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QUASI-UNBIASED HADAMARD MATRICES AND WEAKLY UNBIASED HADAMARD MATRICES: A CODING-THEORETIC APPROACH

MAKOTO ARAYA, MASAAKI HARADA, AND SHO SUDA

Dedicated to Professor Satoshi Yoshiara on his 60th birthday

Abstract. This paper is concerned with quasi-unbiased Hadamard matrices and weakly unbiased Hadamard matrices, which are generalizations of unbiased Hadamard matrices, equivalently unbiased bases. These matrices are studied from the viewpoint of coding theory. As a consequence of a coding-theoretic approach, we provide upper bounds on the number of mutually quasi-unbiased Hadamard matrices. We give classifications of a certain class of self-complementary codes for modest lengths. These codes give quasi-unbiased Hadamard matrices and weakly unbiased Hadamard matrices. Some modification of the notion of weakly unbiased Hadamard matrices is also provided.

1. Introduction

Two Hadamard matrices H, K of order n are said to be *unbiased* if $(1/\sqrt{n})HK^T$ is also a Hadamard matrix of order n, where K^T denotes the transpose of K. This means that the absolute value of any entry of HK^T is $\sqrt{n}$. The notion of unbiased Hadamard matrices is essentially the same as that of unbiased bases in $\mathbb{R}^n$. It is a fundamental problem to determine the maximum size among sets of mutually unbiased Hadamard matrices. Much work has been done concerning this fundamental problem (see [5], [8], [10], [11], [15], [19], [21], [23], [29]).

Recently, the notion of unbiased Hadamard matrices has been generalized in [5], [19] and [26] (see also Section 2.1 for the motivation). Two weighing matrices W_1, W_2 of order n and weight k are *unbiased* if $(1/\sqrt{k})W_1W_2^T$ is a weighing matrix of order n and weight k [19]. As a natural generalization, quasi-unbiased weighing matrices are defined in [26] as follows: W_1, W_2 are *quasi-unbiased for parameters* (n, k, l, a) if $(1/\sqrt{a})W_1W_2^T$ is a weighing matrix of weight l. In this paper, we restrict our investigation to the case where W_1, W_2 are Hadamard in order to adopt a coding-theoretic approach. We say that Hadamard matrices H, K are *quasi-unbiased* Hadamard matrices with parameters (l, a) if $(1/\sqrt{a})HK^T$ is a weighing matrix of weight l. Note that the absolute value of any entry of HK^T is 0 or $\sqrt{a}$. Two Hadamard matrices H, K are *weakly unbiased* if $a_{ij} \equiv 2 \pmod 4$ for $i, j \in \{1, 2, \dots, n\}$ and $|\{|a_{ij}| \mid i, j \in \{1, 2, \dots, n\}\}| \le 2$, where a_{ij} denotes the (i, j)-entry of HK^T [5]. Hadamard matrices $H_1, H_2, \dots, H_f$ are said to be *mutually* unbiased (resp. quasi-unbiased and weakly unbiased) Hadamard matrices if any pair of two

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distinct Hadamard matrices is unbiased (resp. quasi-unbiased and weakly unbiased) Hadamard matrices. In this paper, by adopting a coding-theoretic approach, we study the maximum size among sets of mutually quasi-unbiased Hadamard matrices and weakly unbiased Hadamard matrices.

This paper is organized as follows. In Section 2, we give definitions and some known results of Hadamard matrices, codes and association schemes used in this paper. In Section 3, we give two upper bounds on the number of codewords of binary self-complementary codes (Theorems 3.2 and 3.4). In Sections 4 and 5, we study the existence of mutually quasi-unbiased Hadamard matrices. In Section 5, we characterize binary self-complementary $(n, 2fn)$ codes whose existence is equivalent to that of a set of f mutually quasi-unbiased Hadamard matrices of order n (Theorem 5.1). By Theorems 3.2 and 3.4, this characterization derives upper bounds on the size of sets of mutually quasi-unbiased Hadamard matrices (Theorem 5.4). For modest lengths, we also give classifications of some binary self-complementary codes satisfying the conditions in Theorem 5.1 in order to construct mutually quasi-unbiased Hadamard matrices. In analogy to the case of quasi-unbiased Hadamard matrices, Sections 6 and 7 study the existence of weakly unbiased Hadamard matrices. Theorem 6.1 shows that the size of a set of mutually weakly unbiased Hadamard matrices is at most 2. Similarly to Theorem 5.1, we characterize binary self-complementary codes whose existence is equivalent to that of a pair of weakly unbiased Hadamard matrices of order n (Theorem 7.1). For modest lengths, we also give classifications of some binary self-complementary codes satisfying the conditions in Theorem 7.1 in order to construct weakly unbiased Hadamard matrices. Finally, in Section 8, as a modification of the notion of weakly unbiased Hadamard matrices, we introduce the notion of Type II weakly unbiased Hadamard matrices. We establish results which are analogous to those of quasi-unbiased Hadamard matrices and weakly unbiased Hadamard matrices.

All computer calculations in this paper were done by programs in the algebra software Magma [7] and programs in the language C.

2. Preliminaries

In this section, we give definitions and some known results of Hadamard matrices, codes and association schemes used in this paper.

2.1. **Hadamard matrices.** A *Hadamard matrix* of order n is an $n \times n$ $(1, -1)$-matrix H such that $HH^T = nI_n$, where I_n is the identity matrix of order n. It is well known that the order n is necessarily $1, 2$, or a multiple of 4. Throughout this paper, we assume that $n \geq 2$ unless otherwise specified. A *weighing matrix* of order n and weight k is an $n \times n$ $(1, -1, 0)$-matrix W such that $WW^T = kI_n$. Of course, a weighing matrix of order n and weight n is a Hadamard matrix. The two distinct rows r_i, r_j $(i \neq j)$ of a weighing matrix W of order n and weight k are orthogonal under the standard inner product $r_i \cdot r_j$ and W contains exactly k nonzero entries in each row and each column. Two Hadamard matrices H, K are said to be *equivalent* if there exist $(1, -1, 0)$-monomial matrices P, Q with $K = PHQ$. All Hadamard matrices of orders up to 32 have been classified (see [18, Chap. 7] for orders up to 28 and [22] for order 32; see also [28]). The numbers of inequivalent Hadamard matrices of orders $4, 8, 12, 16, 20, 24, 28, 32$ are $1, 1, 1, 5, 3, 60, 487, 13710027$, respectively.

Two Hadamard matrices H, K of order n are said to be *unbiased* if $(1/\sqrt{n})HK^T$ is also a Hadamard matrix of order n, where K^T denotes the transpose of K. This means that the absolute value of any entry of HK^T is $\sqrt{n}$. Hadamard matrices are said to be *mutually* unbiased Hadamard matrices if any pair of two distinct Hadamard matrices are unbiased Hadamard matrices. The existence of f mutually unbiased Hadamard matrices of order n is equivalent to that of $f+1$ mutually unbiased bases in $\mathbb{R}^n$ [8, Observation 2.1]. It is a fundamental problem to determine the maximum size among sets of mutually unbiased Hadamard matrices of order n. For example, it follows from [8, Observation 2.1] and [15, Table 1] that $f \leq n/2$.

Recently, generalizations of unbiased Hadamard matrices have been presented in [5], [19] and [26]. Two weighing matrices W_1, W_2 of order n and weight k are *unbiased* if $(1/\sqrt{k})W_1W_2^T$ is a weighing matrix of weight k [19]. As a natural generalization, quasi-unbiased weighing matrices are defined in [26] as follows: W_1, W_2 are *quasi-unbiased for parameters* (n,k,l,a) if $(1/\sqrt{a})W_1W_2^T$ is a weighing matrix of order n and weight l. This notion was introduced to show that Conjecture 32 in [6] is true. In addition, a set of f mutually quasi-unbiased weighing matrices for parameters (n,k,l,a) implies a set of $f-1$ mutually unbiased weighing matrices of order n and weight l. In this paper, we restrict our investigation to the case where W_1, W_2 are Hadamard in the definition of quasi-unbiased weighing matrices in order to adopt a coding-theoretic approach. Our restriction is also natural for a consideration of a certain generalization of the situation in [6, Conjecture 32]. We say that Hadamard matrices H, K of order n are *quasi-unbiased* Hadamard matrices with parameters (l,a) if $(1/\sqrt{a})HK^T$ is a weighing matrix of weight l. Equivalently, the absolute value of any entry of HK^T is 0 or $\sqrt{a}$. Two Hadamard matrices H, K are *weakly unbiased* if $a_{ij} \equiv 2 \pmod 4$ for $i,j \in \{1,2,\dots,n\}$ and $|\{|a_{ij}| \mid i,j \in \{1,2,\dots,n\}\}| \leq 2$, where a_{ij} denotes the (i,j)-entry of HK^T [5]. A pair of weakly unbiased Hadamard matrices is constructed from that of unbiased quaternary complex Hadamard matrices satisfying a certain condition [5, Theorem 14].

Throughout this paper, in the presentation of Hadamard matrices, we use $+, -$ to denote $1, -1$, respectively.

2.2. **Binary codes and $\mathbb{Z}_4$-codes.** Let $\mathbb{Z}_{2k}$ $(=\{0,1,\dots,2k-1\})$ denote the ring of integers modulo $2k$. A $\mathbb{Z}_{2k}$*-code* C of length n is a subset of $\mathbb{Z}_{2k}^n$. A $\mathbb{Z}_{2k}$-code C is called *linear* if C is a $\mathbb{Z}_{2k}$-submodule of $\mathbb{Z}_{2k}^n$. Usually $\mathbb{Z}_2$-codes are called *binary*. In this paper, we deal with binary codes and $\mathbb{Z}_4$-codes. In addition, codes mean binary codes unless otherwise specified.

The (Hamming) *distance* $d(x,y)$ between two vectors x and y of $\mathbb{Z}_{2k}^n$ is the number of components in which they differ. Let C be a $\mathbb{Z}_{2k}$-code of length n. A vector of C is called a *codeword* of C. The *minimum* (Hamming) *distance* $d_H(C)$ of C is the smallest (Hamming) distance among all pairs of two distinct codewords of C. A *generator matrix* of a linear $\mathbb{Z}_{2k}$-code is a matrix such that the rows generate the code and no proper subset of the rows of the matrix generates the code. For a linear $\mathbb{Z}_{2k}$-code C of length n and vectors $x_1, x_2, \dots, x_s \in \mathbb{Z}_{2k}^n$, we denote by $\langle C, x_1, x_2, \dots, x_s\rangle$ the linear $\mathbb{Z}_{2k}$-code generated by the codewords of C and $x_1, x_2, \dots, x_s$. Let S_n denote the symmetric group of degree n. For $x \in \mathbb{Z}_{2k}^n$ and $\sigma \in S_n$, let $\sigma(x)$ denote the vector obtained from x by the permutation σ of the coordinates. For $j \in \{1,2,\dots,n\}$, let $\tau_j(x)$ denote the vector obtained from x by changing the sign of the j-th coordinate. In addition, set $\sigma(C) = \{\sigma(c) \mid c \in C\}$ and $\tau_j(C) = \{\tau_j(c) \mid c \in C\}$.

A binary (n, M) code is a binary code of length n with M codewords. A binary (n, M, d) code is a binary (n, M) code with minimum distance d. A binary $[n, k]$ code means a binary linear code of length n with 2^k codewords. A binary $[n, k, d]$ code means a binary $[n, k]$ code with minimum distance d. The *distance distribution* of a binary code C of length n is defined as $(A_0(C), A_1(C), \dots, A_n(C))$, where

$$A_i(C) = \frac{1}{|C|}|\{(x, x') \mid x, x' \in C, d(x, x') = i\}| \quad (i = 0, 1, \dots, n).$$

A binary code C is called *self-complementary* if $x + \mathbf{1} \in C$ for any $x \in C$, where $\mathbf{1}$ denotes the all-one vector. Two binary (n, M, d) codes C, D are *equivalent* if there exist a permutation $\sigma \in S_n$ and a vector $x \in \mathbb{Z}_2^n$ such that $D = x + \sigma(C)$.

A Hadamard matrix is *normalized* if all entries in the first row and the first column are 1. Let H be a normalized Hadamard matrix of order n. Throughout this paper, we denote by $C(H)$ the binary $(n, 2n)$ code consisting of the $2n$ row vectors of $(1, 0)$-matrices $(H + J_n)/2$ and $(-H + J_n)/2$, where J_n denotes the $n \times n$ all-one matrix. The code $C(H)$ is often called a Hadamard code. It is trivial that $C(H)$ is a self-complementary code with distance distribution $(A_0(C), A_{n/2}(C), A_n(C)) = (1, 2n - 2, 1)$.

The *Lee weight* $\mathrm{wt}_L(x)$ of a vector $x = (x_1, x_2, \dots, x_n)$ of $\mathbb{Z}_4^n$ is $n_1(x) + 2n_2(x) + n_3(x)$, where $n_\alpha(x)$ denotes the number of components i with $x_i = \alpha$ $(\alpha = 0, 1, 2, 3)$. The *Lee distance* $d_L(x, y)$ between two vectors x and y of $\mathbb{Z}_4^n$ is $\mathrm{wt}_L(x - y)$. The *minimum Lee distance* $d_L(\mathcal{C})$ of a $\mathbb{Z}_4$-code $\mathcal{C}$ is the smallest Lee distance among all pairs of two distinct codewords of $\mathcal{C}$. The *Gray map* ϕ is defined as a map from $\mathbb{Z}_4^n$ to $\mathbb{Z}_2^{2n}$ mapping $(x_1, x_2, \dots, x_n)$ to $(\phi(x_1), \phi(x_2), \dots, \phi(x_n))$, where $\phi(0) = (0, 0)$, $\phi(1) = (0, 1)$, $\phi(2) = (1, 1)$ and $\phi(3) = (1, 0)$. If $\mathcal{C}$ is a $\mathbb{Z}_4$-code of length n and minimum Lee distance $d_L(\mathcal{C})$, then the Gray image $\phi(\mathcal{C})$ is a binary $(2n, |\mathcal{C}|, d_L(\mathcal{C}))$ code. The *Lee distance distribution* of a $\mathbb{Z}_4$-code $\mathcal{C}$ of length n is defined as $(A_0(\mathcal{C}), A_1(\mathcal{C}), \dots, A_{2n}(\mathcal{C}))$, where

$$A_i(\mathcal{C}) = \frac{1}{|\mathcal{C}|}|\{(x, x') \mid x, x' \in \mathcal{C}, d_L(x, x') = i\}| \quad (i = 0, 1, \dots, 2n).$$

Two linear $\mathbb{Z}_4$-codes $\mathcal{C}, \mathcal{C}'$ of length n are *equivalent* if there exist $\sigma \in S_n$ and $j_1, j_2, \dots, j_k \in \{1, 2, \dots, n\}$ such that $\mathcal{C} = \tau_{j_1}\tau_{j_2}\cdots\tau_{j_k}\sigma(\mathcal{C}')$. Let $G(1, m)$ denote a generator matrix of the first order binary Reed–Muller code $RM(1, m)$ of length 2^m. The first order *Reed–Muller* $\mathbb{Z}_4$*-code* $ZRM(1, m)$ is defined as the linear $\mathbb{Z}_4$-code of length 2^m, which is generated by the rows of the matrix $\left(\begin{smallmatrix} 11 & \cdots & 11 \\ & 2G(1,m) & \end{smallmatrix}\right)$, where we regard $2G(1, m)$ as a $\mathbb{Z}_4$-matrix [16].

2.3. **Association schemes.** Let X be a finite set and $\{R_0, R_1, \dots, R_n\}$ be a set of nonempty subsets of $X \times X$. Let A_i denote the adjacency matrix of the digraph with vertex set X and arc set R_i for $i = 0, 1, \dots, n$. The pair $(X, \{R_i\}_{i=0}^n)$ is called a *symmetric association scheme* of class n if the following conditions hold:

- $A_0 = I_{|X|}$,
- $\sum_{i=0}^n A_i = J_{|X|}$,
- $A_i^T = A_i$ for $i \in \{1, 2, \dots, n\}$,
- $A_iA_j = \sum_{k=0}^n p_{i,j}^k A_k$, where $p_{i,j}^k$ are nonnegative integers $(i, j \in \{0, 1, \dots, n\})$.

The vector space $\mathcal{A}$ over $\mathbb{R}$ spanned by the matrices A_i forms an algebra. Since $\mathcal{A}$ is commutative and semisimple, $\mathcal{A}$ has a unique basis of primitive idempotents $E_0 = \frac{1}{|X|}J_{|X|}, E_1, \ldots, E_n$. The algebra $\mathcal{A}$ is closed under the ordinary multiplication and entry-wise multiplication denoted by $\circ$. We define the Krein numbers $q_{i,j}^k$ for $i, j, k \in \{0, 1, \ldots, n\}$ as $E_i \circ E_j = \frac{1}{|X|}\sum_{k=0}^n q_{i,j}^k E_k$. It is known that the Krein numbers are nonnegative real numbers (see [13, Lemma 2.4]). Since $\{A_0, A_1, \ldots, A_n\}$ forms a basis of $\mathcal{A}$, there exists a matrix $Q = (q_{ij})$ with $E_i = \frac{1}{|X|}\sum_{j=0}^n q_{ji}A_j$. A symmetric association scheme $(X, \{R_i\}_{i=0}^n)$ is said to be *Q-polynomial* if for each $i \in \{0, 1, \ldots, n\}$, there exists a polynomial $v_i(z)$ of degree i such that $q_{ji} = v_i(q_{j1})$ for all $j \in \{0, 1, \ldots, n\}$. We say that a Q-polynomial association scheme is *Q-bipartite* if $q_{i,j}^k = 0$ for all $i, j, k \in \{1, 2, \ldots, n\}$ such that $i + j + k$ is odd.

There exists a matrix $S = (S_0\ S_1\ \cdots\ S_n)$ whose rows and columns are indexed by X, satisfying that $SS^T = |X|I_{|X|}$ and S diagonalizes the adjacency matrices, where $E_i = \frac{1}{|X|}S_iS_i^T$ for $i \in \{0, 1, \ldots, n\}$ [13, p. 11]. We then define the i-th *characteristic matrix* G_i of a subset C of X as the submatrix of S_i that lies in the rows indexed by C.

Suppose that $X = \mathbb{Z}_2^n$ and $R_i = \{(x, y) \mid x, y \in X, d(x, y) = i\}$ for $i = 0, 1, \ldots, n$. Then the pair $(X, \{R_i\}_{i=0}^n)$ is a symmetric association scheme, which is called the *binary Hamming* association scheme. The binary Hamming association scheme is a Q-bipartite Q-polynomial association scheme with the polynomials $v_i(z) = K_i(n - 2z)$, where $K_i(z)$ is the Krawtchouk polynomial of degree i defined as $K_i(z) = \sum_{j=0}^i (-1)^j \binom{z}{j}\binom{n-z}{i-j}$. By [14, Theorem 2.5], the Krawtchouk polynomials satisfy the following recursion:

$$K_1(z)K_i(z) = (n - i + 1)K_{i-1}(z) + (i + 1)K_{i+1}(z), \tag{1}$$

for $i = 0, 1, \ldots, n - 1$, where $K_{-1}(z)$ is defined as 0.

Recently, by generalizing the result in [1], it has been shown in [23] that there exists a set of f mutually unbiased Hadamard matrices of order n if and only if there exists a Q-polynomial association scheme of class 4 which is both Q-antipodal and Q-bipartite with f Q-antipodal classes (see [23] for undefined terms).

3. Bounds for self-complementary codes

For a code C of length n, set $S(C) = \{i \in \{1, 2, \ldots, n\} \mid A_i(C) \neq 0\}$. The size of $S(C)$ is said to be the *degree* of C. The *annihilator polynomial* of C is defined as follows:

$$\alpha_C(z) = |C| \prod_{i \in S(C)} \left(1 - \frac{z}{i}\right).$$

By considering annihilator polynomials, in this section, we give two upper bounds on the number of codewords of binary self-complementary codes. The two bounds are used to give upper bounds on the size of sets of mutually quasi-unbiased (resp. Type II weakly unbiased) Hadamard matrices in Theorem 5.4 (resp. Theorem 8.6). We also consider the condition of equality of the first bound.

Lemma 3.1. *Let S be a subset of $\{1, 2, \ldots, n\}$ such that $|S| = s$, $n \in S$, and if $a \in S \setminus \{n\}$, then $n - a \in S$. Then $\overline{\alpha}(z) = \prod_{i \in S \setminus \{n\}}(1 - \frac{z}{i})$ has the following* *expansion by the Krawtchouk polynomials:*

$$\overline{\alpha}(z) = \sum_{\substack{i=0,1,\ldots,s-1 \\ i\equiv s-1 \pmod 2}} \alpha_i K_i(z), \tag{2}$$

where $\alpha_i \in \mathbb{Q}$.

Proof. When s is odd, we may write $S = \{a_1, a_2, \ldots, a_{(s-1)/2}, n-a_1, n-a_2, \ldots, n-a_{(s-1)/2}, n\}$, where $0 < a_1 < a_2 < \cdots < a_{(s-1)/2} < n/2$. Then we have

$$\begin{aligned}\overline{\alpha}(z) &= \prod_{i=1}^{(s-1)/2} \left(\left(1-\frac{z}{a_i}\right)\left(1-\frac{z}{n-a_i}\right)\right) \\ &= \prod_{i=1}^{(s-1)/2} \frac{1}{a_i(n-a_i)} \prod_{i=1}^{(s-1)/2} \left(-\left(a_i-\frac{n}{2}\right)^2+\frac{x^2}{4}\right),\end{aligned}$$

where $x = n-2z$. Thus, $\overline{\alpha}(z) = \overline{\alpha}(n/2-x/2)$ is an even polynomial in variable x.

When s is even, we may write $S = \{a_1, a_2, \ldots, a_{s/2-1}, n/2, n-a_1, n-a_2, \ldots, n-a_{s/2-1}, n\}$, where $0 < a_1 < a_2 < \cdots < a_{s/2-1} < n/2$. Similarly to the case where s is odd, we have

$$\overline{\alpha}(z) = \left(\prod_{i=1}^{s/2-1} \frac{1}{a_i(n-a_i)}\right) \frac{1}{\frac{n}{2}} \left(\prod_{i=1}^{s/2-1} \left(-\left(a_i-\frac{n}{2}\right)^2+\frac{x^2}{4}\right)\right) \frac{x}{2},$$

where $x = n-2z$. Thus, $\overline{\alpha}(z) = \overline{\alpha}(n/2-x/2)$ is an odd polynomial in variable x.

It can be shown that $K_i(z) = K_i(n/2-x/2)$ is an even (resp. odd) polynomial of degree i in variable x if i is even (resp. odd), from which the expansion of $\overline{\alpha}(z)$ by the Krawtchouk polynomials has the desired form (2). □

Theorem 3.2. *Let C be a self-complementary code of length n and degree s. Then*

$$|C| \le 2 \sum_{\substack{i=0,1,\ldots,s-1 \\ i\equiv s-1 \pmod 2}} \binom{n}{i}.$$

Proof. We consider a subcode C' of C such that $C = C' \cup (C'+\mathbf{1})$, $C' \cap (C'+\mathbf{1}) = \emptyset$. Then $|C| = 2|C'|$ and C' satisfies that $S(C') \subset S(C) \setminus \{n\}$. Since C is self-complementary, the annihilator polynomial $\alpha_{C'}(z)$ of C' has the following expansion by Lemma 3.1:

$$\alpha_{C'}(z) = \sum_{\substack{i=0,1,\ldots,s-1 \\ i\equiv s-1 \pmod 2}} \alpha_i K_i(z).$$

Set $K = \begin{pmatrix} G_0 & G_2 & \cdots & G_{s-1} \end{pmatrix}$ if s is odd and $K = \begin{pmatrix} G_1 & G_3 & \cdots & G_{s-1} \end{pmatrix}$ if s is even, where G_i is the i-th characteristic matrix of C, and set

$$\Gamma = \bigoplus_{\substack{i=0,1,\ldots,s-1 \\ i\equiv s-1 \pmod 2}} \alpha_i I_{K_i(0)}.$$

By [13, Theorem 3.13], we have $K\Gamma K^T = |C'|I_{|C'|}$. Taking the rank of the above equation yields that

$$|C'| = \operatorname{rank}(K\Gamma K^T) \leq \operatorname{rank}(K) \leq \min\left\{ |C'|, \sum_{\substack{i=0,1,\dots,s-1 \\ i\equiv s-1 \pmod 2}} K_i(0) \right\},$$

as desired. □

Remark 3.3. The above upper bound depends on the degrees. An upper bound, which depends on the minimum distances, can be found in [24].

If $|C| = 2\sum_{\substack{i=0,1,\dots,s-1 \\ i\equiv s-1 \pmod 2}} \binom{n}{i}$, then the matrix K is a square matrix and invertible. Thus, $\bigoplus_{\substack{i=0,1,\dots,s-1 \\ i\equiv s-1 \pmod 2}} \alpha_i I_{K_i(0)}$ is a scalar multiple of the identity matrix, which implies that α_i are all equal.

Let C be a self-complementary code of length n and degree s. By Lemma 3.1, we may suppose that the expansion of $\overline{\alpha}_C(z) = \prod_{i\in S(C)\setminus\{n\}}(1-\frac{z}{i})$ by the Krawtchouk polynomials is as follows:

$$\overline{\alpha}_C(z) = \sum_{\substack{i=0,1,\dots,s-1 \\ i\equiv s-1 \pmod 2}} \alpha_i K_i(z). \tag{3}$$

Theorem 3.4. *Suppose that $\alpha_\delta = \alpha_0$ if s is odd and $\alpha_\delta = \alpha_1$ if s is even. If α_i in* (3) *are all nonnegative and α_δ is positive, then*

$$|C| \leq \left\lfloor \frac{2}{\alpha_\delta} \right\rfloor.$$

Proof. The annihilator polynomial of C is written as $\alpha_C(z) = |C|\left(1-\frac{z}{n}\right)\overline{\alpha}_C(z)$.

By $K_1(z) = n-2z$ and (1),

$$\begin{aligned}
\alpha_C(z) &= \frac{|C|}{2}\left(1+\frac{1}{n}K_1(z)\right)\overline{\alpha}_C(z) \\
&= \frac{|C|}{2}\sum_{\substack{i=0,1,\dots,s-1 \\ i\equiv s-1 \pmod 2}}\left(\alpha_i K_i(z) + \frac{\alpha_i K_1(z)K_i(z)}{n}\right) \\
&= \frac{|C|}{2}\sum_{\substack{i=0,1,\dots,s-1 \\ i\equiv s-1 \pmod 2}}\left(\alpha_i K_i(z) + \frac{\alpha_i((n-i+1)K_{i-1}(z)+(i+1)K_{i+1}(z))}{n}\right),
\end{aligned}$$

where $K_{-1}(z) = 0$. Hence, the coefficient of $K_0(z)$ is $|C|\alpha_\delta/2$. By the assumption on α_i, the linear programming bound [13, Theorem 5.23 (ii)] shows that the coefficient of $K_0(z)$ is at most 1. Therefore, the desired bound follows. □

The above two bounds are referred to as the absolute bounds and the linear programming bounds, respectively. As a consequence, upper bounds on the maximum size among sets of mutually quasi-unbiased (resp. Type II weakly unbiased) Hadamard matrices are given in Section 5 (resp. Section 8).

4. Quasi-unbiased Hadamard matrices

In this section, we study quasi-unbiased Hadamard matrices. All feasible parameter sets for quasi-unbiased Hadamard matrices are examined for orders up to 48.

4.1. Basic properties and feasible parameters.

Proposition 4.1. *If there exists a pair of quasi-unbiased Hadamard matrices of order n with parameters (l, a), then*

$$l = \left(\frac{n}{2\alpha}\right)^2, \quad a = 4\alpha^2 \tag{4}$$

for some positive integer α satisfying that $n \equiv 0 \pmod{2\alpha}$ and $n \le 4\alpha^2$.

Proof. Let (H_1, H_2) be a pair of quasi-unbiased Hadamard matrices of order n with parameters (l, a). From the definition, a must be a square, say, $a = b^2$, where b is a positive integer. Let h_1 (resp. h_2) be a row of H_1 (resp. H_2). Let $n_{\pm}(h_1, h_2)$ denote the number of components which are different in h_1 and h_2. Then $2n_{\pm}(h_1, h_2) = n - b$ and $n + b$ if $h_1 \cdot h_2 = b$ and $-b$, respectively. Since $n = 2$ or $n \equiv 0 \pmod 4$, b is even. Therefore, $a = 4\alpha^2$ for some positive integer α; then $l = (n/2\alpha)^2$. Since $(1/\sqrt{a})H_1 H_2^T$ is a weighing matrix of weight l, it is trivial that $l \le n$. Hence, $n \le 4\alpha^2$. □

From now on we assume that α is a positive integer for parameters $((n/2\alpha)^2, 4\alpha^2)$. We say that parameters (l, a) satisfying (4) are *feasible*. Since $(l, a) = (1, n^2)$ satisfies (4), the parameters $(1, n^2)$ are feasible for each order n.

Proposition 4.2. *If there exists a Hadamard matrix of order n, then there exists a set of $2^n n!$ mutually quasi-unbiased Hadamard matrices with parameters $(1, n^2)$, where $2^n n!$ is the maximum size among sets of such matrices.*

Proof. Let H, K be Hadamard matrices of order n. It is easy to see that (H, K) is a pair of quasi-unbiased Hadamard matrices with parameters $(1, n^2)$ if and only if there exists a monomial $(1, -1, 0)$-matrix P such that $K = PH$. In addition, for any monomial $(1, -1, 0)$-matrices P and Q, (PH, QH) is a pair of quasi-unbiased Hadamard matrices with parameters $(1, n^2)$. □

For $n = 4, 8, \dots, 48$, we give in Table 1 feasible parameters (l, a) and our present state of knowledge about the maximum size f_{max} among sets of mutually quasi-unbiased Hadamard matrices of order n with parameters (l, a) except $(1, n^2)$. In the third column of the table, "-" means that there exists no pair of quasi-unbiased Hadamard matrices. The last two columns provide references for the lower and upper bounds on f_{max}.

Proposition 4.3. *Suppose that there exists a pair of quasi-unbiased Hadamard matrices of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$. If $n \ne 4\alpha^2$, then α must be even.*

Proof. Let H be a Hadamard matrix of order n and let h_i be the i-th row of H. Let x be a vector of $\{1, -1\}^n$. Then it is easy to see that $h_i \cdot x \equiv h_j \cdot x \pmod 4$ for $i, j \in \{1, 2, \dots, n\}$.

Let (H, K) be a pair of quasi-unbiased Hadamard matrices with parameters $((n/2\alpha)^2, 4\alpha^2)$. Since $(1/2\alpha)HK^T$ is a weighing matrix of weight $(n/2\alpha)^2$, any row

TABLE 1. Quasi-unbiased Hadamard matrices ($n = 4, 8, \dots, 48$).

n	(l, a)	f_{max}	Reference	
4	$(4, 4)$	2	[11, Proposition 6]	[15, Table 1]
8	$(4, 16)$	8	[26, Theorem 4.4]	[26, Theorem 4.1]
12	$(4, 36)$	-		Corollary 4.4
	$(9, 16)$	2	Section 4.2	Section 4.2
16	$(4, 64)$	$8 - 35$	[17, Section 3]	Table 2
	$(16, 16)$	8	[11, Proposition 6]	[15, Table 1]
20	$(4, 100)$	-		Corollary 4.5
24	$(4, 144)$	$2 - 85$	Section 4.2	Table 2
	$(9, 64)$	$16 - 85$	Section 5.2	Table 2
	$(16, 36)$	-		Proposition 4.3
28	$(4, 196)$	-		Corollary 4.5
32	$(4, 256)$	$8 - 155$	Proposition 4.6	Table 2
	$(16, 64)$	32	[26, Theorem 4.4]	[26, Theorem 4.1]
36	$(4, 324)$	-		Corollary 4.4
	$(9, 144)$	≤ 199		Table 2
	$(36, 36)$	2	[19, Theorem 1.5]	[8, Lemma 3.3]
40	$(4, 400)$	≤ 247		Table 2
	$(16, 100)$	-		Proposition 4.3
	$(25, 64)$	≤ 28		Table 2
44	$(4, 484)$	-		Corollary 4.5
48	$(4, 576)$	$2 - 361$	Proposition 4.6	Table 2
	$(9, 256)$	$16 - 361$	Proposition 4.6	Table 2
	$(16, 144)$	≤ 361		Table 2
	$(36, 64)$	$2 - 28$	Proposition 4.6	Table 2

x of K satisfies that $h_i \cdot x \in \{0, \pm 2\alpha\}$ for $i = 1, 2, \dots, n$. Hence, if $(1/2\alpha)HK^T$ is not Hadamard, equivalently $n \neq 4\alpha^2$, then α must be even. ☐

Corollary 4.4. *Suppose that $n \equiv 4 \pmod 8$ and $n \geq 12$. Then there exists no pair of quasi-unbiased Hadamard matrices of order n with parameters $(4, (n/2)^2)$.*

Proof. Follows from Proposition 4.3 by considering the case $\alpha = n/4$. ☐

Corollary 4.5. *Suppose that $n = 4p$, where p is an odd prime with $p \geq 5$. Then there exists no pair of quasi-unbiased Hadamard matrices of order n with parameters $(l, a) \neq (1, n^2)$.*

Proof. From $p \geq 5$, the only feasible parameters are $(4, 4p^2)$ and $(1, 16p^2)$. By Proposition 4.3, there exists no pair of quasi-unbiased Hadamard matrices with parameters $(4, 4p^2)$. ☐

Proposition 4.6. *Let $\{H_1, H_2, \dots, H_f\}$ (resp. $\{K_1, K_2, \dots, K_f\}$) be a set of f mutually quasi-unbiased Hadamard matrices of order n (resp. n') with parameters (l, a) (resp. (l', a')). Then $\{H_1 \otimes K_1, H_2 \otimes K_2, \dots, H_f \otimes K_f\}$ is a set of f mutually quasi-unbiased Hadamard matrices of order nn' with parameters (ll', aa').*

Proof. It is sufficient to give a proof for the case $f = 2$. Using some $(1, -1, 0)$-matrices L and L', the matrices $H_1 H_2^T$ and $K_1 K_2^T$ are written as $\sqrt{a}L$ and $\sqrt{a'}L'$, respectively. Then $(H_1 \otimes K_1)(H_2 \otimes K_2)^T = \sqrt{aa'} L \otimes L'$. The result follows. ☐

Let (H, K) be a pair of quasi-unbiased Hadamard matrices of order n with parameters (l, a). We denote the unique Hadamard matrix of order 2 by H_2. There exists a pair (H_4, K_4) of unbiased Hadamard matrices of order 4 [11, Proposition 6]. By the above proposition, $(H \otimes H_2, K \otimes H_2)$ is a pair of quasi-unbiased Hadamard matrices of order $2n$ with parameters $(l, 4a)$, and $(H \otimes H_4, K \otimes K_4)$ is a pair of quasi-unbiased Hadamard matrices of order $4n$ with parameters $(4l, 4a)$.

If there exist Hadamard matrices of orders $4m$ and $4n$, then there exists a Hadamard matrix of order $8mn$ [2, Statement 4.10] (see also [12, Theorem 1] and [20, Theorem 4.2.5]). The explicit construction given in [12, Theorem 1] and [20, Theorem 4.2.5] is as follows. Let H be a Hadamard matrix of order $4m$ and K be a Hadamard matrix of order $4n$. Let H_i $(i = 1, 2)$ be the $4m \times 2m$ matrices and K_i $(i = 1, 2)$ be the $2n \times 4n$ matrices such that $H = \begin{pmatrix} H_1 & H_2 \end{pmatrix}$, $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$. The matrix

$$M(H, K) = \frac{1}{2}(H_1 + H_2) \otimes K_1 + \frac{1}{2}(H_1 - H_2) \otimes K_2$$

is a Hadamard matrix of order $8mn$.

Proposition 4.7. *Let $\{H_1, H_2, \dots, H_f\}$ be a set of f mutually quasi-unbiased Hadamard matrices of order $4m$ with parameters (l, a) and let K be a Hadamard matrix of order $4n$. Then $\{M(H_1, K), M(H_2, K), \dots, M(H_f, K)\}$ is a set of f mutually quasi-unbiased Hadamard matrices of order $8mn$ with parameters $(l, 4an^2)$.*

Proof. Similar to that of the above proposition. The tedious but straightforward proof is omitted. ☐

4.2. **Observations by straightforward construction.** From the definition of quasi-unbiased Hadamard matrices, we immediately have the following observation.

Proposition 4.8. *Let P, Q, R be $n \times n$ $(1, -1, 0)$-monomial matrices. Then (H, K) is a pair of quasi-unbiased Hadamard matrices of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$ if and only if (PHQ, RKQ) is a pair of quasi-unbiased Hadamard matrices of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$.*

Suppose that $n \geq 4$. For a given (n, α), when attempting to determine whether there exists a pair of quasi-unbiased Hadamard matrices H, K of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$, it is sufficient to consider only the inequivalent Hadamard matrices of order n as possible choices for H and only the Hadamard matrices $\overline{K}$ of order n as possible choices for K, where the first three columns c_1, c_2, c_3 of $\overline{K}$ satisfy the following:

$$\begin{aligned} c_1^T &= (\quad +\cdots+ \quad +\cdots+ \quad +\cdots+ \quad +\cdots+ \quad), \\ c_2^T &= (\quad +\cdots+ \quad +\cdots+ \quad -\cdots- \quad -\cdots- \quad), \\ c_3^T &= (\quad \underbrace{+\cdots+}_{\frac{n}{4}\text{ rows}} \quad \underbrace{-\cdots-}_{\frac{n}{4}\text{ rows}} \quad \underbrace{+\cdots+}_{\frac{n}{4}\text{ rows}} \quad \underbrace{-\cdots-}_{\frac{n}{4}\text{ rows}} \quad). \end{aligned} \tag{5}$$

This substantially reduces the number of pairs of Hadamard matrices to be checked as possible pairs (H, K).

Let H_{12} be the Hadamard matrix of order 12 having the following form:

$$\begin{pmatrix} + & + & \cdots & + \\ + & & & \\ \vdots & & R & \\ + & & & \end{pmatrix}, \tag{6}$$

where R is the 11×11 circulant matrix with first row:

$$(-+-+++---+-).$$

We determine the maximum size f among sets of mutually quasi-unbiased Hadamard matrices $H_{12,1}, H_{12,2}, \ldots, H_{12,f}$ of order 12 with parameters $(9,16)$ as follows. By Proposition 4.8, without loss of generality, we may assume that $H_{12,1} = H_{12}$. Our exhaustive computer search under the above condition (5) on K found 1485 distinct Hadamard matrices $\overline{K_{12,i}}$ $(i = 1, 2, \ldots, 1485)$ such that $(H_{12}, \overline{K_{12,i}})$ is a pair of quasi-unbiased Hadamard matrices with the parameters. In addition, our exhaustive computer search verified that there exists no pair $(\overline{K_{12,i}}, \overline{K_{12,j}})$ $(i \neq j)$ such that $\{H_{12}, \overline{K_{12,i}}, \overline{K_{12,j}}\}$ is a set of three mutually quasi-unbiased Hadamard matrices. This means that $f = 2$. In Figure 1, we list $\overline{K_{12}}$, which is one of the 1485 Hadamard matrices.

$$\overline{K_{12}} = \left(\begin{array}{c}
+++--++++-+- \\
+++-+------- \\
++++-+-+-+-+ \\
++--+++--+++ \\
++-++--++++- \\
++-+--+-+--+ \\
+-+++++-++-- \\
+-+-----++++ \\
+-+++-++--++ \\
+--+-+----+- \\
+-----++-+-- \\
+---++-++--+
\end{array}\right)$$

FIGURE 1. The matrix $\overline{K_{12}}$

Our computer search under the condition (5) on K found a Hadamard matrix $\overline{K_{24,1}}$ of order 24 such that $(H_{24,1}, \overline{K_{24,1}})$ is a pair of quasi-unbiased Hadamard matrices of order 24 with parameters $(4,144)$, where $H_{24,1}$ is `had.24.1` in [28]. The matrix $\overline{K_{24,1}}$ is listed in Figure 2.

$$\overline{K_{24,1}} = \left(\begin{array}{c}
+++++-+++-+--+-------+-+ \\
++++-+----+--+++-++-+--- \\
+++---++-+--++-+---++-+- \\
+++-+++++-+-+-+-+++++++- \\
+++----++--+---++++---++ \\
++++++-+-+-++-+-+------- \\
++-+++++-+-+-+-+++++++-+ \\
++-+--+-+--++-++--++-+-- \\
++-++----++-+----+++--++ \\
++--+---++-+-++--+--+++- \\
++---+--+++-+--++---++-+ \\
++---++---++-++-+--+--++ \\
+-+-+++-++++---+-+-+---- \\
+-+++-+-+++++++++-+-+-++ \\
+-+----+-+++--+---++++-+ \\
+-++-++----++----+--++++ \\
+-+-+-------++++++-+-+-+ \\
+-++-+--++---+--+-++-++- \\
+----+++++--+++--++----+ \\
+---++-+--++++-+--+--++- \\
+--+---++-++++--++-++--- \\
+---+-+---------+-+-+--- \\
+--+++-++-----++---++-++ \\
+--+--++-++---++++---++-
\end{array}\right)$$

FIGURE 2. The matrix $\overline{K_{24,1}}$

5. A coding-theoretic approach to quasi-unbiased Hadamard matrices

In this section, we give a coding-theoretic approach to mutually quasi-unbiased Hadamard matrices. As an application, upper bounds on the size of sets of mutually quasi-unbiased Hadamard matrices are derived. For modest lengths, we also give classifications of some binary self-complementary codes in order to construct mutually quasi-unbiased Hadamard matrices.

5.1. Binary codes and quasi-unbiased Hadamard matrices.

Theorem 5.1. *Let α be an integer with $0 < \alpha < n/2$. There exists a self-complementary $(n, 2fn)$ code C satisfying the following conditions:*

$$\{i \in \{0, 1, \dots, n\} \mid A_i(C) \neq 0\} = \{0, n/2 \pm \alpha, n/2, n\}, \tag{7}$$

$$C = C_1 \cup C_2 \cup \cdots \cup C_f, \tag{8}$$

where each C_i has distance distribution $(A_0(C_i), A_{n/2}(C_i), A_n(C_i)) = (1, 2n-2, 1)$ if and only if there exists a set of f mutually quasi-unbiased Hadamard matrices of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$.

Proof. Suppose that there exists an $(n, 2fn)$ code C satisfying (7) and (8). Define ψ as a map from $\mathbb{Z}_2^n$ to $\{1, -1\}^n$ ($\subset \mathbb{Z}^n$) by $\psi((x_1, x_2, \dots, x_n)) = (x'_1, x'_2, \dots, x'_n)$, where $x'_i = -1$ if $x_i = 1$ and $x'_i = 1$ if $x_i = 0$. It follows from the distance distribution of C_i that $C_i + \mathbf{1} = C_i$ for $i = 1, 2, \dots, f$. Thus, $\psi(C_i)$ is antipodal, that is, $-\psi(C_i) = \psi(C_i)$ for $i = 1, 2, \dots, f$. Hence, there exists a subset X_i of $\psi(C_i)$ such that $X_i \cup (-X_i) = \psi(C_i)$ and $X_i \cap (-X_i) = \emptyset$. Note that $\psi(x) \cdot \psi(y) = n - 2d(x, y)$ for $x, y \in \mathbb{Z}_2^n$. The distance distribution of C_i implies that $d(x, y) \in \{0, n/2, n\}$ for $x, y \in C_i$. Thus, $\psi(x) \cdot \psi(y) \in \{-n, 0, n\}$ for $x, y \in C_i$. This means that any two different vectors of X_i are orthogonal for $i = 1, 2, \dots, f$. Hence, one may define a Hadamard matrix H_i of order n whose rows are the vectors of X_i for $i = 1, 2, \dots, f$.

Let v_i be a vector of X_i for $i = 1, 2, \dots, f$. The assumption of (7) implies that $d(\psi^{-1}(v_i), \psi^{-1}(v_j)) = n/2, n/2 \pm \alpha$ $(i \neq j)$; namely, $v_i \cdot v_j$ $(i \neq j)$ is $0, \mp 2\alpha$ respectively, where α is the integer given in (7). This shows that for any distinct $i, j \in \{1, 2, \dots, f\}$, $(1/2\alpha) H_i H_j^T$ is a $(1, -1, 0)$-matrix, and thus it is a weighing matrix of weight $(n/2\alpha)^2$. Therefore, $\{H_1, H_2, \dots, H_f\}$ is a set of f mutually quasi-unbiased Hadamard matrices of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$.

The converse assertion follows by reversing the above argument. □

Remark 5.2. The "only if" part in the above proposition was proved in [26] for a specific case; namely, C is a linear code of length $n = 2^m$ satisfying (7) and containing $RM(1, m)$ as a subcode.

Now, as the case $s = 4$ of Theorems 3.2 and 3.4, we have two upper bounds on the number of the codewords of self-complementary codes satisfying (7).

Lemma 5.3. *Let C be a self-complementary code of length n satisfying* (7). *Then*

- (i) $|C| \leq \frac{n(n^2-3n+8)}{3}$. *If equality holds, then $4\alpha^2 = 3n - 8$.*
- (ii) *If $3n - 4\alpha^2 - 2 > 0$, then $|C| \leq \lfloor \frac{2n(n^2-4\alpha^2)}{3n-4\alpha^2-2} \rfloor$. If $|C| = \frac{2n(n^2-4\alpha^2)}{3n-4\alpha^2-2}$, then a pair $(C, \{R_i\}_{i=0}^4)$ is a Q-polynomial association scheme, where $R_i = \{(x, y) \mid x, y \in C, d(x, y) = \beta_i\}$ and $\{i \in \{0, 1, \dots, n\} \mid A_i(C) \neq 0\} = \{\beta_0, \beta_1, \dots, \beta_4\}$ with $0 = \beta_0 < \beta_1 < \cdots < \beta_4$.*

Proof. (i) The upper bound is the case $s = 4$ of Theorem 3.2.

Suppose that equality holds. From the observation after Theorem 3.2, $\alpha_{C'}(z) = \beta(K_1(z) + K_3(z))$ for some β. Since $n/2 \pm \alpha$ are roots of $K_1(z) + K_3(z)$, we have $4\alpha^2 = 3n - 8$.

(ii) Expanding by the Krawtchouk polynomials, we have

$$\begin{aligned}\overline{\alpha}_C(z) =& \Big(1 - \frac{2z}{2\alpha + n}\Big)\Big(1 - \frac{2z}{n}\Big)\Big(1 - \frac{2z}{-2\alpha + n}\Big)\\ =& \frac{3n - 4\alpha^2 - 2}{n(n^2 - 4\alpha^2)} K_1(z) + \frac{6}{n(n^2 - 4\alpha^2)} K_3(z).\end{aligned}$$

By the assumption on α and n, both $\frac{3n-4\alpha^2-2}{n(n^2-4\alpha^2)}$ and $\frac{6}{n(n^2-4\alpha^2)}$ are positive. Thus, Theorem 3.4 implies the desired bound.

Suppose that $|C| = \frac{2n(n^2-4\alpha^2)}{3n-2-4\alpha^2}$. By following the same line as in the proof of [3, Theorems 1.1, 1.2 (5)], we may prove that $(C, \{R_i\}_{i=0}^4)$ is a Q-polynomial association scheme. A detailed proof is given in Appendix A. □

By Theorem 5.1, we immediately have the following two upper bounds on the maximum size among sets of mutually quasi-unbiased Hadamard matrices, one of which depends only on n, and the other depends on n, α. This is one of the main results of this paper.

Theorem 5.4. *Suppose that there exists a set of f mutually quasi-unbiased Hadamard matrices of order n with parameters $((n/2\alpha)^2, 4\alpha^2)$. Then*

(i) $f \leq \lfloor \frac{n^2-3n+8}{6} \rfloor$. *If* $f = \frac{n^2-3n+8}{6}$, *then* $4\alpha^2 = 3n - 8$.

(ii) *If* $3n - 4\alpha^2 - 2 > 0$, *then* $f \leq \lfloor \frac{n^2-4\alpha^2}{3n-4\alpha^2-2} \rfloor$.

Remark 5.5. It is known that $f \leq n/2$ if $n = 4\alpha^2$ and α is even [15, Table 1], $f \leq 2$ if $n = 4\alpha^2$ and α is odd [8, Lemma 3.3], and $f \leq n$ if $2n = 4\alpha^2$ [26, Theorem 4.1]. For the first and third cases, the bounds are the same as (ii).

TABLE 2. Absolute and linear programming bounds in Theorem 5.4.

n	(l, a)	Absolute bound	Linear programming bound
4	$(4, 4)$	2	2
8	$(4, 16)$	8	8
12	$(9, 16)$	$\lfloor 58/3 \rfloor = 19$	$\lfloor 64/9 \rfloor = 7$
16	$(4, 64)$	35	$*$
	$(16, 16)$	36	8
24	$(4, 144)$	$\lfloor 256/3 \rfloor = 85$	$*$
	$(9, 64)$	85	$\lfloor 256/3 \rfloor = 85$
32	$(4, 256)$	155	$*$
	$(16, 64)$	156	32
36	$(9, 144)$	$\lfloor 598/3 \rfloor = 199$	$*$
	$(36, 36)$	199	18
40	$(4, 400)$	247	$*$
	$(25, 64)$	248	$\lfloor 256/9 \rfloor = 28$
48	$(4, 576)$	$\lfloor 1084/3 \rfloor = 361$	$*$
	$(9, 256)$	361	$*$
	$(16, 144)$	361	$*$
	$(36, 64)$	361	$\lfloor 1120/39 \rfloor = 28$

For the feasible parameters given in Table 1, we list in Table 2 the maximum possible sizes among sets of mutually quasi-unbiased Hadamard matrices, which are obtained by the two upper bounds. We do no list the maximum possible sizes when there exists no pair of quasi-unbiased Hadamard matrices. In the table, "$*$" means that the assumption of Theorem 5.4 (ii) is not satisfied. By Theorem 5.4 (i), if $4\alpha^2 \neq 3n-8$, then $f < \frac{n^2-3n+8}{6}$. Suppose that $n = 4\alpha$. Then $4\alpha^2 = 3n-8$ if and only if $\alpha = 1, 2$. As an example, for the cases $(n, l, a) = (16, 4, 64)$, $(32, 4, 256)$, $(40, 4, 400)$ in Table 2, the upper bound can decrease from that of Theorem 5.4 (i) by 1.

The following proposition was proved in [17] for a specific case; namely, C is a linear code of length $n = 2^m$ satisfying (7) and containing $RM(1, m)$ as a subcode. Although the proof can be easily applied to all codes satisfying (7) and (8), we give a proof for the sake of completeness.

Proposition 5.6. *Let C be an $(n, 2fn)$ code satisfying* (7) *and* (8)*. Then the distance distribution of C is given by*

$$(A_0(C), A_{n/2-\alpha}(C), A_{n/2}(C), A_{n/2+\alpha}(C), A_n(C)) \\ = (1, (f-1)l, 2n-2+(f-1)(2n-2l), (f-1)l, 1),$$

where $l = (n/2\alpha)^2$.

Proof. Let H_i be the Hadamard matrix and let C_i be the code as in the proof of Theorem 5.1 for $i = 1, 2, \ldots, f$. Let x_i be a codeword of C_i for $i = 1, 2, \ldots, f$. The distance distribution of C_i implies that there exist $2n-2$ codewords y of C_i such that $d(x_i, y) = n/2$. Now, suppose that $i, j \in \{1, 2, \ldots, f\}$ with $i \neq j$. Since $(1/2\alpha)H_iH_j^T$ is a weighing matrix of weight l, the number of 0's in each row of $(1/2\alpha)H_iH_j^T$ is $n-l$. That is, for a fixed row r_i of H_i, there exist $n-l$ rows r of H_j such that $r_i \cdot r = 0$. Hence, since C is self-complementary, there exist $2(n-l)$ codewords $y \in C_j$ such that $d(x_i, y) = n/2$. Therefore, we have

$$\begin{aligned} A_{n/2}(C) =& (2fn(2n-2) + f(f-1)2n(2n-2l))/|C| \\ =& (2n-2) + (f-1)(2n-2l). \end{aligned}$$

Since C is self-complementary, we have the desired distance distribution. □

Remark 5.7. The minimum distance of C implies the distance distribution of C.

5.2. **Binary codes satisfying** (7) **and** (8)**.** For some $(n, 2n)$ codes C_1 ($n = 8, 12, 16, 20, 24$), we give a classification of $(n, 2fn)$ codes of the following form:

$$C_1 \cup (u_2 + C_1) \cup (u_3 + C_1) \cup \cdots \cup (u_f + C_1), \tag{9}$$

satisfying (7) and (8). Although our method for the classifications is straightforward, we describe it for the sake of completeness. Let C be an $(n, 2(f-1)n)$ code of the form (9) satisfying (7) and (8). Every $(n, 2fn)$ code $\overline{C}$ of the form (9) satisfying (7) and (8) and that $\overline{C} \supset C$ can be constructed as $C \cup (u_f + C_1)$, where $u_f \in \mathbb{Z}_2^n$. By considering all vectors of $\mathbb{Z}_2^n \setminus C$, all $(n, 2fn)$ codes $\overline{C}$ of the form (9) satisfying (7), (8) and that $\overline{C} \supset C$ can be obtained. In addition, by considering all inequivalent $(n, 2(f-1)n)$ codes C of the form (9) satisfying (7) and (8), all $(n, 2fn)$ codes $\overline{C}$ of the form (9) satisfying (7) and (8), which must be checked further for equivalences, can be obtained. By checking equivalences among these codes, one can complete the classification of codes of the form (9) satisfying (7) and (8) for a fixed C_1.

Let C, D be two binary (n, M) codes containing the zero vector $\mathbf{0}$. Two codes C, D are equivalent if and only if there exist a permutation $\sigma \in S_n$ and a vector $x \in C$ such that $D = \{\sigma(c+x) \mid c \in C\}$. For an (n, M) code C, we have an $M \times n$ $(1,0)$-matrix $m(C)$ with rows composed of the codewords of C. To test equivalence, we checked whether there exists a vector $x \in C$ such that the incidence structures with incidence matrices $m(D), m(\{\sigma(c+x) \mid c \in C\})$ are isomorphic. The Magma function `IsIsomorphic` was used to find out whether the incidence structures are isomorphic.

In this way, for some $(n, 2n)$ codes C_1 $(n = 8, 12, 16, 20, 24)$, by a computer calculation, we completed the classification of codes of the form (9) satisfying (7) and (8). We list the number $N_2(C_1, 2fn)$ of the inequivalent $(n, 2fn)$ codes of the form (9) satisfying (7) and (8). We mention that a classification of linear codes of length 2^m satisfying (7) and containing $RM(1,m)$ as a subcode has been recently done in [17] under the equivalence of linear codes for $m = 3, 4, 5$.

Proposition 5.8. *$N_2(RM(1,3), 16f) = 1$ $(f = 2,3,5,6,7,8)$, $N_2(RM(1,3), 16f) = 2$ $(f = 4)$, and $N_2(RM(1,3), 16f) = 0$ $(f = 9)$.*

Table 3. Complete representatives of $\mathbb{Z}_2^8/RM(1,3)$.

i	supp(x_i)	i	supp(x_i)	i	supp(x_i)	i	supp(x_i)
1	$\emptyset$	5	$\{6\}$	9	$\{4\}$	13	$\{4,6\}$
2	$\{8\}$	6	$\{6,8\}$	10	$\{4,8\}$	14	$\{4,6,8\}$
3	$\{7\}$	7	$\{6,7\}$	11	$\{4,7\}$	15	$\{4,6,7\}$
4	$\{7,8\}$	8	$\{6,7,8\}$	12	$\{4,7,8\}$	16	$\{4,6,7,8\}$

To list the result of the classification, we fix the generator matrix of $RM(1,3)$ as $\begin{pmatrix} 11111111 \\ 01010101 \\ 00110011 \\ 00001111 \end{pmatrix}$, and we list the 16 vectors x_i, which give the set of complete representatives of $\mathbb{Z}_2^8/RM(1,3)$. To save space, we list the supports supp(x_i) in Table 3, where supp$(v) = \{i \mid v_i \neq 0\}$ for a vector $v = (v_1, v_2, \dots, v_n)$. The set was found by the Magma function `Transversal`. The unique $(8, 32)$ code $B_{8,1,1}$, the unique $(8, 48)$ code $B_{8,2,1}$, the two $(8, 64)$ codes $B_{8,3,i}$ $(i = 1, 2)$, the unique $(8, 80)$ code $B_{8,4,1}$, the unique $(8, 96)$ code $B_{8,5,1}$, the unique $(8, 112)$ code $B_{8,6,1}$, and the unique $(8, 128)$ code $B_{8,7,1}$ are constructed via $\bigcup_{k \in X(B_{8,j,i})}(x_k + RM(1,3))$, where $X(B_{8,j,i})$ are listed in Table 4. By a computer calculation, we verified that the minimum distances of the eight codes are 2.

Table 4. Codes of length 8 satisfying (7) and (8).

C	$X(C)$	C	$X(C)$
$B_{8,1,1}$	$\{1,4\}$	$B_{8,4,1}$	$\{1,4,6,7,10\}$
$B_{8,2,1}$	$\{1,4,6\}$	$B_{8,5,1}$	$\{1,4,6,7,10,11\}$
$B_{8,3,1}$	$\{1,4,6,7\}$	$B_{8,6,1}$	$\{1,4,6,7,10,11,13\}$
$B_{8,3,2}$	$\{1,4,6,10\}$	$B_{8,7,1}$	$\{1,4,6,7,10,11,13,16\}$

Proposition 5.9. *$N_2(C(H_{12}), 24f) = 0$ ($f = 2$).*

Proposition 5.10. *$N_2(RM(1,4), 32f) = 2$ ($f = 2,3$), $N_2(RM(1,4), 32f) = 5$ ($f = 4$), $N_2(RM(1,4), 32f) = 3$ ($f = 5,6,7,8$), and $N_2(RM(1,4), 32f) = 0$ ($f = 9$).*

Table 5. Codes of length 16 satisfying (7) and (8).

C	$X(C)$	$d_H(C)$	C	$X(C)$	$d_H(C)$
$B_{16,1,1}$	$\{1,2\}$	4	$B_{16,4,3}$	$\{1,5,6,8,9\}$	6
$B_{16,1,2}$	$\{1,5\}$	6	$B_{16,5,1}$	$\{1,2,3,4,12,13\}$	4
$B_{16,2,1}$	$\{1,2,3\}$	4	$B_{16,5,2}$	$\{1,2,3,4,17,18\}$	4
$B_{16,2,2}$	$\{1,5,6\}$	6	$B_{16,5,3}$	$\{1,5,6,8,9,10\}$	6
$B_{16,3,1}$	$\{1,2,3,4\}$	4	$B_{16,6,1}$	$\{1,2,3,4,12,13,14\}$	4
$B_{16,3,2}$	$\{1,2,3,12\}$	4	$B_{16,6,2}$	$\{1,2,3,4,17,18,19\}$	4
$B_{16,3,3}$	$\{1,2,3,17\}$	4	$B_{16,6,3}$	$\{1,5,6,8,9,10,11\}$	6
$B_{16,3,4}$	$\{1,5,6,7\}$	6	$B_{16,7,1}$	$\{1,2,3,4,12,13,14,15\}$	4
$B_{16,3,5}$	$\{1,5,6,8\}$	6	$B_{16,7,2}$	$\{1,2,3,4,17,18,19,20\}$	4
$B_{16,4,1}$	$\{1,2,3,4,12\}$	4	$B_{16,7,3}$	$\{1,5,6,8,9,10,11,16\}$	6
$B_{16,4,2}$	$\{1,2,3,4,17\}$	4			

To list the result of the classification, we fix the generator matrix of $RM(1,4)$ as follows:

$$\begin{pmatrix} 1111111111111111 \\ 0101010101010101 \\ 0011001100110011 \\ 0000111100001111 \\ 0000000011111111 \end{pmatrix}.$$

The two $(16,32)$ codes $B_{16,1,i}$ ($i = 1,2$), the two $(16,64)$ codes $B_{16,2,i}$ ($i = 1,2$), the five $(16,96)$ codes $B_{16,3,i}$ ($i = 1,2,\dots,5$), the three $(16,128)$ codes $B_{16,4,i}$ ($i = 1,2,3$), the three $(16,160)$ codes $B_{16,5,i}$ ($i = 1,2,3$), the three $(16,192)$ codes $B_{16,6,i}$ ($i = 1,2,3$), and the three $(16,224)$ codes $B_{16,7,i}$ ($i = 1,2,3$), are constructed via $\bigcup_{k \in X(B_{16,j,i})}(x_k + RM(1,4))$, where $X(B_{16,j,i})$ are listed in Table 5, and $\mathrm{supp}(x_m)$ ($m = 1,2,\dots,20$) are listed in Table 6. By a computer calculation, we determined the minimum distances $d_H(B_{16,j,i})$, which are also listed in Table 5.

Table 6. Some representatives of $\mathbb{Z}_2^{16}/RM(1,4)$.

i	$\mathrm{supp}(x_i)$	i	$\mathrm{supp}(x_i)$	i	$\mathrm{supp}(x_i)$
1	$\emptyset$	8	$\{7,8,10,12,14,15\}$	15	$\{6,8,10,12,13,14,15,16\}$
2	$\{6,7,10,11\}$	9	$\{4,7,11,13,14,15\}$	16	$\{6,7,11,12,14,16\}$
3	$\{7,8,11,12\}$	10	$\{4,6,10,11,12,15\}$	17	$\{4,8,12,16\}$
4	$\{6,8,10,12\}$	11	$\{4,8,10,13,15,16\}$	18	$\{4,6,7,8,10,11,12,16\}$
5	$\{4,6,7,8,12,13\}$	12	$\{13,14,15,16\}$	19	$\{4,7,11,16\}$
6	$\{6,8,10,11,13,14\}$	13	$\{6,7,10,11,13,14,15,16\}$	20	$\{4,6,10,16\}$
7	$\{4,7,10,11,12,14\}$	14	$\{7,8,11,12,13,14,15,16\}$		

We denote by $H_{16,1}, H_{16,2}, H_{16,3}, H_{16,4}$ `had.16.1`, `had.16.2`, `had.16.3`, `had.16.4` in [28], respectively, which are the remaining four normalized Hadamard matrices. To save space, we only list the numbers $N_2(C(H_{16,i}), 32f)$ in Table 7 for $i = 1,2,3,4$.

TABLE 7. $N_2(C(H_{16,i}), 32f)$ $(i = 1, 2, 3, 4)$.

f	2	3	4	5	6	7	8	9
$N_2(C(H_{16,1}), 32f)$	4	13	47	24	9	3	2	0
$N_2(C(H_{16,2}), 32f)$	7	18	62	34	14	3	2	0
$N_2(C(H_{16,3}), 32f)$	2	3	10	3	3	1	1	0
$N_2(C(H_{16,4}), 32f)$	2	9	22	16	4	1	1	0

Let $H_{24,2}$ be the Paley Hadamard matrix of order 24 having the form (6), where R is the 23×23 circulant matrix with first row:

$$(- - - - - + - + - - + + - - + + - + - + + + +).$$

Our computer search found a $(24, 768, 8)$ code $C_{24} = \bigcup_{i=1}^{16}(u_i + C(H_{24,2}))$ satisfying (7) and (8). The vector u_1 is $\mathbf{0}$ and $\mathrm{supp}(u_i)$ $(i = 2, 3, \ldots, 16)$ are listed in Table 8. This gives a set of 16 mutually quasi-unbiased Hadamard matrices of order 24 with parameters $(9, 64)$ by Theorem 5.1.

TABLE 8. Vectors u_i for C_{24}.

i	$\mathrm{supp}(u_i)$	i	$\mathrm{supp}(u_i)$
2	$\{3,4,5,6,7,10,13,15\}$	10	$\{3,6,7,9,11,14,15,18\}$
3	$\{4,5,6,7,8,11,14,16\}$	11	$\{6,7,8,9,10,13,16,18\}$
4	$\{3,8,10,11,13,14,15,16\}$	12	$\{3,4,5,8,9,15,16,18\}$
5	$\{3,4,8,9,10,12,13,17\}$	13	$\{3,5,8,11,12,14,17,18\}$
6	$\{5,6,7,8,9,12,15,17\}$	14	$\{4,6,7,8,10,11,12,13,14,15,17,18\}$
7	$\{3,5,6,7,9,10,11,12,13,14,16,17\}$	15	$\{3,4,6,7,12,16,17,18\}$
8	$\{4,9,11,12,14,15,16,17\}$	16	$\{5,10,12,13,15,16,17,18\}$
9	$\{4,5,9,10,11,13,14,18\}$		

5.3. **Binary codes satisfying** (7) **and** (8) **from** $\mathbb{Z}_4$**-codes.** In order to construct binary codes satisfying (7) and (8) systematically, we consider $\mathbb{Z}_4$-codes $\mathcal{C}$ of length n with $|\mathcal{C}| = 4fn$ satisfying the following conditions:

$$\{i \in \{0, 1, \ldots, n\} \mid A_i(\mathcal{C}) \neq 0\} = \{0, n \pm \beta, n, 2n\}, \tag{10}$$

$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_f, \tag{11}$$

where β is an integer with $0 < \beta < n$, and each $\mathcal{C}_i$ has Lee distance distribution $(A_0(\mathcal{C}_i), A_n(\mathcal{C}_i), A_{2n}(\mathcal{C}_i)) = (1, 4n - 2, 1)$.

Proposition 5.11. *Let $\mathcal{C}$ be a $\mathbb{Z}_4$-code of length n satisfying* (10) *and* (11). *Then there exists a set of f mutually quasi-unbiased Hadamard matrices of order $2n$ with parameters $(n^2/\beta^2, 4\beta^2)$.*

Proof. Since the Lee distance distribution of $\mathcal{C}$ is the same as the distance distribution of $\phi(\mathcal{C})$, $\phi(\mathcal{C})$ satisfies (7). In addition, $\phi(\mathcal{C}_i)$ has the same distance distribution as $RM(1, m + 1)$ for $i = 1, 2, \ldots, f$. Since $\phi(\mathcal{C}) = \phi(\mathcal{C}_1) \cup \phi(\mathcal{C}_2) \cup \cdots \cup \phi(\mathcal{C}_f)$, $\phi(\mathcal{C})$ satisfies (8). The result follows from Theorem 5.1. $\square$

Now, we restrict our attention to linear $\mathbb{Z}_4$-codes $\mathcal{C}$ of length $n = 2^m$ satisfying the following conditions:

$$\{(n_0(x) - n_2(x))^2 \mid x \in \mathcal{C}\} = \{0, \beta^2, n^2\}, \tag{12}$$

$$\mathcal{C} \text{ contains } ZRM(1, m) \text{ as a subcode}, \tag{13}$$

where β is an integer with $0 < \beta < n$. Let x be a codeword of $\mathcal{C}$. Since $n_1(x) + 2n_2(x) + n_3(x) = n - (n_0(x) - n_2(x))$, $\{\mathrm{wt}_L(x) \mid x \in \mathcal{C}\} = \{0, n \pm \beta, n, 2n\}$. This means that $\mathcal{C}$ satisfies (10). Let $\{t_1, t_2, \ldots, t_f\}$ be a set of complete representatives of $\mathcal{C}/ZRM(1,m)$. It is trivial that $t_i + ZRM(1,m)$ has the same Lee distance distribution as $ZRM(1,m)$ for $i = 1, 2, \ldots, f$. Hence, $\mathcal{C}$ satisfies (11). We note that the Kerdock $\mathbb{Z}_4$-code $\mathcal{K}(m)$ of length 2^m defined in [16] satisfies (12) and (13) for $m \geq 2$.

Remark 5.12. The above method is a slight generalization of that given in [26].

In the rest of this section, we study classifications of linear $\mathbb{Z}_4$-codes of length 2^m satisfying (12) and (13). Note that the conditions (12) and (13) are invariant under equivalences of linear $\mathbb{Z}_4$-codes.

Although our method for classifications of linear $\mathbb{Z}_4$-codes of length 2^m satisfying (12) and (13) is straightforward, we describe it for the sake of completeness. Let $\mathcal{C}$ be a linear $\mathbb{Z}_4$-code with $|\mathcal{C}| = 2^k$ satisfying (12) and (13). Every linear $\mathbb{Z}_4$-code $\overline{\mathcal{C}}$ such that $|\overline{\mathcal{C}}| = 2^{k+1}$ and $\overline{\mathcal{C}} \supset \mathcal{C}$ satisfying (12) and (13) can be constructed as $\langle \mathcal{C}, x\rangle$, where x is some vector of a set R_m of complete representatives of $\mathbb{Z}_4^{2^m}/\mathcal{C}$. By considering all vectors of R_m, all linear $\mathbb{Z}_4$-codes $\overline{\mathcal{C}}$ which must be checked further for equivalence can be obtained. In addition, by considering all inequivalent linear $\mathbb{Z}_4$-codes $\mathcal{C}$ with $|\mathcal{C}| = 2^k$ satisfying (12) and (13), all linear $\mathbb{Z}_4$-codes $\overline{\mathcal{C}}$ with $|\overline{\mathcal{C}}| = 2^{k+1}$ satisfying (12) and (13), which must be checked further for equivalences, can be obtained. By checking equivalences among these codes, one can complete the classification of linear $\mathbb{Z}_4$-codes $\overline{\mathcal{C}}$ with $|\overline{\mathcal{C}}| = 2^{k+1}$ satisfying (12) and (13).

We now describe how to test equivalences of linear $\mathbb{Z}_4$-codes. In this paper, we modify the method for linear codes over a finite field, which is given in [27]. For a linear $\mathbb{Z}_4$-code $\mathcal{C}$ of length n, we define the digraph $\Gamma(\mathcal{C})$ with the following vertex set $V(\Gamma(\mathcal{C}))$ and arc set $A(\Gamma(\mathcal{C}))$:

$$\begin{aligned} V(\Gamma(\mathcal{C})) =& \mathcal{C}^{\#} \cup (\mathcal{P} \times \mathbb{Z}_4^{\#}),\\ A(\Gamma(\mathcal{C})) =& \{(c, (j, c_j)) \mid c = (c_1, c_2, \ldots, c_n) \in \mathcal{C}^{\#}, c_j \neq 0, j \in \mathcal{P}\}\\ & \cup \{((j,x),(j,2)), ((j,2),(j,x)) \mid j \in \mathcal{P}, x \in \{1,3\}\}, \end{aligned}$$

where $\mathcal{C}^{\#} = \mathcal{C} \setminus \{\mathbf{0}\}$, $\mathcal{P} = \{1, 2, \ldots, n\}$ and $\mathbb{Z}_4^{\#} = \mathbb{Z}_4 \setminus \{0\}$. By an argument similar to that in [27], the following characterization is obtained.

Proposition 5.13. *Two linear $\mathbb{Z}_4$-codes $\mathcal{C}, \mathcal{C}'$ are equivalent if and only if $\Gamma(\mathcal{C})$, $\Gamma(\mathcal{C}')$ are isomorphic.*

Proof. Suppose that two linear $\mathbb{Z}_4$-codes $\mathcal{C}, \mathcal{C}'$ of length n are equivalent. Then there exist $\sigma \in S_n$ and $j_1, j_2, \ldots, j_\ell \in \mathcal{P}$ such that $\tau_{j_1}\tau_{j_2}\cdots\tau_{j_\ell}\sigma(\mathcal{C}) = \mathcal{C}'$ (see Section 2.2 for the notation). For $\sigma \in S_n$, define a map f_σ from $V(\Gamma(\mathcal{C}))$ to $V(\Gamma(\sigma(\mathcal{C})))$ mapping $(j,x) \in \mathcal{P} \times \mathbb{Z}_4^{\#}$ to $(\sigma(j), x)$, and $c \in \mathcal{C}^{\#}$ to $\sigma(c)$. Then the map f_σ is an isomorphism from $\Gamma(\mathcal{C})$ to $\Gamma(\sigma(C))$. Now, for $j \in \mathcal{P}$, define a map g_j from $V(\Gamma(\mathcal{C}))$ to $V(\Gamma(\tau_j(\mathcal{C})))$ mapping (j,x) to $(j,-x)$, $(i,x) \in (\mathcal{P} \setminus \{j\}) \times \mathbb{Z}_4^{\#}$ to (i,x), and $c \in C^{\#}$ to $\tau_j(c)$. Then the map g_j is an isomorphism from $\Gamma(\mathcal{C})$ to $\Gamma(\tau_j(\mathcal{C}))$. Hence, $g_{j_1}g_{j_2}\cdots g_{j_\ell}f_\sigma$ is an isomorphism from $\Gamma(\mathcal{C})$ to $\Gamma(\mathcal{C}')$.

Conversely, we suppose that two digraphs $\Gamma(\mathcal{C}), \Gamma(\mathcal{C}')$ are isomorphic. Then there exists a bijection f from $V(\Gamma(\mathcal{C}))$ to $V(\Gamma(\mathcal{C}'))$ such that $(x,y) \in A(\Gamma(\mathcal{C}))$ if and only if $(f(x), f(y)) \in A(\Gamma(\mathcal{C}'))$. By the definition of $A(\Gamma(\mathcal{C}))$, the subsets $\mathcal{C}^{\#}$ and $\mathcal{P} \times \mathbb{Z}_4^{\#}$

of $V(\Gamma(\mathcal{C}))$ are characterized as follows:

$$\mathcal{C}^{\#} = \{v \in V(\Gamma(\mathcal{C})) \mid \text{the indegree of } v \text{ is equal to } 0\},$$
$$\mathcal{P} \times \mathbb{Z}_4^{\#} = V(\Gamma(\mathcal{C})) \setminus \mathcal{C}^{\#}.$$

We have a similar characterization for $\Gamma(\mathcal{C}')$. Thus, we have

$$f(\mathcal{C}^{\#}) = \mathcal{C}'^{\#}, \quad f(\mathcal{P} \times \mathbb{Z}_4^{\#}) = \mathcal{P} \times \mathbb{Z}_4^{\#}.$$

We put $f((j,x)) = (j',x')$ for $(j,x) \in \mathcal{P} \times \mathbb{Z}_4^{\#}$. There exists a permutation $\sigma_f \in S_n$ with $\sigma_f(j) = j'$ for $j \in \mathcal{P}$. The set of the vertices of $\Gamma(\mathcal{C})$ (resp. $\Gamma(\mathcal{C}')$) whose indegrees are at least 2 and outdegrees are equal to 2 is $\{(j,2) \mid j \in \mathcal{P}\}$ (resp. $\{(j',2) \mid j' \in \mathcal{P}\}$). Hence, $(j',2') = (j',2)$ for each $j' \in \mathcal{P}$. Also, we have either that $(j',1') = (j',1)$ and $(j',3') = (j',3)$ or that $(j',1') = (j',3)$ and $(j',3') = (j',1)$ for each $j' \in \mathcal{P}$. Hence, we have

$$\tau_{j_1}\tau_{j_2}\cdots\tau_{j_\ell}\sigma_f(\mathcal{C}) = f(\mathcal{C}^{\#}) \cup \{\mathbf{0}\} = \mathcal{C}',$$

where $\{j_1, j_2, \dots, j_\ell\} = \{j \in \mathcal{P} \mid (j',1') = (j',3), (j',3') = (j',1)\}$ with $|\{j_1, j_2, \dots, j_\ell\}| = \ell$. Therefore, two linear $\mathbb{Z}_4$-codes $\mathcal{C}, \mathcal{C}'$ are equivalent. □

Using the above method, by a computer calculation, we completed the classification of linear $\mathbb{Z}_4$-codes of length 16 satisfying (12) and (13). By the Magma function `IsIsomorphic`, we determined whether $\Gamma(\mathcal{C}), \Gamma(\mathcal{C}')$ are isomorphic.

Proposition 5.14. *Let $N_4(16,k)$ denote the number of inequivalent linear $\mathbb{Z}_4$-codes $\mathcal{C}$ of length* 16 *with $|\mathcal{C}| = 2^k$ satisfying* (12) *and* (13)*. Then $N_4(16,7) = 5$, $N_4(16,8) = 21$, $N_4(16,9) = 62$, $N_4(16,10) = 28$, $N_4(16,11) = 2$ and $N_4(16,12) = 0$.*

To list the result of the classification, we fix the generator matrix of $ZRM(1,4)$ as follows:

$$\begin{pmatrix} 1111111111111111 \\ 0202020202020202 \\ 0022002200220022 \\ 0000222200002222 \\ 0000000022222222 \end{pmatrix}.$$

To save space, we only list the maximal linear $\mathbb{Z}_4$-codes (with respect to the subset relation) given in the above proposition. The seven maximal linear $\mathbb{Z}_4$-codes $\mathcal{C} = \mathcal{C}_{16,3,i}$ $(i = 1, 2, \dots, 7)$ with $|\mathcal{C}| = 2^9$ are constructed as $\langle ZRM(1,4), x_1, x_2, x_3 \rangle$, where x_1, x_2, x_3 are listed in Table 9. The 19 maximal linear $\mathbb{Z}_4$-codes $\mathcal{C} = \mathcal{C}_{16,4,i}$ $(i = 1, 2, \dots, 19)$ with $|\mathcal{C}| = 2^{10}$ are constructed as $\langle ZRM(1,4), x_1, x_2, x_3, x_4 \rangle$, where x_1, x_2, x_3, x_4 are listed in Table 10. The two maximal linear $\mathbb{Z}_4$-codes $\mathcal{C} = \mathcal{C}_{16,5,i}$ $(i = 1, 2)$ with $|\mathcal{C}| = 2^{11}$ are constructed as $\langle ZRM(1,4), x_1, x_2, \dots, x_5 \rangle$, where $x_1, x_2, \dots, x_5$ are listed in Table 11. For each code $\mathcal{C}$, by a computer calculation, we determined the value β^2 in (12), the minimum Hamming distance $d_H(\mathcal{C})$ and the minimum Lee distance $d_L(\mathcal{C})$, which are listed in Table 12.

Remark 5.15. By a computer calculation, we verified that $\Gamma(\mathcal{C}_{16,4,16})$ and $\Gamma(\mathcal{K}(4))$ are isomorphic.

Table 9. Vectors x_j for $\mathcal{C}_{16,3,i}$ $(i = 1, 2, \dots, 7)$.

Code	x_j $(j = 1, 2, 3)$
$\mathcal{C}_{16,3,1}$	$(1,0,0,3,0,1,3,0,0,1,3,0,1,0,0,3), (0,1,1,2,0,1,1,2,0,1,1,2,0,1,1,2),$ $(0,0,0,0,1,1,3,3,1,1,3,3,0,0,0,0)$
$\mathcal{C}_{16,3,2}$	$(1,0,0,1,0,1,1,2,0,1,3,0,1,0,2,3), (0,1,0,1,1,2,3,0,1,0,1,0,0,3,2,1),$ $(0,0,1,1,1,3,0,2,1,3,2,0,2,2,1,1)$
$\mathcal{C}_{16,3,3}$	$(1,0,0,1,1,2,2,3,0,1,1,0,0,1,3,0), (0,1,0,1,0,1,2,3,1,0,3,0,3,0,3,2),$ $(0,0,1,1,1,3,0,2,1,3,2,0,0,0,3,3)$
$\mathcal{C}_{16,3,4}$	$(1,0,0,1,0,3,1,2,0,1,3,2,1,2,2,1), (0,1,0,1,1,0,3,0,0,1,0,3,1,2,1,2),$ $(0,0,1,1,1,1,0,2,0,2,1,1,3,3,0,0)$
$\mathcal{C}_{16,3,5}$	$(1,0,0,1,0,3,3,0,0,1,1,0,1,2,2,1), (0,1,1,0,0,1,1,2,1,2,2,1,3,2,0,3),$ $(0,0,2,0,0,0,0,2,1,1,1,1,1,3,3,1)$
$\mathcal{C}_{16,3,6}$	$(1,0,0,1,0,3,1,2,0,1,3,2,1,2,2,1), (0,1,0,1,0,1,2,3,1,2,3,0,3,0,3,0),$ $(0,0,1,1,0,2,3,1,1,3,0,2,1,1,2,2)$
$\mathcal{C}_{16,3,7}$	$(0,2,0,2,0,0,2,2,0,0,2,2,0,2,0,2), (0,0,0,0,2,0,0,2,0,0,0,0,0,2,2,0),$ $(0,0,0,0,0,2,2,0,0,0,0,0,0,2,2,0)$

Table 10. Vectors x_j for $\mathcal{C}_{16,4,i}$ $(i = 1, 2, \dots, 19)$.

Code	x_j $(j = 1, 2, 3, 4)$
$\mathcal{C}_{16,4,1}$	$(1,0,0,1,0,3,1,0,1,0,2,3,0,1,1,2), (0,1,0,1,0,1,2,1,0,3,2,1,2,3,2,1),$ $(0,0,1,1,0,2,3,3,1,3,0,0,3,1,0,2), (0,0,0,2,0,2,2,2,0,0,0,2,2,0,0,0)$
$\mathcal{C}_{16,4,2}$	$(1,0,0,1,0,3,1,2,1,2,2,3,2,1,1,0), (0,1,0,1,0,3,0,3,0,1,2,3,0,3,2,1),$ $(0,0,1,1,0,2,1,1,1,1,2,2,3,3,0,2), (0,0,0,2,0,0,0,2,1,1,1,1,1,3,1,3)$
$\mathcal{C}_{16,4,3}$	$(1,0,0,1,0,1,3,2,0,1,1,0,1,0,2,3), (0,1,0,1,1,0,1,0,1,0,1,0,2,3,2,3),$ $(0,0,1,1,1,1,2,2,0,0,3,3,1,1,0,0), (0,0,0,0,2,0,0,2,1,1,1,1,1,3,3,1)$
$\mathcal{C}_{16,4,4}$	$(1,0,0,1,0,1,1,0,1,0,2,1,0,3,3,2), (0,1,1,0,0,3,1,0,0,3,1,0,2,1,1,2),$ $(0,0,2,0,0,0,0,2,0,0,0,2,2,2,0,2), (0,0,0,2,0,0,0,2,1,1,3,3,3,3,3,3)$
$\mathcal{C}_{16,4,5}$	$(1,0,0,1,0,1,3,2,1,2,2,1,2,1,3,0), (0,1,0,1,1,0,1,0,0,3,0,3,1,2,1,2),$ $(0,0,1,1,1,1,0,0,0,2,3,1,3,1,0,2), (0,0,0,0,2,0,2,0,1,3,1,3,3,3,3,3)$
$\mathcal{C}_{16,4,6}$	$(1,0,0,1,0,1,3,2,0,1,1,0,1,0,2,3), (0,1,0,1,0,3,2,1,0,1,0,1,2,1,0,3),$ $(0,0,1,1,0,2,1,3,0,2,3,1,0,0,3,3), (0,0,0,2,0,2,0,0,0,2,0,0,2,2,2,0)$
$\mathcal{C}_{16,4,7}$	$(1,0,0,1,0,1,1,0,0,3,1,2,3,0,2,1), (0,1,0,1,1,0,1,0,1,2,3,0,0,3,2,1),$ $(0,0,1,1,1,3,0,2,0,2,3,1,3,3,0,0), (0,0,0,0,2,2,0,0,1,3,3,1,3,1,3,1)$
$\mathcal{C}_{16,4,8}$	$(1,0,0,1,0,1,3,2,0,3,3,0,1,2,0,3), (0,1,0,1,1,0,1,0,1,2,3,0,2,1,0,3),$ $(0,0,1,1,1,1,2,2,0,2,3,1,3,1,2,0), (0,0,0,0,2,0,0,2,1,3,1,3,3,3,1,1)$
$\mathcal{C}_{16,4,9}$	$(1,0,0,1,0,1,1,0,0,1,3,2,1,0,2,3), (0,1,0,1,1,0,1,0,0,3,0,3,1,2,1,2),$ $(0,0,1,1,1,3,0,2,1,1,0,0,0,2,1,3), (0,0,0,0,2,2,0,0,1,1,3,3,3,3,3,3)$
$\mathcal{C}_{16,4,10}$	$(1,0,0,1,0,1,3,2,1,0,2,1,2,1,3,2), (0,1,0,1,1,0,3,0,1,2,1,0,0,3,2,1),$ $(0,0,1,1,1,1,2,2,0,2,3,3,1,1,0,2), (0,0,0,0,2,0,0,2,1,1,3,1,1,1,1,3)$
$\mathcal{C}_{16,4,11}$	$(1,0,0,1,0,1,3,0,0,1,1,2,1,2,0,3), (0,1,1,0,0,1,3,2,0,3,1,2,2,3,3,2),$ $(0,0,2,0,0,0,2,0,1,1,1,1,3,1,1,3), (0,0,0,0,1,1,3,1,0,2,2,0,3,3,1,3)$
$\mathcal{C}_{16,4,12}$	$(1,1,0,0,0,0,1,3,1,3,0,2,0,2,3,3), (0,2,0,0,0,0,2,0,1,3,3,1,3,3,1,1),$ $(0,0,1,1,0,0,1,1,1,1,0,2,3,3,0,2), (0,0,0,2,0,0,0,2,0,0,2,0,2,2,0,2)$
$\mathcal{C}_{16,4,13}$	$(1,0,1,0,0,1,0,1,1,0,1,2,2,3,0,3), (0,1,0,1,0,1,0,3,0,3,2,3,0,3,2,1),$ $(0,0,2,0,0,0,2,0,0,0,0,2,2,2,2,0), (0,0,0,0,1,1,3,1,1,3,3,3,0,2,0,2)$
$\mathcal{C}_{16,4,14}$	$(1,0,0,1,0,3,3,0,0,1,1,0,1,2,2,1), (0,1,0,1,0,3,2,1,1,2,3,2,1,2,1,0),$ $(0,0,1,1,0,2,1,1,1,1,2,2,3,3,0,2), (0,0,0,2,0,0,0,2,1,1,1,1,1,3,1,3)$
$\mathcal{C}_{16,4,15}$	$(2,0,0,0,0,0,0,2,2,0,0,0,2,2,2,0), (3,0,0,0,0,2,0,1,3,2,0,0,3,1,1,2),$ $(0,1,0,1,0,3,0,3,0,3,2,3,2,3,0,3), (0,0,1,0,0,3,2,2,0,2,3,0,1,0,1,3)$
$\mathcal{C}_{16,4,16}$	$(1,0,0,1,0,1,3,2,1,2,2,1,2,1,3,0), (0,1,0,1,0,1,2,3,0,1,2,3,0,1,0,1),$ $(0,0,1,1,0,0,1,1,1,3,2,0,3,1,0,2), (0,0,0,2,0,0,0,2,1,3,3,3,1,3,3,3)$
$\mathcal{C}_{16,4,17}$	$(1,1,0,0,0,2,1,3,1,3,0,2,0,0,3,3), (0,2,0,0,0,0,0,2,0,0,0,2,2,0,2,2),$ $(0,0,1,1,0,2,1,1,1,1,0,2,3,3,2,2), (0,0,0,0,1,1,1,3,1,1,3,1,0,2,0,2)$
$\mathcal{C}_{16,4,18}$	$(1,0,0,1,1,0,0,1,1,0,0,1,1,0,0,1), (0,0,2,0,0,0,2,0,0,0,2,0,0,0,2,0),$ $(0,0,0,0,2,0,0,2,0,0,0,0,2,0,0,2), (0,0,0,0,0,0,0,0,2,0,0,2,2,0,0,2)$
$\mathcal{C}_{16,4,19}$	$(1,0,0,1,0,1,3,2,0,3,1,2,3,2,2,3), (0,2,0,0,0,2,2,2,0,2,2,2,2,0,2,2),$ $(0,3,0,0,0,3,3,3,0,3,1,3,1,2,3,1), (0,0,1,0,0,2,0,3,0,2,2,3,3,3,0,3)$

TABLE 11. Vectors x_j for $\mathcal{C}_{16,5,i}$ $(i = 1, 2)$.

Code	x_j $(j = 1, 2, \ldots, 5)$
$\mathcal{C}_{16,5,1}$	$(1,0,0,1,0,1,3,0,1,0,2,3,0,3,3,2), (0,1,0,1,0,1,2,1,1,0,3,0,3,2,1,0),$ $(0,0,1,1,0,0,1,3,0,2,3,1,2,0,3,3), (0,0,0,2,0,0,0,2,1,1,1,1,3,1,1,3),$ $(0,0,0,0,1,1,1,3,1,1,3,1,0,2,0,2)$
$\mathcal{C}_{16,5,2}$	$(1,0,0,1,0,1,1,2,1,0,0,1,0,3,3,2), (0,1,0,1,0,1,2,1,1,2,3,2,3,0,1,2),$ $(0,0,1,1,0,0,3,1,0,2,3,1,0,2,3,3), (0,0,0,2,0,0,2,0,1,1,1,1,1,3,1,3),$ $(0,0,0,0,1,1,1,3,1,3,1,1,2,2,0,0)$

TABLE 12. Maximal linear $\mathbb{Z}_4$-codes of length 16 satisfying (12) and (13).

$\mathcal{C}$	β^2	$(d_H(\mathcal{C}), d_L(\mathcal{C}))$	$\mathcal{C}$	β^2	$(d_H(\mathcal{C}), d_L(\mathcal{C}))$
$\mathcal{C}_{16,3,1}$	64	$(4, 8)$	$\mathcal{C}_{16,3,2}$	16	$(8, 12)$
$\mathcal{C}_{16,3,3}$	16	$(8, 12)$	$\mathcal{C}_{16,3,4}$	16	$(8, 12)$
$\mathcal{C}_{16,3,5}$	16	$(8, 12)$	$\mathcal{C}_{16,3,6}$	16	$(8, 12)$
$\mathcal{C}_{16,3,7}$	64	$(4, 8)$			
$\mathcal{C}_{16,4,1}$	16	$(6, 12)$	$\mathcal{C}_{16,4,2}$	16	$(8, 12)$
$\mathcal{C}_{16,4,3}$	16	$(8, 12)$	$\mathcal{C}_{16,4,4}$	16	$(6, 12)$
$\mathcal{C}_{16,4,5}$	16	$(8, 12)$	$\mathcal{C}_{16,4,6}$	16	$(6, 12)$
$\mathcal{C}_{16,4,7}$	16	$(8, 12)$	$\mathcal{C}_{16,4,8}$	16	$(8, 12)$
$\mathcal{C}_{16,4,9}$	16	$(8, 12)$	$\mathcal{C}_{16,4,10}$	16	$(8, 12)$
$\mathcal{C}_{16,4,11}$	16	$(6, 12)$	$\mathcal{C}_{16,4,12}$	16	$(6, 12)$
$\mathcal{C}_{16,4,13}$	16	$(6, 12)$	$\mathcal{C}_{16,4,14}$	16	$(8, 12)$
$\mathcal{C}_{16,4,15}$	16	$(6, 12)$	$\mathcal{C}_{16,4,16}$	16	$(8, 12)$
$\mathcal{C}_{16,4,17}$	16	$(6, 12)$	$\mathcal{C}_{16,4,18}$	64	$(4, 8)$
$\mathcal{C}_{16,4,19}$	16	$(6, 12)$			
$\mathcal{C}_{16,5,1}$	16	$(6, 12)$	$\mathcal{C}_{16,5,2}$	16	$(6, 12)$

6. Weakly unbiased Hadamard matrices

In analogy to the case of quasi-unbiased Hadamard matrices, this section studies weakly unbiased Hadamard matrices. All feasible parameter sets for weakly unbiased Hadamard matrices are examined for orders up to 48. It is also shown that the size of a set of mutually weakly unbiased Hadamard matrices of order n is at most 2.

6.1. **Basic properties and feasible parameters.** Let H, K be Hadamard matrices of order n. Let a_{ij} denote the (i, j)-entry of HK^T. Recall that H, K are weakly unbiased if $a_{ij} \equiv 2 \pmod 4$ for $i, j \in \{1, 2, \ldots, n\}$ and $|\{|a_{ij}| \mid i, j \in \{1, 2, \ldots, n\}\}| \le 2$. In this paper, we exclude unbiased Hadamard matrices from weakly unbiased Hadamard matrices. This implies that $|\{|a_{ij}| \mid i, j \in \{1, 2, \ldots, n\}\}| = 2$. It follows immediately from the definition that $n \ge 8$.

Let (H, K) be a pair of weakly unbiased Hadamard matrices of order n. Suppose that a, b are positive integers satisfying $\{|a_{ij}| \mid i, j \in \{1, 2, \ldots, n\}\} = \{a, b\}$. We denote the set $\{a, b\}$ by $\sigma(H, K)$. Let $n(a)$ be the number of components j with $a_{ij} = \pm a$ for $i = 1, 2, \ldots, n$. From now on, we assume that $a < b$. The value $n(a)$ does not depend on i. Indeed, it follows from $(HK^T)(HK^T)^T = n^2 I_n$ that

$$a^2 n(a) + b^2(n - n(a)) = n^2. \tag{14}$$

We say that parameters (a, b) satisfying (14) are *feasible.* Since $(a, b, n(a)) = (2, n-2, n-1)$ satisfies (14), the parameters $(a, b) = (2, n-2)$ are feasible for each order n. The following theorem gives an upper bound on the size of a set of mutually weakly unbiased Hadamard matrices, which is one of the main results of this paper.

Theorem 6.1. *The size of a set of mutually weakly unbiased Hadamard matrices of order n is at most 2.*

Proof. Note that $n \geq 8$ by the definition. Suppose that $\{H_1, H_2, H_3\}$ is a set of three mutually weakly unbiased Hadamard matrices of order n. Let h_i denote the first row of H_i ($i = 1, 2, 3$). By an argument similar to that in Proposition 4.8, (H, K) is a pair of weakly unbiased Hadamard matrices if and only if (HP, KP) is a pair of weakly unbiased Hadamard matrices for any monomial $(1, -1, 0)$-matrix P. Hence, without loss of generality, we may assume the following:

$$\begin{array}{rccccccl} h_1 = & (& +\cdots+ & +\cdots+ & +\cdots+ & +\cdots+ &), \\ h_2 = & (& +\cdots+ & +\cdots+ & -\cdots- & -\cdots- &), \\ h_3 = & (& \underbrace{+\cdots+}_{s \text{ columns}} & -\cdots- & +\cdots+ & -\cdots- &). \end{array}$$

It follows that $4s = n + h_1 \cdot h_2 + h_1 \cdot h_3 + h_2 \cdot h_3$. This gives a contradiction to the fact that $h_1 \cdot h_2 \equiv h_1 \cdot h_3 \equiv h_2 \cdot h_3 \equiv 2 \pmod 4$. $\square$

Remark 6.2. The above theorem is known for $n \equiv 4 \pmod 8$ [5, Lemma 13].

For $n = 4, 8, \ldots, 48$, we give in Table 13 the feasible parameters (a, b) along with $n(a)$. The third column of the table indicates our present state of knowledge about the existence of a pair of weakly unbiased Hadamard matrices for n and $(a, b, n(a))$.

Now, we give two methods for constructing weakly unbiased Hadamard matrices. Let $H = \begin{pmatrix} a & y^T \\ x & H_1 \end{pmatrix}$, $H' = \begin{pmatrix} a' & y'^T \\ x' & H_1' \end{pmatrix}$ be Hadamard matrices of order n, where H_1, H_1' are $(n-1) \times (n-1)$ matrices, x, x', y, y' are $(n-1) \times 1$ matrices and $a, a' \in \{1, -1\}$. Let K be the Hadamard matrix obtained from H' by negating the first column. Then we have

$$HK^T = HH'^T + \begin{pmatrix} -2aa' & -2ax'^T \\ -2a'x & -2xx'^T \end{pmatrix}. \tag{15}$$

Proposition 6.3. *If there exists a Hadamard matrix of order $n \geq 8$, then there exists a pair of weakly unbiased Hadamard matrices H, K of order n with $\sigma(H, K) = \{2, n-2\}$.*

Proof. Suppose that $H' = H$. From (15), the entries of HK^T are $n - 2, \pm 2$. The result follows. $\square$

Proposition 6.4. *Suppose that $n = 4k^2$, where k is even. If there exists a pair of unbiased Hadamard matrices of order n, then there exists a pair of weakly unbiased Hadamard matrices H, K of order n with $\sigma(H, K) = \{\sqrt{n} - 2, \sqrt{n} + 2\}$.*

Proof. Suppose that (H, H') is a pair of unbiased Hadamard matrices of order n. From (15), the entries of HK^T are $\pm\sqrt{n} \pm 2$. The result follows. $\square$

Since there exists a pair of unbiased Hadamard matrices of order 4^k for a positive integer k [11], the above proposition implies the existence of a pair of weakly unbiased Hadamard matrices H, K of order 4^k with $\sigma(H, K) = \{2^k - 2, 2^k + 2\}$ for $k \geq 2$.

TABLE 13. Weakly unbiased Hadamard matrices ($n = 4, 8, \ldots, 48$).

n	$(a, b, n(a))$	Existence	Reference
8	$(2, 6, 7)$	Yes	Proposition 6.3
12	$(2, 6, 9)$	Yes	[5, Table 3], Section 7.2
	$(2, 10, 11)$	Yes	Proposition 6.3, Section 7.2
16	$(2, 6, 10)$	Yes	Proposition 6.4, Section 7.2
	$(2, 10, 14)$	No	Section 6.2
	$(2, 14, 15)$	Yes	Proposition 6.3, Section 7.2
20	$(2, 6, 10)$	Yes	[5, Table 6]
	$(2, 18, 19)$	Yes	Proposition 6.3, Section 7.2
24	$(2, 6, 9)$	Yes	Section 6.2
	$(2, 10, 19)$	No	Section 6.2
	$(2, 22, 23)$	Yes	Proposition 6.3, Section 7.2
28	$(2, 6, 7)$	No	Section 6.2
	$(2, 10, 21)$	No	Section 6.2
	$(2, 26, 27)$	Yes	Proposition 6.3
32	$(2, 6, 4)$	?	
	$(2, 30, 31)$	Yes	Proposition 6.3, Section 7.2
36	$(2, 10, 24)$	?	
	$(2, 14, 30)$	?	
	$(2, 34, 35)$	Yes	Proposition 6.3
40	$(2, 10, 25)$	?	
	$(2, 22, 37)$	?	
	$(2, 38, 39)$	Yes	Proposition 6.3
	$(6, 14, 39)$	?	
44	$(2, 42, 43)$	Yes	Proposition 6.3
48	$(2, 10, 26)$	?	
	$(2, 14, 37)$	?	
	$(2, 46, 47)$	Yes	Proposition 6.3
	$(6, 10, 39)$	?	
	$(6, 18, 46)$	?	

6.2. **Observations by straightforward construction.** For each H of the five inequivalent Hadamard matrices of order 16 and the 60 inequivalent Hadamard matrices of order 24, our exhaustive computer search verified that there exists no $(1, -1)$-vector x of lengths 16 and 24, respectively, such that $|x \cdot r| \in \{2, 10\}$ for all rows r of H. This means that there exists no pair of weakly unbiased Hadamard matrices H, K of orders 16 and 24 with $\sigma(H, K) = \{2, 10\}$. We denote by $H_{24,3}$ `had.24.8` in [28], which is a Hadamard matrix of order 24. Our computer search under the condition (5) on K found a Hadamard matrix $\overline{K_{24,3}}$ such that $(H_{24,3}, \overline{K_{24,3}})$ is a pair of weakly unbiased Hadamard matrices with $\sigma(H_{24,3}, \overline{K_{24,3}}) = \{2, 6\}$, where $\overline{K_{24,3}}$ is listed in Figure 3.

Now, for a Hadamard matrix H of order n, we consider the following graph $\Gamma(H, \{a, b\})$ in order to convert the problem of finding K such that (H, K) is a pair of weakly unbiased Hadamard matrices of order n with $\sigma(H, K) = \{a, b\}$ into that of finding an n-clique in the graph. Let h_i be the i-th row of H. Set

$$V_j = \{x \in X_j \mid |x \cdot h_i| \in \{a, b\} \ (i = 1, 2, \ldots, n)\} \quad (j = 1, 2, 3, 4),$$

where $X_j = \{(x_1, x_2, \ldots, x_n) \in \{1, -1\}^n \mid (x_1, x_2, x_3) = Y_j\}$ with $Y_1 = (1, 1, 1)$, $Y_2 = (1, 1, -1)$, $Y_3 = (1, -1, 1)$ and $Y_4 = (1, -1, -1)$. We define the simple graph $\Gamma(H, \{a, b\})$, whose set of vertices is $V = V_1 \cup V_2 \cup V_3 \cup V_4$ and two vertices $x, y \in V$

$$
\overline{K_{24,3}} = \begin{pmatrix}
+++-+--++---+--++--+--+- \\
++++-+++++--++--+++-+--- \\
+++---+---+++++--------- \\
++++-----++-+-++++--++++ \\
++++++--+++--+++--++---+ \\
+++++-++--++-----++++-++ \\
++-------+---+----+++++- \\
++--++---+-+----++-----+ \\
++--+-++++++--+++-+-++-- \\
++--++++----++++-++--+++ \\
++-+-++-+--++--+---+++-+ \\
++-+-+-++-++-++-++-+-++- \\
+-+--+--+--+--++-++-+-+- \\
+-+--+++-++----+-+-+-+-- \\
+-+-+-+-+----++-++-+++-+ \\
+-+++++--+-++-+-+-++-++- \\
+-+-++-+++++++------++++ \\
+-++---+---+-+-++-+--+-+ \\
+--++++---+--+-++---+-+- \\
+--+--++++----+-------++ \\
+--++--+-+-+++++-+-++--- \\
+----+-+--+-+-+-+-+++--+ \\
+-----+-++++++-+++++--++ \\
+--++---+-+-+----++--+--
\end{pmatrix}
$$

FIGURE 3. The matrix $\overline{K_{24,3}}$

are adjacent if $x \cdot y = 0$. It follows that the graph $\Gamma(H, \{a, b\})$ contains an n-clique if and only if there exists a Hadamard matrix K of order n such that (H, K) is a pair of weakly unbiased Hadamard matrices of order n with $\sigma(H, K) = \{a, b\}$. We denote by $H_{28,1}, H_{28,2}, \dots, H_{28,487}$ `had.28.1`, `had.28.2`, ..., `had.28.487` in [28], respectively, which are the 487 inequivalent Hadamard matrices of order 28. By a computer calculation, we verified that each of the four induced subgraphs on V_j ($j = 1, 2, 3, 4$) of $\Gamma(H_{28,i}, \{2, 6\})$ (resp. $\Gamma(H_{28,i}, \{2, 10\})$) contains a 7-clique for only $i = 54, 295, 456, 479, 484, 487$ (resp. $i = 128, 197, 295, 297, 374, 445, 453, 456, 476, 477, 478, 479, 481, 485$). For these i, we list in Table 14 the sizes $mc(i)$ of the maximum cliques of $\Gamma(H_{28,i}, \{2, 6\})$ and $\Gamma(H_{28,i}, \{2, 10\})$. From Table 14, there exists no pair of weakly unbiased Hadamard matrices H, K of order 28 with $\sigma(H, K) = \{2, 6\}$ and $\{2, 10\}$. Our calculations for finding cliques in this section were done by a computer calculation using the CLIQUER software [25].

TABLE 14. Maximum cliques of $\Gamma(H_{28,i}, \{2, 6\})$ and $\Gamma(H_{28,i}, \{2, 10\})$.

Graph	$(i, mc(i))$					
$\Gamma(H_{28,i}, \{2, 6\})$	$(54, 12)$	$(295, 14)$	$(456, 12)$	$(479, 26)$	$(484, 26)$	$(487, 16)$
$\Gamma(H_{28,i}, \{2, 10\})$	$(128, 9)$	$(197, 10)$	$(295, 16)$	$(297, 12)$	$(374, 10)$	$(445, 12)$
	$(453, 10)$	$(456, 12)$	$(476, 12)$	$(477, 10)$	$(478, 12)$	$(479, 14)$
	$(481, 12)$	$(485, 12)$				

7. A CODING-THEORETIC APPROACH TO WEAKLY UNBIASED HADAMARD MATRICES

In this section we give a coding-theoretic approach to weakly unbiased Hadamard matrices. For modest lengths, we give classifications of some binary self-complementary codes in order to construct weakly unbiased Hadamard matrices.

7.1. **Binary codes and weakly unbiased Hadamard matrices.** Similarly to Theorem 5.1, we give a coding-theoretic approach to weakly unbiased Hadamard matrices.

Theorem 7.1. *Let a, b be odd integers with $0 < a < b < n/2$. There exists a self-complementary $(n, 4n)$ code C satisfying the following conditions:*

$$\{i \in \{0, 1, \ldots, n\} \mid A_i(C) \neq 0\} = \{0, n/2 \pm a, n/2 \pm b, n/2, n\}, \tag{16}$$

$$A_{n/2}(C) = 2n - 2, \tag{17}$$

$$C = C_1 \cup C_2, \tag{18}$$

where each C_i has distance distribution $(A_0(C_i), A_{n/2}(C_i), A_n(C_i)) = (1, 2n-2, 1)$ if and only if there exists a pair of weakly unbiased Hadamard matrices H, K with $\sigma(H, K) = \{2a, 2b\}$.

Proof. The proof is similar to that of Theorem 5.1. We remark that the condition $A_{n/2}(C) = 2n - 2$ corresponds to the condition that HK^T contains no zero entry. □

7.2. **Binary codes satisfying** (16)–(18)**.** By the method given in Section 5.2, for some $(n, 2n)$ codes C_1 $(n = 8, 12, 16, 20, 24)$, our computer calculation completed the classification of codes of the form $C = C_1 \cup (u + C_1)$ satisfying (16)–(18). Let $N_2(C_1)$ denote the number of inequivalent $(n, 4n)$ codes of the form $C_1 \cup (u + C_1)$ satisfying (16)–(18).

Proposition 7.2. *$N_2(RM(1,3)) = 1$. $N_2(C(H_{12})) = 2$. $N_2(RM(1,4)) = 2$, $N_2(C(H_{16,1})) = 4$, $N_2(C(H_{16,2})) = 6$, $N_2(C(H_{16,3})) = 3$ and $N_2(C(H_{16,4})) = 3$. $N_2(C(H_{20,i})) = 1$ $(i = 1, 2, 3)$. $N_2(C(H_{24,i})) = 1$ $(i = 1, 2)$. $N_2(RM(1,5)) = 1$.*

The unique $(8, 32)$ code $D_{8,1}$ is constructed as $\langle RM(1,3), u_1 \rangle$, where $\mathrm{supp}(u_1) = \{1\}$. The two $(12, 48)$ codes $D_{12,i}$ $(i = 1, 2)$ are constructed as $C(H_{12}) \cup (u_i + C(H_{12}))$, where $\mathrm{supp}(u_1) = \{1\}$ and $\mathrm{supp}(u_2) = \{1, 2, 3\}$. To save space, we only give the two $(16, 64)$ codes $D_{16,0,i}$ $(i = 1, 2)$ corresponding to $N_2(RM(1,4))$. The two codes are constructed as $\langle RM(1,4), u_i \rangle$, where $\mathrm{supp}(u_1) = \{1\}$ and $\mathrm{supp}(u_2) = \{1, 2, 3, 5, 9\}$. Let $H_{20,1}$ be the Paley Hadamard matrix of order 20 having the form (6), where R is the 19×19 circulant matrix with first row:

$$(- + - - + + + + - + - + - - - - + + -).$$

We denote by $H_{20,2}, H_{20,3}$ `had.20.toncheviii`, `had.20.tonchevix` in [28], respectively, which are the remaining two Hadamard matrices of order 20. The unique $(20, 80)$ code $D_{20,i}$ is constructed as $C(H_{20,i}) \cup (u + C(H_{20,i}))$, where $\mathrm{supp}(u) = \{1\}$ $(i = 1, 2, 3)$. The unique $(24, 96)$ code $D_{24,i}$ is constructed as $C(H_{24,i}) \cup (u + C(H_{24,i}))$, where $\mathrm{supp}(u) = \{1\}$ $(i = 1, 2)$. The unique $[32, 7]$ code $D_{32,1}$ is constructed as $\langle RM(1,5), u \rangle$, where $\mathrm{supp}(u) = \{4\}$ and the generator matrix of $RM(1,5)$ is given by:

$$\begin{pmatrix} 10010110011010010110100110010110 \\ 01010101010101010101010101010101 \\ 00110011001100110011001100110011 \\ 00001111000011110000111100001111 \\ 00000000111111110000000011111111 \\ 00000000000000001111111111111111 \end{pmatrix}.$$

All distance distributions are listed in Table 15. The distance distributions were obtained by a computer calculation.

TABLE 15. Distance distributions.

Code	$(A_0, A_1, \dots, A_n)$
$D_{8,1}$	$(1, 1, 0, 7, 14, 7, 0, 1, 1)$
$D_{12,1}$	$(1, 1, 0, 0, 0, 11, 22, 11, 0, 0, 0, 1, 1)$
$D_{12,2}$	$(1, 0, 0, 3, 0, 9, 22, 9, 0, 3, 0, 0, 1)$
$D_{16,0,1}$	$(1, 1, 0, 0, 0, 0, 0, 15, 30, 15, 0, 0, 0, 0, 0, 1, 1)$
$D_{16,0,2}$	$(1, 0, 0, 0, 0, 6, 0, 10, 30, 10, 0, 6, 0, 0, 0, 0, 1)$
$D_{20,i}$ $(i = 1, 2, 3)$	$(1, 1, 0, 0, \dots, 0, 0, 19, 38, 19, 0, 0, \dots, 0, 0, 1, 1)$
$D_{24,i}$ $(i = 1, 2)$	$(1, 1, 0, 0, \dots, 0, 0, 23, 46, 23, 0, 0, \dots, 0, 0, 1, 1)$
$D_{32,1}$	$(1, 1, 0, 0, \dots, 0, 0, 31, 62, 31, 0, 0, \dots, 0, 0, 1, 1)$

Similarly to Section 5.3, we consider linear $\mathbb{Z}_4$-codes $\mathcal{C}$ of length $n = 2^m$ satisfying the following conditions:

$$\{(n_0(x) - n_2(x))^2 \mid x \in \mathcal{C}\} = \{0, a^2, b^2, n^2\}, \tag{19}$$

$$|\{x \in \mathcal{C} \mid n_0(x) = n_2(x)\}| = 4n - 2, \tag{20}$$

$$\mathcal{C} \text{ contains } ZRM(1, m) \text{ as a subcode,} \tag{21}$$

where a, b are odd integers with $0 < a < b < n$. Then $\phi(\mathcal{C})$ satisfies (16)–(18). Our exhaustive computer search based on the method in Section 5.3 verified that there exists no $\mathbb{Z}_4$-code satisfying (19)–(21) for lengths 8 and 16.

8. SOME MODIFICATION OF WEAKLY UNBIASED HADAMARD MATRICES

Finally, some modification of the notion of weakly unbiased Hadamard matrices is given. We derive some results which are an analogy to those of quasi-unbiased Hadamard matrices and weakly unbiased Hadamard matrices.

8.1. **Type II weakly unbiased Hadamard matrices.** Let H, K be Hadamard matrices of order n. Let a_{ij} denote the (i, j)-entry of HK^T. We say that H, K are *Type II weakly unbiased* if $a_{ij} \equiv 0 \pmod 4$ for $i, j \in \{1, 2, \dots, n\}$ and $|\{|a_{ij}| \mid i, j \in \{1, 2, \dots, n\}\}| \le 2$. For an even square n, a pair of unbiased Hadamard matrices of order n is a pair of Type II weakly unbiased Hadamard matrices. Hence, the notion of Type II weakly unbiased Hadamard matrices of order n is some natural extension of the notion of unbiased Hadamard matrices for an even square n. Similarly to weakly unbiased Hadamard matrices, in this paper, we exclude unbiased Hadamard matrices from Type II weakly unbiased Hadamard matrices. It follows immediately from the definition that $n \ge 8$.

8.2. **Basic properties and feasible parameters.** Let (H, K) be a pair of Type II weakly unbiased Hadamard matrices of order n. Suppose that a, b are positive integers satisfying $\{|a_{ij}| \mid i, j \in \{1, 2, \dots, n\}\} = \{a, b\}$. We denote the set $\{a, b\}$ by $\sigma(H, K)$. Let $n(a)$ be the number of components j with $a_{ij} = \pm a$ for $i = 1, 2, \dots, n$, where a_{ij} denotes the (i, j)-entry of HK^T. Similarly to weakly unbiased Hadamard matrices, (14) holds. From now on, we assume that $a < b$. We say that parameters (a, b) satisfying (14) are *feasible*. Since $(a, b, n(a)) = (4, n/2 - 4, n - 4)$ satisfies (14), the parameters $(a, b) = (4, n/2 - 4)$ are feasible for each order $n \equiv 0 \pmod 8$.

For $n = 4, 8, \dots, 48$, we give in Table 16 feasible parameters (a, b) along with $n(a)$ and our present state of knowledge about the maximum size f_{max} among sets of mutually Type II weakly unbiased Hadamard matrices of order n for $(a, b, n(a))$. In the third column of the table, "-" means that there exists no pair of Type II weakly unbiased Hadamard matrices.

TABLE 16. Type II weakly unbiased Hadamard matrices ($n = 4, 8, \dots, 48$).

n	$(a, b, n(a))$	f_{max}	Reference	
24	$(4, 8, 20)$	$2-42$	Section 8.3	Table 17
28	$(4, 8, 21)$	-		Section 8.3
32	$(4, 12, 28)$	$4-264$	Section 8.5	Table 17
36	$(4, 8, 21)$	$2-72$	Proposition 8.3	Table 17
	$(4, 16, 33)$	≤ 10671		Table 17
40	$(4, 8, 20)$	≤ 84		Table 17
	$(4, 16, 36)$	≤ 16698		Table 17
48	$(4, 8, 16)$	≤ 112		Table 17
	$(4, 12, 36)$	$2-194$	Proposition 8.1	Table 17
	$(4, 20, 44)$	$2-36034$	Corollary 8.2	Table 17
	$(4, 28, 46)$	≤ 36034		Table 17

Proposition 8.1. *Suppose that there exists a set of f mutually unbiased Hadamard matrices of order m. Assume that one of the following holds:*

(i) *There exists a set of f mutually weakly unbiased Hadamard matrices $H_1, H_2, \dots, H_f$ of order n with $\sigma(H_i, H_j) = \{a, b\}$ $(i, j \in \{1, 2, \dots, f\}$ and $i \neq j)$.*

(ii) *There exists a set of f mutually Type II weakly unbiased Hadamard matrices $H_1, H_2, \dots, H_f$ of order n with $\sigma(H_i, H_j) = \{a, b\}$ $(i, j \in \{1, 2, \dots, f\}$ and $i \neq j)$.*

Then there exists a set of f mutually Type II weakly unbiased Hadamard matrices $L_1, L_2, \dots, L_f$ of order mn with $\sigma(L_i, L_j) = \{\sqrt{m}a, \sqrt{m}b\}$ $(i, j \in \{1, 2, \dots, f\}$ and $i \neq j)$.

Proof. It is sufficient to give a proof for the case $f = 2$. Let (H', K') be a pair of unbiased Hadamard matrices of order m. Then $(H_1 \otimes H', H_2 \otimes K')$ is a pair of Type II weakly unbiased Hadamard matrices of order mn with $\sigma(H_1 \otimes H', H_2 \otimes K') = \{\sqrt{m}a, \sqrt{m}b\}$. $\square$

As an example, a pair of Type II weakly unbiased Hadamard matrices L_1, L_2 of order 48 with $\sigma(L_1, L_2) = \{4, 12\}$ is constructed from a pair of weakly unbiased Hadamard matrices H_1, K_1 of order 12 with $\sigma(H_1, K_1) = \{2, 6\}$ (see Table 13) and a pair of unbiased Hadamard matrices of order 4.

Corollary 8.2. *If there exists a Hadamard matrix of order $n \geq 8$, then there exists a pair of Type II weakly unbiased Hadamard matrices H, K of order $4n$ with $\sigma(H, K) = \{4, 2n - 4\}$.*

Proof. Suppose that there exists a Hadamard matrix of order $n \geq 8$. By Proposition 6.3, there exists a pair of weakly unbiased Hadamard matrices H, K of order n with $\sigma(H, K) = \{2, n-2\}$. Since there exists a pair of unbiased Hadamard matrices of order 4, the result follows from Proposition 8.1. $\square$

Similarly to Proposition 6.4, we immediately have the following:

Proposition 8.3. *Suppose that $n = 4k^2$, where k is odd. If there exists a pair of unbiased Hadamard matrices of order n, then there exists a pair of Type II weakly unbiased Hadamard matrices H, K of order n with $\sigma(H, K) = \{\sqrt{n} - 2, \sqrt{n} + 2\}$.*

As an example, a pair of Type II weakly unbiased Hadamard matrices H, K of order 36 with $\sigma(H, K) = \{4, 8\}$ is constructed from that of unbiased Hadamard matrices of order 36 given in [19].

8.3. **Observations by straightforward construction.** Under the condition (5) on K, our computer search found a Hadamard matrix $\overline{K_{24,4}}$ such that $(H_{24,4}, \overline{K_{24,4}})$ is a pair of Type II weakly unbiased Hadamard matrices with $\sigma(H_{24,4}, \overline{K_{24,4}}) = \{4, 8\}$, where $H_{24,4}$ is `had.24.49` in [28]. The matrix $\overline{K_{24,4}}$ is listed in Figure 4.

$$
\overline{K_{24,4}} = \begin{pmatrix}
+++-+-+----+-++-----+-+- \\
+++--+-+-+--+-+-+-+++++- \\
+++-+-+----++--+++++-+-+ \\
++++-++++-++-+--++++--+- \\
++++-++++-+++-++----++-+ \\
+++--+-+-+---+-+-+-----+ \\
++-+--+--++-++--++--++++ \\
++-+--+--++---++--++---- \\
++-++--++----++++++-++-- \\
++--++--+++++-++++----+- \\
++--++--++++-+----++++-+ \\
++-++--++---+------+--++ \\
+-++----++-+----+----+-- \\
+-+-+-+++++-+++-+-+----+ \\
+-+-+-+++++----+-+-++++- \\
+-++++----+-+----++-+--- \\
+-++----++-+++++-++++-++ \\
+-++++----+--++++--+-+++ \\
+------+--++--+--++--+++ \\
+--+++++-+-++++--+-+-+-- \\
+----++-+---++-+--+--++- \\
+----++-+-----+-++-++--+ \\
+--+++++-+-+---++-+-+-++ \\
+------+--++++-++--++---
\end{pmatrix}
$$

FIGURE 4. The matrix $\overline{K_{24,4}}$

By a computer calculation, we verified that each of the four induced subgraphs on V_j $(j = 1, 2, 3, 4)$ of $\Gamma(H_{28,i}, \{4, 8\})$ contains a 7-clique for 355 Hadamard matrices. In addition, we verified that the induced subgraph on $V_1 \cup V_2$ of only $\Gamma(H_{28,484}, \{4, 8\})$ contains a 14-clique among the 355 graphs $\Gamma(H_{28,i}, \{4, 8\})$. By a computer calculation, we obtained that the size of the maximum cliques of $\Gamma(H_{28,484}, \{4, 8\})$ is 24. Hence, there exists no pair of Type II weakly unbiased Hadamard matrices of order 28 with $\sigma(H, K) = \{4, 8\}$.

8.4. **A coding-theoretic approach.** Similarly to Theorems 5.1 and 7.1, we have the following coding-theoretic approach to Type II weakly unbiased Hadamard matrices.

Theorem 8.4. *Let a, b be even integers with $0 < a < b < n/2$. There exists a self-complementary $(n, 2fn)$ code C satisfying the following conditions:*

$$\{i \in \{0, 1, \ldots, n\} \mid A_i(C) \neq 0\} = \{0, n/2 \pm a, n/2 \pm b, n/2, n\}, \tag{22}$$

$$A_{n/2}(C) = 2n - 2, \tag{23}$$

$$C = C_1 \cup C_2 \cup \cdots \cup C_f, \tag{24}$$

where C_i has distance distribution $(A_0(C_i), A_{n/2}(C_i), A_n(C_i)) = (1, 2n-2, 1)$ if and only if there exists a set of f Type II weakly unbiased Hadamard matrices H, K with $\sigma(H, K) = \{2a, 2b\}$.

Similarly to Lemma 5.3, as in the case $s = 6$ of Theorems 3.2 and 3.4, we have two upper bounds on the number of the codewords of self-complementary codes satisfying (22).

Lemma 8.5. *Let C be a self-complementary code of length n satisfying* (22)*. Then*

(i) $|C| \leq 2(\binom{n}{5} + \binom{n}{3} + \binom{n}{1})$.
(ii) *If* $15n^2 - 30n + 16 - 4(3n-2)(a^2+b^2) + 16a^2b^2 > 0$ *and* $5(n-2) - 2a^2 - 2b^2 \geq 0$, *then* $|C| \leq \lfloor \frac{2n(n^2-4a^2)(n^2-4b^2)}{15n^2-30n+16-4(3n-2)(a^2+b^2)+16a^2b^2} \rfloor$.

Proof. (i) The upper bound is the case $s = 6$ of Theorem 3.2.

(ii) Expanding by the Krawtchouk polynomials, we have

$$\begin{aligned}
\overline{\alpha}_C(z) =& \left(1 - \frac{2z}{2a+n}\right)\left(1 - \frac{2z}{2b+n}\right)\left(1 - \frac{2z}{n}\right)\left(1 - \frac{2z}{-2a+n}\right)\left(1 - \frac{2z}{-2b+n}\right) \\
=& \frac{15n^2 - 30n + 16 - 4(3n-2)(a^2+b^2) + 16a^2b^2}{n(n^2-4a^2)(n^2-4b^2)} K_1(z) \\
&+ \frac{12(5n - 10 - 2a^2 - 2b^2)}{n(n^2-4a^2)(n^2-4b^2)} K_3(z) + \frac{120}{n(n^2-4a^2)(n^2-4b^2)} K_5(z) \\
=& \alpha_1 K_1(z) + \alpha_3 K_3(z) + \alpha_5 K_5(z) \text{ (say)}.
\end{aligned}$$

The assumption on a, b and n yields that α_1 is positive and α_3, α_5 are nonnegative. Therefore, Theorem 3.4 implies the desired bound. □

Similarly to Theorem 5.4, by Theorem 8.4, we immediately have the following two upper bounds on the maximum size among sets of mutually Type II weakly unbiased Hadamard matrices, one of which depends only on n, and the other depends on n, α. This is also one of the main results of this paper.

Theorem 8.6. *Suppose that there exists a set of f mutually Type II weakly unbiased Hadamard matrices H, K of order n with $\sigma(H, K) = \{a, b\}$. Then*

(i) $f \leq \lfloor \frac{n^4 - 10n^3 + 55n^2 - 110n + 184}{5!} \rfloor$.
(ii) *If* $15n^2 - 30n + 16 - 4(3n-2)(a^2+b^2) + 16a^2b^2 > 0$ *and* $5(n-2) - 2a^2 - 2b^2 \geq 0$, *then* $f \leq \lfloor \frac{(n^2-4a^2)(n^2-4b^2)}{15n^2-30n+16-4(3n-2)(a^2+b^2)+16a^2b^2} \rfloor$.

For the feasible parameters given in Table 16, we list in Table 17 the maximum possible sizes among sets of mutually Type II weakly unbiased Hadamard matrices, which are obtained by the two upper bounds. We do not list the maximum possible sizes when there exists no pair of Type II weakly unbiased Hadamard matrices. In the table, "$*$" means that the assumption of Theorem 8.6 (ii) is not satisfied.

TABLE 17. Absolute and linear programming bounds in Theorem 8.6.

n	$(a,b,n(a))$	Absolute bound	Linear programming bound
24	$(4,8,20)$	$\lfloor 5569/3 \rfloor = 1856$	$\lfloor 256/3 \rfloor = 85$
32	$(4,12,28)$	6449	528
36	$(4,8,21)$	$\lfloor 32014/3 \rfloor = 10671$	$\lfloor 5632/39 \rfloor = 144$
	$(4,16,33)$	10671	$*$
40	$(4,8,20)$	$\lfloor 83491/5 \rfloor = 16698$	$\lfloor 4224/25 \rfloor = 168$
	$(4,16,36)$	16698	$*$
48	$(4,8,16)$	$\lfloor 108103/3 \rfloor = 36034$	$\lfloor 64064/285 \rfloor = 224$
	$(4,12,36)$	36034	$\lfloor 20592/53 \rfloor = 388$
	$(4,20,44)$	36034	$*$
	$(4,28,46)$	36034	$*$

8.5. **Binary codes satisfying** (22)–(24)**.** In the process of the classification of codes of length 32 satisfying (16)–(18) with $C_1 = RM(1,5)$, our exhaustive computer search verified that there exists no $[32,7]$ code satisfying (22)–(24) with $C_1 = RM(1,5)$.

Suppose that $\mathcal{C}$ is a linear $\mathbb{Z}_4$-code of length $n = 2^m$ satisfying the following conditions:

$$\{(n_0(x) - n_2(x))^2 \mid x \in \mathcal{C}\} = \{0, a^2, b^2, n^2\}, \tag{25}$$

$$|\{x \in \mathcal{C} \mid n_0(x) = n_2(x)\}| = 4n - 2, \tag{26}$$

$$\mathcal{C} \text{ contains } ZRM(1,m) \text{ as a subcode,} \tag{27}$$

where a, b are even integers with $0 < a < b < n$. Then $\phi(\mathcal{C})$ satisfies (22)–(24).

In the process of verifying that there exists no linear $\mathbb{Z}_4$-code of length 16 satisfying (19)–(21), our computer calculation completed the classification of linear $\mathbb{Z}_4$-code of length 16 satisfying (25)–(27). We give the numbers $N_4'(16,k)$ of inequivalent linear $\mathbb{Z}_4$-codes $\mathcal{C}$ of length 16 with $|\mathcal{C}| = 2^k$ satisfying (25)–(27).

Proposition 8.7. *$N_4'(16,7) = 1$, $N_4'(16,8) = 3$ and $N_4'(16,9) = 0$.*

The unique linear $\mathbb{Z}_4$-code $\mathcal{C} = \mathcal{C}'_{16,1}$ with $|\mathcal{C}| = 2^7$ is constructed as $\langle ZRM(1,4), x_1\rangle$, where $x_1 = (0,0,0,2,1,1,1,3,1,3,1,3,0,0,0,0)$. The three linear $\mathbb{Z}_4$-codes $\mathcal{C} = \mathcal{C}'_{16,2,i}$ with $|\mathcal{C}| = 2^8$ $(i = 1,2,3)$ are constructed as $\langle ZRM(1,4), x_1, x_{2,i}\rangle$, where

$$\begin{aligned}
x_{2,1} &= (0,0,1,3,0,2,3,3,1,3,0,0,1,3,0,0),\\
x_{2,2} &= (0,0,1,3,0,0,3,3,1,3,2,0,1,3,0,0),\\
x_{2,3} &= (0,0,1,1,0,0,3,3,1,3,0,2,3,3,0,0).
\end{aligned}$$

This gives a set of four mutually Type II weakly unbiased Hadamard matrices of order 32 with $\sigma(H,K) = \{4,12\}$ by Theorem 8.4. By a computer calculation, we verified that the above linear $\mathbb{Z}_4$-codes $\mathcal{C}$ have $(a^2,b^2) = (4,36)$ and have $(d_H(\mathcal{C}), d_L(\mathcal{C})) = (8,10)$.

APPENDIX A

In Lemma 5.3, we did not give a detailed proof of the fact that $(C, \{R_i\}_{i=0}^4)$ is a Q-polynomial association scheme when $|C| = \frac{2n(n^2-4\alpha^2)}{3n-2-4\alpha^2}$. In this appendix, we give a detailed proof.

Suppose that C and R_i are as given in Lemma 5.3. Assume that $|C| = \frac{2n(n^2-4\alpha^2)}{3n-2-4\alpha^2}$. For $i \in \{0,1,\dots,4\}$, A_i denotes the adjacency matrix of the graph with vertex set C and edge set R_i. Let $\mathcal{A}$ denote the vector space over $\mathbb{R}$ spanned by $A_0 = I_{|C|}, A_1, \dots, A_4$, which forms an algebra. Let $\{E_0, E_1, \dots, E_4\}$ denote the set of the primitive idempotents of $\mathcal{A}$. Then the matrix $P = (p_{ij})$ is defined by $A_i = \sum_{j=0}^{4} p_{ji} E_i$.

Lemma A.1. $(C, \{R_i\}_{i=0}^4)$ *is a symmetric association scheme.*

Proof. It is sufficient to show that $(C, \{R_i\}_{i=0}^4)$ satisfies the 4-th condition in the definition of a symmetric association scheme given in Section 2.3; namely, we show that $A_iA_j \in \mathcal{A}$ for $i,j \in \{0,1,\dots,4\}$. Since $A_0 = I_{|C|}$, $A_iA_0 = A_0A_i = A_i$ holds for $i \in \{0,1,\dots,4\}$. Since C is self-complementary, $A_iA_4 = A_4A_i = A_{4-i}$ holds for $i \in \{0,1,\dots,4\}$.

Since $|C| = \frac{2n(n^2-4\alpha^2)}{3n-4\alpha^2-2}$, the coefficients of $K_0(z)$ and $K_1(z)$ in $\alpha_C(z)$ are 1 and the other coefficients are positive by the calculation in the proof of Theorem 3.4. By [13, Theorem 5.23 (iii)], C is a 5-design in the binary Hamming scheme; namely, C is an orthogonal array of strength 5.

We denote the Krawtchouk expansion of z^λ by $z^\lambda = \sum_{l=0}^{\lambda} f_{\lambda,l} K_l(z)$, and define a polynomial by $F_{\lambda,\mu}(z) = \sum_{l=0}^{\min\{\lambda,\mu\}} f_{\lambda,l} f_{\mu,l} K_l(z)$. For $\lambda, \mu \in \{0,1,2\}$, expand $(\sum_{k=0}^{\lambda} f_{\lambda,k} G_k G_k^T)(\sum_{l=0}^{\mu} f_{\mu,l} G_l G_l^T)$ in two ways, where G_k denotes the k-th characteristic matrix of C. In the following calculation, define 0^0 to be 1. By [13, Theorem 5.18],

$$\begin{aligned}(\sum_{k=0}^{\lambda} f_{\lambda,k} G_k G_k^T)(\sum_{l=0}^{\mu} f_{\mu,l} G_l G_l^T) &= |C| \sum_{k=0}^{\min\{\lambda,\mu\}} f_{\lambda,k} f_{\mu,k} G_k G_k^T \\ &= |C| \sum_{k=0}^{\min\{\lambda,\mu\}} f_{\lambda,k} f_{\mu,k} \sum_{l=0}^{4} K_k(\beta_l) A_l = |C| \sum_{l=0}^{4} F_{\lambda,\mu}(\beta_l) A_l.\end{aligned}$$

On the other hand, by [13, Theorem 3.13],

$$\begin{aligned}(\sum_{k=0}^{\lambda} f_{\lambda,k} G_k G_k^T)(\sum_{l=0}^{\mu} f_{\mu,l} G_l G_l^T) &= (\sum_{k=0}^{\lambda} f_{\lambda,k} \sum_{i=0}^{4} K_k(\beta_i) A_i)(\sum_{l=0}^{\mu} f_{\mu,l} \sum_{j=0}^{4} K_l(\beta_j) A_j) \\ &= \sum_{k=0}^{\lambda} \sum_{l=0}^{\mu} \sum_{i=0}^{4} \sum_{j=0}^{4} f_{\lambda,k} f_{\mu,l} K_k(\beta_i) K_l(\beta_j) A_i A_j = \sum_{i=0}^{4} \sum_{j=0}^{4} \beta_i^\lambda \beta_j^\mu A_i A_j \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \beta_i^\lambda \beta_j^\mu A_i A_j + \sum_{i=1}^{3} \beta_i^\lambda \beta_0^\mu A_i + \sum_{j=1}^{3} \beta_0^\lambda \beta_j^\mu A_j + \sum_{i=1}^{3} \beta_i^\lambda \beta_4^\mu A_{4-i} \\ &\quad + \sum_{j=1}^{3} \beta_4^\lambda \beta_j^\mu A_{4-j} + \beta_0^{\lambda+\mu} A_0 + \beta_4^{\lambda+\mu} A_0 + \beta_0^\lambda \beta_4^\mu A_4 + \beta_4^\lambda \beta_0^\mu A_4.\end{aligned}$$

Thus, $\sum_{i=1}^{3} \sum_{j=1}^{3} \beta_i^\lambda \beta_j^\mu A_i A_j \in \mathcal{A}$ for $i,j \in \{1,2,3\}$. For $W = \begin{pmatrix} 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 \end{pmatrix}$, $W \otimes W$ is invertible. Hence, $A_iA_j \in \mathcal{A}$ for $i,j \in \{1,2,3\}$. Therefore, $(C, \{R_i\}_{i=0}^4)$ is a symmetric association scheme. $\square$

Lemma A.2. $(C, \{R_i\}_{i=0}^4)$ *is Q-polynomial.*

Proof. Set $F_i = \frac{1}{|C|} G_i G_i^T$ for $i = 0, 1, 2, 3$ and $F_4 = I - \sum_{i=0}^2 F_i$. We claim that $\{F_0, F_1, F_2\}$ is a subset of the set of primitive idempotents of $\mathcal{A}$. Let E_i $(i = 0, 1, \dots, 4)$ be primitive idempotents. Assume that $F_i \notin \{E_0, E_1, \dots, E_4\}$ for some $i \in \{0, 1, 2\}$. Since F_i is an idempotent, we have decomposition $F_i = E + E'$ satisfying $E, E' \neq O$, $E^2 = E$, $E'^2 = E'$ and $EE' = O$, where O denotes the zero matrix. Then $\{F_0, F_1, F_2, E, E'\} \backslash \{F_i\}$ is a set of elements which are linear independent. Thus, $\langle F_3, F_4 \rangle$ has dimension 1. Hence, there exists a nonzero real number c such that $F_4 = cF_3$. Then $A_0 - \frac{1}{|C|} \sum_{i=0}^2 \sum_{j=0}^4 K_i(\beta_j) A_j = \frac{c}{|C|} \sum_{j=0}^4 K_3(\beta_j) A_j$, and thus we obtain $cK_3(\beta_j) + \sum_{i=0}^2 K_i(\beta_j) = 0$ for any $j \in \{1, 2, 3, 4\}$. Since all β_j are distinct and the degree of $cK_3(z) + \sum_{i=0}^2 K_i(z)$ is at most three, this is a contradiction. Therefore, we may assume $E_i = F_i$ for $i = 0, 1, 2$.

For $i = 0, 1, 2$,

$$\begin{aligned} A_4 E_i &= \frac{1}{|C|} \sum_{j=0}^4 K_i(\beta_j) A_4 A_j = \frac{1}{|C|} \sum_{j=0}^4 K_i(\beta_j) A_{4-j} = \frac{1}{|C|} \sum_{j=0}^4 K_i(\beta_{4-j}) A_j \\ &= \frac{1}{|C|} \sum_{j=0}^4 K_i(n - \beta_j) A_j = \frac{1}{|C|} \sum_{j=0}^4 (-1)^i K_i(\beta_j) A_j = (-1)^i E_i. \end{aligned}$$

Thus, $p_{i4} = (-1)^i$ for $i = 0, 1, 2$. By [4, Chap. II, Theorem 4.1 (ii)], $\{p_{04}, p_{14}, \dots, p_{44}\} = \{\gamma_0, \gamma_1, \dots, \gamma_4\}$ as a multiset, where γ_i $(i = 0, 1, \dots, 4)$ are the eigenvalues of the matrix $\begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 1&0&0&0&0 \end{pmatrix}$. Thus, we may assume that $p_{34} = -1$ and $p_{44} = 1$.

By [9, Lemma 2.3.1 (vii)],

$$q_{0i} q_{0j} = \sum_{k=0}^4 q_{i,j}^k q_{0k}, \quad q_{4i} q_{4j} = \sum_{k=0}^4 q_{i,j}^k q_{4k}. \tag{28}$$

By [4, Chap. II, Theorem 3.5 (i)] and $p_{i4} = (-1)^i$, $q_{4i} = (-1)^i q_{0i}$ for $i = 0, 1, \dots, 4$. Substituting these into (28), we obtain

$$(-1)^{i+j} q_{0i} q_{0j} = \sum_{k=0}^4 q_{i,j}^k (-1)^k q_{0k}. \tag{29}$$

By (28) and (29), we have $\sum_{k=0}^4 (1 - (-1)^{i+j+k}) q_{i,j}^k q_{0k} = 0$. Since $q_{0k} > 0$ and $q_{i,j}^k \geq 0$, we obtain

$$q_{i,j}^k = 0 \text{ if } i + j + k \text{ is odd.} \tag{30}$$

For $i = 0, 1$,

$$\begin{aligned} |C| E_1 \circ |C| E_i &= \sum_{l=0}^4 K_1(\beta_l) K_i(\beta_l) A_l \\ &= \sum_{l=0}^4 (n - i + 1) K_{i-1}(\beta_l) A_l + \sum_{l=0}^4 (i + 1) K_{i+1}(\beta_l) A_l \\ &= (n - i + 1)|C| E_{i-1} + (i + 1)|C| E_{i+1}. \end{aligned}$$

Thus, $q_{1,i}^{i-1} = n - i + 1$ and $q_{1,i}^{i+1} = i + 1$ for $i = 0, 1$, and $q_{1,i}^{j} = 0$ for $i = 0, 1, j \neq i - 1, i + 1$. By [4, Chap. II, Proposition 3.7 (v)], $q_{1,2}^{1} = n - 1$, and by [4, Chap. II, Proposition 3.7 (vi)], $q_{1,i}^{j} = 0$ for $(i, j) \in \{(2, 0), (3, 0), (4, 0), (3, 1), (4, 1)\}$. By (30), $q_{1,i}^{j} = 0$ for $(i, j) \in \{(2, 2), (3, 3), (4, 4), (2, 4), (4, 2)\}$. Again by [4, Chap. II, Proposition 3.7 (vi)], $q_{1,i}^{j} > 0$ for $(i, j) \in \{(2, 3), (3, 2), (3, 4), (4, 3)\}$. The Q-polynomiality is equivalent to the condition that the Krein matrix $B_1^* = (q_{1,j}^{k})$ is a tridiagonal matrix with nonzero entries on the superdiagonal and the subdiagonal (see [4, p. 193]). This completes the proof of the fact that the association scheme $(C, \{R_i\}_{i=0}^{4})$ is Q-polynomial. □

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Department of Computer Science, Shizuoka University, Hamamatsu 432–8011, Japan
E-mail address: `araya@inf.shizuoka.ac.jp`

Research Center for Pure and Applied Mathematics, Graduate School of Information Sciences, Tohoku University, Sendai 980–8579, Japan
E-mail address: `mharada@m.tohoku.ac.jp`

Department of Mathematics Education, Aichi University of Education, Kariya 448-8542, Japan
E-mail address: `suda@auecc.aichi-edu.ac.jp`