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THE W_p^1 STABILITY OF THE RITZ PROJECTION ON GRADED MESHES

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ABSTRACT. Consider the Poisson equation on a convex polygonal domain and the finite element method of degree $m \geq 1$ associated with a family of graded meshes for possible singular solutions. We prove the stability of the Ritz projection onto the finite element space in W_p^1, $1 < p \leq \infty$. Consequently, we obtain finite element error estimates in W_p^1 for $1 < p \leq \infty$ and in L^p for $1 < p < \infty$. The key to the analysis is the use of the "index engineering" methodology in modified Kondrat′ev weighted spaces. We also mention possible extensions and applications of these results.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain. We consider the Poisson equation with the Dirichlet boundary condition

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where for simplicity, we assume f is sufficiently smooth. Using continuous piecewise polynomials associated with a triangulation of the domain, the finite element method approximates the solution of (1.1) based on a variational principle. As the Ritz projection of u on the finite dimensional space, the finite element solution u_n is naturally stable in the induced energy norm

$$\|u_n\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

This stability result has been a critical ingredient in analyzing the H^1 approximation properties of the finite element method [8, 10]. In addition to the H^1 energy norm, the finite element approximation in non-energy norms, such as W_p^1 and L^p, is of particular interest and has many applications. For example, see [2, 12, 24] for optimal control problems and [13, 15, 32] for non-linear problems. Unlike the analysis in H^1, the stability estimates in non-energy spaces are technically challenging and usually require additional restrictions on the geometry of the mesh.

It is well known that the solution of (1.1) may possess singularities due to the non-smoothness of the domain. Mesh grading techniques are widely used to improve the accuracy of the finite element method approximating such singular solutions [1, 4, 7, 23, 26]. In particular, meshes with general grading properties (Definition 2.6) have proved to be optimal in the H^1 norm [3, 25, 31]. Beyond the energy norm, it is of both theoretical and practical importance to investigate finite element approximations on these graded meshes in non-energy norms.

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The W^1_p/L^p analysis has a long history and early results can be traced back to 1970s [17, 28–30, 33–35]. Here, we survey some relevant results in the literature. For equation (1.1), under the assumption that the mesh is *quasi-uniform*, the following stability result was announced by Rannacher and Scott [30]

$$\|u_n\|_{W^1_p(\Omega)} \leq C\|u\|_{W^1_p(\Omega)}, \quad 2 \leq p \leq \infty,$$

where u_n is the linear finite element approximation. Let h be the mesh size. Then, assuming the regularity $u \in W^2_p(\Omega)$, they further derived optimal error estimates

$$\begin{aligned} \|u - u_n\|_{W^1_p(\Omega)} &\leq Ch\|u\|_{W^2_p(\Omega)} \quad 2 \leq p \leq \infty, \\ \|u - u_n\|_{L^p(\Omega)} &\leq Ch^2\|u\|_{W^2_p(\Omega)} \quad 2 \leq p < \infty. \end{aligned}$$

When $p = \infty$, the L^∞ estimate becomes [17]

$$\|u - u_n\|_{L^\infty(\Omega)} \leq Ch^2|\ln h|\|u\|_{W^2_\infty(\Omega)}.$$

Note that in addition to the constraint on the mesh, these approximation results require sufficient smoothness of the solution $u \in W^2_p(\Omega)$. It has been difficult to obtain such non-energy estimates without the quasi-uniformity of the mesh. In [5], optimal error estimates in L^∞ were established for a 1D elliptic equation, allowing mesh grading. In [33, 34], using local analytical tools, the authors gave error bounds in L^∞ for a 2D model problem, and special graded meshes were also considered to improve the error in the maximum norm. In [2], graded meshes were developed for the singular solution of a 2D optimal control problem, along with sharp regularity analysis for the solution. These meshes are optimal in the L^∞ norm. In [14] it was shown that the Ritz projection onto the finite element space is bounded in W^1_∞ for equation (1.1) on a class of graded (adaptive) meshes with specific conditions on the mesh geometry.

In this paper, we prove the stability of the Ritz projection onto the finite element space of degree $m \geq 1$ in W^1_p (Theorem 5.5), $1 < p \leq \infty$, on a family of graded meshes (Definition 2.6) for the model problem (1.1). In turn, we establish finite element error estimates in W^1_p, $1 < p \leq \infty$, and in L^p, $1 < p < \infty$, (Corollary 5.6). In contrast to the local analysis in [14], the novelty of our approach lies in the use of the "index engineering" methodology [30] in modified Kondrat′ev spaces ($\mathcal{K}^m_{\vec{\mu}}$) [22]. The $\mathcal{K}^m_{\vec{\mu}}$ space (Definition 2.3) plays an important role in our analysis by providing needed regularity estimates for possible singular solutions. However, the weight ρ in $\mathcal{K}^m_{\vec{\mu}}$ vanishes on vertices of the domain, which raises concerns on the interpolation error estimates in the $\mathcal{K}^m_{\vec{\mu}}$ norm. The remedy is to replace the weight ρ by a new function ϑ (2.9). This modification makes it possible that in the new space, we obtain the analog (Proposition 3.1) of the regularity estimate in the $\mathcal{K}^m_{\vec{\mu}}$ space (Proposition 2.5), and it also gives the new function ϑ a positive lower limit on the domain, which leads to the uniform bound for the interpolation error in the new space (Proposition 3.2). Our further analysis is then built upon these observations.

Note that $u \in W^2_p(\Omega)$ does not always hold for any p even on convex domains. Therefore, mesh grading may be necessary for the finite element approximation to achieve the optimal rate of convergence in W^1_p. On the other hand, the $\mathcal{K}^m_{\vec{\mu}}$ space has good scaling properties and describes full regularity dependence of the solution on the data (Proposition 2.5). With the analysis in weighted spaces, our result can help develop effective graded meshes in non-energy norms. In addition, for 3D polyhedral domains, sharp a priori estimates for anisotropic singular solutions (edge and vertex singularities) of elliptic problems can be formulated in the 3D analog of the weighted space $\mathcal{K}^m_{\vec{\mu}}$ [9, 11, 27]. Therefore, we develop the analytical procedure in this paper also in the hope of preparing technical ingredients for the non-energy norm analysis on 3D anisotropic meshes.

The rest of the paper is organized as follows. In Section 2, we introduce the weighted space $\mathcal{K}^m_{\vec{\mu}}$ and the graded mesh for singular solutions. Then, we obtain a preliminary stability estimate in W^1_p. In Section 3, we derive useful regularity and interpolation error estimates with new weight functions. In Section 4, based on the upper bound of the integral involving different weight functions (Lemma 4.1), we formulate two important weighted regularity estimates (Lemmas 4.3 and 4.4). These results will be used in the W^1_p stability analysis. In Section 5, using weighted finite element analysis, we obtain the main results: the stability and approximation estimates for the finite element solution on graded meshes in the aforementioned non-energy norm. We end with some concluding remarks in Section 6.

Throughout the paper, by $a \simeq b$, we mean that there are constants $C_1, C_2 > 0$, independent of the mesh parameter n, such that $C_1 b \leq a \leq C_2 b$. The generic constant $C > 0$ in our analysis below may be different at different occurrences. It will depend on the computational domain, but not on the functions involved in the estimates or the mesh parameter n in the finite element algorithms.

2. Preliminaries

We introduce necessary notation and definitions in this section. Some preliminary estimates will also be provided.

2.1. Weighted spaces and graded meshes.

For any $\omega \subset \Omega$, we use the standard notation $W^m_p(\omega)$ for the Sobolev spaces. Namely, for $m \geq 0$, when $1 \leq p < \infty$,

$$|v|_{W^m_p(\omega)} := \left(\sum_{|\alpha|=m} \int_\omega |\partial^\alpha v|^p dx\right)^{1/p}, \quad \|v\|_{W^m_p(\omega)} := \left(\sum_{k\leq m} |v|^p_{W^k_p(\omega)}\right)^{1/p};$$

when $p = \infty$,

$$|v|_{W^m_\infty(\omega)} := \max_{|\alpha|=m} \big(\text{ess sup}\{|\partial^\alpha v(x)|,\ x \in \omega\}\big), \quad \|v\|_{W^m_\infty(\omega)} := \max_{k\leq m} \big(|v|_{W^k_\infty(\omega)}\big),$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_{\geq 0}$ is the multi-index and $|\alpha| = \alpha_1 + \alpha_2$. Note that $H^m(\Omega) = W^m_2(\Omega)$ for $m \geq 1$, and $L^2(\Omega) = W^0_2(\Omega)$.

Let $H^1_0(\Omega) \subset H^1(\Omega)$ be the subspace consisting of functions with zero trace on $\partial\Omega$. The variational solution $u \in H^1_0(\Omega)$ of equation (1.1) satisfies

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx = (f, v), \quad \forall v \in H^1_0(\Omega).$$

The regularity of the solution depends on the smoothness of the domain. Near the non-smooth points on the boundary (i.e., vertices of the polygonal domain Ω), u may be singular in certain Sobolev spaces even if the given data f is smooth. These singularities raise issues on both the well-posedness of the solution and on the effectiveness of the numerical approximation.

In particular, we recall the following useful regularity results for (1.1) in Sobolev spaces (Section 2.7 in [20]).

Proposition 2.1. *Let ϕ be the largest interior angle of Ω. The Laplace operator*

$$-\Delta : W_p^{m+2}(\Omega) \cap H_0^1(\Omega) \to W_p^m(\Omega), \quad m \geq 0,$$

defines an isomorphism, provided that the parameter p satisfies $1 < p < \eta_m$, where

$$\begin{cases} \eta_m = \infty & \text{for } \pi/\phi \geq m+2, \\ \eta_m = \frac{2}{m+2-\pi/\phi} & \text{for } \pi/\phi < m+2. \end{cases}$$

Remark 2.2. Since Ω is convex, for $m = 0$, $\eta_0 = 2/(2 - \pi/\phi) > 2$. Thus, by the Sobolev embedding theorem, if f is sufficiently smooth, $u \in W_p^1(\Omega)$ for $1 < p \leq \infty$. Proposition 2.1 also holds on non-convex polygonal domains. Therefore, when $\phi > \pi$, $\eta_0 = 2/(2-\pi/\phi) < 2$. This leads to the well-known *partial regularity* in H^2. Namely, $-\Delta : H^2 \cap H_0^1 \to L^2$ is no longer a bijection on domains with reentrant corners.

Now, we define the weighted Sobolev space in which full regularity estimates can be obtained on polygonal domains.

Definition 2.3 (Kondrat′ev spaces)**.** Let v_i, $1 \leq i \leq l$, be the ith vertex of Ω and $\mathcal{V} := \{v_i\}$ the vertex set. Let $r_i(x)$ be the distance function from $x \in \Omega$ to v_i. Let $\vec{\mu} := (\mu_1, \mu_2, \cdots, \mu_l)$ be an l-dimensional vector. For a constant c, we denote $c \pm \vec{\mu} := (c \pm \mu_1, c \pm \mu_2, \cdots, c \pm \mu_l)$. Then, we define the function

$$\rho(x) := \prod_{1 \leq i \leq l} r_i(x),$$

and its vector exponents

$$\rho^{c \pm \vec{\mu}}(x) := \prod_{1 \leq i \leq l} r_i(x)^{c \pm \mu_i} = \rho^c \prod_{1 \leq i \leq l} r_i(x)^{\pm \mu_i}.$$

Then, the Kondrat′ev space is

$$\mathcal{K}_{\vec{\mu}}^m(\Omega) := \{\rho^{|\alpha| - \vec{\mu}} \partial^\alpha v \in L^2(\Omega) \text{ for all } |\alpha| \leq m\},$$

$$|v|_{\mathcal{K}_{\vec{\mu}}^m(\Omega)} := \left(\sum_{|\alpha|=m} \|\rho^{m-\vec{\mu}} \partial^\alpha v\|_{L^2(\Omega)}^p \right)^{1/2}, \quad \|v\|_{\mathcal{K}_{\vec{\mu}}^m(\Omega)} := \left(\sum_{|\alpha| \leq m} |v|^2_{\mathcal{K}_{\vec{\mu}}^{|\alpha|}(\Omega)} \right)^{1/2}.$$

Remark 2.4. Kondrat′ev-type spaces are widely used in a priori estimates for elliptic equations. See [11, 20, 22, 25, 27] and the references therein. Note that in the neighborhood of the vertex v_i, ρ is equivalent to the distance function r_i. Let $B(x, r)$ be the ball of radius r with center at x. Recall the ith vertex v_i of Ω. Thus, we can choose $0 < \bar{r} < 1$ and define the neighborhood $\omega_i := B(v_i, \bar{r}) \cap \Omega$ of v_i, such that $\omega_i \cap \omega_j = \emptyset$ for $i \neq j$, and the distance from any $x \in \omega_i$ to the vertex set $\mathcal{V}$ is the distance $r_i(x)$ to v_i.

In contrast to Proposition 2.1, we have the following full-regularity estimates in weighted spaces [25].

Proposition 2.5. *Let ϕ_i be the interior angle associated with the ith vertex v_i and $\vec{a} := (a_1, a_2, \cdots, a_l)$. Then, for $-\pi/\phi_i < a_i < \pi/\phi_i$, if $f \in \mathcal{K}_{\vec{a}-1}^m(\Omega)$, the variational solution of equation* (1.1) *satisfies*

$$\|u\|_{\mathcal{K}_{\vec{a}+1}^{m+2}(\Omega)} \leq C \|f\|_{\mathcal{K}_{\vec{a}-1}^m(\Omega)}.$$

In particular, if $a_i \leq 1$, we have

$$\|u\|_{\mathcal{K}^{m+2}_{\vec{a}+1}(\Omega)} \leq C\|f\|_{\mathcal{K}^{m}_{\vec{a}-1}(\Omega)} \leq C\|f\|_{H^m(\Omega)}.$$

Based on a priori estimates in weighted spaces, graded meshes can be used to handle the singular solution from the non-smoothness of the domain. In this paper, we consider graded meshes with the following general properties (see [3] and the references therein).

Definition 2.6 (Graded meshes)**.** Let $\mathcal{T}_n = \{T_k\}$ be a triangulation of Ω with *shape-regular* triangles, where the mesh parameter $n \geq 0$ is an integer, such that

$$h := \max_k \operatorname{diam}(T_k) \simeq 2^{-n}.$$

For a triangle $T \in \mathcal{T}_n$, let $h_T := \operatorname{diam}(T)$. Let $\vec{a} = (a_1, a_2, \cdots, a_l)$, $0 < a_i \leq 1$, be a vector. Recall that ρ is comparable to the distance function to the vertex set $\mathcal{V}$. Then, we assume that the mesh $\mathcal{T}_n$ is graded to each vertex $v_i \in \mathcal{V}$ in the following way:

$$h_T \simeq \begin{cases} h\rho^{1-a_i}, & \text{if } \min_{x\in T}\rho(x) > 0, \\ h^{1/a_i}, & \text{if } \min_{x\in T}\rho(x) = 0. \end{cases} \tag{2.1}$$

Let $S_n \subset H^1_0(\Omega)$ be the Lagrange finite element space of degree $m \geq 1$ associated with the graded triangulation $\mathcal{T}_n$. Namely, $S_n = \{v \in \mathcal{C}(\Omega),\ v|_T \in \mathcal{P}_m,\ \text{for any triangle } T \in \mathcal{T}_n\}$, where $\mathcal{P}_m$ is the space of polynomials of degree m. Then, the finite element solution $u_n \in S_n$ for equation (1.1) is

$$a(u_n, v_n) = (f, v_n), \quad \forall v_n \in S_n. \tag{2.2}$$

Remark 2.7. The graded meshes (Definition 2.6) allow the change of the mesh size based on the distance to the vertex, which is determined by the vector $\vec{a}$. For equation (1.1), a sharp range for the vector $\vec{a}$ can be determined, for which the associated graded meshes lead to the optimal convergence rate for the finite element approximation of singular solutions in the H^1 norm [3,6,25]. Note that for $a_i = 1$, $\mathcal{T}_n$ is quasi-uniform near the vertex v_i; while for $0 < a_i < 1$, the mesh size gradually decreases when approaching v_i. The relation between the mesh size and the distance to the vertex set can be quantified as follows.

Definition 2.8 (Mesh layers)**.** Let $4d$ be the length of the shortest edge of Ω. Recall the distance function r_i to the vertex v_i. Let $\Omega_0 := \{x \in \Omega,\ r_i(x) > 2d,\ 1 \leq i \leq l\}$. Recall $\vec{a}$ from Definition 2.6. Define the vector $\vec{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_l)$, such that

$$\kappa_i = 2^{-1/a_i}, \quad 1 \leq i \leq l. \tag{2.3}$$

Then, in the neighborhood of each vertex $v_i \in \mathcal{V}$, based on the distance to the vertex, we define the subsets $L_{i,j}$, $0 \leq j \leq n$, such that

$$d < r_i|_{L_{i,0}} \leq 2d; \quad d\kappa_i^j < r_i|_{L_{i,j}} \leq d\kappa_i^{j-1} \quad \text{for } 1 \leq j \leq n-1; \quad 0 < r_i|_{L_{i,n}} \leq d\kappa_i^n.$$

Then, we denote the jth layer L_j, $0 \leq j \leq n$, of the mesh $\mathcal{T}_n$ by

$$L_j = \bigcup_{1\leq i\leq l} L_{i,j}.$$

Clearly, $L_{i,j} \cap L_{i,k} = \emptyset$ if $j \neq k$ and $\Omega = \Omega_0 \cup (\bigcup_{0\leq j\leq n} L_j)$. To simplify the notation, in the text below, we also denote the specific neighborhoods of the vertex v_i by

$$\mathbb{T}_{i,j} := \bigcup_{j\leq k\leq n} L_{i,k}, \quad 0 \leq j \leq n. \tag{2.4}$$

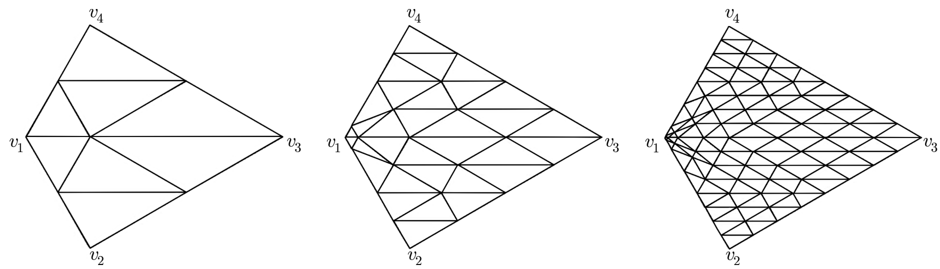

FIGURE 1. Three consecutive graded refinements of a polygonal domain with $\vec{\kappa} = (0.2, 0.5, 0.5, 0.5)$ (left – right): $\mathcal{T}_0$, the initial triangulation; $\mathcal{T}_1$, the mesh after one refinement; $\mathcal{T}_2$, the mesh after two refinements.

Remark 2.9. Based on Definition 2.6 and (2.3), on $\mathcal{T}_n$, the diameter of the triangles in $L_{i,j}$ is

$$\left\{ \begin{array}{ll} h_{i,j} \simeq hr_i^{1-a_i} \simeq h\kappa_i^{j(1-a_i)} \simeq \kappa_i^j 2^{j-n} & \text{for} \quad 0 \leq j \leq n-1, \\ h_{i,n} \simeq h^{1/a_i} \simeq \kappa_i^n. \end{array} \right. \tag{2.5}$$

In addition, by Definition 2.8 and (2.5), since $\text{area}(L_{i,j}) \simeq \kappa_i^{2j}$, the number of triangles in $L_{i,j}$ is $O(\kappa_i^{2j}/h_{i,j}^2) = O(2^{2(n-j)})$ and the number of triangles in Ω_0 is $O(h^{-2}) = O(2^{2n})$. Therefore, the total number of triangles in a graded mesh $\mathcal{T}_n$ (Definition 2.6) is

$$O\big(2^{2n} + \sum_{1 \leq i \leq l,\ 0 \leq j \leq n} 2^{2(n-j)}\big) = O(2^{2n}).$$

Hence, the dimension of the finite element space S_n is $O(4^n)$.

We finish this subsection by giving a simple construction of graded meshes (see [6, 25]) that satisfy the condition (2.1) in Definition 2.6.

Example 2.10. Let $\mathcal{T}$ be a triangulation of Ω whose vertices include $\mathcal{V}$, such that no triangle in $\mathcal{T}$ has more than one of its vertices in $\mathcal{V}$. Recall the vector $\vec{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_l)$ from (2.3), *a $\vec{\kappa}$ refinement of $\mathcal{T}$*, denoted by $\vec{\kappa}(\mathcal{T})$, is obtained by dividing each edge AB of $\mathcal{T}$ in two parts as follows:

- If neither A nor B is in $\mathcal{V}$, then we divide AB into two equal parts.
- Otherwise, if A is v_i, we divide AB into AC and CB such that $|AC| = \kappa_i |AB|$.

This will divide each triangle of $\mathcal{T}$ into four triangles. Given an initial triangulation $\mathcal{T}_0$, the associated family of graded triangulations $\{\mathcal{T}_j : \ j \geq 0\}$ is defined recursively, $\mathcal{T}_{j+1} = \vec{\kappa}(\mathcal{T}_j)$. See Figure 1 for an illustration of this procedure. With a straightforward calculation, it can be shown that this construction leads to a triangulation $\mathcal{T}_n$ with the mesh size (2.5) in the layer $L_{i,j}$, which verifies the condition (2.1).

2.2. **Preliminary W_p^1 analysis.** We begin by introducing new functions that are necessary for the analysis. For a graded mesh $\mathcal{T}_n$, recall that the dimension of the finite element space $N := \dim(S_n)$ is $O(4^n)$. Recall the largest mesh size $h = 2^{-n}$. For any point $z \in \Omega$, let $T_z \in \mathcal{T}_n$ be the triangle, such that $z \in T_z$, and let

$h_z = \mathrm{diam}(T_z)$ be its diameter. If z is on the intersection of multiple triangles, we choose one of the triangles as T_z. The choice is not unique but fixed. Define the function

$$\sigma_z(x) = (|x-z|^2 + t^2 h_z^2)^{1/2}, \quad t \geq 1. \tag{2.6}$$

Therefore, for any triangle $T \in \mathcal{T}_n$, there is a constant C, independent of n, such that

$$\sigma_z(x)|_T \geq C h_T \quad \text{and} \quad \max_{x\in T} \sigma_z^\lambda(x) \leq C \min_{x\in T} \sigma_z^\lambda(x), \tag{2.7}$$

where h_T is the diameter of T and $\lambda \in \mathbb{R}$. The estimate $\sigma_z(x)|_T \geq Ch_T$ can be derived as follows. Suppose $z \in L_{i,j}$, for $1 \leq j \leq n$. Then, based on Definition 2.8, for any $T \subset L_{i,k}$, $j-1 \leq k \leq n$, $h_z \geq Ch_T$, and therefore the estimate holds. For any $T \subset L_{i,k}$, $0 \leq k \leq j-2$, by the definition of mesh layers, we have $|x|_T - z| \geq C\kappa_i^k \geq C\kappa_i^k 2^{k-n} \geq Ch_T$. For $T \subset \Omega \setminus \mathbb{T}_{i,0}$ (see (2.4) for the definition of $\mathbb{T}_{i,0}$), we have $|x|_T - z| \geq C \geq Ch_T$. The estimate thus holds for any $z \in L_{i,j}$, $1 \leq j \leq n$. Now, for $z \in \Omega \setminus \bigcup_{1\leq j\leq n} L_j$, $h_z \simeq 2^{-n}$. Therefore, for any $T \in \mathcal{T}_n$, $\sigma_z(x) \geq h_z \geq h_T$. Hence, we obtain the desired estimate for any $z \in \Omega$.

Let $\delta_z \in \mathcal{C}_0^\infty(T_z)$ be a function such that for $m \geq 1$ and $s = 0, 1, \cdots$,

$$\int_\Omega \delta_z p dx = p(z), \quad \forall p \in \mathcal{P}_{m-1}, \quad \text{and} \quad \|\nabla_s \delta_z\|_{L^\infty(\Omega)} \leq C h_z^{-2-s}, \tag{2.8}$$

where ∇_s denotes the vector of all sth-order derivatives. Recall the weight function $\rho(x)$ from Definition 2.3. Then, we define a modified weight function associated with the triangulation $\mathcal{T}_n$,

$$\vartheta(x) := \prod_{1\leq i\leq l} \left(r_i^2(x) + h_{i,n}^2\right)^{1/2}, \tag{2.9}$$

where $h_{i,n}$ is mesh size (2.5) in $L_{i,n}$ of the nth layer, and its vector exponents

$$\vartheta^{c\pm\vec{\mu}}(x) := \prod_{1\leq i\leq l} \left(r_i^2(x) + h_{i,n}^2\right)^{\frac{1}{2}(c\pm\mu_i)}.$$

Consequently, based on the definitions, for $\lambda \in \mathbb{R}$ and a multi-index α, we have

$$|\partial^\alpha \sigma_z^\lambda(x)| \leq C\sigma_z^{\lambda-|\alpha|}(x) \qquad \text{and} \qquad |\partial^\alpha \vartheta^\lambda(x)| \leq C\vartheta^{\lambda-|\alpha|}(x). \tag{2.10}$$

Remark 2.11. The functions σ_z and δ_z are the analogs of the weight function and the regularized Dirac δ-function defined in [30] for quasi-uniform meshes. We modified their definitions in order to represent the local mesh size on graded triangulations. The parameter $t \geq 1$ in (2.6) is arbitrary but fixed in our subsequent analysis. In Lemma 5.2, we will specifically choose t. We introduce the function ϑ that resembles the distance function ρ in Definition 2.3, except on the triangles touching the vertices of the domain. Namely,

$$\begin{cases} \vartheta(x) \simeq |x - v_i| \simeq \kappa_i^j \text{ on } L_{i,j} \quad \text{for} \quad 0 \leq j < n, \\ \vartheta(x) \simeq h_{i,n} \simeq \kappa_i^n \text{ on } L_{i,n}. \end{cases} \tag{2.11}$$

Note that ϑ does not decay to zero on the last mesh layer L_n. These are the desired properties to carry out our analysis below.

We first give an estimate on the weight function σ_z defined in (2.6).

Lemma 2.12. *Recall $\sigma_z(x) = (|x-z|^2 + t^2 h_z^2)^{1/2}$, $h := 2^{-n}$, and $h_z := diam(T_z)$. Then, for $z \in \Omega$ and $\lambda > 0$, there exists a constant $C > 0$, independent of n and t, such that*

$$\| \int_\Omega \frac{h_z^\lambda}{h^\lambda} \sigma_z^{-2-\lambda}(x) dz \|_{L^\infty(\Omega)} \leq C h^{-\lambda}. \tag{2.12}$$

Proof. Recall the mesh layer $L_{i,j}$ from Definition 2.8. Let h_x be the diameter of the triangle that contains x. We show (2.12) in the following two cases.

Case I ($x \in L_{i,j}$ for $1 \leq j \leq n$). We need to consider the situation for $z \in L_{i,k}$, $0 \leq k \leq n$, and for $z \in \Omega \setminus \mathbb{T}_{i,0}$.

Suppose $z \in L_{i,k}$, for $0 \leq k \leq n$. Thus, by Definition 2.6 and (2.5), there are positive constants c_1, c_1', c_2, and c_2', independent of j, k, and n, such that $c_1' \kappa_i^k 2^{k-n} \leq h_z \leq c_1 \kappa_i^k 2^{k-n}$, and if $j \neq k$, the distance $D_i^{j,k}$ between the layers $L_{i,j}$ and $L_{i,k}$ satisfies $c_2' \kappa_i^{\min(j,k)} \leq D_i^{j,k} \leq c_2 \kappa_i^{\min(j,k)}$.

We first consider the case $|z-x| \leq h_z$, which implies $c_2' \kappa_i^{\min(j,k)} \leq c_1 \kappa_i^k 2^{k-n}$. Thus, if $j > k$, we have $n-k \leq |\log_2(c_1/c_2')|$, and therefore $j-k \leq |\log_2(c_1/c_2')|$. If $j < k$, we have $\kappa_i^{k-j} \geq (c_2'/c_1) 2^{n-k}$, and therefore by (2.3),

$$j - k \geq a_i\big((n-k) + \log_2(c_2'/c_1)\big).$$

This may hold only when $n-k < |\log_2(c_2'/c_1)|$ and $\log_2(c_2'/c_1) < 0$. These two conditions imply $k-j \leq |\log_2(c_2'/c_1)|$. Hence, if $|z-x| \leq h_z$ and $k \neq j$, we have $|k-j| \leq |\log_2(c_2'/c_1)|$. By (2.5), $h_x/h_z \simeq (2\kappa_i)^{j-k}$, which leads to $C(2\kappa_i)^{|\log_2(c_2'/c_1)|} h_z \leq h_x \leq C(2\kappa_i)^{-|\log_2(c_2'/c_1)|} h_z$, and implies $h_x \simeq h_z$, since $|\log_2(c_2'/c_1)|$ represents a constant. If $k = j$, it is clear that $h_x \simeq h_z$. Thus, we have shown $h_z \simeq h_x$ provided that $|z-x| \leq h_z$. Therefore,

$$\int_{|z-x| \leq h_z} \frac{h_z^\lambda}{h^\lambda} \sigma_z^{-2-\lambda} dz \leq C \int_{|z-x| \leq h_z} \frac{h_x^\lambda}{h^\lambda} h_x^{-2-\lambda} dz \leq C h^{-\lambda}. \tag{2.13}$$

We now consider the case $|z-x| > h_z$. First, for $z \in L_{i,j}$, we have $h_z \simeq \kappa_i^j 2^{j-n}$. Thus,

$$\int_{|z-x| > h_z,\ z \in L_{i,j}} \frac{h_z^\lambda}{h^\lambda} \sigma_z^{-2-\lambda} dz \leq \int_{|z-x| > h_z,\ z \in L_{i,j}} \frac{h_z^\lambda}{h^\lambda} |z-x|^{-2-\lambda} dz \leq C h^{-\lambda}. \tag{2.14}$$

For $z \in L_{i,k}$, where $k > j$, we have $|z-x| \simeq \kappa_i^j$ and $h_z \simeq \kappa_i^k 2^{k-n}$. Thus,

$$\begin{aligned} \int_{|z-x| > h_z,\ z \in L_{i,k}} \frac{h_z^\lambda}{h^\lambda} \sigma_z^{-2-\lambda} dz &\leq C \int_{|z-x| > h_z,\ z \in L_{i,k}} \frac{h_z^\lambda}{h^\lambda} \kappa_i^{j(-2-\lambda)} dz \\ &\leq C h^{-\lambda} \kappa_i^{2k + k\lambda - j\lambda - 2j} 2^{\lambda(k-n)} \leq C 2^{-\lambda|k-j|} h^{-\lambda}. \end{aligned} \tag{2.15}$$

For $z \in L_{i,k}$, where $k < j$, we have $|z-x| \simeq \kappa_i^k$ and $h_z \simeq \kappa_i^k 2^{k-n}$. Thus,

$$\begin{aligned} \int_{|z-x| > h_z,\ z \in L_{i,k}} \frac{h_z^\lambda}{h^\lambda} \sigma_z^{-2-\lambda} dz &\leq C \int_{|z-x| > h_z,\ z \in L_{i,k}} \frac{h_z^\lambda}{h^\lambda} \kappa_i^{k(-2-\lambda)} dz \\ &\leq C h^{-\lambda} \kappa_i^{2k + k\lambda - k\lambda - 2k} 2^{\lambda(k-n)} \leq C 2^{-\lambda|k-j|} h^{-\lambda}. \end{aligned} \tag{2.16}$$

Note that if $z \in \Omega \setminus \mathbb{T}_{i,0}$, we have $|z-x| \simeq 1 > h_z$. Therefore, by (2.13), (2.14), (2.15), and (2.16), we have

$$\begin{aligned}\int_\Omega \frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz &= \int_{|z-x|\le h_z} \frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz \\ &\quad + \sum_{k\le n}\int_{|z-x|>h_z,\ z\in L_{i,k}} \frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz + \int_{\Omega\setminus\mathbb{T}_{i,0}} \frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz. \\ &\le Ch^{-\lambda}(1+\sum_{s\le n}2^{-s\lambda}+1) \le Ch^{-\lambda}. \end{aligned} \tag{2.17}$$

Case II ($x \in \Omega \setminus \bigcup_{1\le j\le n} L_j$). By (2.5), we have $h_x \simeq 2^{-n}$ and for any $z \in \Omega$, $h_z \le Ch_x$, where C is independent of n. Note that for $|z-x| \le h_x$, $h_z \simeq h_x \simeq h$. Therefore, we have

$$\begin{aligned}\int_\Omega \frac{h_x^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz &= \int_{|z-x|\le h_x} \frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz + \int_{|z-x|>h_x} \frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz \\ &\le Ch_x^2 h_z^{-2-\lambda} + \int_{|z-x|>h_x}|z-x|^{-2-\lambda}dz \le Ch^{-\lambda}. \end{aligned} \tag{2.18}$$

The lemma is thus proved by (2.17) and (2.18). □

Recall that $u_n \in S_n$ is the finite element solution in (2.2) on the graded mesh $\mathcal{T}_n$ (Definition 2.6). In order to analyze $\|u_n\|_{W_p^1(\Omega)}$, we introduce two functions g and g_n as follows.

Recall the function δ_z in (2.8). Let $g \in H_0^1(\Omega)$ satisfy

$$a(g,w) = (-\nu\cdot\nabla\delta_z, w), \quad \forall w \in H_0^1(\Omega), \tag{2.19}$$

where ν is an arbitrary direction vector. Thus, g can be considered as a "derivative" of the regularized Green's function. Note that by the usual regularity estimate, $\|g\|_{H^2(\Omega)} \le C\|\delta_z\|_{H^1(\Omega)}$. Denote by $g_n \in S_n$ the finite element approximation of g, such that

$$a(g_n,w) = (-\nu\cdot\nabla\delta_z, w), \quad \forall w \in S_n. \tag{2.20}$$

Recall that for a point $z \in \Omega$, we let $T_z \in \mathcal{T}_n$ be the triangle containing z and $h_z = \mathrm{diam}(T_z)$. Thus, we obtain an estimate concerning the upper bound of $\|\nabla u_n\|_{L^p(\Omega)}$.

Lemma 2.13. *Recall $h := 2^{-n}$ and $h_z := \mathrm{diam}(T_z)$ for $z \in \Omega$. Define*

$$M = \max_{z\in\Omega}\big((\int_\Omega \frac{h^\lambda}{h_z^\lambda}\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx)^{\frac12}\big).$$

For $2 < p \le \infty$ and $\lambda > 0$, we have

$$\|\nabla u_n\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}(1+h^{-\frac{\lambda}{2}}M), \tag{2.21}$$

where C is independent of n and t, and $t \ge 1$ is from (2.6).

Proof. Note that by (2.8), (2.19), and (2.20), we have

$$\begin{aligned}\nu\cdot\nabla u_n(z) &= (\nu\cdot\nabla u_n, \delta_z) = (-\nu\cdot\nabla\delta_z, u_n) = (\nabla g, \nabla u_n) \\ &= \big(\nabla g, \nabla(u_n-u)\big) + (\nabla g, \nabla u) \\ &= (\nu\cdot\nabla u, \delta_z) - \big(\nabla(g-g_n), \nabla u\big). \end{aligned} \tag{2.22}$$

We first derive the result (2.21) for $p=\infty$. By (2.8) and Hölder's inequality, we have

$$(\nu\cdot\nabla u,\delta_z)=\int_\Omega \delta_z\nu\cdot\nabla u dx\leq C\|\nabla u\|_{L^\infty(\Omega)}, \tag{2.23}$$

and by Hölder's inequality and (2.6),

$$\begin{aligned}\big(\nabla(g-g_n),\nabla u\big)&=\int_\Omega\nabla(g-g_n)\cdot\nabla u dx\\ &\leq\big(\int_\Omega\frac{h^\lambda}{h_z^\lambda}\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx\big)^{\frac12}\big(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}|\nabla u|^2dx\big)^{\frac12},\\ &\leq M\|\nabla u\|_{L^\infty(\Omega)}\big(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dx\big)^{\frac12}\leq Ch^{-\frac{\lambda}{2}}M\|\nabla u\|_{L^\infty(\Omega)}.\end{aligned}\tag{2.24}$$

Then, the case for $p=\infty$ is proved by combining (2.22), (2.23), and (2.24).

For $2<p<\infty$, using Hölder's inequality and (2.8), the first term in (2.22) leads to

$$\begin{aligned}\int_\Omega(\int_\Omega\delta_z\nu\cdot\nabla u dx)^p dz&\leq C\int_\Omega\big[\|\nabla u\|^p_{L^p(T_z)}(\int_{T_z}\delta_z^{\frac{p}{p-1}}dx)^{p-1}\big]dz\\ &\leq C\int_\Omega h_z^{-2}\|\nabla u\|^p_{L^p(T_z)}dz\leq C\|\nabla u\|^p_{L^p(\Omega)}.\end{aligned}\tag{2.25}$$

For the second term in (2.22), by Hölder's inequality,

$$\begin{aligned}&\int_\Omega\big(\int_\Omega\nabla(g-g_n)\cdot\nabla u dx\big)^p dz\\ &\leq\int_\Omega\big(\int_\Omega\frac{h^\lambda}{h_z^\lambda}\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx\big)^{\frac p2}\big(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}|\nabla u|^2dx\big)^{\frac p2}dz.\end{aligned}\tag{2.26}$$

Using Hölder's inequality, we have

$$\begin{aligned}\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}|\nabla u|^2dx&\leq(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}|\nabla u|^pdx)^{\frac2p}(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dx)^{\frac{p-2}{p}}\\ &\leq C(h^\lambda)^{\frac{2-p}{p}}(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}|\nabla u|^pdx)^{\frac2p}.\end{aligned}\tag{2.27}$$

Then, by (2.22), (2.25), Minkowski's inequality, (2.26), (2.27), Fubini's Theorem, and (2.12), we have

$$\begin{aligned}\|\nabla u_n\|_{L^p(\Omega)}&\leq\Big(\|(\nu\cdot\nabla u_n,\delta_z)\|_{L^p(\Omega)}+\|(\nabla(g-g_n),\nabla u)\|_{L^p(\Omega)}\Big)\\ &\leq C\Big(\|\nabla u\|_{L^p(\Omega)}+\big[\int_\Omega M^p(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}|\nabla u|^2dx)^{\frac p2}dz\big]^{1/p}\Big)\\ &\leq C\Big(\|\nabla u\|_{L^p(\Omega)}+(h^\lambda)^{\frac{2-p}{2p}}M\big[\int_\Omega(\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz)|\nabla u|^pdx\big]^{1/p}\Big)\\ &\leq C\big(\|\nabla u\|_{L^p(\Omega)}+(h^\lambda)^{\frac{2-p}{2p}}M\|\nabla u\|_{L^p(\Omega)}\|\int_\Omega\frac{h_z^\lambda}{h^\lambda}\sigma_z^{-2-\lambda}dz\|^{1/p}_{L^\infty(\Omega)}\big)\\ &\leq C\|\nabla u\|_{L^p(\Omega)}\big(1+h^{-\frac{\lambda}{2}}M\big).\end{aligned}\tag{2.28}$$

Thus, (2.22)–(2.24) and (2.28) together complete the proof of (2.21). $\square$

Remark 2.14. Based on Lemma 2.13, in order to show $\|\nabla u_n\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}$, it suffices to prove

$$M = \max_{z\in\Omega} \big((\int_\Omega \frac{h^\lambda}{h_z^\lambda} \sigma_z^{2+\lambda} |\nabla(g-g_n)|^2 dx)^{\frac{1}{2}} \big) \leq Ch^{\frac{\lambda}{2}}, \tag{2.29}$$

for appropriately chosen $\lambda > 0$. The parameter λ for now is arbitrary but fixed. Due to the local behavior of δ_z in (2.19) and (2.20), motivated by [30], we introduced σ_z to cancel the mesh-dependent singularity in the regularized Green's function. The subsequent sections are dedicated to obtain the estimate in (2.29) through analysis in weighted spaces.

3. Regularity and interpolation analysis

From now on, we start to develop analytical tools for (2.29). In this section, we establish the analog of the full-regularity result (Proposition 2.5) in spaces with a new weight, and in turn give interpolation error estimates in the new space.

We first derive new regularity estimates in norms involving the weight function ϑ defined in (2.9).

Proposition 3.1. *Let $v \in H_0^1(\Omega) \cap H^2(\Omega)$. Recall $\vec{a}$ from Definition* 2.6 *with $0 < a_i \leq 1$. Then, we have*

$$\sum_{0\leq s\leq 2} \int_\Omega \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx \leq C \int_\Omega \vartheta^{2-2\vec{a}} |\Delta v|^2 dx.$$

Proof. Recall L_n in Definition 2.8. Let $\Omega_c := \Omega \setminus L_n$. We first have

$$\sum_{0\leq s\leq 2} \int_\Omega \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx$$
$$= \sum_{0\leq s\leq 2} \int_{\Omega_c} \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx + \sum_{0\leq s\leq 2} \sum_{1\leq i\leq l} \int_{L_{i,n}} \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx. \tag{3.1}$$

Note $\rho(x) \leq \vartheta(x)$ on Ω and $\vartheta(x) \leq C\rho(x)$ on Ω_c. Note that the weighted regularity estimate in Proposition 2.5 holds for any $0 < a_i \leq 1$ on the convex domain Ω. In addition, by (2.9) and Definition 2.8, for $s = 0, 1$, because $\rho \leq \vartheta$ on $L_{i,n}$ and $2s - 2 - 2a_i < 0$, we have $\vartheta^{2s-2-2\vec{a}} \leq C\rho^{2s-2-2\vec{a}}$ on $L_{i,n}$. Thus, by the definition of the weighted space $\mathcal{K}^m_{\vec{\mu}}$ and Proposition 2.5, we have

$$\sum_{0\leq s\leq 2} \int_{\Omega_c} \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx + \sum_{0\leq s\leq 1} \sum_{1\leq i\leq l} \int_{L_{i,n}} \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx$$
$$\leq \sum_{0\leq s\leq 2} \int_{\Omega_c} \vartheta^{2s-2-2\vec{a}} |\nabla_s v|^2 dx + C \sum_{0\leq s\leq 1} \sum_{1\leq i\leq l} \int_{L_{i,n}} \rho^{2s-2-2\vec{a}} |\nabla_s v|^2 dx$$
$$\leq C\|v\|^2_{\mathcal{K}^2_{\vec{a}+1}(\Omega)} \leq C\|\Delta v\|^2_{\mathcal{K}^0_{\vec{a}-1}(\Omega)}$$
$$= C\int_\Omega \rho^{2-2\vec{a}} |\Delta v|^2 dx \leq C\int_\Omega \vartheta^{2-2\vec{a}} |\Delta v|^2 dx. \tag{3.2}$$

Then, it remains to show the upper bound of the second term in (3.1) for the case $s = 2$. Recall the constant $\bar{r}$ and the neighborhood ω_i from Remark 2.4. Let $\chi_i(x) \in \mathcal{C}^\infty(\Omega)$ be a partition of unity of Ω, such that for $1 \leq i \leq l$, $\chi_i(x) = 1$

if $r_i(x) < \bar{r}/2$ and $\chi_i(x) = 0$ if $r_i(x) \geq \bar{r}$; $\chi_0(x) := 1 - \sum_{1\leq i\leq l}\chi_i(x)$. Then $v \in H_0^1(\Omega)\cap H^2(\Omega)$ implies $\chi_i v \in H_0^1(\omega_i)\cap H^2(\omega_i)$ and

$$
\begin{aligned}
\sum_{1\leq i\leq l}\int_{L_{i,n}} \vartheta^{2-2\vec{a}}|\nabla_2 v|^2dx &\leq C\sum_{1\leq i\leq l} h_{i,n}^{2-2a_i}\int_{L_{i,n}}|\nabla_2 v|^2dx\\
&\leq C\sum_{1\leq i\leq l} h_{i,n}^{2-2a_i}\int_{\omega_i}|\nabla_2(\chi_i v)|^2dx \leq C\sum_{1\leq i\leq l} h_{i,n}^{2-2a_i}\int_{\omega_i}|\Delta(\chi_i v)|^2dx\\
&\leq C\sum_{1\leq i\leq l} h_{i,n}^{2-2a_i}\int_{\omega_i}|\Delta v|^2+|\nabla v|^2+|v|^2dx\\
&\leq C\sum_{1\leq i\leq l}\int_{\omega_i}\vartheta^{2-2\vec{a}}(|\Delta v|^2+|\nabla v|^2+|v|^2)dx, \qquad (3.3)
\end{aligned}
$$

where we used the fact that for any $w \in H_0^1(\omega_i)\cap H^2(\omega_i)$, $\|w\|_{H^2(\omega_i)} \leq C\|\Delta w\|_{L^2(\omega_i)}$ [16, 19]. Recall $0 < a_i \leq 1$. Then, by the definition of ω_i, Proposition 2.5, and the fact that $\vartheta \leq C$, we have

$$
\begin{aligned}
\sum_{1\leq i\leq l}\int_{\omega_i}\vartheta^{2-2\vec{a}}(|\nabla v|^2+|v|^2)dx &\leq C\sum_{0\leq s\leq 1}\sum_{1\leq i\leq l}\int_{\omega_i}\vartheta^{2s-2-2\vec{a}}|\nabla_s v|^2dx\\
&\leq C\|v\|^2_{\mathcal{K}^2_{\vec{a}+1}(\Omega)} \leq C\int_\Omega \vartheta^{2-2\vec{a}}|\Delta v|^2dx. \qquad (3.4)
\end{aligned}
$$

Combining (3.1), (3.2), (3.3), and (3.4), we complete the proof. ☐

We are ready to obtain interpolation error estimates in new weighted norms that are needed in subsequent analysis. Recall $N \simeq 4^n$ is the dimension of the finite element space S_n.

Proposition 3.2. *For $v \in H^2(\Omega)$, let $v_I \in S_n$ be its nodal interpolation associated with the triangulation $\mathcal{T}_n$. Recall $\vec{a}$ from Definition 2.6. Then, for $\lambda \in \mathbb{R}$,*

$$
\int_\Omega \sigma_z^\lambda|\nabla_s(v-v_I)|^2dx \leq CN^{-1}\int_\Omega \sigma_z^{2-2s+\lambda}|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx, \quad s=0,1. \qquad (3.5)
$$

Proof. For any triangle $T \in \mathcal{T}_n$, let h_T be its diameter. In the case $T \subset L_{i,j}$, $0 \leq j \leq n$, by (2.11), we have $\vartheta(x) \simeq \kappa_i^j$, for any $x \in T$. Recall $h_T \simeq \kappa_i^j 2^{j-n}$ from (2.5). Then, by (2.7) and (2.3), we have

$$
\begin{aligned}
\int_T \sigma_z^\lambda|\nabla_s(v-v_I)|^2dx &\leq \max_{x\in T}\sigma_z^\lambda(x)\int_T|\nabla_s(v-v_I)|^2dx\\
&\leq Ch_T^{4-2s}\max_{x\in T}\sigma_z^\lambda(x)\int_T|\nabla_2 v|^2dx \leq Ch_T^{4-2s}\min_{x\in T}\sigma_z^\lambda(x)\int_T|\nabla_2 v|^2dx\\
&\leq Ch_T^{4-2s}\int_T\sigma_z^\lambda(x)|\nabla_2 v|^2dx \leq C\kappa_i^{j(4-2s)}2^{(j-n)(4-2s)}\int_T\sigma_z^\lambda(x)|\nabla_2 v|^2dx\\
&\leq CN^{-1}\int_T(\kappa_i^j 2^j)^{(2-2s)}2^{-n(2-2s)}\sigma_z^\lambda|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx\\
&\leq CN^{-1}\int_T h_T^{(2-2s)}\sigma_z^\lambda|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx \leq CN^{-1}\int_T\sigma_z^{2-2s+\lambda}|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx. \qquad (3.6)
\end{aligned}
$$

In the case $T \subset \Omega_0 = \Omega\setminus\bigcup_i \mathbb{T}_{i,0}$, we can follow the same estimates above by replacing κ_i with $1/2$ and noting that ϑ is comparable to a constant. Therefore, we have

$$\int_T \sigma_z^\lambda |\nabla_s(v - v_I)|^2 dx \leq CN^{-1} \int_T h_T^{(2-2s)} \sigma_z^\lambda |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx$$

$$\leq CN^{-1} \int_T \sigma_z^{2-2s+\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx. \tag{3.7}$$

The proof follows by adding up the estimates in (3.6) and (3.7) over all the triangles. □

Remark 3.3. With a convex domain Ω, the solution of equation (1.1) satisfies $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$. Therefore, it is reasonable to assume $v \in H^2(\Omega)$ in Propositions 3.1 and 3.2. Using the new weight function ϑ, we both recovered the regularity result in Proposition 2.5 for $m = 0$ (Proposition 3.1) and obtained a uniform upper bound for the interpolation error (Proposition 3.2). The interpolation error estimate, however, is not obvious in the weighted space $\mathcal{K}_{\vec{\mu}}^m$ (i.e., replacing ϑ by ρ in (3.5)). This is the motivation for us to introduce ϑ, instead of working with the norm involving ρ.

4. More regularity estimates

Recall weight functions $\sigma_z(x)$ and $\vartheta(x)$ in (2.6) and (2.9), respectively. In this section, we derive useful estimates involving these functions. In particular, we obtain important regularity results in Lemmas 4.3 and 4.4.

4.1. **A weighted result.**

Recall $N = \dim(S_n) \simeq 4^n$. We first derive the upper bound of an integral that will frequently appear in the analysis.

Lemma 4.1. *For any point $z \in \Omega$, suppose $z \in T_z \in \mathcal{T}_n$, where T_z is the triangle that contains z chosen as in* (2.6). *Let $\vec{a}$ be from Definition* 2.6 *with $0 < a_i \leq 1$. Let $a_{\min} := \min_{1 \leq i \leq l}(a_i)$. For $0 < |\lambda| < 2a_{\min}$, choose $q > 1$, such that $q(2a_{\min} - \lambda) > 2$. Let $h_z = \operatorname{diam}(T_z)$. Then, for n sufficiently large,*

$$\int_\Omega (\sigma_z^{-2+\lambda} \vartheta^{2-2\vec{a}})^q dx \leq C h_z^{2+\lambda q} N^q,$$

where C is independent of n and t.

Proof. **Case I ($T_z \subset L_{i,j}$, $1 \leq j \leq n$).** Recall from (2.11), $\vartheta \simeq \kappa_i^k$ on $L_{i,k}$, $0 \leq k \leq n$. Define $B_z := \{x \in \Omega : \ |x - x'| \leq th_z, \ \forall x' \in T_z\} \cap \Omega$. Recall $\mathbb{T}_{i,j}$ from Definition 2.8. Thus, for n sufficiently large, $B_z \subset \mathbb{T}_{i,0}$. We establish the estimates on the sets $R_1 := (\mathbb{T}_{i,0} \setminus B_z) \cap (\bigcup_{j-1 \leq k \leq n} L_{i,k})$, $R_2 := \mathbb{T}_{i,0} \setminus (B_z \cup R_1) = (\mathbb{T}_{i,0} \setminus B_z) \cap (\bigcup_{0 \leq k \leq j-2} L_{i,k})$, B_z, and $\Omega \setminus \mathbb{T}_{i,0}$.

In R_1, by (2.9) and the definition of $\mathcal{T}_n$ (Definition 2.6),

$$\sigma_z(x)^{-2+\lambda} \vartheta(x)^{2-2\vec{a}} \leq C\sigma_z(x)^{-2+\lambda} \kappa_i^{j(2-2a_i)}. \tag{4.1}$$

On $L_{i,k}$, $0 \leq k \leq j-2$,

$$\sigma_z(x)^{-2+\lambda} \vartheta(x)^{2-2\vec{a}} \leq C\sigma_z(x)^{-2+\lambda} \kappa_i^{k(2-2a_i)}. \tag{4.2}$$

The conditions $0 < |\lambda| < 2a_{\min} \leq 2$ and $q(2a_{\min} - \lambda) > 2$ imply

$$(-2 + \lambda)q = -(2 - \lambda)q \leq -(2a_{\min} - \lambda)q < -2. \tag{4.3}$$

If the ith vertex $v_i \notin B_z$, then for any $x \in B_z$, $\vartheta(x) \le C\kappa_i^j$. Therefore, by (2.5) and (2.3),

$$\int_{B_z} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx \le C\int_{B_z} (t^{-2+\lambda}h_z^{-2+\lambda}\kappa_i^{j(2-2a_i)})^q dx \le C(t^{-2+\lambda}h_z^{\lambda}N)^q t^2 h_z^2$$
$$\le Ct^{2-2q+\lambda q}h_z^{2+\lambda q}N^q.$$

If $v_i \in B_z$, then for $x \in B_z$, $\vartheta(x) \le Cth_z$. Therefore, by (2.5) and (2.3), we have

$$\begin{aligned}\int_{B_z} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx &\le C\int_{B_z} (t^{-2+\lambda}h_z^{-2+\lambda}(th_z)^{(2-2a_i)})^q dx \\ &\le C(t^{\lambda-2a_i}h_z^{\lambda-2a_i})^q t^2 h_z^2 \le C2^{2q(1-a_i)(j-n)}t^{2-2a_iq+\lambda q}h_z^{2+\lambda q}N^q \\ &\le Ct^{2-2a_iq+\lambda q}h_z^{2+\lambda q}N^q.\end{aligned}$$

Combining these cases, we have a uniform upper bound on B_z,

$$\int_{B_z} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx \le Ct^{2-2a_iq+\lambda q}h_z^{2+\lambda q}N^q. \tag{4.4}$$

Note that $\sigma_z = O(1)$ and $\vartheta \le C$ on $\Omega \setminus \mathbb{T}_{i,0}$. Then, by (4.1), (4.2), (2.3), (2.5), (4.4), and (4.3), we first have

$$\begin{aligned}\int_{\Omega} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx &= \int_{B_z} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx + \int_{R_1} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx \\ &+ \sum_{0\le k\le j-2}\int_{L_{i,k}\cap R_2} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx + \int_{\Omega\setminus\mathbb{T}_{i,0}} (\sigma_z^{-2+\lambda}\vartheta^{2-2\vec{a}})^q dx \\ &\le C\Big(t^{2-2a_iq+\lambda q}h_z^{2+\lambda q}N^q + \int_{R_1} (\kappa_i^{j(2-2a_i)}\sigma_z^{-2+\lambda})^q dx \\ &+ \sum_{0\le k\le j-2}\int_{L_{i,k}\cap R_2} (\kappa_i^{k(2-2a_i)}\sigma_z^{-2+\lambda})^q dx + 1\Big) \\ &\le C\Big(t^{2-2a_iq+\lambda q}h_z^{2+\lambda q}N^q + \kappa_i^{qj(2-2a_i)}(th_z)^{2-q(2-\lambda)} \\ &+ \sum_{0\le k\le j-2}\big(\kappa_i^{qk(2-2a_i)}\kappa_i^{k(2-2q+\lambda q)}\big) + 1\Big) \\ &\le C(t^{2-2a_iq+\lambda q}h_z^{2+\lambda q}N^q + t^{2+\lambda q-2q}h_z^{2+\lambda q}N^q + \sum_{0\le k\le j-2}\kappa_i^{k(2+\lambda q-2a_iq)} + 1).\end{aligned} \tag{4.5}$$

Note that by (2.5) and (2.3),

$$\begin{aligned}t^{2+\lambda q-2q}h_z^{2+\lambda q}N^q &\simeq t^{2+\lambda q-2q}\kappa_i^{j(2+\lambda q)}2^{(j-n)(2+\lambda q)}2^{2nq} \\ &\simeq t^{2+\lambda q-2q}2^{(2q-2-\lambda q)n-(a_i^{-1}-1)(2+\lambda q)j}.\end{aligned}$$

If $\lambda q \ge -2$, by (4.3), we have

$$t^{2+\lambda q-2q}2^{(2q-2-\lambda q)n-(a_i^{-1}-1)(2+\lambda q)j} \ge Ct^{2+\lambda q-2q}2^{(2q-a_i^{-1}(2+\lambda q))n} \ge 1,$$

given that n is sufficiently large. If $\lambda q < -2$, for a sufficiently large n, by (4.3), we have

$$t^{2+\lambda q-2q}2^{(2q-2-\lambda q)n-(a_i^{-1}-1)(2+\lambda q)j} \ge 1.$$

Therefore, under the conditions $0 < |\lambda| < 2a_{\min} \leq 2$, $q(2a_{\min} - \lambda) > 2$, and n sufficiently large, we have

$$t^{2+\lambda q-2q} h_z^{2+\lambda q} N^q \geq 1. \tag{4.6}$$

Now for the term $\sum_{0\leq k\leq j-2} \kappa_i^{k(2+\lambda q-2a_i q)}$ in (4.5), by (2.3) and (2.5),

$$\begin{aligned}\kappa_i^{k(2+\lambda q-2a_i q)} &\leq Ch_z^{2+\lambda q} N^q 2^{-a_i^{-1}k(2+\lambda q)+2kq} 2^{a_i^{-1}j(2+\lambda q)} 2^{(n-j)(2+\lambda q)} 2^{-2nq} \\ &\leq Ch_z^{2+\lambda q} N^q 2^{(a_i^{-1}-1)(j-k)(2+\lambda q)-(2q-2-\lambda q)(n-k)}.\end{aligned}$$

If $\lambda q \geq -2$, by (4.3),

$$2^{(a_i^{-1}-1)(j-k)(2+\lambda q)-(2q-2-\lambda q)(n-k)} \leq 2^{(2a_i^{-1}+a_i^{-1}\lambda q-2q)(n-k)} = 2^{\bar{c}_1(k-n)},$$

where $\bar{c}_1 = 2q - 2a_i^{-1} - a_i^{-1}\lambda q > 0$. If $\lambda q < -2$, by (4.3),

$$2^{(a_i^{-1}-1)(j-k)(2+\lambda q)-(2q-2-\lambda q)(n-k)} \leq 2^{\bar{c}_2(k-n)},$$

where $\bar{c}_2 = 2q - 2 - \lambda q > 0$. Let $\bar{c} = \min(\bar{c}_1, \bar{c}_2)$. Therefore,

$$\kappa_i^{k(2+\lambda q-2a_i q)} \leq C2^{\bar{c}(k-n)} h_z^{2+\lambda q} N^q. \tag{4.7}$$

Note that by (4.3), the exponents of t in (4.5), $2-2a_i q+\lambda q < 0$ and $2+\lambda q-2q < 0$. Thus, by (4.5), (4.6), and (4.7), for $z \in T_z \subset L_{i,j}$, $1 \leq j \leq n$, we have

$$\begin{aligned}\int_\Omega (\sigma_z^{-2-\lambda} \vartheta^{2-2\vec{a}})^q dx \leq C\big(&t^{2-2a_i q+\lambda q} h_z^{2+\lambda q} N^q + t^{2+\lambda q-2q} h_z^{2+\lambda q} N^q \\ &+ \sum_{0\leq k\leq j-2} \kappa_i^{k(2+\lambda q-2a_i q)} + t^{2+\lambda q-2q} h_z^{2+\lambda q} N^q\big) \leq Ch_z^{2+\lambda q} N^q.\end{aligned} \tag{4.8}$$

Case II ($T_z \subset \Omega \setminus (\bigcup_{1\leq j\leq n} L_{i,j})$). Recall $\bar{r}$ from Remark 2.4. For n sufficiently large, we can pick a constant $c > 1$, such that in the region $R_3 = \{x \in \Omega,\ \sigma_z(x) \leq \bar{r}/c\}$, $\vartheta^{2-2\vec{a}}(x) = O(1)$; in $\Omega \setminus R_3$, $\sigma_z(x) = O(1)$. Recall that in this case, $h_z^2 \simeq 2^{-2n} \simeq N^{-1}$. Thus, we have

$$\begin{aligned}\int_\Omega (\sigma_z^{-2+\lambda} \vartheta^{2-2\vec{a}})^q dx &= \int_{B_z} (\sigma_z^{-2+\lambda} \vartheta^{2-2\vec{a}})^q dx + \int_{R_3\setminus B_z} (\sigma_z^{-2+\lambda} \vartheta^{2-2\vec{a}})^q dx \\ &\qquad + \int_{\Omega\setminus R_3} (\sigma_z^{-2+\lambda} \vartheta^{2-2\vec{a}})^q dx \\ &\leq C\Big((t^{-2+\lambda} h_z^{-2+\lambda})^q t^2 h_z^2 + \int_{R_2\setminus B_z} (\sigma_z^{-2+\lambda})^q dx + 1\Big) \\ &\leq C\Big(t^{2-2q+\lambda q} h_z^{2-2q+\lambda q} + (th_z)^{2-2q+\lambda q} + 1\Big) \leq Ct^{2-2q+\lambda q} h_z^{2+\lambda q} N^q.\end{aligned} \tag{4.9}$$

The proof is complete by combining (4.8) and (4.9). □

Remark 4.2. Lemma 4.1 holds regardless of the sign of λ. For $\lambda < 0$, we shall use a convenient form of this estimate. Namely, let $\lambda' = -\lambda > 0$. Then, for $0 < \lambda' < 2a_{\min}$, $q(2a_{\min} + \lambda') > 2$, and n sufficiently large,

$$\int_\Omega (\sigma_z^{-2-\lambda'} \vartheta^{2-2\vec{a}})^q dx \leq Ch_z^{2-\lambda' q} N^q. \tag{4.10}$$

4.2. **Intermediate regularity estimates.** We derive two important weighted regularity results for further analysis. Recall $N := \dim(S_n) \simeq 4^n$ and the parameter η_m in Proposition 2.1.

Lemma 4.3. *Recall that $z \in T_z$, $\vec{a}$, and $a_{\min}$ from Lemma 4.1. Let $0 < \lambda < \min(2a_{\min}, 2 - 4/\eta_0)$. Choose $q > 1$ such that $q(2a_{\min} + \lambda) > 2$ and $\lambda q < 2$. Let $v \in H_0^1(\Omega)$ be a function such that $\Delta v \in H_0^1(\Omega)$. Then, for n sufficiently large, we have*

$$\int_\Omega \sigma_z^{-2-\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx \leq C t^{\lambda - 2q^{-1}} N \int_\Omega \sigma_z^{2-\lambda} |\nabla \Delta v|^2 dx.$$

Proof. Let $p > 1$ be such that $1/p + 1/q = 1$. Then, by Hölder's inequality

$$\int_\Omega \sigma_z^{-2-\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx \leq \big(\int_\Omega (\sigma_z^{-2-\lambda} \vartheta^{2-2\vec{a}})^q dx \big)^{1/q} \|\nabla_2 v\|^2_{L^{2p}(\Omega)}. \tag{4.11}$$

Note that $\lambda q < 2$ implies $p > 2/(2-\lambda)$. Then, by (4.11), (4.10), Proposition 2.1, and the Sobolev embedding theorem, we have

$$\begin{aligned}\int_\Omega \sigma_z^{-2-\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx &\leq C N h_z^{2q^{-1}-\lambda} \|\nabla_2 v\|^2_{L^{2p}(\Omega)} \\ &\leq C N h_z^{2q^{-1}-\lambda} \|\Delta v\|^2_{L^{2p}(\Omega)} \leq C N h_z^{2q^{-1}-\lambda} \|\nabla \Delta v\|^2_{L^{4p/(2+2p)}(\Omega)},\end{aligned} \tag{4.12}$$

where we needed $2p < \eta_0$ to be able to use Proposition 2.1. Note that $\lambda < 2 - 4/\eta_0$ implies $2/(2-\lambda) < \eta_0/2$. Therefore, there is an open interval $(2/(2-\lambda), \eta_0/2)$ in which p can be chosen in order to obtain (4.12). Let $s = (1+p)/p$ and $1/s + 1/s' = 1$. Note that by Hölder's inequality and $p > 2/(2-\lambda)$,

$$\begin{aligned}\|\nabla \Delta v\|^{2/s}_{L^{4p/(2+2p)}(\Omega)} &= \int_\Omega |\nabla \Delta v|^{2/s} dx \\ &\leq \big(\int_\Omega \sigma_z^{2-\lambda} |\nabla \Delta v|^2 dx \big)^{1/s} \big(\int_\Omega \sigma_z^{(\lambda-2)s'/s} dx \big)^{1/s'} \\ &\leq C \big(\int_\Omega \sigma_z^{(\lambda-2)/(s-1)} dx \big)^{(s-1)/s} \big(\int_\Omega \sigma_z^{2-\lambda} |\nabla \Delta v|^2 dx \big)^{1/s} \\ &\leq C (t h_z)^{s^{-1}(\lambda - 2 + 2p^{-1})} \big(\int_\Omega \sigma_z^{2-\lambda} |\nabla \Delta v|^2 dx \big)^{1/s}.\end{aligned} \tag{4.13}$$

Combining (4.11), (4.12), and (4.13), we have

$$\int_\Omega \sigma_z^{-2-\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx \leq C t^{\lambda - 2q^{-1}} N \int_\Omega \sigma_z^{2-\lambda} |\nabla \Delta v|^2 dx,$$

which completes the proof. □

We also need the following lemma.

Lemma 4.4. *Recall $\vec{a}$ and $a_{\min}$ from Lemma 4.1. Let $f \in H_0^1(\Omega)$ and ν be a direction vector. Let $v \in H_0^1(\Omega)$ be the solution of*

$$a(v, w) = (-\nu \cdot \nabla f, w), \quad \forall w \in H_0^1(\Omega). \tag{4.14}$$

For $0 < \lambda < \min(2a_{\min}, 2 - 4/\eta_0)$, choose $q > 1$, such that $q(2a_{\min} - \lambda) > 2$. Then,

$$\int_\Omega \sigma_z^{2+\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx \leq C \Big(\int_\Omega \sigma_z^{2+\lambda} |\vartheta^{1-\vec{a}} \nabla f|^2 dx + t^{-\lambda - \frac{2}{q}} N \int_\Omega \sigma_z^{2+\lambda} |f|^2 dx \Big).$$

Proof. Recall $0 < a_i \leq 1$. The usual regularity estimate [16,19] for equation (4.14) gives $v \in H_0^1(\Omega) \cap H^2(\Omega)$. Then, by the triangle inequality, the regularity estimate in Proposition 3.1, (2.10), and (4.14), we first have

$$\begin{aligned}
\int_\Omega \sigma_z^{2+\lambda}|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx &\leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}}|\nabla_2(\sigma_z^{1+\frac{\lambda}{2}}v)|^2dx \\
&\qquad + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx\Big) \\
&\leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{2+\lambda}|\Delta v|^2dx + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx\Big) \\
&\leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{2+\lambda}|\nu\cdot\nabla f|^2dx + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx\Big). \qquad (4.15)
\end{aligned}$$

For the second term in (4.15), by (4.14), (2.10), and Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx &= \int_\Omega \nabla v\cdot\nabla(\vartheta^{2-2\vec{a}}\sigma_z^{\lambda}v)dx \\
&\qquad - \int_\Omega \sigma_z^{\lambda}v\nabla v\cdot\nabla(\vartheta^{2-2\vec{a}})dx - \int_\Omega \vartheta^{2-2\vec{a}}v\nabla v\cdot\nabla(\sigma_z^{\lambda})dx \\
&\leq \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}v(\nu\cdot\nabla f)dx \\
&\qquad +(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx)^{\frac{1}{2}}(\int_\Omega \vartheta^{2\vec{a}-2}\sigma_z^{\lambda}|\nabla(\vartheta^{2-2\vec{a}})|^2v^2dx)^{\frac{1}{2}} \\
&\qquad +(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx)^{\frac{1}{2}}(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx)^{\frac{1}{2}}.
\end{aligned}$$

Thus, using Hölder's inequality and Young's inequality, for $\epsilon > 0$ small, we have

$$\begin{aligned}
\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx &\leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{2+\lambda}|\nu\cdot\nabla f|^2dx + 2\epsilon\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx \\
&\qquad +\epsilon^{-1}\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx + \epsilon^{-1}\int_\Omega \vartheta^{2\vec{a}-2}\sigma_z^{\lambda}|\nabla(\vartheta^{2-2\vec{a}})|^2v^2dx\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda}|\nabla v|^2dx &\leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{2+\lambda}|\nu\cdot\nabla f|^2dx \\
&\qquad + \int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx + \int_\Omega \vartheta^{2\vec{a}-2}\sigma_z^{\lambda}|\nabla(\vartheta^{2-2\vec{a}})|^2v^2dx\Big). \qquad (4.16)
\end{aligned}$$

In addition, by (2.10), Hölder's inequality and Young's inequality, we have for ϵ small,

$$\begin{aligned}
\int_\Omega \vartheta^{2\vec{a}-2}\sigma_z^{\lambda}|\nabla(\vartheta^{2-2\vec{a}})|^2v^2dx &\leq C\int_\Omega \vartheta^{-2\vec{a}}\sigma_z^{\lambda}v^2dx \\
&\leq C(\int_\Omega \vartheta^{-2-2\vec{a}}\sigma_z^{2+\lambda}v^2dx)^{\frac{1}{2}}(\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx)^{\frac{1}{2}} \\
&\leq C\big(\epsilon^2\int_\Omega \vartheta^{-2-2\vec{a}}\sigma_z^{2+\lambda}v^2dx + \epsilon^{-2}\int_\Omega \vartheta^{2-2\vec{a}}\sigma_z^{\lambda-2}v^2dx\big). \qquad (4.17)
\end{aligned}$$

Using the regularity estimate in Proposition 3.1 for $(\sigma_z^{1+\frac{\lambda}{2}} v)$, (4.14), and (2.10), we have

$$\begin{aligned}\epsilon^2 \int_\Omega \vartheta^{-2-2\vec{a}}(\sigma_z^{1+\frac{\lambda}{2}} v)^2 dx \leq C\epsilon^2 (\int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{2+\lambda} |\nu \cdot \nabla f|^2 dx \\ + \int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{\lambda} |\nabla v|^2 dx + \int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{\lambda-2} v^2 dx).\end{aligned} \tag{4.18}$$

Therefore, for ϵ sufficiently small, by (4.16), (4.17), and (4.18),

$$\int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{\lambda} |\nabla v|^2 dx \leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{2+\lambda} |\nu \cdot \nabla f|^2 dx + \int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{\lambda-2} v^2 dx\Big). \tag{4.19}$$

Hence, by (4.15) and (4.19),

$$\int_\Omega \sigma_z^{2+\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx \leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{2+\lambda} |\nu \cdot \nabla f|^2 dx + \int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{\lambda-2} v^2 dx\Big). \tag{4.20}$$

Let $p > 1$ be such that $1/q + 1/p = 1$. By Hölder's inequality and Lemma 4.1, we have

$$\begin{aligned}\int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{\lambda-2} v^2 dx &\leq \Big(\int_\Omega (\vartheta^{2-2\vec{a}} \sigma_z^{\lambda-2})^q dx\Big)^{1/q} \|v\|^2_{L^{2p}(\Omega)} \\ &\leq C h_z^{2/q+\lambda} N \|v\|^2_{L^{2p}(\Omega)}.\end{aligned} \tag{4.21}$$

Let $y \in H_0^1(\Omega)$ be the solution of

$$-\Delta y = \operatorname{sign}(v)|v|^{2p-1}.$$

Let $s = 2p/(p+1)$ and $s' = 2p/(p-1)$. Then, by Hölder's inequality, the Sobolev embedding theorem, and Proposition 2.1, we have

$$\begin{aligned}\|v\|^{2p}_{L^{2p}(\Omega)} &= (\nabla y, \nabla v) = (f, \nu \cdot \nabla y) \leq \|f\|_{L^s(\Omega)} \|\nabla y\|_{L^{s'}(\Omega)} \\ &\leq C\|f\|_{L^s(\Omega)} \|y\|_{W^2_{2p/(2p-1)}(\Omega)} \leq C\|f\|_{L^s(\Omega)} \|v^{2p-1}\|_{L^{2p/(2p-1)}(\Omega)} \\ &= C\|f\|_{L^s(\Omega)} \|v\|^{2p-1}_{L^{2p}(\Omega)},\end{aligned}$$

provided that $2p/(2p-1) < \eta_0$. Note that the condition $q(2a_{\min} - \lambda) > 2$ implies $s < 4/(4 - 2a_{\min} + \lambda)$; the condition $2p/(2p-1) < \eta_0$ implies that $s > 2\eta_0/(3\eta_0 - 2)$. Since $\eta_0 > 2$ on Ω and $\lambda < 2a_{\min}$, there is an open interval from which s can be chosen to have the estimates above. Therefore, by Hölder's inequality,

$$\begin{aligned}\|v\|_{L^{2p}(\Omega)} &\leq C\|f\|_{L^s(\Omega)} \leq \Big(\int_\Omega \sigma_z^{2+\lambda} |f|^2 dx\Big)^{\frac{1}{2}} \Big(\int_\Omega \sigma_z^{-(2+\lambda)s/(2-s)} dx\Big)^{\frac{2-s}{2s}} \\ &\leq C(th_z)^{\frac{4-4s-\lambda s}{2s}} \Big(\int_\Omega \sigma_z^{2+\lambda} |f|^2 dx\Big)^{\frac{1}{2}} = C(th_z)^{-\frac{\lambda}{2}-\frac{1}{q}} \Big(\int_\Omega \sigma_z^{2+\lambda} |f|^2 dx\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.22}$$

Combining (4.20)–(4.22), we complete the proof

$$\int_\Omega \sigma_z^{2+\lambda} |\vartheta^{1-\vec{a}} \nabla_2 v|^2 dx \leq C\Big(\int_\Omega \vartheta^{2-2\vec{a}} \sigma_z^{2+\lambda} |\nabla f|^2 dx + t^{-\lambda-\frac{2}{q}} N \int_\Omega \sigma_z^{2+\lambda} |f|^2 dx\Big).$$

$\square$

Remark 4.5. The key to the development of the regularity estimates in Lemmas 4.3 and 4.4 is the exploration of the intrinsic connection between the modified Kondrat′ev weight function ϑ and the geometry of the graded mesh. This provides the technical results needed for the stability analysis in the next section.

5. Stability analysis

Our primary goal in this paper is to establish the stability result $\|u_n\|_{W_p^1(\Omega)} \leq C\|u\|_{W_p^1(\Omega)}$ (Theorem 5.5) for equation (1.1), where $u_n \in S_n$ is the finite element solution in (2.2) on the graded mesh $\mathcal{T}_n$ (Definition 2.6).

As discussed in Remark 2.14, it is important to obtain an upper bound for

$$M = \max_{z\in\Omega}\big((\int_\Omega \frac{h^\lambda}{h_z^\lambda}\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx)^{\frac{1}{2}}\big).$$

We first have the approximation result below.

Lemma 5.1. *Let g and g_n be defined in* (2.19) *and* (2.20)*. For $\lambda > 0$ and $t > 1$, we have*

$$\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx$$
$$\leq C\big(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_I)|^2dx + \int_\Omega \sigma_z^\lambda(g-g_I)^2dx + \int_\Omega \sigma_z^\lambda(g-g_n)^2dx\big),$$

where $g_I \in S_n$ is the nodal interpolation of g.

Proof. Define $\psi := \sigma_z^{2+\lambda}(g_I - g_n)$. Let $\psi_I \in S_n$ be the nodal interpolation of ψ. By (2.20), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx &= \int_\Omega \nabla(g-g_n)\cdot\big(\sigma_z^{2+\lambda}\nabla(g-g_I)+\nabla\psi\big)dx\\
&\leq \int_\Omega|\nabla(g-g_n)||\sigma_z^{2+\lambda}\nabla(g-g_I)|dx + \int_\Omega\nabla(g-g_n)\cdot\nabla(\psi-\psi_I)dx\\
&\leq (\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx)^{1/2}(\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_I)|^2dx)^{1/2}\\
&\quad +(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx)^{1/2}(\int_\Omega\sigma_z^{-2-\lambda}|\nabla(\psi-\psi_I)|^2dx)^{1/2}\\
&\leq C\big(\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_I)|^2dx + \int_\Omega\sigma_z^{-2-\lambda}|\nabla(\psi-\psi_I)|^2dx\big)\\
&\quad +\frac{1}{2}\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx.\end{aligned}\tag{5.1}$$

Let $T \in \mathcal{T}_n$ be a triangle and $h_T = \mathrm{diam}(T)$. Then, on T, by (2.7), the usual interpolation error estimate, the definition of ψ, (2.10), and the inverse inequality, we have

$$\begin{aligned}\int_T \sigma_z^{-2-\lambda}|\nabla(\psi-\psi_I)|^2dx &\leq \max_{x\in T}\sigma_z^{-2-\lambda}(x)\int_T|\nabla(\psi-\psi_I)|^2dx\\
&\leq C\min_{x\in T}\sigma_z^{-2-\lambda}(x)h_T^{2m}\int_T|\nabla_{m+1}\psi|^2dx \leq Ch_T^{2m}\int_T\sigma_z^{-2-\lambda}|\nabla_{m+1}\psi|^2dx\\
&\leq C\sum_{0\leq s\leq m}h_T^{2m}\int_T\sigma_z^{\lambda-2(m-s)}|\nabla_s(g_I-g_n)|^2dx\\
&\leq C\sum_{0\leq s\leq m}h_T^{2m-2s}\int_T\sigma_z^{\lambda-2(m-s)}|g_I-g_n|^2dx \leq C\int_T\sigma_z^\lambda|g_I-g_n|^2dx.\end{aligned}$$

Therefore,

$$\int_\Omega \sigma_z^{-2-\lambda}|\nabla(\psi-\psi_I)|^2dx = \sum_{T\in\mathcal{T}_n}\int_T \sigma_z^{-2-\lambda}|\nabla(\psi-\psi_I)|^2dx$$

$$\leq C\int_\Omega \sigma_z^{\lambda}|g_I-g_n|^2dx. \tag{5.2}$$

By (5.1) and (5.2), we have

$$\begin{aligned}\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx &\leq C\big(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_I)|^2dx+\int_\Omega\sigma_z^{\lambda}(g_I-g_n)^2dx\big)\\ &\leq C\big(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_I)|^2dx+\int_\Omega\sigma_z^{\lambda}(g-g_I)^2dx+\int_\Omega\sigma_z^{\lambda}(g-g_n)^2dx\big).\end{aligned}$$

This completes the proof. □

Now, we analyze the integral $\int_\Omega \sigma_z^\lambda(g-g_n)^2dx$ from the last lemma.

Lemma 5.2. *Let $0<\lambda<\min(2a_{\min}, 2-4/\eta_0)$. For any $\epsilon>0$, there exists $t>1$, such that for n sufficiently large,*

$$\int_\Omega \sigma_z^{\lambda}(g-g_n)^2dx \leq \epsilon\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx.$$

Proof. Let $v\in H_0^1(\Omega)$ be the solution of $-\Delta v=\sigma_z^\lambda(g-g_n)$. Then, choosing q as in Lemma 4.3, by Lemma 4.3 and (2.10), we have

$$\int_\Omega \sigma_z^{-2-\lambda}|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx \leq Ct^{\lambda-2q^{-1}}N\big(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2+\sigma_z^\lambda(g-g_n)^2dx\big). \tag{5.3}$$

Using Hölder's inequality, Proposition 3.2, Young's inequality, and (5.3), we have

$$\begin{aligned}\int_\Omega \sigma_z^{\lambda}(g-g_n)^2dx &= \int_\Omega \nabla(v-v_I)\cdot\nabla(g-g_n)dx\\ &\leq \big(\int_\Omega \sigma_z^{-2-\lambda}|\nabla(v-v_I)|^2dx\big)^{\frac12}\big(\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx\big)^{\frac12}\\ &\leq CN^{-\frac12}\big(\int_\Omega \sigma_z^{-2-\lambda}|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx\big)^{\frac12}\big(\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx\big)^{\frac12}\\ &\leq C\epsilon^{-1}N^{-1}\int_\Omega \sigma_z^{-2-\lambda}|\vartheta^{1-\vec{a}}\nabla_2 v|^2dx+\frac{\epsilon}{2}\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx\\ &\leq Ct^{\lambda-2q^{-1}}\epsilon^{-1}\big(\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2+\sigma_z^\lambda(g-g_n)^2dx\big)\\ &\quad+\frac{\epsilon}{2}\int_\Omega\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx.\end{aligned}$$

Recall from Lemma 4.3 that $\lambda q<2$. Therefore, $\lambda-2q^{-1}<0$. Thus, we can choose t large enough, such that

$$\int_\Omega \sigma_z^{\lambda}(g-g_n)^2dx \leq \epsilon\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx,$$

which completes the proof. □

We are ready to obtain the stability result for $2 \leq p \leq \infty$.

Lemma 5.3. *For $2 \leq p \leq \infty$ and n sufficiently large, we have*

$$\|\nabla u_n\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)},$$

where C is independent of n.

Proof. Recall $M = \max_{z\in\Omega}\big((\int_\Omega \frac{h^\lambda}{h_z^\lambda}\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx)^{\frac{1}{2}}\big)$ from Lemma 2.13. Let $0 < \lambda < \min(2a_{\min}, 2-4/\eta_0)$. Choose $q > 1$ such that $q(2a_{\min}-\lambda) > 2$. By Lemma 5.1, Lemma 5.2, Proposition 3.2, and Lemma 4.4, we have

$$\begin{aligned}\int_\Omega \frac{h^\lambda}{h_z^\lambda}\sigma_z^{2+\lambda}|\nabla(g-g_n)|^2dx &\leq C\frac{h^\lambda}{h_z^\lambda}\Big(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_I)|^2 + \sigma_z^\lambda(g-g_I)^2\\ &\qquad\qquad + \sigma_z^\lambda(g-g_n)^2dx\Big)\\ &\leq C\frac{h^\lambda}{h_z^\lambda}\Big(\int_\Omega \sigma_z^{2+\lambda}|\nabla(g-g_I)|^2 + \sigma_z^\lambda(g-g_I)^2\Big)\\ &\leq CN^{-1}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{2+\lambda}|\vartheta^{1-\vec{a}}\nabla_2 g|^2dx\\ &\leq C\Big(N^{-1}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{2+\lambda}|\vartheta^{1-\vec{a}}\nabla\delta_z|^2dx + t^{-\lambda-2q^{-1}}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{\lambda+2}|\delta_z|^2dx\Big).\end{aligned} \tag{5.4}$$

Recall $L_{i,j}$ from Definition 2.8. If $T_z \in L_{i,j}$, for $0 \leq j \leq n$, since $\vartheta \simeq \kappa_i^j$ on T_z, by (2.8), (2.5), and (2.3), we have

$$\begin{aligned}N^{-1}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{2+\lambda}|\vartheta^{1-\vec{a}}\nabla\delta_z|^2dx + t^{-\lambda-2q^{-1}}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{\lambda+2}|\delta_z|^2dx&\\ \leq Ch^\lambda(N^{-1}h_z^{-2}\kappa_i^{j(2-2a_i)}+1)&\\ \leq Ch^\lambda(2^{-2n}2^{2n-2j}\kappa_i^{-2ja_i}+1) \leq Ch^\lambda.&\end{aligned} \tag{5.5}$$

If $T_z \in \Omega_0 = \Omega \setminus (\bigcup_i \mathbb{T}_{i,0})$, $\vartheta|_{T_z} = O(1)$ and $h_z \simeq 2^{-n}$. Then, we similarly have

$$\begin{aligned}N^{-1}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{2+\lambda}|\vartheta^{1-\vec{a}}\nabla\delta_z|^2dx + t^{-\lambda-2q^{-1}}\frac{h^\lambda}{h_z^\lambda}\int_\Omega \sigma_z^{\lambda+2}|\delta_z|^2dx&\\ \leq Ch^\lambda(N^{-1}h_z^{-2}+1) \leq Ch^\lambda.&\end{aligned} \tag{5.6}$$

Then, by Lemma 2.13, (5.4), (5.5), and (5.6), we have for $2 < p \leq \infty$,

$$\|\nabla u_n\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}(1 + h^{-\frac{\lambda}{2}}h^{\frac{\lambda}{2}}) \leq C\|\nabla u\|_{L^p(\Omega)}.$$

It remains to show the lemma for $p = 2$. This is the case by

$$\|\nabla u_n\|^2_{L^2\Omega)} = (\nabla u_n, \nabla u_n) = (\nabla u, \nabla u_n) \leq \|\nabla u_n\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}.$$

Thus, the proof is completed. □

To extend the stability result in Lemma 5.3 to the full $W_p^1(\Omega)$-norm of u_n, we first need an upper bound for the finite element solution in $L^p(\Omega)$.

Lemma 5.4. *For $1 < p \leq \infty$ and n sufficiently large, we have*

$$\|u_n\|_{L^p(\Omega)} \leq C\|u\|_{W_p^1(\Omega)}.$$

Proof. For $p \geq 2$, the result follows from the Poincaré inequality and Lemma 5.3. We now show the proof for $1 < p < 2$.

Let $w \in H_0^1(\Omega)$ be the solution of

$$-\Delta w = \text{sign}(u_n)|u_n|^{p-1} \quad \text{in } \Omega,$$

and $w_n \in S_n$ be its finite element solution. Let $q > 2$, such that $1/p + 1/q = 1$. Let $1/r = 1/q + 1/2$, and therefore $1 < r < 2$. Then, by Hölder's inequality, Lemma 5.3, the Sobolev embedding theorem, and Proposition 2.1,

$$\begin{aligned}
\|u_n\|^p_{L^p(\Omega)} &= (\nabla u_n, \nabla w) = (\nabla u_n, \nabla w_n) = (\nabla u, \nabla w_n) \\
&\leq \|\nabla u\|_{L^p(\Omega)}\|\nabla w_n\|_{L^q(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}\|\nabla w\|_{L^q(\Omega)} \\
&\leq C\|\nabla u\|_{L^p(\Omega)}\|w\|_{W^2_r(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}\||u_n|^{p-1}\|_{L^r(\Omega)} \\
&\leq C\|\nabla u\|_{L^p(\Omega)}\||u_n|^{p-1}\|_{L^q(\Omega)} = C\|\nabla u\|_{L^p(\Omega)}\|u_n\|^{p-1}_{L^p(\Omega)}.
\end{aligned}$$

The lemma is hence proved. □

Thus, we have the main stability result for the finite element solution $u_n \in S_n$ of equation (1.1) on the graded mesh $\mathcal{T}_n$.

Theorem 5.5. *For $1 < p \leq \infty$ and n sufficiently large, we have*

$$\|u_n\|_{W^1_p(\Omega)} \leq C\|u\|_{W^1_p(\Omega)},$$

where C is independent of n.

Proof. Using Lemma 5.3 and Lemma 5.4, we obtain the expected estimate for $2 \leq p \leq \infty$. Therefore, it suffices to prove the case $1 < p < 2$. By Theorem 4.32 in [27], Lemma 5.3, and Lemma 5.4, for $1/p + 1/q = 1$ and any $0 \neq v \in \{v \in W^1_q(\Omega) : v|_{\partial\Omega} = 0\}$, we have

$$\begin{aligned}
\|u_n\|_{W^1_p(\Omega)} &\leq C\big(\sup_v \frac{a(u_n, v)}{\|v\|_{W^1_q(\Omega)}} + \|u_n\|_{L^p(\Omega)}\big) = C\big(\sup_v \frac{a(u_n, v_n)}{\|v\|_{W^1_q(\Omega)}} + \|u_n\|_{L^p(\Omega)}\big) \\
&= C\big(\sup_v \frac{a(u, v_n)}{\|v\|_{W^1_q(\Omega)}} + \|u_n\|_{L^p(\Omega)}\big) \leq C\|u\|_{W^1_p(\Omega)},
\end{aligned}$$

where $v_n \in S_n$ is the finite element approximation of v. This completes the proof. □

As a direct consequence of Theorem 5.5, we derive the following approximation property for the finite element solution in non-energy norms.

Corollary 5.6. *For $1 < p \leq \infty$ and n sufficiently large, we have*

$$\|u - u_n\|_{W^1_p(\Omega)} \leq C \inf_{v \in S_n} \|u - v\|_{W^1_p(\Omega)}. \tag{5.7}$$

For $1 < p < \infty$, let $q = p/(p-1)$. For $u \in L^p(\Omega)$, let $w \in W^1_q(\Omega) \cap \{w|_{\partial\Omega} = 0\}$ be the solution of

$$-\Delta w = \text{sign}(u - u_n)|u - u_n|^{p-1} \quad \text{in } \Omega. \tag{5.8}$$

Then,

$$\|u - u_n\|^p_{L^p(\Omega)} \leq C\big(\inf_{v_1 \in S_n} \|u - v_1\|_{W^1_p(\Omega)}\big)\big(\inf_{v_2 \in S_n} \|w - v_2\|_{W^1_q(\Omega)}\big). \tag{5.9}$$

Proof. (5.7) is a direct consequence of Theorem 5.5. For any $v \in S_n$,

$$\|u - u_n\|_{W_p^1(\Omega)} \leq \|u - v\|_{W_p^1(\Omega)} + \|v - u_n\|_{W_p^1(\Omega)} \leq C\|u - v\|_{W_p^1(\Omega)}.$$

To show (5.9), we first verify the solution of (5.8) $w \in W_q^1(\Omega)$ for any $1 < p < \infty$. For $p \geq 2$, $1 < q < 2$. Therefore, by Proposition 2.1,

$$\|w\|_{W_q^1(\Omega)} \leq \|w\|_{W_q^2(\Omega)} \leq C\||u - u_n|^{p-1}\|_{L^q(\Omega)} = C\|u - u_n\|_{L^p(\Omega)}^{p-1}.$$

For $1 < p < 2$, let $1 < r < 2$ be such that $1/r = 1/q + 1/2$. Using the Sobolev embedding theorem, Proposition 2.1, and Hölder's inequality,

$$\begin{aligned} \|w\|_{W_q^1(\Omega)} &\leq \|w\|_{W_r^2(\Omega)} \leq C\||u - u_n|^{p-1}\|_{L^r(\Omega)} \\ &\leq C\||u - u_n|^{p-1}\|_{L^q(\Omega)} = C\|u - u_n\|_{L^p(\Omega)}^{p-1}. \end{aligned}$$

Therefore, $w \in W_q^1(\Omega)$ for $1 < p < \infty$. Then, for any $v_1, v_2 \in S_n$, (5.9) follows immediately from (5.8) and (5.7):

$$\begin{aligned} \|u - u_n\|_{L^p(\Omega)}^p &= a(w, u - u_n) = a(w - v_2, u - u_n) \\ &\leq \|w - v_2\|_{W_q^1(\Omega)}\|u - u_n\|_{W_p^1(\Omega)} \leq C\|w - v_2\|_{W_q^1(\Omega)}\|u - v_1\|_{W_p^1(\Omega)}. \end{aligned}$$

□

Remark 5.7. Corollary 5.6 concerns the approximation property of the finite element solution in the spaces W_p^1 $(1 < p \leq \infty)$ and in L^p $(1 < p < \infty)$. Using the weighted technique in (modified) Kondrat′ev spaces, we expect these approximation results will help develop specific graded meshes that lead to optimal rate of convergence in these non-energy norms. See [3, 4, 6, 23, 25, 31] and the references therein for the design of optimal graded meshes in energy norms. Note that the estimate in L^∞ needs further effort, partially due to the lack of regularity in the L^1 space. We refer to [2] for an L^∞ error estimate on graded meshes.

6. Concluding remarks

In this paper, we developed analytical tools in weighted spaces and proved the stability result

$$\|u_n\|_{W_p^1(\Omega)} \leq C\|u\|_{W_p^1(\Omega)}, \qquad 1 < p \leq \infty, \tag{6.1}$$

for the finite element approximation of the model problem (1.1) on a family of graded meshes. This further led to the error analysis (Corollary 5.6) in non-energy norms. To obtain the resu, we introduced a new weight function to replace the conventional weight in the Kondrat′ev space. This modified weighted space offers not only needed regularity results, but also good approximation properties for the finite element analysis. The intrinsic connection between the new weight and the mesh grading property was studied in detail and played a key role in the analysis. The stability result in (6.1) excludes the case for W_1^1, because the duality argument in the proof of Theorem 5.5 does not hold for $p = 1$.

The result in this paper has several foreseeable important extensions for finite element approximations of elliptic problems. With the interpolation error estimates in [18] for functions of ρ^λ type and the stability estimate (6.1), we expect the development of optimal 2D graded meshes in non-energy norms. Our weighted analysis extends to 3D convex polyhedral domains with isotropic graded meshes for vertex singularities, which we shall include in an forthcoming paper. We also expect our work will lead to new ideas for stability analysis on 3D anisotropic meshes.

Note that the W^1_p stability on non-convex domains is still an open problem; it has been difficult due to the lack of regularity in Sobolev norms. We hope the weighted analysis developed in this paper can be helpful to motivate new techniques in this direction.

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