), & t \leq \frac{5}{2}, \\ \frac{1}{t} - \frac{25}{8} - \ln(t) + \ln(\frac{5}{2}) - \frac{9}{50}\arctan^2(t), & t > \frac{5}{2}, \end{cases}$$

and the exact solution is given by

$$y(t) = \begin{cases} t, & t \leq \frac{5}{2}, \\ \frac{1}{t}, & t > \frac{5}{2}. \end{cases}$$

We use the multistep algorithm in Table 2.1 to resolve equation (5.5) numerically. To this end, we first decompose the interval $[0, 5]$ into two coarse grid cells $[0, \frac{3}{5}\arctan(5)]$ and $[\frac{3}{5}\arctan(5), 5]$, then we refine the coarse grid cells such that the set of fine grid points includes the breaking point $t = \frac{5}{2}$. In Figures 5.15 and 5.16, we list the maximum errors and the discrete L^2-errors of equation (5.5), with $T = 5$, the uniform mode $M_k \equiv M$ and various N_μ^f. They indicate that our algorithm also provides accurate results even for discontinuous solutions.

5.6. **Numerical comparisons.** We use the multistep algorithm in Table 2.1 to resolve equation (5.1) numerically. In Table 5.1 below, we compare the maximum errors at the mesh points of our algorithm with that of the collocation method suggested in [11] (see Tables 2 and 3 of [11]), for which we take $T = 10$, $q = 0.5,\ 0.2$, the uniform mode $M = 7$, and the same number of collocation points as in [11]. More precisely, we take

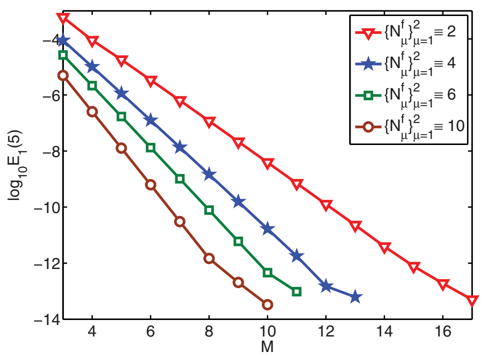

FIGURE 5.15. The maximum errors of equation (5.5).

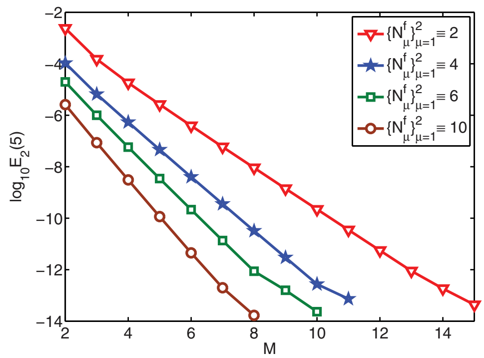

FIGURE 5.16. The discrete L^2-errors of equation (5.5).

(a) $N^c = 5$, $\{N^f_\mu\}^5_{\mu=1} = 5$, the total interval number $N = 25$;
(b) $N^c = 5$, $\{N^f_\mu\}^5_{\mu=1} = 10$, the total interval number $N = 50$;
(c) $N^c = 10$, $\{N^f_\mu\}^{10}_{\mu=1} = 10$, the total interval number $N = 100$;
(d) $N^c = 10$, $\{N^f_\mu\}^{10}_{\mu=1} = 20$, the total interval number $N = 200$;
(e) $N^c = 20$, $\{N^f_\mu\}^{20}_{\mu=1} = 20$, the total interval number $N = 400$.

We observe that our method provides more accurate numerical results by using the same number of collocation points. It should be noted that the meshes of the two methods are different. The method in [11] is based on geometric meshes.

TABLE 5.1. A comparison of numerical errors for equation (5.1).

		$q = 0.5$		$q = 0.2$	
	DOF	The method in [11]	Our method	The method in [11]	Our method
(a)	200	1.94e-07	7.93e-12	1.70e-07	5.14e-09
(b)	400	1.49e-08	3.11e-13	1.78e-08	2.95e-11
(c)	800	1.45e-09	3.90e-14	1.72e-09	2.95e-11
(d)	1600	1.40e-10	9.44e-15	1.60e-10	1.40e-13
(e)	3200	1.23e-11	9.99e-15	1.47e-11	1.40e-13

6. Concluding Remarks

In this paper, we proposed a multistep Legendre-Gauss spectral collocation method for the nonlinear VIEs with vanishing variable delays. We designed a simple iterative algorithm and derived the hp-convergence under L^2-norm. Numerical experiments demonstrated that the suggested method possesses spectral accuracy. In particular, it is very appropriate for various problems with highly oscillating solutions, steep gradient solutions and nonsmooth solutions. It also works well for numerical simulations of long time behaviors.

Appendix A. The proof of the inequality (4.21)

Proof. Let $\nu \in \widetilde{I}_n$ and $\widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}} : \ C(\theta(t_{n-1}), \theta(t_n)) \to \mathcal{P}_{M_{\widehat{n}}}(\theta(t_{n-1}), \theta(t_n))$ be the Legendre-Gauss interpolation operator in the ν-direction. As in equation (2.4), we set

$$\eta = \rho(t,\nu) := \theta(t_{n-1}) + \frac{(\nu - \theta(t_{n-1}))(\theta(t) - \theta(t_{n-1}))}{\theta(t_n) - \theta(t_{n-1})}. \tag{A.1}$$

It is clear that $\eta \in (\theta(t_{n-1}), \theta(t)]$. Denote by $\nu_{n,j}$ the $M_{\widehat{n}} + 1$ Legendre-Gauss quadrature nodes in $\widetilde{I}_n$ (actually, $\nu_{n,j} = \widetilde{t}_{n,j}$, see equation (2.20)) and $\eta_{n,j} := \eta_{n,j}(t) = \rho(t, \nu_{n,j})$. We define a new shifted Legendre-Gauss interpolation operator $\widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} : \ C(\theta(t_{n-1}), \theta(t)) \to \mathcal{P}_{M_{\widehat{n}}}(\theta(t_{n-1}), \theta(t))$ as follows:

$$\widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} v(\eta_{n,j}) = v(\eta_{n,j}), \qquad 0 \le j \le M_{\widehat{n}}.$$

Obviously,

$$\widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} v(\eta_{n,j}) = v(\eta_{n,j}) = v(\rho(t,\nu_{n,j})) = \widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}} v(\rho(t,\nu_{n,j})), \qquad 0 \le j \le M_{\widehat{n}}.$$

Moreover, $\widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} v(\eta)$ and $\widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}} v(\rho(t,\nu))\Big|_{\nu = \theta(t_{n-1}) + \frac{(\theta(t_n) - \theta(t_{n-1}))(\eta - \theta(t_{n-1}))}{\theta(t) - \theta(t_{n-1})}}$ belong to $\mathcal{P}_{M_{\widehat{n}}}(\theta(t_{n-1}), \theta(t))$ in the variable η. Hence,

$$\widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} v(\eta) = \widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}} v(\rho(t,\nu))\Big|_{\nu = \theta(t_{n-1}) + \frac{(\theta(t_n) - \theta(t_{n-1}))(\eta - \theta(t_{n-1}))}{\theta(t) - \theta(t_{n-1})}}. \tag{A.2}$$

Thus, by (A.1), (A.2), (2.24), and (2.22), we obtain

$$\begin{aligned}
\int_{\theta(t_{n-1})}^{\theta(t)} \widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} v(\eta) d\eta &= \frac{\theta(t) - \theta(t_{n-1})}{\theta(t_n) - \theta(t_{n-1})} \int_{\widetilde{I}_n} \widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}} v(\rho(t,\nu)) d\nu \\
&= \frac{\theta(t) - \theta(t_{n-1})}{2} \sum_{j=0}^{M_{\widehat{n}}} v(\rho(t,\nu_{n,j}))\omega_{\widehat{n},j} \\
&= \frac{\theta(t) - \theta(t_{n-1})}{2} \sum_{j=0}^{M_{\widehat{n}}} v(\eta_{n,j})\omega_{\widehat{n},j}.
\end{aligned} \tag{A.3}$$

Similarly,

$$\int_{\theta(t_{n-1})}^{\theta(t)} (\widetilde{\mathcal{I}}^{\eta,t}_{M_{\widehat{n}}} v(\eta))^2 d\eta = \frac{\theta(t) - \theta(t_{n-1})}{2} \sum_{j=0}^{M_{\widehat{n}}} v^2(\eta_{n,j})\omega_{\widehat{n},j}. \tag{A.4}$$

Next, due to the Lagrange's mean value theorem,

$$\theta(t_n) - \theta(t_{n-1}) = \theta'(\zeta) h_n, \qquad \zeta \in (t_n, t_{n-1}). \tag{A.5}$$

We use a similar argument as in Lemma 3.2 to deduce that for any $v \in H^m(\widetilde{I}_n)$ with integer $1 \le m \le M_{\widehat{n}} + 1$,

$$\begin{aligned}
\|v - \widetilde{\mathcal{I}}^{\nu}_{M_{\widehat{n}}} v\|_{\widetilde{I}_n} &\le c M_{\widehat{n}}^{-m} \|\partial_\nu^m v\|_{L^2_{\widetilde{\chi}^m_n}(\widetilde{I}_n)} \\
&\le c(\theta(t_n) - \theta(t_{n-1}))^m M_{\widehat{n}}^{-m} \|\partial_\nu^m v\|_{\widetilde{I}_n} \\
&\le c h_n^m M_{\widehat{n}}^{-m} \|\partial_\nu^m v\|_{\widetilde{I}_n},
\end{aligned} \tag{A.6}$$

where $\widetilde{\chi}_n^m(t) = (\theta(t_n) - t)^m (t - \theta(t_{n-1}))^m$ and c depends on $\max_{t \in I_n} |\theta'(t)|$. Accordingly, we have

$$\begin{aligned}
&\int_{\theta(t_{n-1})}^{\theta(t)} (v(\eta) - \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\eta,t} v(\eta))^2 d\eta \\
&\quad = \frac{\theta(t) - \theta(t_{n-1})}{\theta(t_n) - \theta(t_{n-1})} \int_{\widetilde{I}_n} \Big(v(\rho(t,\nu)) - \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\nu} v(\rho(t,\nu)) \Big)^2 d\nu \\
&\quad \le c h_n^{2m} M_{\widehat{n}}^{-2m} \frac{\theta(t) - \theta(t_{n-1})}{\theta(t_n) - \theta(t_{n-1})} \int_{\widetilde{I}_n} (\partial_\nu^m v(\rho(t,\nu)))^2 d\nu \\
&\quad \le c h_n^{2m} M_{\widehat{n}}^{-2m} \int_{\theta(t_{n-1})}^{\theta(t)} (\partial_\eta^m v(\eta))^2 d\eta.
\end{aligned} \tag{A.7}$$

We now estimate the term $\|B_3\|_{I_n}$. In a manner similar to equation (4.11), we obtain

$$\|B_3\|_{I_n}^2 \le 2(\|B_{3,1}\|_{I_n}^2 + \|B_{3,2}\|_{I_n}^2), \tag{A.8}$$

where

$$\begin{aligned}
B_{3,1} &= \mathcal{I}_{M_n}^t \int_{\theta(t_{n-1})}^{\theta(t)} (\mathcal{I} - \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\eta,t}) \Big(K_2(t,\eta) G_2(\eta, y^{\widehat{n}}(\eta)) \Big) d\eta, \\
B_{3,2} &= \mathcal{I}_{M_n}^t \int_{\theta(t_{n-1})}^{\theta(t)} \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\eta,t} \Big(K_2(t,\eta) \big(G_2(\eta, y^{\widehat{n}}(\eta)) - G_2(\eta, Y^{\widehat{n}}(\eta)) \big) \Big) d\eta.
\end{aligned}$$

Next, by (2.16), (2.14), (A.5) and (A.7), we have that for $\theta(t) \in C^1(I)$,

$$\begin{aligned}
\|B_{3,1}\|_{I_n}^2 &= \Big\| \mathcal{I}_{M_n}^t \int_{\theta(t_{n-1})}^{\theta(t)} (\mathcal{I} - \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\eta,t}) \Big(K_2(t,\eta) G_2(\eta, y^{\widehat{n}}(\eta)) \Big) d\eta \Big\|_{I_n}^2 \\
&= \frac{h_n}{2} \sum_{j=0}^{M_n} \omega_{n,j} \Big[\int_{\theta(t_{n-1})}^{\theta(t_{n,j})} (\mathcal{I} - \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\eta,t_{n,j}}) \Big(K_2(t_{n,j},\eta) G_2(\eta, y^{\widehat{n}}(\eta)) \Big) d\eta \Big]^2 \\
&\le c h_n^2 \sum_{j=0}^{M_n} \omega_{n,j} \int_{\theta(t_{n-1})}^{\theta(t_{n,j})} \Big[(\mathcal{I} - \widetilde{\mathcal{I}}_{M_{\widehat{n}}}^{\eta,t_{n,j}}) \Big(K_2(t_{n,j},\eta) G_2(\eta, y^{\widehat{n}}(\eta)) \Big) \Big]^2 d\eta \\
&\le c h_n^{2m+2} M_{\widehat{n}}^{-2m} \sum_{j=0}^{M_n} \omega_{n,j} \int_{\theta(t_{n-1})}^{\theta(t_{n,j})} \Big[\partial_\eta^m \big(K_2(t_{n,j},\eta) G_2(\eta, y^{\widehat{n}}(\eta)) \big) \Big]^2 d\eta \\
&\le c h_n^{2m+2} M_{\widehat{n}}^{-2m} \sum_{i=0}^{m} \int_{\widetilde{I}_n} (\partial_\eta^i G_2(\eta, y(\eta)))^2 d\eta.
\end{aligned} \tag{A.9}$$

Thanks to (A.3), (2.16) and (2.14), we get

$$
\begin{aligned}
&\|B_{3,2}\|^2_{I_n}\\
&\quad = \frac{h_n}{2}\sum_{i=0}^{M_n}\Big[\frac{\theta(t_{n,i})-\theta(t_{n-1})}{2}\sum_{j=0}^{M_{\widehat{n}}} K_2(t_{n,i},\eta_{n,j})\big(G_2(\eta_{n,j},y^{\widehat{n}}(\eta_{n,j}))\\
&\qquad\qquad - G_2(\eta_{n,j},Y^{\widehat{n}}(\eta_{n,j}))\big)\omega_{\widehat{n},j}\Big]^2\omega_{n,i},
\end{aligned}
$$

where $\eta_{n,j}=\eta_{n,j}(t_{n,i})$. Next, by (4.18), (A.5), (A.4), (A.7) and Hölder's inequality, we deduce readily that

(A.10)

$$
\begin{aligned}
\|B_{3,2}\|^2_{I_n} &\le ch_n^2\sum_{i=0}^{M_n}\Big[\omega_{n,i}\frac{\theta(t_{n,i})-\theta(t_{n-1})}{2}\sum_{j=0}^{M_{\widehat{n}}}\big(y^{\widehat{n}}(\eta_{n,j})-Y^{\widehat{n}}(\eta_{n,j})\big)^2\omega_{\widehat{n},j}\Big]\sum_{j=0}^{M_{\widehat{n}}}\omega_{\widehat{n},j}\\
&\le ch_n^2\sum_{i=0}^{M_n}\omega_{n,i}\int_{\theta(t_{n-1})}^{\theta(t_{n,i})}\Big(\widetilde{\mathcal{I}}^{\eta,t_{n,i}}_{M_{\widehat{n}}}y^{\widehat{n}}(\eta)-Y^{\widehat{n}}(\eta)\Big)^2 d\eta\\
&\le ch_n^2\sum_{i=0}^{M_n}\omega_{n,i}\int_{\theta(t_{n-1})}^{\theta(t_{n,i})}\Big[(\widetilde{\mathcal{I}}^{\eta,t_{n,i}}_{M_{\widehat{n}}}y^{\widehat{n}}(\eta)-y^{\widehat{n}}(\eta))^2+(y^{\widehat{n}}(\eta)-Y^{\widehat{n}}(\eta))^2\Big]d\eta\\
&\le ch_n^2\|y^{\widehat{n}}-Y^{\widehat{n}}\|^2_{\widetilde{I}_n}+ch_n^{2m+2}M_{\widehat{n}}^{-2m}\int_{\widetilde{I}_n}(\partial^m_\eta y(\eta))^2 d\eta.
\end{aligned}
$$

Therefore, a combination of (A.8)-(A.10) leads to the result (4.21). □

Appendix B. The proof of the inequality (4.22)

Proof. Using a similar argument as in equation (4.15), we obtain that

$$\|B_5\|^2_{I_n}\le 2T\sum_{k=1}^{n-1}\frac{1}{h_k}(\|B_{5,1}\|^2_{I_n}+\|B_{5,2}\|^2_{I_n}), \tag{B.1}$$

where

$$
\begin{aligned}
B_{5,1}&=\mathcal{I}^t_{M_n}\int_{\widetilde{I}_k}(\mathcal{I}-\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}})\Big(K_2(t,\varsigma)G_2(\varsigma,y^{\widehat{k}}(\varsigma))\Big)d\varsigma,\\
B_{5,2}&=\mathcal{I}^t_{M_n}\int_{\widetilde{I}_k}\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}}\Big(K_2(t,\varsigma)\big(G_2(\varsigma,y^{\widehat{k}}(\varsigma))-G_2(\varsigma,Y^{\widehat{k}}(\varsigma))\big)\Big)d\varsigma.
\end{aligned}
$$

Next, by (2.16), (2.14), (A.5), (A.6) and the Cauchy-Schwarz inequality, we have that, for $\theta(t)\in C^1(I)$,

$$
\begin{aligned}
\|B_{5,1}\|^2_{I_n}&=\|\mathcal{I}^t_{M_n}\int_{\widetilde{I}_k}(\mathcal{I}-\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}})\Big(K_2(t,\varsigma)G_2(\varsigma,y^{\widehat{k}}(\varsigma))\Big)d\varsigma\|^2_{I_n}\\
&=\frac{h_n}{2}\sum_{j=0}^{M_n}\omega_{n,j}\Big[\int_{\widetilde{I}_k}(\mathcal{I}-\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}})\Big(K_2(t_{n,j},\varsigma)G_2(\varsigma,y^{\widehat{k}}(\varsigma))\Big)d\varsigma\Big]^2\\
&\le ch_nh_k\sum_{j=0}^{M_n}\omega_{n,j}\int_{\widetilde{I}_k}\Big[(\mathcal{I}-\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}})\Big(K_2(t_{n,j},\varsigma)G_2(\varsigma,y^{\widehat{k}}(\varsigma))\Big)\Big]^2d\varsigma\\
&\le ch_nh_k^{2m+1}M_{\widehat{k}}^{-2m}\sum_{j=0}^{M_n}\omega_{n,j}\int_{\widetilde{I}_k}\Big(\partial^m_\varsigma\big(K_2(t_{n,j},\varsigma)G_2(\varsigma,y^{\widehat{k}}(\varsigma))\big)\Big)^2d\varsigma\\
&\le ch_nh_k^{2m+1}M_{\widehat{k}}^{-2m}\sum_{i=0}^{m}\int_{\widetilde{I}_k}(\partial^i_\varsigma G_2(\varsigma,y(\varsigma)))^2d\varsigma.
\end{aligned}
\tag{B.2}
$$

Thanks to (2.16), (2.14), (2.23) and (2.24), we get

$$\|B_{5,2}\|_{I_n}^2 = \frac{h_n}{2}\sum_{i=0}^{M_n}\Big[\frac{\theta(t_k)-\theta(t_{k-1})}{2}\sum_{j=0}^{M_{\widehat{k}}} K_2(t_{n,i},\widetilde{t}_{k,j})\big(G_2(\widetilde{t}_{k,j},y^{\widehat{k}}(\widetilde{t}_{k,j})) - G_2(\widetilde{t}_{k,j},Y^{\widehat{k}}(\widetilde{t}_{k,j}))\big)\omega_{\widehat{k},j}\Big]^2\omega_{n,i}.$$

Next, by (4.18), (A.5), (2.24), (2.22) and Hölder's inequality, we deduce readily that

(B.3)

$$\begin{aligned}
\|B_{5,2}\|_{I_n}^2 &\le ch_kh_n\sum_{i=0}^{M_n}\omega_{n,i}\Big[\frac{\theta(t_k)-\theta(t_{k-1})}{2}\sum_{j=0}^{M_{\widehat{k}}}\big(y^{\widehat{k}}(\widetilde{t}_{k,j})-Y^{\widehat{k}}(\widetilde{t}_{k,j})\big)^2\omega_{\widehat{k},j}\Big]\sum_{j=0}^{M_{\widehat{k}}}\omega_{\widehat{k},j}\\
&\le ch_kh_n\sum_{i=0}^{M_n}\omega_{n,i}\int_{\widetilde{I}_k}\big(\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}}y^{\widehat{k}}(\varsigma)-Y^{\widehat{k}}(\varsigma)\big)^2d\varsigma\\
&\le ch_kh_n\int_{\widetilde{I}_k}\Big[(\widetilde{\mathcal{I}}^{\varsigma}_{M_{\widehat{k}}}y^{\widehat{k}}(\varsigma)-y^{\widehat{k}}(\varsigma))^2+(y^{\widehat{k}}(\varsigma)-Y^{\widehat{k}}(\varsigma))^2\Big]d\varsigma\\
&\le ch_kh_n\|y^{\widehat{k}}-Y^{\widehat{k}}\|_{\widetilde{I}_k}^2+ch_nh_k^{2m+1}M_{\widehat{k}}^{-2m}\int_{\widetilde{I}_k}(\partial_\varsigma^m y(\varsigma))^2d\varsigma.
\end{aligned}$$

Hence, a combination of (B.1)-(B.3) leads to (4.22). □

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