MATHEMATICS OF COMPUTATION
Volume 86, Number 304, March 2017, Pages 855–864
http://dx.doi.org/10.1090/mcom/3181
Article electronically published on June 29, 2016

MONOTONICITY PROPERTIES OF THE ZEROS OF FREUD AND SUB-RANGE FREUD POLYNOMIALS: ANALYTIC AND EMPIRICAL RESULTS

WALTER GAUTSCHI

Abstract. Freud and sub-range Freud polynomials are orthogonal with respect to the weight function $w(t) = |t|^\mu \exp(-|t|^\nu)$, $\mu > -1$, $\nu > 0$, supported on the whole real line $\mathbb{R}$, resp. on strict subintervals thereof. The zeros of these polynomials are studied here as functions of ν and shown, analytically and empirically by computation, to collectively increase or decrease on appropriate intervals of the variable ν.

1. Introduction

Freud polynomials are commonly defined to be orthogonal with respect to the weight function

$$w(t) = w(t; \mu, \nu) = |t|^\mu e^{-|t|^\nu}, \quad \mu > -1, \ \nu > 0, \tag{1.1}$$

supported on the whole real line $\mathbb{R}$. Here we also consider "sub-range" Freud polynomials, which are orthogonal with respect to the same weight function (1.1), but on strict subintervals of $\mathbb{R}$. Specifically, *lower* and *upper symmetric sub-range Freud polynomials* are orthogonal on an interval $[-c, c]$, $0 < c < \infty$, resp. on two disjoint intervals $[-\infty, -c] \cup [c, \infty]$, and become ordinary Freud polynomials when $c \to \infty$, resp. $c \to 0$. Likewise, *lower* and *upper one-sided sub-range Freud polynomials* are orthogonal on $[0, c]$, $0 < c < \infty$, resp. on $[c, \infty]$, and become *half-range Freud polynomials* when $c \to \infty$, resp. $c \to 0$.

Our interest is in the zeros of these polynomials, in particular their monotonicity properties when considered functions of the parameter ν. Analytic results can be derived from a well-known theorem dealing with the dependence of the zeros of orthogonal polynomials on a parameter. While of limited scope, these results contain statements valid for arbitrary parameters $\mu > -1$ and arbitrary degrees n. They are presented and discussed in Section 2. A more comprehensive study of the zeros, at present, is possible only through experimental computation. Results obtained along these lines are described in Section 3. For computational details, however, we must refer to [2].

Received by the editor September 8, 2015.
2010 *Mathematics Subject Classification.* Primary 33C47, 33F05.
Key words and phrases. Freud polynomials, sub-range Freud polynomials, zeros of orthogonal polynomials, Matlab software.

2. Analytic results

The standard result for dealing with zeros of an orthogonal polynomial that depends on a parameter is Markov's theorem (Theorem 6.12.1 of [3]). We apply it here to Freud and sub-range Freud polynomials, where the parameter in question is ν.

2.1. **One-sided sub-range Freud polynomials.**

Theorem 1. (a) *Let* $\nu_0 > 0$ *and* $0 < c \leq e^{-1/\nu_0}$. *Denote by* π_n *the lower one-sided sub-range Freud polynomial of degree* n *orthogonal on* $[0, c]$ *with respect to the weight function*

$$w(t; c, \mu, \nu) = t^\mu e^{-t^\nu}, \quad t \in [0, c]. \tag{2.1}$$

Then all zeros of π_n *are monotonically increasing on* $[\nu_0, \infty)$ *as functions of* ν, *for every* $\mu > -1$ *and* $n = 1, 2, 3, \dots$.

(b) *Let* $\nu_0 > 0$ *and* $c \geq e^{-1/\nu_0}$. *Denote by* π_n *the upper one-sided sub-range Freud polynomial of degree* n *orthogonal on* $[c, \infty]$ *with respect to the weight function*

$$w(t; c, \mu, \nu) = t^\mu e^{-t^\nu}, \quad t \in [c, \infty]. \tag{2.2}$$

Then all zeros of π_n *are monotonically decreasing on* $(0, \nu_0)$ *as functions of* ν, *for every* $\mu > -1$ *and* $n = 1, 2, 3, \dots$.

Proof. (a) Let the zeros of π_n, in decreasing order, be

$$\tau_1(\nu) > \tau_2(\nu) > \cdots > \tau_n(\nu) > 0.$$

Then, according to Theorem 6.12.1 of [3], the regularity assumptions of which being all satisfied, the zero $\tau_k(\nu)$, for k fixed, is an increasing [decreasing] function of ν provided

$$f(t) := \frac{\partial w(t; c, \mu, \nu)/\partial \nu}{w(t; c, \mu, \nu)}, \quad 0 < t < c,$$

is an increasing [decreasing] function of t on $(0, c)$. An elementary computation will show that, irrespective of the value of μ,

$$f(t) = -t^\nu \ln t.$$

Now

$$f'(t) = -t^{\nu-1}(\nu \ln t + 1),$$

which is positive on the interval $(0, t_0)$ and negative on $t > t_0$, where $t_0 = e^{-1/\nu}$. Since by assumption $c \leq e^{-1/\nu_0}$, we have, for $\nu \geq \nu_0$,

$$0 < t < c \leq e^{-1/\nu_0} \leq e^{-1/\nu} = t_0, \tag{2.3}$$

hence $f'(t) > 0$ on $(0, c)$. By the cited theorem, therefore, $\tau_k(\nu)$ for each k is an increasing function of ν on $[\nu_0, \infty)$, as claimed.

(b) We now have, in place of (2.3), when $\nu < \nu_0$,

$$t \geq c \geq e^{-1/\nu_0} > e^{-1/\nu} = t_0,$$

so that $f'(t)$ is negative on (c, ∞), and the assertion follows as in part (a) from Theorem 6.12.1 of [3]. $\square$

Part (a) of Theorem 1 is of limited scope insofar as it deals only with intervals of orthogonality $[0, c]$, $0 < c < 1$. (In the limit $c = 1$, that is, $\nu_0 = \infty$, it provides no information at all.) It is also of limited interest, since the zeros, in this case, are almost constant functions of ν (see Example 1 below). In this regard, part (b) of the theorem has a wider scope, covering intervals $[c, \infty]$, $0 < c < \infty$, and in the case $c \geq 1$, that is, $\nu_0 = \infty$, provides monotonicity information valid on the whole interval $0 < \nu < \infty$. (In the other limit case $c = 0$, that is, $\nu_0 = 0$, it again is devoid of content.)

We illustrate Theorem 1 by numerical examples. To compute the desired zeros, we first compute the first N recurrence coefficients of the respective orthogonal polynomials from the first $2N$ moments of the weight function, using the classical Chebyshev algorithm in sufficiently high precision (cf. [1, §2.1.7]). The moments are always expressible in terms of the gamma and incomplete gamma functions. Thereafter, the zeros of the orthogonal polynomial of degree n can be obtained (in ordinary working precision) for all $n \leq N$ by well-known eigenvalue/vector techniques (cf. [1, §3.1.1]).

Example 1. The zeros of π_n (of Theorem 1(a)) for $n = 15$ and $n = 30$, when $\nu_0 = 3$, $c = e^{-1/3} = .7165\ldots$, $\mu = 0$, and $3 \leq \nu \leq 10$.

Here, the monotone growth of the zeros is extremely slow. When $n = 15$, the slope is as small as 3.64×10^{-7} and never larger than 8.78×10^{-4}. For $n = 30$, the corresponding numbers are 4.77×10^{-8} and 4.52×10^{-4}. Thus, the zeros are practically constant as functions of ν. Plots of them are shown in Figure 1 for $n = 15$ and $n = 30$. It was determined that monotone growth of all zeros holds even

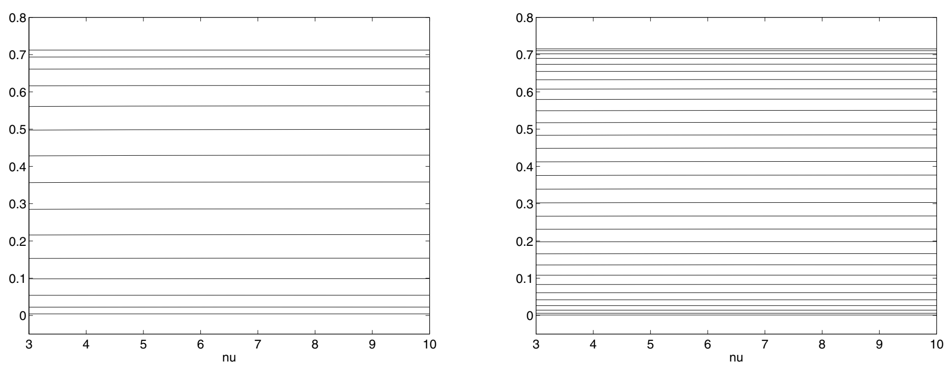

FIGURE 1. The zeros of π_n in the case $\nu_0 = 3$ of Theorem 1(a) for $n = 15$ (on the left) and $n = 30$ (on the right).

for smaller values of ν, namely for $\nu \geq 1.6926$ when $n = 15$, and for $\nu \geq 1.7064$ when $n = 30$. Thus, Theorem 1(a) is not sharp with regard to the interval of monotonicity.

Example 2. The zeros of π_n (of Theorem 1(b)) for $n = 15$ and $c = 1$ ($\nu_0 = \infty$), $\mu = 0$, and $0 < \nu \le 10$.

In Figure 2 the zeros of π_n are shown for $n = 15$, on the left when $0 < \nu < 2$, and on the right when $2 \le \nu \le 10$. In the former case, some of the zeros are very large, so that the plot is logarithmic in the y-axis. It is seen, and has been checked, that all zeros, as predicted by the theorem, are monotonically decreasing.

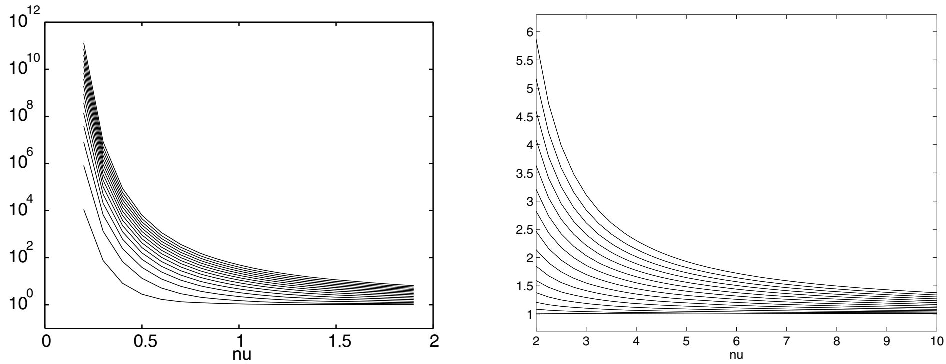

FIGURE 2. The zeros of π_n in the case $c = 1$ of Theorem 1(b) for $n = 15$ on $0 < \nu < 2$ (on the left) and $2 \le \nu \le 10$ (on the right).

The graphs look similar for values of c greater than 1 but, of course, lie above the horizontal line at height c. They require much higher precision (250-digit arithmetic when $c = 6$) to produce.

2.2. **Symmetric sub-range Freud polynomials.**

Theorem 2. (a) *Let $\nu_0 > 0$ and $0 < c \le e^{-1/(2\nu_0)}$. Denote by π_n^* the symmetric sub-range Freud polynomial of degree n orthogonal on $[-c, c]$ with respect to the weight function*

$$w(t; c, \mu, \nu) = |t|^\mu e^{-|t|^\nu}, \quad t \in [-c, c]. \tag{2.4}$$

Then all positive zeros of π_n^* are monotonically increasing on $[2\nu_0, \infty]$ as functions of ν, for every $\mu > -1$ and $n = 2, 3, \ldots$.

(b) *Let $\nu_0 > 0$ and $c \ge e^{-1/(2\nu_0)}$. Denote by π_n^* the symmetric sub-range Freud polynomial of degree n orthogonal on $[-\infty, -c] \cup [c, \infty]$ with respect to the weight function*

$$w(t; c, \mu, \nu) = |t|^\mu e^{-|t|^\nu}, \quad t \in [-\infty, -c] \cup [c, \infty]. \tag{2.5}$$

Then all positive zeros of π_n^* are monotonically decreasing on $(0, 2\nu_0)$ as functions of ν, for every $\mu > -1$ and $n = 2, 3, \ldots$.

Proof. (a) Since in this case the weight function is even and the interval of orthogonality is symmetric with respect to the origin, the orthogonal polynomial of even degree $2n$ is $\pi_{2n}^*(t) = \pi_n^+(t^2)$ and the one of odd degree is $\pi_{2n+1}^*(t) = t\pi_n^-(t^2)$, where $\pi_n^{\pm}$ is orthogonal on $[0, c^2]$ relative to the weight function $w^{\pm}(t) = t^{\mp 1/2}w(t^{1/2})$ (cf. [1, Theorem 1.18]). Thus, the positive zeros of π_{2n}^*, resp. π_{2n+1}^*, are the square root of the zeros of π_n^+, resp. π_n^-. The weight functions for the latter polynomials are $t^{(\mu-1)/2}e^{-t^{\nu/2}}$, resp. $t^{(\mu+1)}e^{-t^{\nu/2}}$. To both of them, part (a) of Theorem 1 can be applied if ν is replaced by $\nu/2$ and c by c^2, showing that the square root of the zeros of $\pi_n^{\pm}$, hence also the zeros themselves, are monotonically increasing on $[\nu_0, \infty)$ if $c^2 \le e^{-1/\nu_0}$ and $\nu/2 \ge \nu_0$, that is, if $c \le e^{-1/(2\nu_0)}$ and $\nu \ge 2\nu_0$.

(b) The polynomials $\pi_n^{\pm}$ are now orthogonal on $[c^2, \infty]$ with respect to the weight function $w^{\pm}$. The proof then proceeds as in part (a), but applying part (b) of Theorem 1, again replacing ν by $\nu/2$ and c by c^2. □

As to the scope and sharpness of Theorem 2, here remarks similar to those after Theorem 1 also apply.

3. Empirical results

For simplicity, we concentrate on the case $\mu = 0$, but will indicate what effect other values of μ may have on our results. Also with regard to the range of ν-values, we will generally assume $0 < \nu \le 10$, which seems to be the interval in which the more interesting monotonicity properties of the zeros play out.

As already noted, there are significant gaps in part (a) of the theorems of Section 2 with regard to intervals of orthogonality covered, and deficiencies in part (b) with regard to sharpness. Here, we fill the gaps and remove the deficiencies by numerical computation.

3.1. **Lower one-sided sub-range Freud polynomials.** The interval of orthogonality $[0, c]$, $0 < c < 1$, is covered by Theorem 1(a) of Section 2.1. It is not a particularly interesting case, since all zeros are essentially constant as functions of ν. The same is still true when $c = 1$ (the limiting case $\nu_0 = \infty$ of Theorem 1(a)), as is shown in Figure 3, depicting the zeros of π_n for $n = 15$ and $n = 30$.

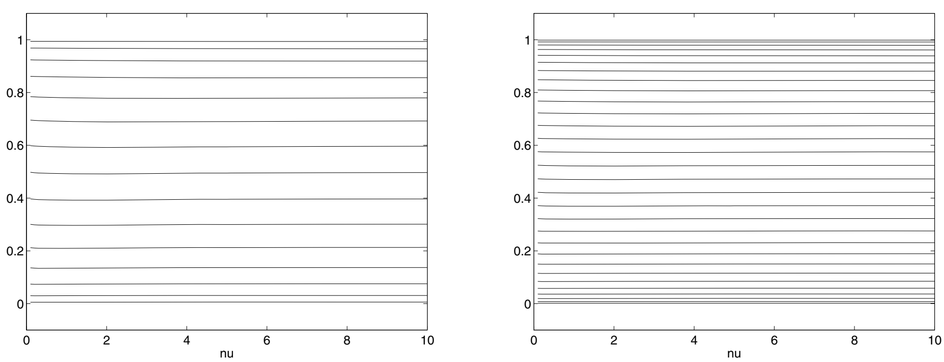

FIGURE 3. The zeros of π_n when $c = 1$ for $n = 15$ (on the left) and $n = 30$ (on the right).

To provide an idea of how the zeros behave when $c > 1$, we look at the case $c = 2$ and show graphs of them in Figure 4 for $n = 1, 7, 15$, and 30. The case $n = 1$ is somewhat special, the zero decreasing to a minimum value and increasing almost imperceptively thereafter. For $n > 1$, the appearance of the graphs resembles that of a waterfall, a gentle one when c is relatively small, and a more precipitous one for larger c; see, e.g., Figure 5, where $c = 6$. Although it may appear that all zeros are collectively decreasing, this is not quite true; there are exceptional intervals early on, when $\nu \le \nu_1$, where $\nu_1 = 1.392, 1.420, 1.420$ for respectively $n = 7, 15, 30$, and also for ν much larger than 10. But all these exceptions occur in the flat parts of the graphs and are quite minute and not visible to the naked eye.

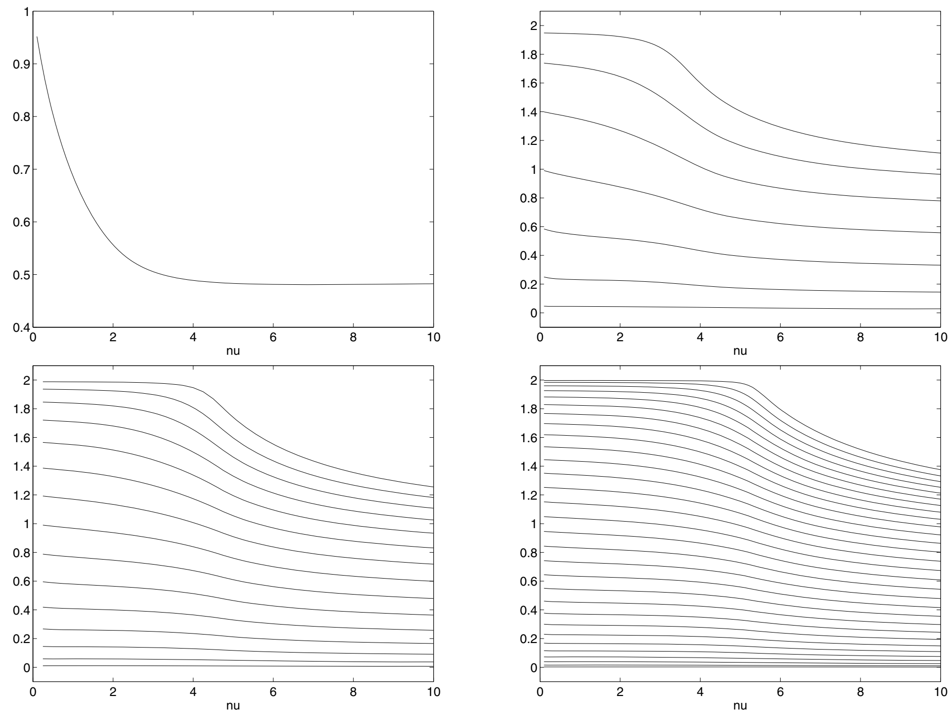

FIGURE 4. The zeros of π_n when $c = 2$ for $n = 1, 7, 15, 30$ (from top left to bottom right).

3.2. **Upper one-sided sub-range Freud polynomials.** Theorem 1(b) covers intervals $[c, \infty]$ with $0 < c < \infty$. It is sharp when $c \ge 1$ ($\nu_0 = \infty$), in which case all zeros decrease monotonically on $0 < \nu < \infty$. Plots of them have been shown in Figure 2, for $0 < \nu \le 10$.

Here we wish to discuss in detail the sharpness of Theorem 1(b) for selected values of $c < 1$; specifically, for given ν_0 we compute the true interval $(0, \nu_0^*)$ on which all zeros decrease monotonically, for all $n \ge 1$ and all $\mu > -1$.

To begin with, we found evidence, by numerical experimentation, that any interval of monotonicity expands as either n, μ, or both, are increased. The least favorable case, therefore, is $n = 1$ and $\mu > -1$ very close to -1, say, $\mu = -.99999$.

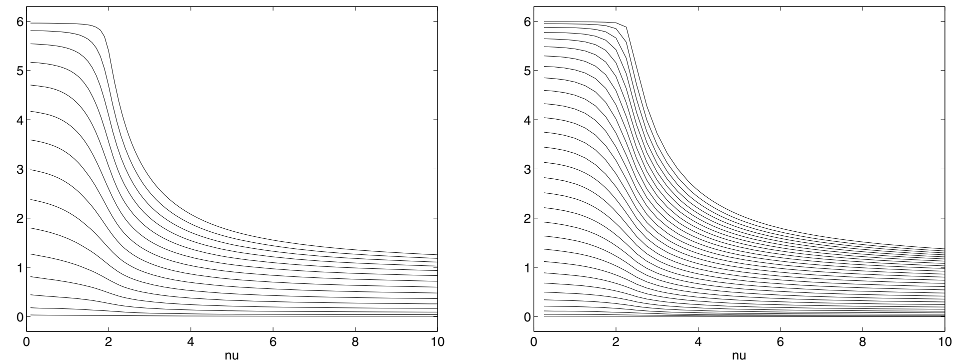

FIGURE 5. The zeros of π_n when $c = 6$ for $n = 15$ (on the left) and $n = 30$ (on the right).

In this case it is relatively straightforward to compute the desired interval $(0, \nu_0^*)$ as a function of ν_0. Results for selected values of ν_0 are shown in Table 1. It can be seen that these intervals are significantly larger than the intervals $(0, \nu_0)$ claimed in Theorem 1(b), but like the latter become smaller with decreasing ν_0, that is, decreasing c.

TABLE 1. Worst-case intervals $(0, \nu_0^*)$ of monotonic decrease of all zeros ($n = 1$ and $\mu \approx -1$).

ν_0	$c = e^{-1/\nu_0}$	ν_0^*	ν_0	$c = e^{-1/\nu_0}$	ν_0^*
6	.8464...	39.336	.6	.1887...	5.1103
4	.7788...	26.594	.5	.1353...	4.5158
2	.6065...	13.877	.4	.0820...	3.9393
1	.3678...	7.5703	.3	.0356...	3.3930
.9	.3291...	6.9481	.2	6.738×10^{-3}	2.8985
.8	.2865...	6.3295	.1	4.540×10^{-5}	2.4861
.7	.2396...	5.7162			

Notice that in the last few entries of Table 1 we are getting very close to the case of half-range Freud polynomials. The fact that the corresponding intervals $(0, \nu_0^*)$ remain finite, and even become a bit smaller, suggests that the zeros of the half-range Freud polynomials are not likely to collectively decrease for arbitrary $n \geq 1$ and $\mu > -1$. We will confirm and quantify this computationally in the next subsection.

3.3. **Half-range Freud polynomials.** Here we explore computationally how ν_0^*, the upper endpoint of the interval $(0, \nu_0^*)$ in which all zeros decrease monotonically, depends on n for $\mu = \mu_- = -.9999999$, about the least favorable value of μ, and also for $\mu = -1/2$, 0, 1/2, and 1. The results are shown in Table 2. Notice the extent of monotonic expansion of the interval $(0, \nu_0^*)$ when n and/or μ are increased. We can see from this table that, for example, all zeros of the half-range Freud polynomial

π_n, for any $\mu > -1$ (more precisely, $\mu \geq \mu_-$), decrease monotonically for all ν in the interval $(0, 10]$ when $n \geq 2$, and for all ν in the interval $(0, 100]$ when $n \geq 6$.

TABLE 2. The intervals $(0, \nu_0^*)$ of monotonic decrease of all zeros of half-range Freud polynomials π_n in dependence of n and μ.

$\mu = \mu_-$		$\mu = -1/2$		$\mu = 0$		$\mu = 1/2$		$\mu = 1$	
n	ν_0^*	n	ν_0^*	n	ν_0^*	n	ν_0^*	n	ν_0^*
1	2.1662	1	4.5574	1	6.8949	1	9.2204	1	11.541
2	11.541	2	15.371	2	19.204	2	23.039	2	26.874
3	26.874	3	32.233	3	37.593	3	42.955	3	48.318
4	48.318	4	55.207	4	62.098	4	68.990	4	75.882
5	75.882	5	84.303	5	92.725	5	101.15		
6	109.57								

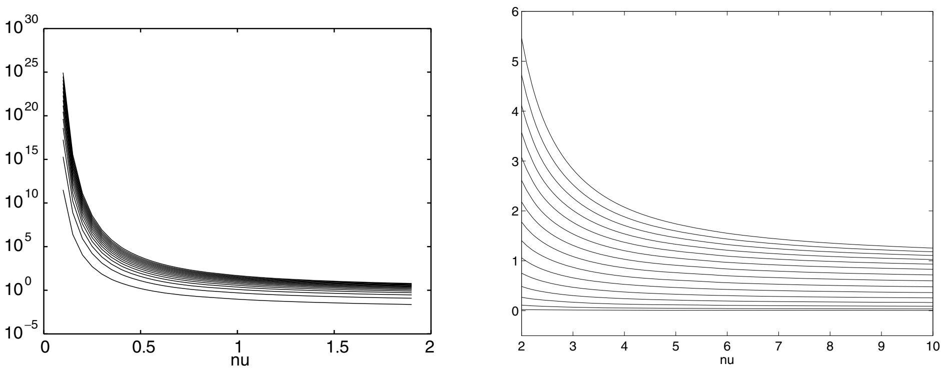

FIGURE 6. The zeros of the half-range Freud polynomial π_n of degree $n = 15$ on the interval $0 < \nu < 2$ (on the left) and $2 \leq \nu \leq 10$ (on the right) when $\mu = 0$.

We show plots of the zeros in Figure 6 for $n = 15$ and $\mu = 0$.

3.4. **Lower symmetric sub-range Freud polynomials.** Symmetric intervals $[-c, c]$ for $0 < c < 1$ are covered by Theorem 2(a). Since all zeros are then practically constant as functions of ν, even in the limit case $c = 1$ (that is, $\nu_0 = \infty$), when the theorem is devoid of content, the case $0 < c \leq 1$ is not of particular interest. For the more interesting cases $c > 1$, we again must rely on computational exploration.

One expects that the behavior of the positive zeros of π_n^* will be similar to that of all zeros of π_n in the asymmetric case. This is indeed borne out by numerical computation. One finds again the waterfall-like descent of all positive zeros, the steepness of the descent being larger the larger the parameter c. It does not seem necessary, therefore, to illustrate this pictorially.

3.5. **Upper symmetric sub-range Freud polynomials.** As in the asymmetric case of Section 3.2, also here in the symmetric case there is a need to sharpen part (b) of Theorem 2, that is, to determine, for given ν_0, the exact interval $(0, \nu_0^*)$ of monotone decrease of all positive zeros in the worst-case scenario of μ very close to -1 and $n = 1$. (Increasing μ and/or n, as in the asymmetric case, yields larger intervals $(0, \nu_0^*)$.) Results analogous to those in Table 1 are shown in Table 3.

TABLE 3. Worst-case intervals $(0, \nu_0^*)$ of monotonic decrease of all positive zeros ($n = 1$ and $\mu \approx -1$).

ν_0	$c = e^{-1/(2\nu_0)}$	ν_0^*	ν_0	$c = e^{-1/(2\nu_0)}$	ν_0^*
6	.9200...	78.671	.6	.4345...	10.221
4	.8824...	53.189	.5	.3678...	9.0315
2	.7788...	27.753	.4	.2865...	7.8786
1	.6065...	15.141	.3	.1888...	6.7860
.9	.5737...	13.896	.2	$8.208{\times}10^{-2}$	5.7970
.8	.5352...	12.659	.1	$6.737{\times}10^{-3}$	4.9722
.7	.4895...	11.432			

Here again, the last few entries, pertaining to cases very close to ordinary Freud polynomials, suggest that also the zeros of Freud polynomials are not likely to collectively decrease without some qualifications.

3.6. **Freud polynomials.** Computations analogous to those carried out in Section 3.3 have been made for the case of Freud polynomials. With notation as in Section 3.3, the results are shown in Table 4. Notice again the monotonic behavior of the intervals $(0, \nu_0^*)$ for increasing n and/or μ. It can be seen that all positive zeros of the Freud polynomial π_n^* decrease monotonically on the interval $(0, 10]$ if $n \geq 3$, and on the interval $(0, 100]$ if $n \geq 9$.

TABLE 4. The intervals $(0, \nu_0^*)$ of monotonic decrease of all positive zeros of Freud polynomials π_n^* in dependence of n and μ.

$\mu = \mu_-$		$\mu = -1/2$		$\mu = 0$		$\mu = 1/2$		$\mu = 1$	
n	ν_0^*	n	ν_0^*	n	ν_0^*	n	ν_0^*	n	ν_0^*
2	4.3325	2	6.7519	2	9.1147	2	11.457	2	13.790
3	13.790	3	16.117	3	18.441	3	20.762	3	23.082
4	23.082	4	26.910	4	30.741	4	34.574	4	38.408
5	38.408	5	42.242	5	46.077	5	49.913	5	53.749
6	53.749	6	59.106	6	64.465	6	69.825	6	75.186
7	75.186	7	80.548	7	85.910	7	91.273	7	96.636
8	96.636	8	103.52						
9	124.20								

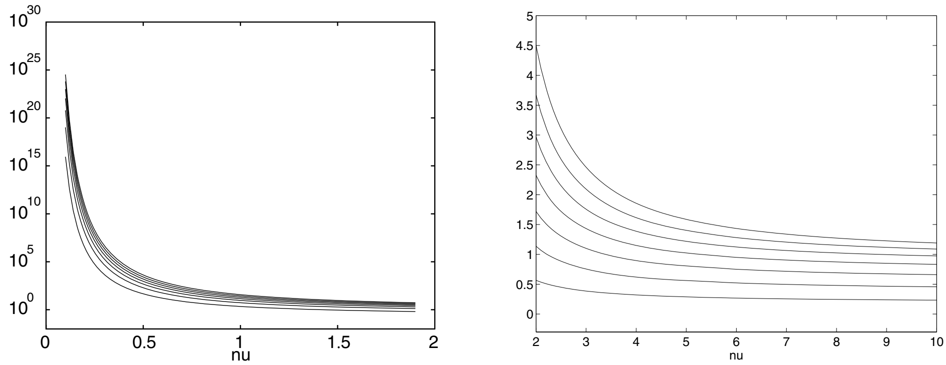

FIGURE 7. The positive zeros of the Freud polynomial π_n^* for $n = 15$ on $0 < \nu < 2$ (on the left) and $2 \leq \nu \leq 10$ (on the right).

Plots of the zeros for $n = 15$ are shown in Figure 7. In the process of producing these plots it was checked that all zeros indeed decrease on $(0, 10]$.

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DEPARTMENT OF COMPUTER SCIENCE, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907-2066

E-mail address: wgautschi@purdue.edu