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ERROR ESTIMATES FOR THE STANDARD GALERKIN-FINITE ELEMENT METHOD FOR THE SHALLOW WATER EQUATIONS

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ABSTRACT. We consider a simple initial-boundary-value problem for the shallow water equations in one space dimension and also the analogous problem for a symmetric variant of the system. Assuming smoothness of solutions, we discretize these problems in space using standard Galerkin-finite element methods and prove L^2-error estimates for the semidiscrete problems for quasiuniform and uniform meshes. In particular we show that in the case of spatial discretizations with piecewise linear continuous functions on a uniform mesh, suitable compatibility conditions at the boundary and superaccuracy properties of the L^2 projection on the finite element subspaces lead to an optimal-order $O(h^2)$ L^2-error estimate. We also examine the temporal discretization of the semidiscrete problems by a third-order explicit Runge-Kutta method due to Shu and Osher and prove L^2-error estimates of optimal order in the temporal variable under a Courant-number stability condition. In a final section of remarks we prove optimal-order L^2-error estimates for smooth spline spatial discretizations of the periodic initial-value problem for the systems. We also prove that small-amplitude, appropriately transformed solutions of the symmetric system are close to the corresponding solutions of the usual system while they are both smooth, thus providing a justification of the symmetric system.

1. Introduction

In this paper we will analyze standard Galerkin approximations to the system of *shallow water equations* (also known as *Saint-Venant equations*)

$$\begin{aligned} &\eta_t + u_x + (\eta u)_x = 0, \\ &u_t + \eta_x + u u_x = 0, \end{aligned} \tag{1.1}$$

which is an approximation of the two-dimensional Euler equations of water-wave theory that models two-way propagation of long waves of finite amplitude on the surface of an ideal fluid in a uniform horizontal channel of finite depth, [21], [13]. The variables in (1.1) are nondimensional and unscaled; $x \in \mathbb{R}$ and $t \geq 0$ are proportional to position along the channel and time, respectively, and $\eta = \eta(x,t)$ and $u = u(x,t)$ are proportional to the elevation of the free surface above a level of rest corresponding to $\eta = 0$ and to the horizontal velocity of the fluid, respectively. (In these variables the bottom of the channel lies at a depth equal to -1.)

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At http://arxiv.org/abs/1403.5699 interested readers may find an extended version of the present paper ([3]), including additional results as well as details of proofs omitted herein.

It is well known that, given smooth initial conditions $\eta(x,0)=\eta_0(x)$, $u(x,0)=u_0(x)$, $x\in\mathbb{R}$, the initial-value problem for (1.1) has smooth solutions in general only locally in t. The existence of smooth solutions may be studied by standard methods of the theory of nonlinear hyperbolic systems; cf. e.g. [12, Ch. 2] and [18, Ch. 16].

In this paper we shall consider the following initial-boundary-value problem (ibvp) for (1.1) posed on the spatial interval $[0,1]$. We seek $\eta=\eta(x,t)$, $u=u(x,t)$, $0\le x\le 1$, $0\le t\le T$, satisfying

$$\text{(SW)}\qquad \begin{aligned} &\left.\begin{aligned} &\eta_t+u_x+(\eta u)_x=0,\\ &u_t+\eta_x+uu_x=0, \end{aligned}\right.\quad 0\le x\le 1,\ \ 0\le t\le T,\\ &\eta(x,0)=\eta_0(x),\quad u(x,0)=u_0(x),\quad 0\le x\le 1,\\ &u(0,t)=0,\quad u(1,t)=0,\quad 0\le t\le T. \end{aligned}$$

In [14] Petcu and Temam established the local existence-uniqueness of H^2-solutions of (SW). Specifically, they proved that given u_0, $\eta_0\in H^2$ such that $1+\eta_0(x)\ge 2\alpha$ for some constant $\alpha>0$ for $x\in[0,1]$, there exists a $T_*=T_*(\|\eta_0\|_2,\|u_0\|_2)>0$ and a unique solution (u,η) of (SW) for $0\le t\le T_*$ such that $(u,\eta)\in L^\infty(0,T_*;H^2)$ and $1+\eta(x,t)\ge\alpha$ for $x\in[0,1]$, $t\in(0,T_*]$. In the course of the proof it is also shown that $u\in L^\infty(0,T_*;H^2\cap H_0^1)$, that u_t, $\eta_t\in L^\infty(0,T_*;H^1)$ and that $\eta_x(0,t)=\eta_x(1,t)=0$ for $0<t<T_*$; it is also assumed that $u_0(0)=u_0(1)=0$ and $\eta_0'(0)=\eta_0'(1)=0$. (Here and in the sequel, for integer $k\ge 0$, H^k, $\|\cdot\|_k$ will denote the usual L^2-based Sobolev space of classes of functions on $[0,1]$ and its associated norm and H_0^1 the subspace of H^1 whose elements are zero at $x=0,1$. For a Banach space X of functions on $[0,1]$, $L^\infty(0,T;X)$ will denote the space of L^∞ maps from $[0,T]$ into X.)

We shall also consider the analogous ibvp for a *symmetric* variant of the shallow water equations, posed again on $[0,1]$. For this purpose we seek $\eta=\eta(x,t)$, $u=u(x,t)$, $0\le x\le 1$, $0\le t\le T$, satisfying

$$\text{(SSW)}\qquad \begin{aligned} &\left.\begin{aligned} &\eta_t+u_x+\tfrac12(\eta u)_x=0,\\ &u_t+\eta_x+\tfrac32 uu_x+\tfrac12\eta\eta_x=0, \end{aligned}\right.\quad 0\le x\le 1,\ \ 0\le t\le T,\\ &\eta(x,0)=\eta_0(x),\quad u(x,0)=u_0(x),\quad 0\le x\le 1,\\ &u(0,t)=0,\quad u(1,t)=0,\quad 0\le t\le T. \end{aligned}$$

Here, the nonlinear hyperbolic system is symmetric; existence-uniqueness of H^2-solutions of the ibvp (SSW) for T sufficiently small may be established if one follows the argument of [14]. Specifically, it may be shown that if $u_0\in H^2\cap H_0^1$, $\eta_0\in H^2$ with $\eta_0'(0)=\eta_0'(1)=0$, and α is a constant such that $1+\frac12\eta_0(x)\ge 2\alpha>0$, $x\in[0,1]$, then there exist a $T_*=T_*(\|u_0\|_2,\|\eta_0\|_2)>0$ and a unique solution (u,η) of (SSW) for $0\le t\le T_*$ such that $u\in L^\infty(0,T_*;H^2\cap H_0^1)$, $\eta\in L^\infty(0,T_*;H^2)$, u_t, $\eta_t\in L^\infty(0,T_*;H^1)$. Moreover $1+\frac12\eta(x,t)\ge\alpha>0$ for $(x,t)\in[0,1]\times(0,T_*]$ and $\eta_x(0,t)=\eta_x(1,t)=0$ for $0\le t\le T_*$.

We chose this symmetric system motivated by the work of Bona, Colin, and Lannes, [6], on completely symmetric Boussinesq-type dispersive approximations of small-amplitude, long-wave solutions of the Euler equations. In Section 6.2 we derive the symmetric system in the context of the small-amplitude, scaled shallow water equations and study its relation to the usual shallow water system by analytical and numerical means.

In the analysis of the Galerkin approximations that we pursue in this paper we generally prove in parallel error estimates for both (SW) and (SSW). It will be seen that as a result of the symmetry of the latter system, the proofs for (SSW) are more straightforward and generally hold under less stringent hypotheses compared to their (SW) analogs. Let us also mention that it is easy to see that the solution of (SSW) satisfies the L^2-conservation equation

$$\int_0^1 \big(\eta^2(x,t) + u^2(x,t)\big)dx = \int_0^1 \big(\eta_0^2(x) + u_0^2(x)\big)dx \tag{1.2}$$

for $0 \leq t \leq T$.

We begin the error analysis in Section 2 by first considering the standard Galerkin semidiscretizations of (SW) and (SSW) using for the spatial approximation piecewise polynomial functions of order $r \geq 2$ (i.e. of degree $r-1 \geq 1$) with respect to a quasiuniform mesh on $[0,1]$ of maximum meshlength h; the spaces consist of C^k functions, where $0 \leq k \leq r-2$. We assume throughout that the solutions of (SW) and (SSW) are sufficiently smooth for the purposes of the error estimates. In the case of (SSW) the error analysis is straightforward due to the symmetry of the system and yields, for $r \geq 2$, an L^2-error estimate of $O(h^{r-1})$ for the Galerkin approximations of η and u. It is well known that this is the expected best order of convergence in L^2 for the standard Galerkin semidiscretization of first-order hyperbolic problems on general quasiuniform meshes. (In this proof and in subsequent error estimates in this paper we compare the Galerkin approximation with the L^2 projection of the solution of the p.d.e. problem onto the finite element subspaces and estimate their difference.) For (SW) the proof is more complicated: we use a symmetrizing choice of test function in the error equation corresponding to the second p.d.e. of (SW), a 'superapproximation' property of the finite element subspaces, and the positivity of $1+\eta$ in order to establish the expected $O(h^{r-1})$ L^2-error estimates for η and u assuming now that $r \geq 3$. This last assumption is needed in the proof for the control of the $W^{1,\infty}$ norm of an intermediate error term. Thus our proof for (SW) and its assumptions resemble those of the analogous proof of Dupont, [9], in the case of a 2×2 nonlinear hyperbolic system which is a close relative of (SW). It is worth noting that numerical experiments, the results of which are presented at the end of Section 2, suggest that for $r = 2$, i.e. for piecewise linear continuous functions on a quasiuniform mesh, the L^2- and L^∞-errors of the Galerkin approximations to η and u have $O(h)$ bounds, i.e. that the assumption $r \geq 3$ may not be needed. In fact, for special quasiuniform meshes, e.g. for piecewise uniform or gradually varying meshes, numerical experiments in [3] indicate that the error bounds are of $O(h^2)$, resembling those of the uniform mesh case (see below.)

In Sections 3 and 4 we examine the error of the standard Galerkin semidiscretization of (SW) and (SSW) in the special case of subspaces of continuous, piecewise linear functions on a uniform mesh on $[0,1]$. It is well known that for *linear*, first-order hyperbolic equations in the uniform mesh case the standard Galerkin approximations may enjoy optimal-order L^2-convergence, i.e. of $O(h^r)$, as a result of superaccuracy due to cancellations in the interior mesh intervals and to suitable compatibility conditions at the boundary, provided the solutions of the continuous problem are smooth enough. Early evidence of this were the classic results of Dupont, [10], in the case of $r = 2$ and $r = 4$ (with $k = 2$, i.e. cubic splines) and e.g. of Thomée and Wendroff, [20], for problems with variable coefficients in the case of subspaces consisting of smooth splines of arbitrary order ($k = r-2$, $r \geq 2$). In these works the *periodic* initial-value problem was under consideration; the spatial periodicity and the assumed smoothness of solutions automatically furnish the requisite compatibility conditions at the boundary that yield superaccuracy. In Section 6.1 of [2] we pointed out how compatibility at the boundary for smooth solutions of a simple initial-boundary-value problem for a first-order linear hyperbolic equation gives the superaccuracy estimate in the case $r = 2$ for uniform mesh. We also refer the reader to the papers [11] and [23] for results and references to the Chinese literature on related topics.

In order to treat the nonlinear case, in Section 3 of the paper at hand we prove some superconvergence properties of the L^2 projections of smooth functions on $[0,1]$ satisfying suitable boundary conditions onto spaces of piecewise linear, continuous functions defined on a uniform mesh in $[0,1]$. The key results are Lemmas 3.3 and 3.6, in which it is shown that integrals of the form $\int_{I_i} we dx$, where w is a C^2 function and e is the error of the L^2 projection of a C^4 function satisfying suitable boundary conditions at 0 and 1, are, for any mesh interval I_i, of $O(h^5)$. These results are used in Section 4, where optimal-order $O(h^2)$ L^2-error estimates for the Galerkin semidiscretizations of (SSW) and (SW) are established. It is assumed that the ibvp's have classical, sufficiently smooth solutions, which, as a consequence of their smoothness, must satisfy natural compatibility conditions at 0 and 1. Again the proof for (SSW) is relatively straightforward, while in the case of (SW) some additional twists are needed. These theoretical results are confirmed in numerical experiments at the end of Section 4. These also indicate that the analogous L^2-errors for spatial discretizations with cubic splines ($k = 2$, $r = 4$) on uniform meshes have convergence rates which are practically equal to 4, i.e. optimal.

In Section 5 we turn to the temporal discretization of the o.d.e. systems represented by the semidiscretizations considered in Sections 2 and 4. In [9] Dupont analyzed, in the case of a system similar to the shallow water equations, the convergence of a linearized Crank-Nicolson scheme. In the paper at hand we analyze a fully discrete scheme for the (SW) system in which the standard Galerkin semidiscretization is coupled with an explicit, third-order accurate Runge-Kutta time-stepping method due to Shu and Osher, [17], that has been extensively used as a time-stepping scheme for the numerical approximation of hyperbolic systems in conservation law form with finite-volume or DG spatial discretizations. Since our emphasis in the proof is on the temporal discretization aspect of the fully discrete problem, we chose the most straightforward to treat spatial discretization, i.e. piecewise polynomial functions of order $r \geq 3$ on a quasiuniform mesh. Thus, as was mentioned previously, the expected spatial rate of convergence in L^2 is of $O(h^{r-1})$. We prove that there exists a constant λ_0 such that if $k/h \leq \lambda_0$ (here k is the time step), then the L^2-error estimate of the fully discrete scheme is of $O(k^3 + h^{r-1})$. An analogous result holds for (SSW); cf. [3].

As is well known, the explicit Euler scheme is not suitable for approximating in time first-order hyperbolic problems discretized in space by the standard Galerkin method. These semidiscretizations lead to stiff systems of o.d.e.'s; for example, in the case of the initial-periodic boundary-value problem for $u_t + u_x = 0$ on $[0,1]$, the standard Galerkin semidiscretization with splines on uniform meshes leads to systems of o.d.e.'s having imaginary eigenvalues of magnitude of $O(1/h)$. As a result, as we prove in [3], the fully discrete scheme with explicit Euler time stepping has an L^2-error estimate of $O(k + h^{r-1})$ for (SSW) under the restrictive mesh condition $k = O(h^2)$. In [3] we also analyze time stepping for (SSW) with the 'improved Euler' method, a two-stage, explicit, second-order accurate Runge-Kutta scheme and show that it has an L^2-error estimate of $O(k^2+h^{r-1})$ provided the (still restrictive) stability condition $k = O(h^{4/3})$ holds. Both these explicit schemes have no absolute stability intervals on the imaginary axis, as opposed to the Shu-Osher scheme, whose region of absolute stability includes the interval $[-\sqrt{3}, \sqrt{3}]$ on the imaginary axis. The latter fact allows us to show that the Shu-Osher scheme is stable for linear hyperbolic problems, discretized in space by the standard Galerkin method, under a Courant-number stability restriction, a property that persists in the case of (SW), as was mentioned previously.

We should point out that in recent years there have appeared a number of papers with proofs of error estimates of full discretizations of Galerkin-type methods with explicit Runge-Kutta methods for first-order hyperbolic problems. For example, Zhang and Shu have analyzed discontinuous Galerkin methods for scalar conservation laws in [24] and for symmetrizable systems of conservation laws in [25] using a second-order explicit Runge-Kutta method (the explicit trapezoidal rule) for time stepping. For the DG methods analyzed in these papers this full discretization turns out to be stable under a Courant-number restriction $k \leq \alpha h$ for a $\mathbb{P}_1$ spatial discretization but needs k to be of $O(h^{4/3})$ for higher-order polynomial spaces. The same Runge-Kutta scheme is proved by Ying, [22], to yield a stable full discretization and the expected error estimates for a standard Galerkin method for scalar conservation laws in several space dimensions under the condition $k = O(h^{4/3})$. In [26] Zhang and Shu prove error estimates for a fully discrete DG-3$^{\mathrm{d}}$ order Shu-Osher scheme for scalar conservation laws under Courant-number restriction. In [7] Burman *et al.* consider initial-boundary-value problems for first-order linear hyperbolic systems of Friedrichs-type in several space dimensions, discretized in space by a class of symmetrically stabilized finite element methods that includes DG schemes, and in time by explicit Runge-Kutta schemes of second (RK2) and third (RK3) order of accuracy. They prove L^2-error estimates of optimal order in time and quasioptimal in space under Courant-number restrictions for RK2 schemes with $\mathbb{P}_1$ elements and under the condition $k = O(h^{4/3})$ for higher-order elements and under Courant-number restrictions for RK3 schemes. Let us also mention that for a closely related to the shallow water equations *dispersive* system (the 'classical' Boussinesq equations), we proved error estimates in [2], [1], for the classical, four-stage, fourth-order explicit Runge-Kutta temporal discretization of standard Galerkin methods with cubic splines; the error bounds had an $O(k^4)$ dependence under a Courant-number stability condition.

We close the paper with some supplementary remarks in Section 6. In Section 6.1 we consider the *periodic* initial-value problem for the shallow water system and its symmetric version and discretize it in space using the standard Galerkin method with smooth periodic splines of order $r \geq 2$ on a uniform mesh. Using suitable *quasiinterpolants* in the space of periodic splines (cf. [20]), we prove optimal-order, i.e. $O(h^r)$, L^2-error estimates for both systems. In Section 6.2 we first recall the nondimensional *scaled* form of the shallow water equations in the case of long surface waves of *small* amplitude (in which the nonlinear terms of the system are multiplied by the small parameter $\varepsilon = a/h_0$, where a is a typical wave amplitude and h_0 the depth of the channel) and derive the analogous scaled form of the symmetric shallow water equations using the nonlinear change of variables of Bona, Colin, and Lannes, [6]. In view of the classical theory of local existence of solutions of initial-value problems of quasilinear hyperbolic systems and the results of [6], we argue that the difference in suitable norms of appropriately transformed solutions of the Cauchy problems for the two systems is of $O(\varepsilon^2 t)$ for times t up to $O(1/\varepsilon)$. Given that initially smooth solutions of both systems are expected in general to develop singularities after times of $O(1/\varepsilon)$, this result indicates that appropriately transformed, smooth, small-amplitude solutions of the symmetric system remain close to corresponding smooth solutions of the usual system within their life span and provides a justification for the symmetric system. Section 6.3 closes with some numerical experiments which suggest that the difference of the solutions of (SW) and (SSW) (i.e. of the ibvp's) also behaves like $\varepsilon^2 t$ for times up to $O(1/\varepsilon)$.

In summary, the main contributions of the paper at hand are as follows. We consider the initial-boundary-value problem (SW) for the system of shallow water equations, a well-known example of a nonlinear hyperbolic system in one space dimension, in the case of smooth solutions. We first analyze the convergence of its standard Galerkin spatial discretization on quasiuniform meshes and obtain the expected $O(h^{r-1})$ L^2-error estimate. In the case of a general quasiuniform mesh we are not able to dispense with the need for using in the proof at least quadratics (i.e. $r \geq 3$) for the usual, nonsymmetric SW in order to achieve convergence, thus not advancing beyond Dupont's [9] analysis. (We point out how *symmetry*, as e.g. in the case of (SSW), removes this obstacle and allows taking $r = 2$ also.) However, in the case of *uniform mesh* we show $O(h^2)$, i.e. optimal convergence in L^2 for both types of systems for $\mathbb{P}_1$ elements by exploiting cancellation properties in the errors of the L^2 projection of the solution and stressing the role that the compatibility of the boundary conditions plays in the analysis. We then consider full discretizations of the problem using explicit Runge-Kutta schemes in time. Our main analysis concerns the Shu-Osher third-order RK scheme that has been widely used for time stepping in conservation laws. We prove the optimal $O(k^3)$ temporal rate of convergence for this scheme under a Courant-number restriction. We finally point out that in the case of smooth periodic solutions one has $O(h^r)$, i.e. optimal-rate, spatial discretization L^2-error estimates for smooth splines on uniform meshes for both types of systems. We also justify the use of SSW as an equivalent model for SW under a simple nonlinear change of variables in the case of small-amplitude solutions, following the analogous argument put forward for dispersive systems in [6].

In addition to previously introduced notation, in this paper we let $C^k = C^k[0,1]$, $k = 0, 1, 2, \ldots$, denote the space of k times continuously differentiable functions on $[0,1]$ and define $C_0^k = \{\phi \in C^k : \phi(0) = \phi(1) = 0\}$. The inner product and norm on $L^2 = L^2(0,1)$ are denoted by $\|\cdot\|$, $(\cdot,\cdot)$, respectively, while the norms on $L^\infty = L^\infty(0,1)$ and on the L^∞-based Sobolev space $W_\infty^k = W_\infty^k(0,1)$ are denoted by $\|\cdot\|_\infty$, $\|\cdot\|_{k,\infty}$, respectively. We let $\mathbb{P}_r$ be the polynomials of degree $\leq r$ and $\langle\cdot,\cdot\rangle$, $|\cdot|$ be the Euclidean inner product and norm on $\mathbb{R}^N$.

2. Semidiscretization on quasiuniform meshes

Let $0 = x_1 < x_2 < \cdots < x_{N+1} = 1$ denote a quasiuniform partition of $[0,1]$ with $h := \max_i(x_{i+1} - x_i)$, and for integers r, k such that $r \geq 2$, $0 \leq k \leq r-2$, let $S_h = S_h^{r,k} := \{\phi \in C^k : \phi\big|_{[x_j,x_{j+1}]} \in \mathbb{P}_{r-1}, 1 \leq j \leq N\}$ and $S_{h,0} = S_{h,0}^{k,r} =$

$\{\phi \in S_h^{k,r}, \phi(0) = \phi(1) = 0\}$. It is well known that given $w \in H^r$ there exists an element $\chi \in S_h$ such that

$$\|w - \chi\| + h\|w' - \chi'\| \le Ch^r \|w^{(r)}\|, \tag{2.1a}$$

and if $r \ge 3$ in addition (cf. [16]),

$$\|w - \chi\|_2 \le Ch^{r-2}\|w^{(r)}\|, \tag{2.1b}$$

for some constant C independent of h and w, and that a similar property holds in $S_{h,0}$ if $w \in H^r \cap H_0^1$. Let P, P_0 denote the L^2 projection operators onto S_h, $S_{h,0}$, respectively. Then (cf. [8]), there holds that

$$\|Pv\|_\infty \le C\|v\|_\infty \quad \text{if} \quad v \in L^\infty, \tag{2.2a}$$

$$\|Pv - v\|_\infty \le Ch^r\|v\|_{r,\infty} \quad \text{if} \quad v \in W^{r,\infty} \tag{2.2b}$$

and that a similar property holds for P_0 if $v \in W^{r,\infty} \cap H_0^1$. (Here and in the sequel we will denote by C generic constants independent of discretization parameters.)

As a consequence of the quasiuniformity of the mesh the inverse inequalities

$$\|\chi\|_1 \le Ch^{-1}\|\chi\|, \tag{2.3}$$

$$\|\chi\|_{j,\infty} \le Ch^{-(j+1/2)}\|\chi\|, \quad j = 0, 1, \tag{2.4}$$

hold for $\chi \in S_h$. (In (2.4) $\|\chi\|_{0,\infty} = \|\chi\|_\infty$.)

We let the *standard Galerkin semidiscretization* of (SW) be defined as follows: We seek $\eta_h : [0,T] \to S_h$, $u_h : [0,T] \to S_{h,0}$, such that for $t \in [0,T]$,

$$\begin{aligned} (\eta_{ht}, \phi) + (u_{hx}, \phi) + ((\eta_h u_h)_x, \phi) &= 0, \quad \forall \phi \in S_h\,, \\ (u_{ht}, \chi) + (\eta_{hx}, \chi) + (u_h u_{hx}, \chi) &= 0, \quad \forall \chi \in S_{h,0}\,, \end{aligned} \tag{2.5}$$

with initial conditions

$$\eta_h(0) = P\eta_0\,, \qquad u_h(0) = P_0 u_0\,. \tag{2.6}$$

Similarly, we define the analogous semidiscretization of (SSW), given for $t \in [0,T]$ by

$$\begin{aligned} (\eta_{ht}, \phi) + (u_{hx}, \phi) + \tfrac{1}{2}((\eta_h u_h)_x, \phi) &= 0, \quad \forall \phi \in S_h, \\ (u_{ht}, \chi) + (\eta_{hx}, \chi) + \tfrac{3}{2}(u_h u_{hx}, \chi) + \tfrac{1}{2}(\eta_h \eta_{hx}, \chi) &= 0, \quad \forall \chi \in S_{h,0}, \end{aligned} \tag{2.7}$$

$$\eta_h(0) = P\eta_0\,, \qquad u_h(0) = P_0 u_0. \tag{2.8}$$

Upon choice of bases for S_h, S_h^0, it is seen that the semidiscrete problems (2.5)-(2.6) and (2.7)-(2.8) represent initial-value problems (ivp's) for systems of o.d.e's. Clearly, these ivp's have unique solutions at least locally in time. One conclusion of Propositions 2.1 and 2.2 is that they possess unique solutions up to at least $t = T$, where $[0,T]$ is the interval of existence of smooth solutions of (SW) or (SSW) as the case may be. We start with the error analysis of the semidiscrete symmetric system (2.7)-(2.8), which is quite straightforward, due to the symmetry of (SSW).

Proposition 2.1. *Let (η, u) be the solution of* (SSW). *Then the semidiscrete ivp* (2.7)-(2.8) *has a unique solution (η_h, u_h) for $0 \le t \le T$ satisfying*

$$\max_{0\le t\le T}\big(\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|\big) \le Ch^{r-1}. \tag{2.9}$$

Proof. Setting $\phi = \eta_h$ and $\chi = u_h$ in (2.7) and adding the resulting equations we obtain the discrete analog of (1.2), i.e. that the conservation property

$$\|\eta_h(t)\|^2 + \|u_h(t)\|^2 = \|\eta_h(0)\|^2 + \|u_h(0)\|^2 \tag{2.10}$$

holds in the interval of existence of solutions of (2.7)-(2.8). By standard o.d.e. theory we conclude that the ivp (2.7)-(2.8) possesses unique solutions in any finite temporal interval $[0, t^*]$ and in particular in $[0, T]$.

We now let $\rho := \eta - P\eta$, $\theta := P\eta - \eta_h$, $\sigma := u - P_0 u$, $\xi := P_0 u - u_h$. Using (SSW) and (2.7)-(2.8) we obtain for $0 \le t \le T$,

$$\begin{aligned} &(\theta_t, \phi) + (\sigma_x + \xi_x, \phi) \\ &\qquad + \tfrac{1}{2}((\eta u - \eta_h u_h)_x, \phi) = 0, \quad \forall \phi \in S_h, \end{aligned} \tag{2.11}$$

$$\begin{aligned} &(\xi_t, \chi) + (\rho_x + \theta_x, \chi) \\ &\qquad + \tfrac{3}{2}(u u_x - u_h u_{hx}, \chi) + \tfrac{1}{2}(\eta\eta_x - \eta_h \eta_{hx}, \chi) = 0, \quad \forall \chi \in S_{h,0}. \end{aligned} \tag{2.12}$$

Note that $\eta u - \eta_h u_h = \eta(\sigma + \xi) + u(\rho + \theta) - (\rho + \theta)(\sigma + \xi)$, $u u_x - u_h u_{hx} = (u\sigma)_x + (u\xi)_x - (\sigma\xi)_x - \sigma\sigma_x - \xi\xi_x$, $\eta\eta_x - \eta_h\eta_{hx} = (\eta\rho)_x - \theta\theta_x + (\eta\theta)_x - (\rho\theta)_x - \rho\rho_x$. Take $\phi = \theta$ in (2.11) and obtain, for $0 \le t \le T$, using integration by parts,

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 + \big([(1 + \tfrac{\eta}{2})\xi]_x, \theta\big) &= -(\sigma_x, \theta) - \tfrac{1}{2}((\eta\sigma)_x, \theta) - \tfrac{1}{2}((u\rho)_x, \theta) \\ &\quad - \tfrac{1}{2}((u\theta)_x, \theta) + \tfrac{1}{2}((\rho\sigma)_x, \theta) + \tfrac{1}{2}((\theta\sigma)_x, \theta) + \tfrac{1}{2}((\rho\xi)_x, \theta) + \tfrac{1}{2}((\theta\xi)_x, \theta). \end{aligned} \tag{2.13}$$

We now examine the various terms in the r.h.s. of (2.13). Integration by parts yields that $((\theta\xi)_x, \theta) = \frac{1}{2}(\xi_x\theta, \theta)$. Using now the approximation and inverse properties of S_h and $S_{h,0}$ and integration by parts we have

$$\begin{aligned} &|(\sigma_x, \theta)| \le \|\sigma_x\|\|\theta\| \le Ch^{r-1}\|\theta\|, \qquad |((\eta\sigma)_x, \theta)| \le C\|\sigma\|_1\|\theta\| \le Ch^{r-1}\|\theta\|, \\ &|((u\rho)_x, \theta)| \le C\|\rho\|_1\|\theta\| \le Ch^{r-1}\|\theta\|, \qquad |((u\theta)_x, \theta)| = \tfrac{1}{2}|(u_x\theta, \theta)| \le C\|\theta\|^2, \\ &|((\rho\sigma)_x, \theta)| \le \|\rho\|_\infty\|\sigma_x\|\|\theta\| + \|\sigma\|_\infty\|\rho_x\|\|\theta\| \le Ch^{2r-1}\|\theta\|, \\ &|((\theta\sigma)_x, \theta)| = \tfrac{1}{2}|(\sigma_x\theta, \theta)| \le C\|\theta\|^2, \\ &|((\rho\xi)_x, \theta)| \le \|\rho_x\|_\infty\|\xi\|\|\theta\| + \|\rho\|_\infty\|\xi_x\|\|\theta\| \le C\|\xi\|\|\theta\|. \end{aligned}$$

Therefore (2.13) and the above yield for $0 \le t \le T$,

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 + \big([(1 + \tfrac{\eta}{2})\xi]_x, \theta\big) \le \tfrac{1}{4}(\xi_x\theta, \theta) + C(h^{r-1}\|\theta\| + \|\theta\|^2 + \|\xi\|^2). \tag{2.14}$$

Now take $\chi = \xi$ in (2.12). Then for $0 \le t \le T$ using integration by parts we have

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\|\xi\|^2 - \big([(1 + \tfrac{\eta}{2})\xi]_x, \theta\big) \\ &\quad = -\tfrac{1}{4}(\xi_x\theta, \theta) - (\rho_x, \xi) - \tfrac{3}{2}((u\sigma)_x, \xi) - \tfrac{3}{2}((u\xi)_x, \xi) + \tfrac{3}{2}((\sigma\xi)_x, \xi) \\ &\qquad + \tfrac{3}{2}(\sigma\sigma_x, \xi) - \tfrac{1}{2}((\eta\rho)_x, \xi) - \tfrac{1}{2}(\eta_x\theta, \xi) + \tfrac{1}{2}((\rho\theta)_x, \xi) + \tfrac{1}{2}(\rho\rho_x, \xi). \end{aligned} \tag{2.15}$$

Again using (2.1a)-(2.4) and integration by parts we see that

$$\begin{aligned} &|(\rho_x, \xi)| \le \|\rho_x\|\|\xi\| \le Ch^{r-1}\|\xi\|, \qquad |((u\sigma)_x, \xi)| \le C\|\sigma\|_1\|\xi\| \le Ch^{r-1}\|\xi\|, \\ &|((u\xi)_x, \xi)| = \tfrac{1}{2}|(u_x\xi, \xi)| \le C\|\xi\|^2, \\ &|((\sigma\xi)_x, \xi)| = \tfrac{1}{2}|(\sigma_x\xi, \xi)| \le C\|\sigma_x\|_\infty\|\xi\|^2 \le C\|\xi\|^2, \\ &|(\sigma\sigma_x, \xi)| \le \|\sigma\|_\infty\|\sigma_x\|\|\xi\| \le Ch^{2r-1}\|\xi\|, \qquad |((\eta\rho)_x, \xi)| \le C\|\rho\|_1\|\xi\| \le Ch^{r-1}\|\xi\|, \\ &|(\eta_x\theta, \xi)| \le C\|\theta\|\|\xi\|, \qquad |((\rho\theta)_x, \xi)| \le C\|\theta\|\|\xi\|, \qquad |(\rho\rho_x, \xi)| \le Ch^{2r-1}\|\xi\|. \end{aligned}$$

Therefore, by (2.15), for $0 \le t \le T$,

$$\tfrac{1}{2}\tfrac{d}{dt}\|\xi(t)\|^2 - \big([(1+\tfrac{\eta}{2})\xi]_x, \theta\big) \le -\tfrac{1}{4}(\xi_x\theta,\theta) + C(h^{r-1}\|\xi\| + \|\theta\|^2 + \|\xi\|^2). \tag{2.16}$$

Adding (2.14) and (2.16) gives

$$\tfrac{d}{dt}(\|\xi\|^2 + \|\theta\|^2) \le C[h^{r-1}(\|\theta\| + \|\xi\|) + \|\theta\|^2 + \|\xi\|^2], \quad 0 \le t \le T.$$

Therefore, by Gronwall's inequality and (2.6) we see that $\|\theta\| + \|\xi\| \le Ch^{r-1}$, $0 \le t \le T$, from which (2.9) follows. ☐

We turn now to the semidiscrete approximation of (SW). The error analysis that follows is similar to that of Dupont, [9], and the proof assumes that $r \ge 3$ and that the solution of (SW) satisfies $1 + \eta > 0$; cf. [14] and the remarks in the Introduction.

Proposition 2.2. *Let (η, u) be the solution of* (SW)*, satisfying $1 + \eta > 0$ for $t \in [0,T]$, $r \ge 3$, and let h be sufficiently small. Then the semidiscrete ivp* (2.5)-(2.6) *has a unique solution (η_h, u_h) for $0 \le t \le T$ satisfying*

$$\max_{0\le t\le T}\big(\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|\big) \le Ch^{r-1}. \tag{2.17}$$

Proof. We use the same notation as in the proof of Proposition 2.1. While the solution of (2.5)-(2.6) exists we have

$$(\theta_t, \phi) + (\xi_x + \sigma_x, \phi) + ((\eta u)_x - (\eta_h u_h)_x, \phi) = 0, \quad \forall \phi \in S_h, \tag{2.18}$$

$$(\xi_t, \chi) + (\theta_x + \rho_x, \chi) + (uu_x - u_h u_{hx}, \chi) = 0, \quad \forall \chi \in S_{h,0}. \tag{2.19}$$

Taking $\phi = \theta$ in (2.18) and using integration by parts we have

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 + \big([(1+\eta)\xi]_x, \theta\big) = &-(\sigma_x,\theta) - ((\eta\sigma)_x,\theta) - ((u\rho)_x,\theta)\\ &- ((u\theta)_x,\theta) + ((\rho\sigma)_x,\theta) + ((\theta\sigma)_x,\theta) + ((\rho\xi)_x,\theta) + ((\theta\xi)_x,\theta).\end{aligned} \tag{2.20}$$

In view of (2.6), by continuity we conclude that there exists a maximal temporal instance $t_h > 0$ such that (η_h, u_h) exist and $\|\xi_x\|_\infty \le 1$ for $t \le t_h$. Suppose that $t_h < T$. Using the approximation and inverse properties of S_h and $S_{h,0}$ and integration by parts we may then estimate the various terms in the r.h.s. of (2.20) for $t \in [0, t_h]$ as follows:

$$\begin{gathered}|(\sigma_x,\theta)| \le \|\sigma_x\|\|\theta\| \le Ch^{r-1}\|\theta\|, \qquad |((\eta\sigma)_x,\theta)| \le C\|\sigma\|_1\|\theta\| \le Ch^{r-1}\|\theta\|,\\ |((u\rho)_x,\theta)| \le C\|\rho\|_1\|\theta\| \le Ch^{r-1}\|\theta\|, \qquad |((u\theta)_x,\theta)| = \tfrac{1}{2}|(u_x\theta,\theta)| \le C\|\theta\|^2,\\ |((\rho\sigma)_x,\theta)| \le Ch^{2r-1}\|\theta\|, \qquad |((\theta\sigma)_x,\theta)| = \tfrac{1}{2}|(\sigma_x\theta,\theta)| \le C\|\theta\|^2,\\ |((\rho\xi)_x,\theta)| \le C\|\xi\|\|\theta\|, \qquad |((\theta\xi)_x,\theta)| = \tfrac{1}{2}|(\xi_x\theta,\theta)| \le \tfrac{1}{2}\|\xi_x\|_\infty\|\theta\|^2 \le \tfrac{1}{2}\|\theta\|^2.\end{gathered}$$

Hence, we conclude from (2.20) that for $t \in [0, t_h]$,

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 - (\gamma, \theta_x) \le C(h^{r-1}\|\theta\| + \|\theta\|^2 + \|\xi\|^2), \tag{2.21}$$

where we have put $\gamma := (1+\eta)\xi$.

We turn now to (2.19), in which we set $\chi = P_0\gamma = P_0[(1+\eta)\xi]$. Then for $0 \le t \le t_h$ it holds that

$$\begin{aligned}(\xi_t,\gamma) + (\theta_x, P_0\gamma) = &-(\rho_x, P_0\gamma) - ((u\sigma)_x, P_0\gamma) - ((u\xi)_x, P_0\gamma)\\ &+ ((\sigma\xi)_x, P_0\gamma) + (\sigma\sigma_x, P_0\gamma) + (\xi\xi_x, P_0\gamma).\end{aligned} \tag{2.22}$$

For the first two terms in the r.h.s. of (2.22) we have

$$|(\rho_x, P_0\gamma)| \leq \|\rho_x\|\|P_0\gamma\| \leq Ch^{r-1}\|\gamma\| \leq Ch^{r-1}\|\xi\|,$$
$$|((u\sigma)_x, P_0\gamma)| \leq C\|\sigma\|_1\|P_0\gamma\| \leq Ch^{r-1}\|\xi\|.$$

Note now that

$$\begin{aligned}((u\xi)_x, P_0\gamma) &= ((u\xi)_x, P_0\gamma - \gamma) + ((u\xi)_x, \gamma)\\ &= ((u\xi)_x, P_0\gamma - \gamma) + (u_x(1+\eta), \xi^2) - \tfrac{1}{2}\big([(1+\eta)u]_x, \xi^2\big).\end{aligned}$$

We now use a well-known *superapproximation* property of $S_{h,0}$ (cf. [9], [8]) (which holds for $r = 2$ as well) to estimate the term $P_0\gamma - \gamma$:

$$\|P_0\gamma - \gamma\| = \|P_0[(1+\eta)\xi] - (1+\eta)\xi\| \leq Ch\|\xi\|. \tag{2.23}$$

Therefore, by (2.3)

$$|((u\xi)_x, P_0\gamma)| \leq Ch\|\xi\|_1\|\xi\| + C\|\xi\|^2 \leq C\|\xi\|^2.$$

Similarly, using the approximation and inverse properties of S_h, $S_{h,0}$ and (2.23) we have

$$\begin{aligned}|((\sigma\xi)_x, P_0\gamma)| &\leq |(\sigma_x\xi, P_0\gamma - \gamma)| + |(\sigma\xi_x, P_0\gamma - \gamma)| + |((\sigma\xi)_x, \gamma)|\\ &\leq C\|\sigma_x\|_\infty h\|\xi\|^2 + C\|\sigma\|_\infty\|\xi_x\|h\|\xi\|\\ &\quad + C\|\sigma_x\|_\infty\|\xi\|^2 + C\|\sigma\|_\infty\|\xi_x\|\|\xi\| \leq C\|\xi\|^2,\end{aligned}$$

$$\begin{aligned}|(\sigma\sigma_x, P_0\gamma)| &\leq |(\sigma\sigma_x, P_0\gamma - \gamma)| + |(\sigma\sigma_x, \gamma)|\\ &\leq C\|\sigma\|_\infty\|\sigma_x\|h\|\xi\| + C\|\sigma\|_\infty\|\sigma_x\|\|\xi\| \leq Ch^{2r-1}\|\xi\|,\end{aligned}$$

$$\begin{aligned}|(\xi\xi_x, P_0\gamma)| &\leq |(\xi\xi_x, P_0\gamma - \gamma)| + |(\xi\xi_x, (1+\eta)\xi)|\\ &\leq C\|\xi_x\|_\infty h\|\xi\|^2 + C\|\xi_x\|_\infty\|\xi\|^2 \leq C\|\xi\|^2.\end{aligned}$$

Therefore, using (2.22) we have for $0 \leq t \leq t_h$,

$$(\xi_t, (1+\eta)\xi) + (\theta_x, P_0\gamma) \leq C(h^{r-1}\|\xi\| + \|\xi\|^2). \tag{2.24}$$

Now adding (2.21) and (2.24) we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 + (\xi_t, (1+\eta)\xi) + (\theta_x, P_0\gamma - \gamma) \leq C\big[h^{r-1}(\|\theta\| + \|\xi\|) + \|\theta\|^2 + \|\xi\|^2\big].$$

But $(\xi_t, (1+\eta)\xi) = \frac{1}{2}\frac{d}{dt}((1+\eta)\xi, \xi) - \frac{1}{2}(\eta_t\xi, \xi)$. Therefore, for $0 \leq t \leq t_h$,

$$\tfrac{1}{2}\tfrac{d}{dt}\big[\|\theta\|^2 + ((1+\eta)\xi, \xi)\big] \leq C\big[h^{r-1}(\|\theta\| + \|\xi\|) + \|\theta\|^2 + \|\xi\|^2\big],$$

for a constant C independent of h and t_h. Since $1+\eta > 0$, the norm $((1+\eta)\cdot, \cdot)^{1/2}$ is equivalent to that of L^2 uniformly for $t \in [0, T]$. Hence, Gronwall's inequality and (2.6) yield, for a constant $C = C(T)$,

$$\|\theta\| + \|\xi\| \leq Ch^{r-1} \quad \text{for} \quad 0 \leq t \leq t_h. \tag{2.25}$$

We conclude from (2.4) that $\|\xi_x\|_\infty \leq Ch^{r-5/2}$ for $0 \leq t \leq t_h$, and, since $r \geq 3$, if h is sufficiently small, we see that t_h is not maximal. Hence we may take $t_h = T$, and (2.17) follows from (2.25). $\square$

The proof of Proposition 2.2 needs the assumption that $r \geq 3$. The following numerical experiment suggests that the result holds for $r = 2$, i.e. for piecewise linear continuous functions as well. Table 2.1 shows the errors and associated orders of convergence in the L^2 and L^∞ norms at $t = 1$ of the standard Galerkin approximation with piecewise linear continuous functions of (SW) with suitable right-hand side and initial conditions so that its exact solution is $\eta = \exp(2t)(\cos(\pi x) + x + 2)$, $u = \exp(-xt)\sin(\pi x)$. The semidiscrete ivp was integrated in time with the 'classical', four-stage, fourth-order explicit Runge-Kutta (RK) method, which may be shown to be stable for systems like (SW) under a Courant-number restriction. (To obtain the results of Table 2.1 we took a small time step, namely $k = \Delta x/20$, to ensure that the temporal error of the discretization would be very small compared with the spatial error, so that the errors and rates of convergence shown are essentially those of the semidiscrete problem.) On the spatial interval $[0, 1]$ we used the quasiuniform mesh given by $h_{2i-1} = 0.75\Delta x$, $h_{2i} = 0.5\Delta x$, $i = 1, \ldots, N/2$, where $h_i = x_{i+1} - x_i$ and $\Delta x = 1.6/N$. The table suggests that the L^2-errors for η and u are of $O(h)$. It also suggests that the L^∞-errors are also $O(h)$. (The H^1-errors were found to be of $O(1)$.)

TABLE 2.1. Errors and orders of convergence. (SW) system, standard Galerkin semidiscretization with piecewise linear, continuous elements on a quasiuniform mesh, $t = 1$.

	$L^2 - errors$				$L^\infty - errors$			
N	η	$order$	u	$order$	η	$order$	u	$order$
40	0.1216		0.1749(−2)		0.2099		0.4090(−2)	
80	0.5973(−1)	1.0256	0.8259(−3)	1.0825	0.1051	0.9979	0.1935(−2)	1.0798
160	0.2959(−1)	1.0133	0.4092(−3)	1.0132	0.5188(−1)	1.0185	0.1015(−2)	0.9309
320	0.1473(−1)	1.0064	0.2041(−3)	1.0035	0.2587(−1)	1.0039	0.5115(−3)	0.9887
480	0.9804(−2)	1.0040	0.1359(−3)	1.0030	0.1723(−1)	1.0024	0.3418(−3)	0.9942
640	0.7347(−2)	1.0028	0.1019(−3)	1.0009	0.1291(−1)	1.0034	0.2562(−3)	1.0020

3. SOME SUPERACCURACY PROPERTIES OF THE L^2 PROJECTION ON SPACES OF CONTINUOUS, PIECEWISE LINEAR FUNCTIONS

In this section we will prove in a series of lemmas some *superaccuracy (superconvergence)* properties of the L^2 projection of smooth functions that satisfy suitable boundary conditions onto spaces of piecewise linear, continuous functions defined on a *uniform mesh* in $[0, 1]$. These properties will be used in Section 4 to establish optimal-order L^2-error estimates for the semidiscrete approximations of (SW) and (SSW) in these finite element spaces.

For the purposes of this section (and of Section 4) for integer $N \geq 2$ we let $h = 1/N$, $x_i = (i - 1)h$, $i = 1, \ldots, N + 1$, be a uniform partition of $[0, 1]$ and $I_i = x_{i+1} - x_i$, $1 \leq i \leq N$. We put $x_{i+1/2} = (x_i + x_{i+1})/2$. We also let $S_h = S_h^{0,2} := \{\phi \in C^0 : \phi\big|_{[x_j, x_{j+1}]} \in \mathbb{P}_1 \, , 1 \leq j \leq N\}$ and $S_{h,0} = S_{h,0}^{0,2} = \{\phi \in S_h : \phi(0) = \phi(1) = 0\}$. We equip S_h with the basis $\{\phi_i\}_{i=1}^{N+1}$, where $\phi_i \in S_h$ and $\phi_i(x_j) = \delta_{ij}$, $1 \leq i, j \leq N+1$, and $S_{h,0}$ with the basis $\{\chi_i\}_{i=1}^{N-1}$, where $\chi_i = \phi_{i+1}$, $1 \leq i \leq N - 1$. We again let P, P_0 be the L^2 projection operators onto S_h, $S_{h,0}$, respectively.

Lemma 3.1. *Let $u \in C_0^4$, $u''(0) = u''(1) = 0$, and $\sigma = u - P_0 u$. Then, there exists a constant $C = C(\|u^{(4)}\|_\infty)$ such that*

$$\max_{1\le i\le N}\left|\int_{I_i}\sigma dx\right| \le Ch^5. \tag{3.1}$$

Proof. By the definition of P_0 we have for $x \in I_i$, $2 \le i \le N-1$, $\sigma = u - P_0 u = u - (d_{i-1}\chi_{i-1} + d_i\chi_i)$, giving for $\varepsilon_i = \int_{I_i}\sigma dx$, $1 \le i \le N$,

$$\varepsilon_i = \int_{I_i} udx - \tfrac{h}{2}(d_{i-1}+d_i), \quad 2 \le i \le N-1, \quad \varepsilon_1 = \int_{I_1} udx - \tfrac{h}{2}d_1, \tag{3.2}$$
$$\varepsilon_N = \int_{I_N} udx - \tfrac{h}{2}d_{N-1}.$$

Here we have denoted by $d = (d_1, d_2, \dots, d_{N-1})^T$ the coefficients of $P_0 u$ with respect to the basis $\{\chi_i\}_{i=1}^{N-1}$, i.e. the solution of the linear system $G^0 d = b$, where $G^0_{ij} = (\chi_j, \chi_i)$, $1 \le i, j \le N-1$, and $b_i = (u, \chi_i)$, $1 \le i \le N-1$. The equations of this system may be written explicitly as $\widetilde{G}^0 d = \widetilde{b}$, where $\widetilde{G}^0$ is the $(N-1)\times(N-1)$ tridiagonal matrix with elements $\widetilde{G}^0_{ii} = 4$, $\widetilde{G}^0_{ij} = 1$ if $|i-j| = 1$, and $\widetilde{b}_i = 6b_i/h$, $1 \le i \le N-1$. If we combine these equations with (3.2) it is straightforward to infer that $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N)^T$ is the solution of the linear system $\Gamma\varepsilon = r$, where Γ is the $N \times N$ tridiagonal matrix with elements $\Gamma_{11} = \Gamma_{NN} = 3$, $\Gamma_{ii} = 4$, $2 \le i \le N-1$, and $\Gamma_{ij} = 1$ if $|i-j| = 1$, and $r = (r_1 \dots, r_N)^T$ is given by

$$\begin{aligned} r_i &= \int_{I_{i-1}} udx + 4\int_{I_i} udx + \int_{I_{i+1}} udx - 3(b_{i-1}+b_i), \quad 2 \le i \le N-1, \\ r_1 &= 3\int_{I_1} udx + \int_{I_2} udx - 3b_1, \quad r_N = \int_{I_{N-1}} udx + 3\int_{I_N} udx - 3b_{N-1}. \end{aligned} \tag{3.3}$$

We will show that $r_i = O(h^5)$, $1 \le i \le N$. For r_1 we have by the above that $r_1 = 3\int_{I_1} udx + \int_{I_2} udx - \frac{3}{h}\int_{I_1} xudx - \frac{3}{h}\int_{I_2}(2h-x)udx$, from which, by Taylor's theorem and our hypotheses on u, we obtain that $r_1 = O(h^5)$. For $2 \le i \le N-1$, we have

$$r_i = \int_{I_{i-1}\cup I_i\cup I_{i+1}} udx - 3\int_{I_{i-1}} \frac{x - x_{i-1}}{h} udx - 3\int_{I_{i+1}} \frac{x_{i+2}-x}{h} udx.$$

Since $u \in C^4$, it follows from Simpson's rule and Taylor's theorem, as in the first part of the proof of Lemma 5.7 of [2], that $r_i = O(h^5)$, $2 \le i \le N-1$. Finally, since $r_N = 3\int_{I_1} vdx + \int_{I_2} vdx - 3(v, \chi_1)$, where $v(x) := u(1-x)$, we see that r_N is the same as r_1 with v replacing u. It follows that $r_N = O(h^5)$. Note that $\frac{1}{4}\Gamma\varepsilon = \frac{1}{4}r$, where $\frac{1}{4}\Gamma = I - E$, and E is an $N \times N$ matrix with $\|E\|_\infty = 1/2$. Hence, $\|(I-E)^{-1}\|_\infty \le 2$, and thus $\max_{1\le i\le N}|\varepsilon_i| \le \frac{1}{2}\max_{1\le i\le N}|r_i| \le Ch^5$. ∎

Lemma 3.2. *Let $u \in C_0^3$, $u''(0) = u''(1) = 0$, and $\sigma = u - P_0 u$. Then, there exists a constant $C = C(\|u^{(3)}\|_\infty)$ such that*

$$\max_{1\le i\le N}\left|\sigma'(x_{i+1/2})\right| \le Ch^2. \tag{3.4}$$

Proof. Let $d = (d_1, \dots, d_{N-1})^T$ be defined as in the proof of Lemma 3.1. Then

$$\sigma'(x) = u'(x) - \frac{d_1}{h},\ x \in I_1,\ \sigma'(x) = u'(x) - \frac{d_i - d_{i-1}}{h},\ x \in I_i,\ 2 \le i \le N-1,$$
$$\sigma'(x) = u'(x) - \frac{-d_{N-1}}{h},\ x \in I_N.$$

It is straightforword to check as in Lemma 3.1 that if $b_i = (u, \chi_i)$, $1 \le i \le N-1$, and $\varepsilon_i' := \sigma'(x_{i+1/2})$, $1 \le i \le N$, then the vector $\varepsilon' = (\varepsilon_1', \dots, \varepsilon_N')^T$ is the solution of the system $A\varepsilon' = r'$, where A is the $N \times N$ tridiagonal matrix with elements $A_{11} = A_{NN} = 5$, $A_{ii} = 4$, $2 \le i \le N-1$, and $A_{ij} = 1$ if $|i-j| = 1$, and $r' = (r_1', \dots, r_N')^T$ is given by

$$\begin{aligned} r_i' &= u'(x_{i-1/2}) + 4u'(x_{i+1/2}) + u'(x_{i+3/2}) - 6(b_i - b_{i-1})/h^2, \quad 2 \le i \le N-1,\\ r_1' &= 5u'(x_{1+1/2}) + u'(x_{2+1/2}) - 6b_1/h^2,\\ r_N' &= u'(x_{N-1/2}) + 5u'(x_{N+1/2}) + 6b_{N-1}/h^2. \end{aligned}$$

We will show that $r_i' = O(h^2)$, $1 \le i \le N$. By Taylor's theorem and our assumptions on u we first have $b_1 = \frac{1}{4}\int_{I_1} xudx + \frac{1}{h}\int_{I_2}(2h-x)udx = h^2u'(0) + O(h^4)$. Therefore, $r_1' = 5u'(0) + u'(0) - 6u'(0) + O(h^2) = O(h^2)$. For r_i', $2 \le i \le N-1$, we have, since $\phi_i = \chi_{i-1}$, $r_i' = u'(x_{i-1/2}) + 4u'(x_{i+1/2}) + u'(x_{i+3/2}) - \frac{6}{h^2}\big((u, \phi_{i+1}) - (u, \phi_i)\big)$. It then follows from the relations (5.13)-(5.17) et seq. in the proof of Lemma 5.5 of [2] that $r_i' = O(h^2)$, $2 \le i \le N-1$. Finally, since $r_N' = -\big[v'(x_{2+1/2}) + 5v'(x_{1+1/2}) - 6(v, \chi_1)/h^2\big]$, where we have denoted $v(x) := u(1-x)$, we see that r_N' is given by $-r_1'$ with u replaced by v. It follows that $r_N' = O(h^2)$ as well. Obviously $r_i' = O(h^2)$, $1 \le i \le N$, implies that $\varepsilon_i' = O(h^2)$ in view of the properties of the matrix A. ☐

Lemma 3.3. *Suppose that $v \in C^2$, $u \in C_0^4$, $u''(0) = u''(1) = 0$, and $\sigma = u - P_0u$. Then there exists a constant C independent of h such that*

$$\max_{1 \le i \le N} \left| \int_{I_i} v\sigma dx \right| \le Ch^5. \tag{3.5}$$

Proof. Since $\|\sigma\|_\infty = O(h^2)$ by (2.2b), a Taylor expansion of v gives

$$\int_{I_i} v\sigma dx = v(x_{i+1/2}) \int_{I_i} \sigma dx + v'(x_{i+1/2}) \int_{I_i} (x - x_{i+1/2})\sigma dx + O(h^5).$$

For the second integral in the right-hand side of this relation a Taylor expansion of σ and the fact that $P_0u\big|_{I_i} \in \mathbb{P}_1$ yield $\int_{I_i}(x - x_{i+1/2})\sigma dx = \frac{h^3}{12}\sigma'(x_{i+1/2}) + O(h^5)$. The estimate (3.5) now follows from (3.1) and (3.4). ☐

Lemma 3.4. *Let $\eta \in C^4$ with $\eta'(0) = \eta'(1) = 0$ and $\eta'''(0) = \eta'''(1) = 0$. If $\rho = \eta - P\eta$, then there exists a constant $C = C(\|\eta^{(4)}\|_{L^\infty})$ such that*

$$\max_{1 \le i \le N} \left| \int_{I_i} \rho dx \right| \le Ch^5. \tag{3.6}$$

Proof. The proof is similar to that of Lemma 3.1, mutatis mutandis; for full details cf. [3]. ☐

Lemma 3.5. *Let $\eta \in C^3$ and $\rho = \eta - P\eta$. Then, there exists a constant $C = C(\|\eta'''\|_\infty)$ such that*

$$\max_{1\le i\le N} \left|\rho'(x_{i+1/2})\right| \le Ch^2. \tag{3.7}$$

Proof. See Lemma 5.5 of [2]. ☐

Lemma 3.6. *Let $w \in C^2$, $\eta \in C^4$ with $\eta'(0) = \eta'(1) = 0$, $\eta'''(0) = \eta'''(1) = 0$. If $\rho = \eta - P\eta$, there exists a constant C independent of h such that*

$$\max_{1\le i\le N} \left| \int_{I_i} w\rho dx \right| \le Ch^5. \tag{3.8}$$

Proof. The proof is similar to that of Lemma 3.3 if (3.6) and (3.7) are taken into account. ☐

Lemma 3.7. *Consider the mass matrices $G_{ij} = (\phi_j, \phi_i)$, $1 \le i, j \le N+1$, and $G^0_{ij} = (\chi_j, \chi_i)$, $1 \le i, j \le N-1$.*
(i) There exist constants c_i, $1 \le i \le 4$, independent of h such that

$$c_1 h|\beta|^2 \le \langle G\beta, \beta\rangle \le c_2 h|\beta|^2 \quad \forall \beta \in \mathbb{R}^{N+1},$$
$$c_3 h|\beta|^2 \le \langle G^0\beta, \beta\rangle \le c_4 h|\beta|^2 \quad \forall \beta \in \mathbb{R}^{N-1}.$$

(ii) Let $b \in \mathbb{R}^{N+1}$, $G\beta = b$, and $\zeta = \sum_{j=1}^{N+1} \beta_j\phi_j$. Then $\|\zeta\| \le (c_1 h)^{-1/2}|b|$.
If $b \in \mathbb{R}^{N-1}$, $G^0\beta = b$, and $\zeta = \sum_{j=1}^{N-1} \beta_j\chi_j$, then $\|\zeta\| \le (c_3 h)^{-1/2}|b|$.

Proof. The proofs of (i) and (ii) are given in Dupont, [10], when the elements of the finite element subspace satisfy periodic boundary conditions. In our case, the proof of (i) follows again from Gerschgorin's lemma, and (ii) is a consequence of (i). ☐

Lemma 3.8. *Let $w \in C^2_0$, $v \in C^2$, $\eta \in C^4$ with $\eta'(0) = \eta'(1) = 0$, $\eta'''(0) = \eta'''(1) = 0$, $u \in C^4_0$ with $u''(0) = u''(1) = 0$, $\rho = \eta - P\eta$, $\sigma = u - P_0 u$. Then, for constants C independent of h:*
(i) If $\zeta_1 \in S_{h,0}$ is defined by $(\zeta_1, \chi) = (\rho', \chi)$, $\forall \chi \in S_{h,0}$, then $\|\zeta_1\| \le Ch^3$.
(ii) If $\zeta_2 \in S_h$ is defined by $(\zeta_2, \phi) = (\sigma', \phi)$, $\forall \phi \in S_h$, then $\|\zeta_2\| \le Ch^3$.
(iii) If $\zeta_3 \in S_h$ is defined by $(\zeta_3, \phi) = ((w\rho)', \phi)$, $\forall \phi \in S_h$, then $\|\zeta_3\| \le Ch^3$.
(iv) If $\zeta_4 \in S_h$ is defined by $(\zeta_4, \phi) = ((v\sigma)', \phi)$, $\forall \phi \in S_h$, then $\|\zeta_4\| \le Ch^3$.
(v) If $\zeta_5 \in S_{h,0}$ is defined by $(\zeta_5, \chi) = ((v\sigma)', \chi)$, $\forall \chi \in S_{h,0}$, then $\|\zeta_5\| \le Ch^3$.
(vi) If $\zeta_6 \in S_{h,0}$ is defined by $(\zeta_6, \chi) = ((v\rho)', \chi)$, $\forall \chi \in S_{h,0}$, then $\|\zeta_6\| \le Ch^3$.

Proof. (i) If $b_i = (\rho', \chi_i)$, $1 \le i \le N-1$, then $b_i = -(\rho, \chi_i')$, i.e. $b_i = -\frac{1}{h}\int_{I_i} \rho dx + \frac{1}{h}\int_{I_{i+1}} \rho dx$, $1 \le i \le N-1$. By (3.6), $|b_i| \le Ch^4$. Hence $|b| \le Ch^{3.5}$ and (i) follows by Lemma 3.7(ii).

The proof of (ii) is similar and takes into account (3.1).

(iii) If now $b_i = ((w\rho)', \phi_i)$, $1 \le i \le N+1$, then $b_i = -(w\rho, \phi_i')$, i.e. $b_1 = \frac{1}{h}\int_{I_1} w\rho dx$, $b_i = -\frac{1}{h}\int_{I_{i-1}} w\rho dx + \frac{1}{h}\int_{I_i} w\rho dx$, $2 \le i \le N$, $b_{N+1} = -\frac{1}{h}\int_{I_N} w\rho dx$. By (3.8) $\max_{1\le i\le N}|b_i| \le Ch^4$, so that $|b| \le Ch^{3.5}$ and (iii) follows from Lemma 3.7(ii).

The proofs of (iv) and (v) are similar to that of (iii) if we take into account (3.5). Finally, if $b_i = ((v\rho)', \chi_i)$, $1 \le i \le N-1$, then $b_i = -(v\rho, \chi_i') = -\frac{1}{h}\int_{I_i} v\rho dx + \frac{1}{h}\int_{I_{i+1}} v\rho dx$, $1 \le i \le N-1$. By (3.8), $|b_i| \le Ch^4$. Hence, $|b| \le Ch^{3.5}$ and (vi) follows from Lemma 3.7(ii). $\square$

4. Semidiscretization with continuous, piecewise linear functions on uniform meshes

In this section we will prove optimal-order L^2-error estimates for the solutions of the semidiscrete problems (2.5)-(2.6) and (2.7)-(2.8) that approximate the ibvp's (SW) and (SSW), respectively, in the spaces $S_h = S_h^{0,2}$, $S_{h,0} = S_{h,0}^{0,2}$ of piecewise linear continuous functions on a uniform spatial mesh, using the notation and results of Section 3. The proof of optimality of the order of convergence in the error estimates uses, in addition to the superaccuracy properties of the L^2 projection, *compatibility conditions* at the boundary $\partial I = \{0, 1\}$ that smooth solutions of (SW) and (SSW) satisfy.

We will assume that the ibvp (SW) has a unique solution (η, u) such that $\eta \in C(0,T;C^4)$, $u \in C(0,T;C_0^4)$ for some $0 < T < \infty$. We will also assume that for some $\alpha > 0$, $\min_{0\le x\le 1}(1+\eta_0(x)) \ge \frac{\alpha}{2}$, so that by the theory of [14], $\min_{0\le x\le 1}(1+\eta(x,t)) \ge \alpha > 0$, for all $t \in [0,T]$. In addition to the hypothesis $(\eta_0, u_0) \in C^4 \times C_0^4$, we assume that $\eta_0' \in C_0^3$, $\eta_0''' \in C_0^1$, $u_0'' \in C_0^2$. Then, from the second p.d.e. of (SW) and the b.c. $u|_{\partial I} = 0$, it follows that $\eta_x|_{\partial I} = 0$ for $t \in [0,T]$. Differentiating the first p.d.e. with respect to x and using the positivity of $1+\eta$ we also conclude that $u_{xx}|_{\partial I} = 0$ for $t \in [0,T]$. Finally, differentiating the second p.d.e. twice with respect to x we see that for $0 \le t \le T$, $\eta_{xxx}|_{\partial I} = 0$ as well. We will make the same hypotheses, leading to the same compatibility conditions for the solution (η, u) of (SSW), under the assumption that

$$\min_{\substack{0\le x\le 1\\ 0\le t\le T}} (1 + \frac{1}{2}\eta(x,t)) \ge \beta$$

for some positive constant $\beta > 0$, which may also be similarly justified; cf. the remarks in the Introduction.

We begin with the error estimate for (SSW), which is again simpler due to the symmetry of this system.

Theorem 4.1. *Let (η, u) be the solution of* (SSW) *and suppose that $\eta \in C(0,T;C^4)$, $u \in C(0,T;C_0^4)$, $\eta_0' \in C_0^3$, $\eta_0''' \in C_0^1$, $u_0'' \in C_0^2$ and*

$$\min_{\substack{0\le x\le 1\\ 0\le t\le T}} (1 + \frac{1}{2}\eta(x,t)) \ge \beta > 0$$

for some constant β. Let $x_i = (i-1)h$, $1 \le i \le N+1$, $Nh = 1$, and (η_h, u_h) be the solution of (2.7)-(2.8) *for $t \in [0,T]$ in the space of piecewise linear continuous functions $S_h \times S_{h,0}$. Then*

$$\max_{0\le t\le T} (\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|) \le Ch^2 \tag{4.1}$$

and

$$\max_{0\le t\le T} (\|\eta(t) - \eta_h(t)\|_\infty + \|u(t) - u_h(t)\|_\infty) \le Ch^2. \tag{4.1$'$}$$

Proof. We refer to the analogous proof (Proposition 2.1) in the quasiuniform mesh case for notation. We again let $\theta = P\eta - \eta_h$, $\xi = P_0 u - u_h$, $\rho = \eta - P\eta$, $\sigma = u - P_0 u$. The identity (2.13) still holds and we write it, using integration by parts, in the form

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \big([(1+\tfrac{\eta}{2})\xi]_x, \theta\big) = \tfrac{1}{4}(\xi_x\theta, \theta) + A_1 + A_2, \tag{4.2}$$

where

$$A_1 := -(\sigma_x, \theta) - \tfrac{1}{2}((\eta\sigma)_x, \theta) - \tfrac{1}{2}((u\rho)_x, \theta), \tag{4.3}$$

$$A_2 := -\tfrac{1}{4}(u_x\theta, \theta) + \tfrac{1}{4}(\sigma_x\theta, \theta) + \tfrac{1}{2}((\rho\sigma)_x, \theta) + \tfrac{1}{2}((\rho\xi)_x, \theta). \tag{4.4}$$

We will estimate the terms of A_1 using the superaccuracy properties of Section 3 in view of the compatibility conditions on η and u for $0 \le t \le T$ implied by our hypotheses, as was previously explained.

By Lemma 3.8(ii), (iv) with $v = \eta$, and (iii) with $w = u$, we have $|(\sigma_x, \theta)| \le Ch^3\|\theta\|$, $|((\eta\sigma)_x, \theta)| \le Ch^3\|\theta\|$, $|((u\rho)_x, \theta)| \le Ch^3\|\theta\|$, and we conclude by (4.3) that

$$|A_1| \le Ch^3\|\theta\|. \tag{4.5}$$

The terms of A_2 are estimated as in the proof of Proposition 2.1, immediately after (2.13), in the case $r = 2$. As a result we have

$$|A_2| \le C(h^3\|\theta\| + \|\theta\|^2 + \|\xi\|^2). \tag{4.6}$$

Therefore, by (4.2), (4.5), and (4.6), there holds for $t \in [0, T]$ that

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \big([(1+\tfrac{\eta}{2})\xi]_x, \theta\big) \le \tfrac{1}{4}(\xi_x\theta, \theta) + C(h^3\|\theta\| + \|\theta\|^2 + \|\xi\|^2). \tag{4.7}$$

In addition, the identity (2.15) still holds. Using integration by parts we write it for $t \in [0, T]$ in the form

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 - \big([(1+\tfrac{\eta}{2})\xi]_x, \theta\big) = -\tfrac{1}{4}(\xi_x\theta, \theta) + B_1 + B_2, \tag{4.8}$$

where

$$B_1 := -(\rho_x, \xi) - \tfrac{1}{2}((\eta\rho)_x, \xi) - \tfrac{3}{2}((u\sigma)_x, \xi), \tag{4.9}$$

(4.10)
$$B_2 := -\tfrac{3}{2}((u\xi)_x, \xi) + \tfrac{3}{2}((\sigma\xi)_x, \xi) + \tfrac{3}{2}(\sigma\sigma_x, \xi) - \tfrac{1}{2}(\eta_x\theta, \xi) + \tfrac{1}{2}((\rho\theta)_x, \xi) + \tfrac{1}{2}(\rho\rho_x, \xi).$$

Using again the compatibility properties of η and u for $0 \le t \le T$, by Lemma 3.8(i), (vi) with $v = \eta$, and (v) with $v = u$ we have $|(\rho_x, \xi)| \le Ch^3\|\xi\|$, $|((\eta\rho)_x, \xi)| \le Ch^3\|\xi\|$, $|((u\sigma)_x, \xi)| \le Ch^3\|\xi\|$, so that by (4.9)

$$|B_1| \le Ch^3\|\xi\|. \tag{4.11}$$

The terms of B_2 are estimated again as in the proof of Proposition 2.1, after (2.15), in the case $r = 2$. We have therefore

$$|B_2| \le C(h^3\|\xi\| + \|\theta\|^2 + \|\xi\|^2), \tag{4.12}$$

and by (4.8), (4.11), and (4.12), for $t \in [0, T]$:

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 - \big([(1+\tfrac{\eta}{2})\xi]_x, \theta\big) \le -\tfrac{1}{4}(\xi_x\theta, \theta) + C(h^3\|\xi\| + \|\theta\|^2 + \|\xi\|^2). \tag{4.13}$$

Adding (4.7) and (4.13) we get for $t \in [0, T]$ that

$$\frac{d}{dt}(\|\theta\|^2 + \|\xi\|^2) \le Ch^3(\|\xi\| + \|\theta\|) + C(\|\theta\|^2 + \|\xi\|^2).$$

Therefore, since $\theta(0) = 0$, $\xi(0) = 0$, Gronwall's lemma gives the superaccurate estimate

$$\|\theta\| + \|\xi\| \le Ch^3, \quad 0 \le t \le T, \tag{4.14}$$

from which (4.1) follows. In view of (2.4) and (2.2b), (4.14) implies the L^∞ estimate $(4.1')$ as well. $\square$

We prove now the analogous optimal-order L^2-error estimate for (SW).

Theorem 4.2. *Let (η, u) be the solution of* (SW) *and suppose that $\eta \in C(0,T;C^4)$, $u \in C(0,T;C_0^4)$, $\eta_0' \in C_0^3$, $u_0'' \in C_0^2$, and $\min_{\substack{0\le x\le 1\\ 0\le t\le T}}(1+\eta(x,t)) \ge \alpha > 0$ for some positive constant α. Let $x_i = (i-1)h$, $1 \le i \le N+1$, $Nh = 1$, and (η_h, u_h) be the solution of* (2.5)-(2.6) *for $t \in [0,T]$ in the space of piecewise linear continuous functions $S_h \times S_{h,0}$. Then*

$$\max_{0\le t\le T}(\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|) \le Ch^2 \tag{4.15}$$

and

$$\max_{0\le t\le T}(\|\eta(t) - \eta_h(t)\|_\infty + \|u(t) - u_h(t)\|_\infty) \le Ch^2. \tag{4.15$'$}$$

Proof. We refer again to the analogous proof (Proposition 2.2) in the quasiuniform case for notation. In particular we again let $\theta = P\eta - \eta_h$, $\xi = P_0 u - u_h$, $\rho = \eta - P\eta$, $\sigma = u - P_0 u$. The identity (2.20) still holds and we write it, using integration by parts, in the form

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 + \big([(1+\eta)\xi]_x, \theta\big) = \tfrac{1}{2}(\xi_x\theta, \theta) + A_3 + A_4, \tag{4.16}$$

where

$$A_3 = -(\sigma_x, \theta) - ((\eta\sigma)_x, \theta) - ((u\rho)_x, \theta), \tag{4.17}$$

$$A_4 = -\tfrac{1}{2}(u_x\theta, \theta) + \tfrac{1}{2}(\sigma_x\theta, \theta) + ((\rho\sigma)_x, \theta) + ((\rho\xi)_x, \theta). \tag{4.18}$$

Using the compatibility conditions on η and u implied by our hypotheses, we have, by Lemma 3.8 (ii), (iv) with $v = \eta$, and (iii) with $w = u$, that $|(\sigma_x, \theta)| \le Ch^3\|\theta\|$, $|((\eta\sigma)_x, \theta)| \le Ch^3\|\theta\|$, $|((u\rho)_x, \theta)| \le Ch^3\|\theta\|$. Hence, by (4.17),

$$|A_3| \le Ch^3\|\theta\|. \tag{4.19}$$

The terms of A_4 are estimated as in the proof of Proposition 2.2 in various inequalities after (2.20), which hold for $r = 2$ as well. As a result, we have

$$|A_4| \le C(h^3\|\theta\| + \|\theta\|^2 + \|\xi\|^2). \tag{4.20}$$

As in Proposition 2.2, we let t_h be such that $\|\xi_x\|_\infty \le 1$ for $t \le t_h$ and suppose that $t_h < T$. Then we have that $|(\xi_x\theta, \theta)| \le \|\theta\|^2$ and (4.16), (4.19) and (4.20) imply that for $0 \le t \le t_h$,

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 - (\gamma, \theta_x) \le C(h^3\|\theta\| + \|\theta\|^2 + \|\xi\|^2), \tag{4.21}$$

where $\gamma = (1+\eta)\xi$. The identity (2.22) still holds. We write it in the form

$$(\xi_t, \gamma) + (\theta_x, P_0\gamma) = (\xi\xi_x, P_0\gamma) + B_3 + B_4, \tag{4.22}$$

where

$$B_3 = -(\rho_x, P_0\gamma) - ((u\sigma)_x, P_0\gamma), \tag{4.23}$$

$$B_4 = -((u\xi)_x, P_0\gamma) + ((\sigma\xi)_x, P_0\gamma) + (\sigma\sigma_x, P_0\gamma). \tag{4.24}$$

By Lemma 3.8(i), and (v) with $v = u$, we have $|(\rho_x, P_0\gamma)| \le Ch^3\|P_0\gamma\| \le Ch^3\|\gamma\| \le Ch^3\|\xi\|$ and $|((u\sigma)_x, P_0\gamma)| \le Ch^3\|P_0\gamma\| \le Ch^3\|\xi\|$. Therefore,

$$|B_3| \le Ch^3\|\xi\|. \tag{4.25}$$

Now, using the superapproximation property (2.23) we have, as in the proof of Proposition 2.2, that $|((u\xi)_x, P_0\gamma)| \le C\|\xi\|^2$, $|((\sigma\xi)_x, P_0\gamma)| \le C\|\xi\|^2$, and $|(\sigma\sigma_x, P_0\gamma)| \le C\|\sigma\|_\infty\|\sigma_x\|\|\xi\| \le Ch^3\|\xi\|$. (These estimates hold in the case $r = 2$ as well.) Hence,

$$|B_4| \le C(h^3\|\xi\| + \|\xi\|^2). \tag{4.26}$$

Finally, as in the proof of Proposition 2.2, we have for $0 \le t \le t_h$ that

$$|(\xi\xi_x, P_0\gamma)| \le C\|\xi\|^2. \tag{4.27}$$

From (4.22), (4.25), (4.26), and (4.27) it follows that for $0 \le t \le t_h$,

$$(\xi_t, \gamma) + (\theta_x, P_0\gamma) \le C(h^3\|\xi\| + \|\xi\|^2). \tag{4.28}$$

From (4.21) and (4.28) we have, as in the proof of Proposition 2.2, that for $0 \le t \le t_h$, $\frac{d}{dt}\left[\|\theta\|^2 + ((1+\eta)\xi, \xi)\right] \le C(h^6 + \|\theta\|^2 + \|\xi\|^2)$, from which, since $\theta(0) = \xi(0) = 0$, we see from Gronwall's lemma that for a constant $C = C(T)$ it holds that

$$\|\theta\| + \|\xi\| \le Ch^3, \quad 0 \le t \le t_h. \tag{4.29}$$

Hence $\|\xi_x\|_\infty \le Ch^{3/2}$ for $0 \le t \le t_h$ in view of (2.4). It follows that t_h is not maximal; thus we may take $t_h = T$ in (4.29). The conclusion of the theorem follows. $\square$

We close this section by presenting the results of some relevant numerical experiments. We solve the nonhomogeneous (SSW) and (SW) with the standard Galerkin method with piecewise linear continuous functions on a uniform mesh on $[0, 1]$ with $h = 1/N$ using as exact solutions the functions $\eta = \exp(2t)(\cos(\pi x) + x + 2)$ and $u = \exp(-xt)\sin(\pi x)$. As in Section 2, the fourth-order explicit classical RK method is used for time stepping with $k = h/10$. Table 4.1 shows the L^2-errors

TABLE 4.1. L^2-errors and orders of convergence, continuous, piecewise linear functions, uniform mesh.

	$L^2 - errors$: SW				$L^2 - errors$: SSW			
N	η	$order$	u	$order$	η	$order$	u	$order$
40	0.4721(−2)		0.1859(−3)		0.2883(−2)		0.1772(−3)	
80	0.1179(−2)	2.0015	0.4627(−4)	2.0064	0.7203(−3)	2.0009	0.4415(−4)	2.0049
160	0.2948(−3)	1.9998	0.1155(−4)	2.0022	0.1800(−3)	2.0006	0.1105(−4)	1.9984
320	0.7369(−4)	2.0002	0.2888(−5)	1.9998	0.4501(−4)	1.9997	0.2762(−5)	2.0003
480	0.3275(−4)	2.0001	0.1284(−5)	1.9991	0.2000(−4)	2.0005	0.1228(−5)	1.9991
640	0.1842(−4)	2.0004	0.7221(−6)	2.0007	0.1125(−4)	2.0000	0.6905(−6)	2.0013

at $t = 1$ and their order of convergence, which is essentially the spatial order of convergence for this problem for both systems. As predicted by the theory of the present section the order of convergence is equal to 2. In addition, the L^∞-errors (not shown here) converge at the same rate. It is also worth noting that for special quasiuniform meshes, e.g. for piecewise uniform or gradually varying meshes, numerical evidence suggests that the L^2- and L^∞-errors are again of $O(h^2)$. We refer to the relevant numerical results at the end of Section 2 of [3]. In Table 4.2 we present the L^2-errors for the same problems for the analogous Galerkin method that uses cubic splines on a uniform mesh for the spatial discretization. The convergence of this scheme was not analyzed here, but its order of convergence appears to be equal to 4, i.e. optimal in L^2. (In order to render the temporal errors negligible and essentially approximate the spatial error, we took $k/h = 1/10$ for (SW) and $k/h = 1/80$ for (SSW) in the numerical experiment presented in this table.)

TABLE 4.2. L^2-errors and orders of convergence, cubic splines, uniform mesh.

	$L^2 - errors$: SW				$L^2 - errors$: SSW			
N	η	$order$	u	$order$	η	$order$	u	$order$
40	0.4877(−6)		0.2307(−7)		0.3287(−6)		0.2280(−7)	
80	0.2938(−7)	4.0531	0.1444(−8)	3.9979	0.1997(−7)	4.0409	0.1412(−8)	4.0132
160	0.1802(−8)	4.0272	0.9015(−10)	4.0016	0.1230(−8)	4.0211	0.8821(−10)	4.0007
320	0.1115(−9)	4.0145	0.5640(−11)	3.9986	0.7632(−10)	4.0105	0.5523(−11)	3.9974
480	0.2196(−10)	4.0073	0.1116(−11)	3.9957	0.1504(−10)	4.0058	0.1096(−11)	3.9886
600	0.8982(−11)	4.0064	0.4576(−12)	3.9952	0.6155(−11)	4.0039	0.4617(−12)	3.8742

5. Full discretization with the third-order Shu-Osher scheme

In this section we turn to temporal discretizations of the o.d.e. systems represented by the standard Galerkin semidiscretizations of (SW) and (SSW) that were studied in Sections 2 and 4. In [9] Dupont analyzed, in the case of a system similar to (SW), the convergence of a linearized Crank-Nicolson scheme. Here we will consider explicit Runge-Kutta schemes, and since our main focus will be on the time-stepping aspect, we will analyze the convergence of the fully discrete schemes in the case of semidiscretizations based on a quasiuniform spatial mesh, whose treatment is more straightforward. Thus the expected spatial rate of convergence in L^2 (see Section 2) is of $O(h^{r-1})$. We will use the notation of Section 2 letting $S_h = S_h^{k,r}$, $S_{h,0} = S_{h,0}^{k,r}$, $r \geq 2$, on a quasiuniform mesh. For a positive integer M we let $k = T/M$ be the time step, put $t^n = nk$ for $n = 0, 1, 2, \ldots, M$, and denote the fully discrete approximations of $\eta(\cdot, t^n)$, $u(\cdot, t^n)$, by $H_h^n \in S_h$, $U_h^n \in S_{h,0}$, respectively, using as initial values the L^2-projections $H_h^0 = P\eta_0$, $U_h^0 = P_0 u_0$ of the initial data.

As is well known, the explicit Euler method is not suitable for discretizing in time the o.d.e. systems generated by standard Galerkin spatial discretizations of linear, first-order hyperbolic p.d.e.'s, since its region of absolute stability does not include an interval of the imaginary axis. In [3] we analyzed the fully discrete explicit Euler-standard Galerkin scheme for (SSW) and proved (for subspaces with $r \geq 2$) that if $\mu = k/h^2$, then there exists a constant $C = C(\mu)$ such that

$$\max_{0 \leq n \leq M} (\|H_h^n - \eta(t^n)\| + \|U_h^n - u(t^n)\|) \leq C(k + h^{r-1}).$$

(In the case of (SW) the analogous proof requires that $r \geq 3$.) Thus the explicit Euler scheme needs the restrictive mesh condition $k = O(h^2)$ for convergence. The situation is only marginally improved in the case of second-order accurate, two-stage explicit Runge-Kutta schemes. For example, in [3] we considered the analogous fully discrete method for the 'improved Euler' scheme (the explicit midpoint method), which for the o.d.e. $y' = f(t, y)$ may be written in the form

$$\begin{aligned} y^{n,1} &= y^n + \tfrac{k}{2} f(t^n, y^n), \\ y^{n+1} &= y^n + k f(t^n + \tfrac{k}{2}, y^{n,1}). \end{aligned}$$

This scheme possesses no interval of absolute stability on the imaginary axis and is not expected therefore to be suitable for the temporal discretization of first-order hyperbolic problems. In [3] we proved that if $\mu = k/h^{4/3}$, there exists a constant $C = C(\mu)$ such that

$$\max_{0\le n\le M}(\|H_h^n - \eta(t^n)\| + \|U_h^n - u(t^n)\|) \le C(k^2 + h^{r-1}),$$

where (H_h^n, U_h^n) is the fully discrete approximation of the solution $(\eta(t^n), u(t^n))$ of (SSW) for $r \ge 3$. (A similar result holds for (SW).) Hence this scheme requires the, still restrictive, mesh condition $k = O(h^{4/3})$ for convergence.

We now examine a practically useful method, namely a third-order accurate explicit Runge-Kutta scheme due to Shu and Osher, [17]. Written in the standard Butcher notation, it is a three-stage scheme corresponding to the tableau:

$$\begin{array}{c|c} A & \tau \\ \hline b^T & \end{array} = \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1/4 & 1/4 & 0 & 1/2 \\ \hline 1/6 & 1/6 & 2/3 & \end{array}$$

One may simplify the scheme and write it as a two-stage method approximating the o.d.e. $y' = f(t, y)$ in the form

$$\begin{aligned} y^{n,1} &= y^n + kf(t^n, y^n), \\ y^{n,2} &= y^n + \tfrac{k}{4}f(t^n, y^n) + \tfrac{k}{4}f(t^{n+1}, y^{n,1}), \\ y^{n+1} &= y^n + \tfrac{k}{6}f(t^n, y^n) + \tfrac{k}{6}f(t^{n+1}, y^{n,1}) + \tfrac{2k}{3}f(t^n + k/2, y^{n,2}); \end{aligned}$$

this is precisely the explicit scheme (2.19) in [17]. It is easy to check that the absolute stability interval of this scheme on the imaginary axis is $[-\sqrt{3}, \sqrt{3}]$; thus it is suitable for integrating in time semidiscretizations of e.g. linear, first-order hyperbolic problems, such as the periodic initial-value problem for $u_t + u_x = 0$, under a Courant-number restriction. It is also well known, [17], that this scheme has good nonlinear stability properties such as the TVD property and has been extensively used as a time-stepping scheme for the numerical approximation of hyperbolic systems in conservation law form with finite volume or DG spatial discretizations. In the rest of this section we will use it to discretize in time the semidiscrete (SW) initial-value problem (2.5)-(2.6). (Its application to (SSW) is analyzed in [3].)

We first define the fully discrete Shu-Osher scheme. Using the notation of Section 2, we let $S_h = S_h^{k,r}$, $S_{h,0} = S_{h,0}^{k,r}$ for $r \ge 3$. Let $(\eta(t), u(t))$, $0 \le t \le T$, be the solution of (SW). We denote $H(t) = P\eta(t)$, $U(t) = P_0u(t)$, $H^n = H(t^n)$, $U^n = U(t^n)$, and define

$$\Phi = U + HU\,, \qquad \Phi^n = \Phi(t^n), \tag{5.1}$$

$$F = H_x + UU_x\,, \qquad F^n = F(t^n). \tag{5.2}$$

The Shu-Osher time-stepping scheme for the semidiscrete problem (2.5)-(2.6) is the following: We seek $H_h^n \in S_h$, $U_h^n \in S_{h,0}$ for $0 \le n \le M$ and $H_h^{n,1}$, $H_h^{n,2} \in S_h$, $U_h^{n,1}$,

$U_h^{n,2} \in S_{h,0}$ for $0 \le n \le M-1$ such that for $0 \le n \le M-1$,

$$\begin{aligned} H_h^{n,1} - H_h^n + kP\Phi_{hx}^n &= 0, \\ U_h^{n,1} - U_h^n + kP_0F_h^n &= 0, \\ H_h^{n,2} - H_h^n + \tfrac{k}{4}P\Phi_{hx}^n + \tfrac{k}{4}P\Phi_{hx}^{n,1} &= 0, \\ U_h^{n,2} - U_h^n + \tfrac{k}{4}P_0F_h^n + \tfrac{k}{4}P_0F_h^{n,1} &= 0, \\ H_h^{n+1} - H_h^n + \tfrac{k}{6}P\Phi_{hx}^n + \tfrac{k}{6}P\Phi_{hx}^{n,1} + \tfrac{2k}{3}P\Phi_{hx}^{n,2} &= 0, \\ U_h^{n+1} - U_h^n + \tfrac{k}{6}P_0F_h^n + \tfrac{k}{6}P_0F_h^{n,1} + \tfrac{2k}{3}P_0F_h^{n,2} &= 0, \end{aligned} \tag{5.3}$$

and $H_h^0 = \eta_h(0) = P\eta_0$, $U_h^0 = u_h(0) = P_0u_0$, where

$$\Phi_h^n = U_h^n + H_h^nU_h^n, \tag{5.4}$$

$$F_h^n = H_{hx}^n + U_h^nU_{hx}^n, \tag{5.5}$$

and for $j = 1, 2$,

$$\Phi_h^{n,j} = U_h^{n,j} + H_h^{n,j}U_h^{n,j}, \tag{5.6}$$

$$F_h^{n,j} = H_{hx}^{n,j} + U_h^{n,j}U_{hx}^{n,j}. \tag{5.7}$$

For the purposes of the error analysis of the fully discrete scheme (5.3) we first prove two preliminary lemmas.

Lemma 5.1. *Let $H = P\eta$. Then there exist constants C such that*

(i)

$$\|P_0[(1+H)\xi] - (1+H)\xi\| \le Ch\|\xi\|, \quad \forall \xi \in S_{h,0}, \tag{5.8}$$

(ii) and for $f \in L^2$,

$$((1+H)\xi, P_0f) = ((1+H)\xi, f) + b(\xi, f)\,, \quad \xi \in S_{h,0}\,, \tag{5.9}$$

where $|b(\xi, f)| \le Ch\|\xi\|\|f\|$.

Proof. (i) We have

$$P_0[(1+H)\xi] - (1+H)\xi = P_0[(H-\eta)\xi] + P_0[(1+\eta)\xi] - (1+\eta)\xi - (H-\eta)\xi.$$

Hence by (2.2a) and the superapproximation property (2.23) (which holds for any $\xi \in S_{h,0}$) we obtain

$$\|P_0[(1+H)\xi] - (1+H)\xi\| \le C(\|H-\eta\|_\infty\|\xi\| + h\|\xi\|),$$

and therefore (5.8) follows from (2.2b).

(ii) Since for $\xi \in S_{h,0}$, $b(\xi, f) = (P_0[(1+H)\xi] - (1+H)\xi, f)$, it follows from (5.8) that $|b(\xi, f)| \le Ch\|\xi\|\|f\|$. ☐

Lemma 5.2. *Let η be the first component of the solution of* (SW) *for which we suppose that $1+\eta \ge \alpha > 0$ for $t \in [0, T]$. Let $H = P\eta$. If $\eta \in W_\infty^r$, then for h small enough it holds that $1 + H \ge \alpha/2$, for $0 \le t \le T$. In addition, if $f \in L^2$, then*

$$\frac{\alpha}{2}\|f\|^2 \le \big((1+H)f, f\big) \le C'\|f\|^2\,, \tag{5.10}$$

for some constant C' that depends on η.

Proof. From (2.2b) we have $1+\eta-C_1h^r \le 1+H \le 1+\eta+C_1h^r$, for some constant C_1. Therefore, if $h \le (\alpha/(2C_1))^{1/r}$, then $\alpha/2 \le 1+H \le C'$, and (5.10) follows. ☐

For the purposes of the error analysis we define 'intermediate' stages $V^{n,j} \in S_h$, $W^{n,j} \in S_{h,0}$ for $j = 1, 2$ and $0 \le n \le M-1$, starting from H^n and U^n, as follows:

$$V^{n,1} - H^n + kP\Phi_x^n = 0, \tag{5.11}$$

$$W^{n,1} - U^n + kP_0F^n = 0, \tag{5.12}$$

$$V^{n,2} - H^n + \tfrac{k}{4}P\Phi_x^n + \tfrac{k}{4}P\Phi_x^{n,1} = 0, \tag{5.13}$$

$$W^{n,2} - U^n + \tfrac{k}{4}P_0F^n + \tfrac{k}{4}P_0F^{n,1} = 0, \tag{5.14}$$

where, for $j = 1, 2$,

$$\Phi^{n,j} = W^{n,j} + V^{n,j}W^{n,j}, \tag{5.15}$$

$$F^{n,j} = V_x^{n,j} + W^{n,j}W_x^{n,j}. \tag{5.16}$$

We are now in a position to estimate the continuous time truncation errors using L^2 projections.

Lemma 5.3. *Let (η, u) be the solution of* (SW) *on* $[0,T]$*. If $H(t) = P\eta(t)$, $U(t) = P_0u(t)$, and $\psi = \psi(t) \in S_h$, $\zeta = \zeta(t) \in S_{h,0}$ are such that*

$$H_t + P\Phi_x = \psi, \tag{5.17}$$

$$U_t + P_0F = \zeta, \tag{5.18}$$

for $0 \le t \le T$, then there exists a constant C such that for $j = 0, 1, 2$, it holds that

$$\max_{0\le t\le T}\big(\|\partial_t^j\psi\| + \|\partial_t^j\zeta\|\big) \le Ch^{r-1}.$$

Proof. Subtracting both sides of the equations $P\eta_t + Pu_x + P(\eta u)_x = 0$, $H_t + PU_x + P(HU)_x = \psi$, and putting $\rho = \eta - H$, $\sigma = u - U$, we have $P\sigma_x + P(\eta u - HU)_x = -\psi$. Since $\eta u - HU = \eta\sigma + u\rho - \rho\sigma$, it follows that $P\sigma_x + P(\eta\sigma)_x + P(u\rho)_x - P(\rho\sigma)_x = -\psi$, and, as a consequence of the approximation properties of S_h and $S_{h,0}$, for $j = 0, 1, 2$, $\|\partial_t^j\psi\| \le Ch^{r-1}$. Subtracting now both sides of the equations $P_0u_t + P_0\eta_x + P_0(uu_x) = 0$, $U_t + P_0H_x + P_0(UU_x) = \zeta$, we obtain $P_0\rho_x + P_0(uu_x - UU_x) = -\zeta$. Since $uu_x - UU_x = (u\sigma)_x - \sigma\sigma_x$, it follows that $P_0\rho_x + P_0(u\sigma)_x - P_0(\sigma\sigma_x) = -\zeta$, and, as in the ψ case, we see that for $j = 0, 1, 2$, $\|\partial_t^j\zeta\| \le Ch^{r-1}$. □

We now prove consistency estimates for the scheme (5.3).

Lemma 5.4. *Let $\lambda = k/h$. If*

$$\delta_1^n = H^{n+1} - H^n + \tfrac{k}{6}P\Phi_x^n + \tfrac{k}{6}P\Phi_x^{n,1} + \tfrac{2k}{3}P\Phi_x^{n,2}, \tag{5.19}$$

$$\delta_2^n = U^{n+1} - U^n + \tfrac{k}{6}P_0F^n + \tfrac{k}{6}P_0F^{n,1} + \tfrac{2k}{3}P_0F^{n,2}, \tag{5.20}$$

then there exists a constant C_λ which is a polynomial of λ with positive coefficients such that

$$\max_{0\le n\le M-1}(\|\delta_1^n\| + \|\delta_2^n\|) \le C_\lambda k(k^3 + h^{r-1}).$$

Proof. From (5.11), (5.17) and (5.12), (5.18) we see that $V^{n,1} = H^n + kH_t^n - k\psi^n$, $W^{n,1} = U^n + kU_t^n - k\zeta^n$, and hence that $V^{n,1}W^{n,1} = H^nU^n + k(HU)_t^n + k^2H_t^nU_t^n + v_1^n$, where, by (2.2a) and the approximation properties of S_h it holds that $\|v_1^n\| \le Ckh^{r-1}$, $\|v_1^n\|_1 \le C_\lambda h^{r-1}$. Thus

$$\Phi^{n,1} = W^{n,1} + V^{n,1}W^{n,1} = \Phi^n + k\Phi_t^n + k^2H_t^nU_t^n + v_2^n, \tag{5.21}$$

with

$$\|v_2^n\| \le Ckh^{r-1}, \qquad \|v_2^n\|_1 \le C_\lambda h^{r-1}.$$

Also, since $W^{n,1}W_x^{n,1} = U^nU_x^n + k(UU_x)_t^n + k^2U_t^nU_{tx}^n + \omega_2^n$, with $\|\omega_2^n\| \le C_\lambda h^{r-1}$, $k\|\omega_2^n\|_1 \le Ck^3 + C_\lambda h^{r-1}$, we have

$$F^{n,1} = F^n + kF_t^n + k^2U_t^nU_{tx}^n + \omega_3^n, \tag{5.22}$$

with $\|\omega_3^n\| \le C_\lambda h^{r-1}$, $k\|\omega_3^n\|_1 \le Ck^3 + C_\lambda h^{r-1}$. Now

$$V^{n,2} = H^n + \tfrac{k}{2}H_t^n - \tfrac{k}{2}\psi^n + \tfrac{k^2}{4}H_{tt}^n - \tfrac{k^2}{4}\psi_t^n - \tfrac{k^3}{4}P(H_t^nU_t^n)_x - \tfrac{k}{4}Pv_{2x}^n,$$

and finally

$$V^{n,2} = H^n + \tfrac{k}{2}H_t^n + \tfrac{k^2}{4}H_{tt}^n + \psi_1^n, \tag{5.23}$$

with

$$\|\psi_1^n\| \le Ck^3 + C_\lambda h^{r-1}, \qquad \|\psi_1^n\|_1 \le Ck^3 + C_\lambda h^{r-1},$$

where we used the stability of the L^2 projection in the H^1 norm (cf. [19]) and the inverse and approximation properties of S_h. Now $W^{n,2} = U^n + \frac{k}{2}U_t^n - \frac{k}{2}\zeta^n + \frac{k^2}{4}U_{tt}^n - \frac{k^2}{4}\zeta_t^n - \frac{k^3}{4}P_0(U_t^nU_{tx}^n) - \frac{k}{4}P_0\omega_3^n$, and therefore

$$W^{n,2} = U^n + \tfrac{k}{2}U_t^n + \tfrac{k^2}{4}U_{tt}^n + \zeta_1^n, \tag{5.24}$$

with $\|\zeta_1^n\| \le Ck^3 + C_\lambda kh^{r-1}$, $\|\zeta_1^n\|_1 \le Ck^3 + C_\lambda h^{r-1}$, where we took into account the approximation and inverse properties of $S_{h,0}$ and the stability of the L^2 projection in H^1. It follows that $V^{n,2}W^{n,2} = H^nU^n + \frac{k}{2}(HU)_t^n + \frac{k^2}{4}(HU)_{tt}^n - \frac{k^2}{4}H_t^nU_t^n + v_3^n$, with $\|v_3^n\| \le Ck^3 + C_\lambda kh^{r-1}$, $\|v_3^n\|_1 \le Ck^3 + C_\lambda h^{r-1}$. Thus

$$\Phi^{n,2} = \Phi^n + \tfrac{k}{2}\Phi_t^n + \tfrac{k^2}{4}\Phi_{tt}^n - \tfrac{k^2}{4}H_t^nU_t^n + v_4^n, \tag{5.25}$$

with $\|v_4^n\| \le Ck^3 + C_\lambda kh^{r-1}$, $\|v_4^n\|_1 \le Ck^3 + C_\lambda h^{r-1}$. From (5.21), (5.25) we conclude that $\frac{1}{6}\Phi^n + \frac{1}{6}\Phi^{n,1} + \frac{2}{3}\Phi^{n,2} = \Phi^n + \frac{k}{2}\Phi_t^n + \frac{k^2}{6}\Phi_{tt}^n + \frac{1}{6}v_2^n + \frac{2}{3}v_4^n$. Hence $\delta_1^n = H^{n+1} - H^n + kP\Phi_x^n + \frac{k^2}{2}P\Phi_{tx}^n + \frac{k^3}{6}P\Phi_{ttx}^n + \frac{k}{2}Pv_{2x}^n + \frac{2k}{3}Pv_{4x}^n = H^{n+1} - H^n - kH_t^n - \frac{k^2}{2}H_{tt}^n - \frac{k^3}{6}H_{ttt}^n + \alpha^n$, with $\|\alpha^n\| \le Ckh^{r-1} + C_\lambda kh^{r-1} + C_\lambda k(k^3 + h^{r-1}) \le C_\lambda k(k^3 + h^{r-1})$. Therefore

$$\|\delta_1^n\| \le Ck(k^3 + C_\lambda h^{r-1}).$$

From (5.24) we obtain $W^{n,2}W_x^{n,2} = U^nU_x^n + \frac{k}{2}(UU_x)_t^n + \frac{k^2}{4}(UU_x)_{tt}^n - \frac{k^2}{4}U_t^nU_{tx}^n + \omega_5^n$, with $\|\omega_5^n\| \le Ck^3 + C_\lambda h^{r-1}$. Thus, using (5.23), $F^{n,2} = F^n + \frac{k}{2}F_t^n + \frac{k^2}{4}F_{tt}^n + \psi_{1x}^n - \frac{k^2}{4}U_t^nU_{tx}^n + \omega_5^n$, i.e.

$$F^{n,2} = F^n + \tfrac{k}{2}F_t^n + \tfrac{k^2}{4}F_{tt}^n - \tfrac{k^2}{4}U_t^nU_{tx}^n + \omega_6^n, \tag{5.26}$$

where $\|\omega_6^n\| \le Ck^3 + C_\lambda h^{r-1}$. From (5.22), (5.26) we now obtain $\frac{1}{6}F^n + \frac{1}{6}F^{n,1} + \frac{2}{3}F^{n,2} = F^n + \frac{k}{2}F_t^n + \frac{k^2}{6}F_{tt}^n + \frac{1}{6}\omega_3^n + \frac{2}{3}\omega_6^n$, and therefore $\delta_2^n = U^{n+1} - U^n - kU_t^n - \frac{k^2}{2}U_{tt}^n - \frac{k^3}{6}U_{ttt}^n + k\beta^n$, where $\|\beta^n\| \le Ckh^{r-1} + Ck^3 + C_\lambda h^{r-1} \le Ck^3 + C_\lambda h^{r-1}$. Hence $\|\delta_2^n\| \le Ck(k^3 + C_\lambda h^{r-1})$. From this and the analogous estimate for δ_1^n the result of the lemma follows. ∎

We now proceed with the proof of convergence of the scheme.

Proposition 5.1. *Let* (H_h^n, U_h^n) *be the solution of* (5.3) *and* (η, u) *the solution of* (SW) *for which we suppose that* $1 + \eta \ge \alpha > 0$ *for* $t \in [0, T]$. *Let* h *be sufficiently* *small. Then if $\lambda = k/h$, there exists a constant λ_0 and a constant C independent of k, h such that for $\lambda \leq \lambda_0$,*

$$\max_{0\leq n\leq M}(\|\eta(t^n) - H_h^n\| + \|u(t^n) - U_h^n\|) \leq C(k^3 + h^{r-1}).$$

Proof. It suffices to show that

$$\max_{0\leq n\leq M}(\|H^n - H_h^n\| + \|U^n - U_h^n\|) \leq C(k^3 + h^{r-1}).$$

We let

$$\varepsilon^n = H^n - H_h^n,\ e^n = U^n - U_h^n,\ \varepsilon^{n,j} = V^{n,j} - H_h^{n,j},\ e^{n,j} = W^{n,j} - U_h^{n,j},\ j = 1, 2.$$

Then from (5.3), (5.11)-(5.14) if follows that

$$\varepsilon^{n,1} = \varepsilon^n - kP(\Phi^n - \Phi_h^n)_x, \tag{5.27}$$

$$e^{n,1} = e^n - kP_0(F^n - F_h^n), \tag{5.28}$$

$$\varepsilon^{n,2} = \varepsilon^n - \tfrac{k}{4}P(\Phi^n - \Phi_h^n)_x - \tfrac{k}{4}P(\Phi^{n,1} - \Phi_h^{n,1})_x, \tag{5.29}$$

$$e^{n,2} = e^n - \tfrac{k}{4}P_0(F^n - F_h^n) - \tfrac{k}{4}P_0(F^{n,1} - F_h^{n,1}), \tag{5.30}$$

so that from the last two equations of (5.3) and also from (5.19), (5.20) we have

$$\varepsilon^{n+1} = \varepsilon^n - \tfrac{k}{6}P(\Phi^n - \Phi_h^n)_x - \tfrac{k}{6}P(\Phi^{n,1} - \Phi_h^{n,1})_x - \tfrac{2k}{3}P(\Phi^{n,2} - \Phi_h^{n,2})_x + \delta_1^n, \tag{5.31}$$

$$e^{n+1} = e^n - \tfrac{k}{6}P_0(F^n - F_h^n) - \tfrac{k}{6}P_0(F^{n,1} - F_h^{n,1}) - \tfrac{2k}{3}P_0(F^{n,2} - F_h^{n,2}) + \delta_2^n. \tag{5.32}$$

From (5.1), (5.4) it follows that $\Phi^n - \Phi_h^n = e^n + (H^nU^n - H_h^nU_h^n)$, and since $H^nU^n - H_h^nU_h^n = H^ne^n + U^n\varepsilon^n - \varepsilon^ne^n$, we see that $\Phi^n - \Phi_h^n = (1 + H^n)e^n + U^n\varepsilon^n - \varepsilon^ne^n$ or

$$\Phi^n - \Phi_h^n = \rho^n + \rho_1^n, \tag{5.33}$$

where

$$\rho^n = (1 + H^n)e^n + U^n\varepsilon^n \tag{5.34}$$

and $\rho_1^n = -\varepsilon^ne^n$. Hence

$$\|\rho^n\| \leq C(\|\varepsilon^n\| + \|e^n\|), \tag{5.35}$$

$$\|\rho_x^n\| \leq \tfrac{C}{h}(\|\varepsilon^n\| + \|e^n\|). \tag{5.36}$$

Now let $0 \leq n^* \leq M - 1$ be the maximal integer for which $\|\varepsilon^n\|_{1,\infty} + \|e^n\|_{1,\infty} \leq 1$, $0 \leq n \leq n^*$. Then, for $0 \leq n \leq n^*$,

$$\|\rho_x^n\|_\infty \leq C\,, \qquad \|\rho_{1x}^n\|_\infty \leq 1\,, \tag{5.37}$$

$$\|\rho_{1x}^n\| \leq \|\varepsilon^n\| + \|e^n\|. \tag{5.38}$$

Now, from (5.37) and (5.33) $\varepsilon^{n,1} = \varepsilon^n - kP\rho_x^n - kP\rho_{1x}^n$, and by (5.37), for $0 \leq n \leq n^*$, we have

$$\|\varepsilon^{n,1}\| \leq C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.39}$$

$$\|\varepsilon^{n,1}\|_{1,\infty} \leq C_\lambda, \tag{5.40}$$

where we used the inverse properties of S_h and the stability of P in the L^∞ norm. Since now $F^n - F_h^n = \varepsilon_x^n + U^n U_x^n - U_h^n U_{hx}^n$ and $U^n U_x^n - U_h^n U_{hx}^n = (U^n e^n)_x - e^n e_x^n$, we will have $F^n - F_h^n = \varepsilon_x^n + (U^n e^n)_x - e^n e_x^n$ or

$$F^n - F_h^n = r_x^n + r_1^n, \tag{5.41}$$

where

$$r^n = \varepsilon^n + U^n e^n, \quad r_1^n = -e^n e_x^n, \tag{5.42}$$

with

$$\|r^n\| \le C(\|\varepsilon^n\| + \|e^n\|), \tag{5.43}$$

$$\|r_x^n\| \le \tfrac{C}{h}(\|\varepsilon^n\| + \|e^n\|), \tag{5.44}$$

and, for $0 \le n \le n^*$,

$$\|r_x^n\|_\infty \le C\,, \qquad \|r_1^n\|_\infty \le 1, \tag{5.45}$$

$$\|r_1^n\| \le \|e^n\|. \tag{5.46}$$

From (5.28), (5.41) it follows that $e^{n,1} = e^n - kP_0 r_x^n - kP_0 r_1^n$, and therefore, for $0 \le n \le n^*$,

$$\|e^{n,1}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.47}$$

$$\|e^{n,1}\|_{1,\infty} \le C_\lambda. \tag{5.48}$$

Now, from (5.15), (5.6) (for $j = 1$) we obtain $\Phi^{n,1} - \Phi_h^{n,1} = e^{n,1} + V^{n,1}e^{n,1} + W^{n,1}\varepsilon^{n,1} - \varepsilon^{n,1}e^{n,1}$, and $V^{n,1}e^{n,1} = H^n e^n - kH^n P_0 r_x^n - kH^n P_0 r_1^n - kP\Phi_x^n \cdot e^{n,1}$, $W^{n,1}\varepsilon^{n,1} = U^n\varepsilon^n - kU^n P\rho_x^n - kU^n P\rho_{1x}^n - kP_0 F^n \cdot \varepsilon^{n,1}$. Thus, $\Phi^{n,1} - \Phi_h^{n,1} = (e^n + H^n e^n + U^n \varepsilon^n) - k(P_0 r_x^n + H^n P_0 r_x^n + U^n P\rho_x^n) - kP_0 r_1^n - k(U^n P\rho_{1x}^n + P_0 F^n \cdot \varepsilon^{n,1} + H^n P_0 r_1^n + P\Phi_x^n \cdot e^{n,1}) - \varepsilon^{n,1}e^{n,1}$ or

$$\Phi^{n,1} - \Phi_h^{n,1} = \rho^n - kf^n + \rho_1^{n,1}, \tag{5.49}$$

where

$$f^n = (1 + H^n)P_0 r_x^n + U^n P\rho_x^n \tag{5.50}$$

and $\rho_1^{n,1} = -kP_0 r_1^n - k(U^n P\rho_{1x}^n + P_0 F^n \cdot \varepsilon^{n,1} + H^n P_0 r_1^n + P\Phi_x^n \cdot e^{n,1}) - \varepsilon^{n,1}e^{n,1}$. From the inverse properties of S_h, $S_{h,0}$ the estimates (5.46), (5.38), the stability of the L^2 projection in the L^∞ norm, the fact that $\|F^n\|_\infty \le C$, and that $\|\Phi_x^n\| \le C$, and the inequalities (5.47), (5.40), (5.48) and (5.45), (5.37), we obtain

$$\|\rho_{1x}^{n,1}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.51}$$

$$\|\rho_{1x}^{n,1}\|_\infty \le C_\lambda. \tag{5.52}$$

In addition, from the inverse properties of S_h, $S_{h,0}$ and (5.37), (5.36), (5.45) we see that

$$\|f_x^n\| \le \tfrac{C}{h^2}(\|\varepsilon^n\| + \|e^n\|), \tag{5.53}$$

$$\|f_x^n\|_\infty \le \tfrac{C}{h}. \tag{5.54}$$

Now, from (5.29), (5.33), (5.49) if follows that $\varepsilon^{n,2} = \varepsilon^n - \frac{k}{2}P\rho_x^n + \frac{k^2}{4}Pf_x^n - \frac{k}{4}(P\rho_{1x}^n + P\rho_{1x}^{n,1})$. Hence, from (5.36), (5.53), (5.38), (5.51), (5.37), (5.54) (5.52), and the inverse properties of S_h, we obtain, for $0 \le n \le n^*$,

$$\|\varepsilon^{n,2}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.55}$$

$$\|\varepsilon^{n,2}\|_{1,\infty} \le C_\lambda. \tag{5.56}$$

We also have $F^{n,1} - F_h^{n,1} = \varepsilon_x^{n,1} + (W^{n,1}e^{n,1})_x - e^{n,1}e_x^{n,1}$ and $W^{n,1}e^{n,1} = U^n e^n - kU^n P_0 r_x^n - kU^n P_0 r_1^n - kP_0 F^n \cdot e^{n,1}$. Hence, $F^{n,1} - F_h^{n,1} = (\varepsilon^n + U^n e^n)_x - k(P\rho_x^n + U^n P_0 r_x^n)_x - k(P\rho_{1x}^n + U^n P_0 r_1^n + P_0 F^n \cdot e^{n,1})_x - e^{n,1}e_x^{n,1}$, i.e.

$$F^{n,1} - F_h^{n,1} = r_x^n - kg_x^n + r_1^{n,1}, \tag{5.57}$$

where

$$g^n = P\rho_x^n + U^n P_0 r_x^n \tag{5.58}$$

and $r_1^{n,1} = -k(P\rho_{1x}^n + U^n P_0 r_1^n + P_0 F^n \cdot e^{n,1})_x - e^{n,1}e_x^{n,1}$. From the inverse properties of S_h, $S_{h,0}$, (5.40), (5.48), (5.37), (5.39), (5.46), (5.47), (5.40), (5.45), we obtain, for $0 \le n \le n^*$,

$$\|r_1^{n,1}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.59}$$

$$\|r_1^{n,1}\|_\infty \le C_\lambda. \tag{5.60}$$

In addition, from the inverse properties of S_h, $S_{h,0}$, (5.36), (5.44) and (5.37), (5.45), we obtain, for $0 \le n \le n^*$,

$$\|g_x^n\| \le \tfrac{C}{h^2}(\|\varepsilon^n\| + \|e^n\|), \tag{5.61}$$

$$\|g_x^n\|_\infty \le \tfrac{C}{h}. \tag{5.62}$$

Now from (5.30), (5.41), (5.57), it follows that $e^{n,2} = e^n - \frac{k}{2}P_0 r_x^n + \frac{k^2}{4}P_0 g_x^n - \frac{k}{4}(P_0 r_1^n + P_0 r_1^{n,1})$. Hence, from (5.44), (5.58), (5.46), (5.59), the inverse properties of $S_{h,0}$, and (5.45), (5.62), (5.60), we obtain, for $0 \le n \le n^*$,

$$\|e^{n,2}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.63}$$

$$\|e^{n,2}\|_{1,\infty} \le C_\lambda. \tag{5.64}$$

In order to derive expressions for $\Phi^n - \Phi_h^n$, $F^n - F_h^n$, we note from (5.13), (5.14) that $H^n - V^{n,2} = \frac{k}{4}(P\Phi_x^n + P\Phi_x^{n,1})$, $U^n - W^{n,2} = \frac{k}{4}(P_0 F^n + P_0 F^{n,1})$, and therefore

$$\|H^n - V^{n,2}\|_\infty \le C_\lambda k, \qquad \|H^n - V^{n,2}\|_{1,\infty} \le C_\lambda, \tag{5.65}$$

$$\|U^n - W^{n,2}\|_\infty \le C_\lambda k, \qquad \|U^n - W^{n,2}\|_{1,\infty} \le C_\lambda. \tag{5.66}$$

We now have $\Phi^{n,2} - \Phi_h^{n,2} = e^{n,2} + V^{n,2}e^{n,2} + W^{n,2}\varepsilon^{n,2} - \varepsilon^{n,2}e^{n,2}$ and $V^{n,2}e^{n,2} = H^n e^n - \frac{k}{2}H^n P_0 r_x^n + \frac{k^2}{4}H^n P_0 g_x^n - \frac{k}{4}H^n P_0(r_1^n + r_1^{n,1}) - (H^n - V^{n,2})e^{n,2}$, $W^{n,2}\varepsilon^{n,2} = U^n\varepsilon^n - \frac{k}{2}U^n P\rho_x^n + \frac{k^2}{4}U^n P f_x^n - \frac{k}{4}U^n P(\rho_{1x}^n + \rho_{1x}^{n,1}) - (U^n - W^{n,2})\varepsilon^{n,2}$. Hence, according to (5.34), (5.50),

$$\begin{aligned}\Phi^{n,2} - \Phi_h^{n,2} = \rho^n - \tfrac{k}{2}f^n &+ \tfrac{k^2}{4}[(1 + H^n)P_0 g_x^n + U^n P f_x^n] \\ &- \tfrac{k}{4}[(1 + H^n)P_0(r_1^n + r_1^{n,1}) + U^n P(\rho_{1x}^n + \rho_{1x}^{n,1})] \\ &- (H^n - V^{n,2})e^{n,2} - (U^n - W^{n,2})\varepsilon^{n,2} - \varepsilon^{n,2}e^{n,2},\end{aligned}$$

i.e.

$$\Phi^{n,2} - \Phi_h^{n,2} = \rho^n - \tfrac{k}{2}f^n + \tfrac{k^2}{4}[(1 + H^n)P_0 g_x^n + U^n P f_x^n] + \rho_1^{n,2}, \tag{5.67}$$

where

$$\rho_1^{n,2} = -\tfrac{k}{4}[(1+H^n)P_0(r_1^n + r_1^{n,1}) + U^n P(\rho_{1x}^n + \rho_{1x}^{n,1})] \\ - (H^n - V^{n,2})e^{n,2} - (U^n - W^{n,2})\varepsilon^{n,2} - \varepsilon^{n,2}e^{n,2}.$$

From the inverse properties of S_h, $S_{h,0}$ and (5.46), (5.59), (5.38), (5.51), (5.65), (5.63), (5.66), (5.55), (5.56), (5.64), we obtain for $0 \le n \le n^*$,

$$\|\rho_{1x}^{n,2}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|). \tag{5.68}$$

From (5.31) and (5.33), (5.49), (5.67) we see that

$$\varepsilon^{n+1} = \varepsilon^n - kP\rho_x^n + \tfrac{k^2}{2}Pf_x^n - \tfrac{k^3}{6}P[(1+H^n)P_0 g_x^n + U^n P f_x^n]_x + \tfrac{k}{6}\omega^n + \delta_1^n, \tag{5.69}$$

where $\omega^n = -P(\rho_{1x}^n + \rho_{1x}^{n,1} + 4\rho_{1x}^{n,2})$, for which it holds that

$$\|\omega^n\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.70}$$

for $0 \le n \le n^*$, by (5.38), (5.51), (5.68). Also, $F^{n,2} - F_h^{n,2} = \varepsilon_x^{n,2} + (W^{n,2}e^{n,2})_x - e^{n,2}e_x^{n,2}$ and $W^{n,2}e^{n,2} = U^n e^n - \tfrac{k}{2}U^n P_0 r_x^n + \tfrac{k^2}{4}U^n P_0 g_x^n - \tfrac{k}{4}U^n P_0(r_1^n + r_1^{n,1}) - (U^n - W^{n,2})e^{n,2}$. Hence, from (5.42), (5.58) it follows that

$$F^{n,2} - F_h^{n,2} = r_x^n - \tfrac{k}{2}g_x^n + \tfrac{k^2}{4}[Pf_x^n + U^n P_0 g_x^n]_x \\ - \tfrac{k}{4}[P(\rho_{1x}^n + \rho_{1x}^{n,1}) + U^n P_0(r_1^n + r_1^{n,1})]_x \\ - [(U^n - W^{n,2})e^{n,2}]_x - e^{n,2}e_x^{n,2}$$

or

$$F^{n,2} - F_h^{n,2} = r_x^n - \tfrac{k}{2}g_x^n + \tfrac{k^2}{4}[Pf_x^n + U^n P_0 g_x^n]_x + r_1^{n,2}, \tag{5.71}$$

where

$$r_1^{n,2} = -\tfrac{k}{4}[P(\rho_{1x}^n + \rho_{1x}^{n,1}) + U^n P_0(r_1^n + r_1^{n,1})]_x - [(U^n - W^{n,2})e^{n,2}]_x - e^{n,2}e_x^{n,2}.$$

From the inverse properties of S_h, $S_{h,0}$, (5.38), (5.51), (5.46), (5.59), (5.55), (5.66), (5.55), (5.63), (5.38), we have, for $0 \le n \le n^*$,

$$\|r_1^{n,2}\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|). \tag{5.72}$$

Also, from (5.32) and (5.41), (5.51), (5.71) we see that

$$e^{n+1} = e^n - kP_0 r_x^n + \tfrac{k^2}{2}P_0 g_x^n - \tfrac{k^3}{6}P_0[Pf_x^n + U^n P_0 g_x^n]_x + \tfrac{k}{6}w^n + \delta_2^n, \tag{5.73}$$

where $w^n = -P_0(r_1^n + r_1^{n,1} + 4r_1^{n,2})$ satisfies

$$\|w^n\| \le C_\lambda(\|\varepsilon^n\| + \|e^n\|), \tag{5.74}$$

for $0 \le n \le n^*$, in view of (5.36), (5.59), (5.72). We now write (5.69), (5.73) in the form

$$\varepsilon^{n+1} = \gamma^n + \tfrac{k}{6}\omega^n + \delta_1^n, \tag{5.75}$$

$$e^{n+1} = \sigma^n + \tfrac{k}{6}w^n + \delta_2^n, \tag{5.76}$$

where $\gamma^n = \varepsilon^n - kP\rho_x^n + \tfrac{k^2}{2}Pf_x^n - \tfrac{k^3}{6}P\widetilde{f}_x^n$, $\sigma^n = e^n - kP_0 r_x^n + \tfrac{k^2}{2}P_0 g_x^n - \tfrac{k^3}{6}P_0\widetilde{g}_x^n$, and

$$\widetilde{f}^n = (1+H^n)P_0 g_x^n + U^n P f_x^n, \tag{5.77}$$

$$\widetilde{g}^n = Pf_x^n + U^n P_0 g_x^n. \tag{5.78}$$

Therefore we have the identity

$$\begin{aligned}\|\gamma^n\|^2 + ((1+H^n)\sigma^n, \sigma^n) = {}& \|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n) + ka_1^n \\ &+ k^2 a_2^n + k^3 a_3^n + k^4 a_4^n + k^5 a_5^n + k^6 a_6^n.\end{aligned} \tag{5.79}$$

We next compute and estimate the coefficients a_j^n, $1 \le j \le 6$. For a_1^n we obtain $a_1^n = -2(\varepsilon^n, P\rho_x^n) - 2((1+H^n)e^n, P_0 r_x^n)$. It follows from Lemma 5.1(ii) that $a_1^n = 2(\varepsilon_x^n, \rho^n) - 2((1+H^n)e^n, r_x^n) + b(e^n, r_x^n)$, where $|b(e^n, r_x^n)| \le Ch\|e^n\|\|r_x^n\|$. Using now the definitions of ρ^n, r^n in (5.34), (5.42), we see that $a_1^n = -(U_x^n \varepsilon^n, \varepsilon^n) - 2((1+H^n)e^n, U_x^n e^n) + ([(1+H^n)U^n]_x e^n, e^n) + b(e^n, r_x^n)$. Therefore, using (5.44) we conclude that

$$|a_1^n| \le C(\|\varepsilon^n\|^2 + \|e^n\|^2). \tag{5.80}$$

For a_2^n we obtain $a_2^n = (\varepsilon^n, Pf_x^n) + ((1+H^n)e^n, P_0 g_x^n) + \|P\rho_x^n\|^2 + ((1+H^n)P_0 r_x^n, P_0 r_x^n)$, so that according to Lemma 5.1(ii), $a_2^n = (\varepsilon^n, Pf_x^n) + ((1+H^n)e^n, g_x^n) + \|P\rho_x^n\|^2 + ((1+H^n)P_0 r_x^n, r_x^n) + a_{21}^n$, where $a_{21}^n = b(e^n, g_x^n) + b(P_0 r_x^n, r_x^n)$. Using now the definitions of f^n and r^n from (5.50) and (5.42) we have $(\varepsilon^n, Pf_x^n) = -(r_x^n, (1+H^n)P_0 r_x^n) + ((U^n e^n)_x, (1+H^n)P_0 r_x^n) - ((U^n\varepsilon^n)_x, P\rho_x^n) + (U_x^n \varepsilon^n, P\rho_x^n)$. Hence, by the definition of ρ^n, (5.34),

$$\begin{aligned}(\varepsilon^n, Pf_x^n) = {}& -(r_x^n, (1+H^n)P_0 r_x^n) + ((U^n e^n)_x, (1+H^n)P_0 r_x^n) \\ &- (\rho_x^n, P\rho_x^n) + \big([(1+H^n)e^n]_x, P\rho_x^n\big) + (U_x^n \varepsilon^n, P\rho_x^n).\end{aligned} \tag{5.81}$$

In addition, using the definition of g^n in (5.58) we see that $((1+H^n)e^n, g_x^n) = -\big([(1+H^n)e^n]_x, P\rho_x^n\big) - \big([(1+H^n)e^n]_x, U^n P_0 r_x^n\big)$. From this and (5.81) we have finally $a_2^n = ((1+H^n)U_x^n e^n, P_0 r_x^n) - (H_x^n U^n e^n, P_0 r_x^n) + (U_x^n \varepsilon^n, P\rho_x^n) + a_{21}^n$, and therefore, from (5.36), (5.44), Lemma 5.1(ii), and (5.61), we see that

$$|a_2^n| \le \frac{C}{h}(\|\varepsilon^n\|^2 + \|e^n\|^2), \tag{5.82}$$

for $0 \le n \le n^*$. For a_3^n we have $a_3^n = -\frac{1}{3}(\varepsilon^n, P\widetilde{f}_x^n) - \frac{1}{3}((1+H^n)e^n, P_0 \widetilde{g}_x^n) - (P\rho_x^n, Pf_x^n) - ((1+H^n)P_0 r_x^n, P_0 g_x^n)$, whence, taking also into account Lemma 5.1(ii),

$$a_3^n = -\tfrac{1}{3}(\varepsilon^n, \widetilde{f}_x^n) - \tfrac{1}{3}((1+H^n)e^n, \widetilde{g}_x^n) - (P\rho_x^n, f_x^n) - ((1+H^n)P_0 r_x^n, g_x^n) + a_{31}^n, \tag{5.83}$$

where

$$a_{31}^n = -\tfrac{1}{3} b(e^n, P_0 \widetilde{g}_x^n) - b(P_0 r_x^n, g_x^n). \tag{5.84}$$

Now, it follows by the definition of $\widetilde{f}^n$, (5.77), that $(\varepsilon^n, \widetilde{f}_x^n) = -((1+H^n)\varepsilon_x^n, P_0 g_x^n) - ((U^n \varepsilon^n)_x, Pf_x^n) + (U_x^n \varepsilon^n, Pf_x^n)$, and, by the definition of $\widetilde{g}^n$, (5.78), that

$$\begin{aligned}((1+H^n)e^n, \widetilde{g}_x^n) = {}& -\big([(1+H^n)e^n]_x, Pf_x^n\big) - ((1+H^n)(U^n e^n)_x, P_0 g_x^n) \\ &+ ((1+H^n)U_x^n e^n, P_0 g_x^n) - (H_x^n U^n e^n, P_0 g_x^n).\end{aligned}$$

So, by the definitions of ρ^n, r^n, (5.34), (5.42), $(\varepsilon^n, \widetilde{f}_x^n) + ((1+H^n)e^n, \widetilde{g}_x^n) = -(\rho_x^n, Pf_x^n) - ((1+H^n)r_x^n, P_0 g_x^n) + a_{32}^n$, where

$$a_{32}^n = (U_x^n \varepsilon^n, Pf_x^n) + ((1+H^n)U_x^n e^n, P_0 g_x^n) - (H_x^n U^n e^n, P_0 g_x^n). \tag{5.85}$$

Again using Lemma 5.1(ii) we see that $((1+H^n)r_x^n, P_0 g_x^n) = ((1+H^n)P_0 r_x^n, g_x^n) + b(P_0 r_x^n, g_x^n) - b(P_0 g_x^n, r_x^n)$. Thus $(\varepsilon^n, \widetilde{f}_x^n) + ((1+H^n)e^n, \widetilde{g}_x^n) = -(\rho_x^n, Pf_x^n) - ((1+H^n)P_0 r_x^n, g_x^n) + a_{33}^n$, where

$$a_{33}^n = a_{32}^n - b(P_0 r_x^n, g_x^n) + b(P_0 g_x^n, r_x^n). \tag{5.86}$$

Hence, from (5.83), $a_3^n = -\frac{2}{3}(P\rho_x^n, f_x^n) - \frac{2}{3}((1+H^n)P_0 r_x^n, g_x^n) + a_{34}^n$, where

$$a_{34}^n = a_{31}^n - \tfrac{1}{3}a_{33}^n. \tag{5.87}$$

Using the definition of f^n, g^n, we have

$$\begin{aligned}(P\rho_x^n, f_x^n) &= \big(P\rho_x^n, [(1+H^n)P_0 r_x^n]_x + (U^n P\rho_x^n)_x\big), ((1+H^n)P_0 r_x^n, g_x^n) \\ &= -\big([(1+H^n)P_0 r_x^n]_x, P\rho_x^n + U^n P_0 r_x^n\big).\end{aligned}$$

Hence

$$\begin{aligned}&(P\rho_x^n, f_x^n) + ((1+H^n)P_0 r_x^n, g_x^n) \\ &\quad = \tfrac{1}{2}(U_x^n P\rho_x^n, P\rho_x^n) - (H_x^n U^n P_0 r_x^n, P_0 r_x^n) + \tfrac{1}{2}\big([(1+H^n)U^n]_x P_0 r_x^n, P_0 r_x^n\big).\end{aligned}$$

So,

$$a_3^n = -\tfrac{1}{3}(U_x^n P\rho_x^n, P\rho_x^n) + \tfrac{2}{3}(H_x^n U^n P_0 r_x^n, P_0 r_x^n) - \tfrac{1}{3}\big([(1+H^n)U^n]_x P_0 r_x^n, P_0 r_x^n\big) + a_{34}^n,$$

and by (5.36), (5.44) we obtain $|a_3^n| \le \frac{C}{h^2}(\|\varepsilon^n\|^2 + \|e^n\|^2) + |a_{34}^n|$. Now, by (5.84), Lemma 5.1(ii), and (5.78), (5.53), (5.61), (5.44) we have $|a_{31}^n| \le \frac{C}{h^2}(\|\varepsilon^n\|^2 + \|e^n\|^2)$. In addition, from (5.85), and (5.53), (5.61) we see that $|a_{32}^n| \le \frac{C}{h^2}(\|\varepsilon^n\|^2 + \|e^n\|^2)$. From this estimate and (5.86), Lemma 5.1(ii), (5.44), (5.61) we obtain $|a_{33}^n| \le \frac{C}{h^2}(\|\varepsilon^n\|^2 + \|e^n\|^2)$. Thus, using (5.87), it follows that $|a_{34}^n| \le \frac{C}{h^2}(\|\varepsilon^n\|^2 + \|e^n\|^2)$ and finally, for $0 \le n \le n^*$, that

$$|a_3^n| \le \frac{C}{h^2}(\|\varepsilon^n\|^2 + \|e^n\|^2). \tag{5.88}$$

For a_4^n we have $a_4^n = \frac{1}{3}(P\rho_x^n, P\widetilde{f}_x^n) + \frac{1}{3}\big((1+H^n)P_0 r_x^n, P_0\widetilde{g}_x^n\big) + \frac{1}{4}\|Pf_x^n\|^2 + \frac{1}{4}((1+H^n)P_0 g_x^n, P_0 g_x^n)$. We note that $(P\rho_x^n, P\widetilde{f}_x^n) + \big((1+H^n)P_0 r_x^n, P_0\widetilde{g}_x^n\big) = (P\rho_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 r_x^n, \widetilde{g}_x^n) + b(P_0 r_x^n, \widetilde{g}_x^n)$. Now using the definitions of $\widetilde{f}^n$, $\widetilde{g}^n$, (5.77), (5.78) we see that $(P\rho_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 r_x^n, \widetilde{g}_x^n) = -(U^n(P\rho_x^n)_x, Pf_x^n) - \big([(1+H^n)P_0 r_x^n]_x, Pf_x^n\big) - \big((1+H^n)(P\rho_x^n)_x, P_0 g_x^n\big) - \big(U^n[(1+H^n)P_0 r_x^n]_x, P_0 g_x^n\big)$, so that $(P\rho_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 r_x^n, \widetilde{g}_x^n) = -\big([U^n P\rho_x^n + (1+H^n)P_0 r_x^n]_x, Pf_x^n\big) - \big((1+H^n)[P\rho_x^n + U^n P_0 r_x^n]_x, P_0 g_x^n\big) + a_{41}^n$, where

$$a_{41}^n = (U_x^n P\rho_x^n, Pf_x^n) + ((1+H^n)U_x^n P_0 r_x^n, P_0 g_x^n) - (H_x^n U^n P_0 r_x^n, P_0 g_x^n). \tag{5.89}$$

Using the definitions of f^n, g^n we have $(P\rho_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 r_x^n, \widetilde{g}_x^n) = -(f_x^n, Pf_x^n) - ((1+H^n)g_x^n, P_0 g_x^n) + a_{41}^n$, and since, by Lemma 5.1(ii) $((1+H^n)g_x^n, P_0 g_x^n) = ((1+H^n)P_0 g_x^n, g_x^n) = ((1+H^n)P_0 g_x^n, P_0 g_x^n) - b(P_0 g_x^n, g_x^n)$, we obtain $(P\rho_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 r_x^n, \widetilde{g}_x^n) = -(f_x^n, Pf_x^n) - ((1+H^n)P_0 g_x^n, P_0 g_x^n) + a_{41}^n + b(P_0 g_x^n, g_x^n)$. Hence

$$a_4^n = -\tfrac{1}{12}\big[\|Pf_x^n\|^2 + ((1+H^n)P_0 g_x^n, P_0 g_x^n)\big] + a_{42}^n, \tag{5.90}$$

where $a_{42}^n = \frac{1}{3}a_{41}^n + \frac{1}{3}b(P_0 g_x^n, g_x^n)$. Therefore, by (5.89), (5.36), (5.53), (5.44), (5.61) and Lemma 5.1(ii), we see, for $0 \le n \le n^*$, that

$$|a_{42}^n| \le \frac{C}{h^3}(\|\varepsilon^n\|^2 + \|e^n\|^2). \tag{5.91}$$

For a_5^n it holds that $a_5^n = -\frac{1}{6}(Pf_x^n, P\widetilde{f}_x^n) - \frac{1}{6}((1+H^n)P_0 g_x^n, P_0\widetilde{g}_x^n)$. Since $(Pf_x^n, P\widetilde{f}_x^n) + ((1+H^n)P_0 g_x^n, P_0\widetilde{g}_x^n) = (Pf_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 g_x^n, \widetilde{g}_x^n) + b(P_0 g_x^n, \widetilde{g}_x^n)$, and

$(Pf_x^n, \widetilde{f}_x^n) + ((1+H^n)P_0 g_x^n, \widetilde{g}_x^n) = \frac{1}{2}(U_x^n Pf_x^n, Pf_x^n) + \frac{1}{2}\big((1+H^n)U_x^n P_0 g_x^n, P_0 g_x^n\big) - \frac{1}{2}(H_x^n U^n P_0 g_x^n, P_0 g_x^n)$, we conclude that

$$\begin{aligned} a_5^n = &- \tfrac{1}{12}(U_x^n Pf_x^n, Pf_x^n) - \tfrac{1}{12}\big((1+H^n)U_x^n P_0 g_x^n, P_0 g_x^n\big) + \tfrac{1}{12}(H_x^n U^n P_0 g_x^n, P_0 g_x^n) \\ &- \tfrac{1}{6} b(P_0 g_x^n, \widetilde{g}_x^n). \end{aligned}$$

Hence, from (5.53), (5.61), Lemma 5.1(ii), and (5.78) we obtain, for $0 \le n \le n^*$,

$$|a_5^n| \le \frac{C}{h^4}(\|\varepsilon^n\|^2 + \|e^n\|^2). \tag{5.92}$$

For a_6^n we have $a_6^n = \frac{1}{36}\|P\widetilde{f}_x^n\|^2 + \frac{1}{36}\big((1+H^n)P_0\widetilde{g}_x^n, P_0\widetilde{g}_x^n\big)$. Since $\widetilde{f}_x^n = [(1+H^n)P_0 g_x^n]_x + (U^n Pf_x^n)_x$, we see that $\|\widetilde{f}_x^n\| \le \frac{C}{h}(\|P_0 g_x^n\| + \|Pf_x^n\|)$. Similarly, since $\widetilde{g}_x^n = (Pf_x^n)_x + (U^n P_0 g_x^n)_x$, we obtain $\|\widetilde{g}_x^n\| \le \frac{C}{h}(\|Pf_x^n\| + \|P_0 g_x^n\|)$. Therefore, according to Lemma 5.2,

$$|a_6^n| \le \frac{C}{h^2}(\|Pf_x^n\|^2 + \|P_0 g_x^n\|^2) \le \frac{C}{h^2}\Big(\|Pf_x^n\|^2 + \frac{2}{\alpha}((1+H^n)P_0 g_x^n, P_0 g_x^n)\Big).$$

Hence it holds that

$$|a_6^n| \le \frac{\widetilde{C}_0}{h^2}\big(\|Pf_x^n\|^2 + ((1+H^n)P_0 g_x^n, P_0 g_x^n)\big), \tag{5.93}$$

for $0 \le n \le n^*$ and for some constant $\widetilde{C}_0 = \widetilde{C}_0(u,\eta)$ independent of h and k. Hence, by (5.79), (5.80), (5.82), (5.88), (5.89)-(5.93), and taking into account Lemma 5.2, we have $\|\gamma^n\|^2 + ((1+H^n)\sigma^n, \sigma^n) \le (1+\widetilde{C}_\lambda k)\big(\|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n)\big) + k^4(\widetilde{C}_0\lambda^2 - \frac{1}{12})\big(\|Pf_x^n\|^2 + ((1+H^n)P_0 g_x^n, P_0 g_x^n)\big)$, and, therefore, for $\lambda \le \lambda_0 = \sqrt{1/(12\widetilde{C}_0)}$ it holds that

$$\|\gamma^n\|^2 + ((1+H^n)\sigma^n, \sigma^n) \le (1+\widetilde{C}_\lambda k)\big(\|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n)\big). \tag{5.94}$$

Hence, according to Lemma 5.2, for some constant $\widetilde{C}$ independent of h and k, there holds

$$\|\gamma^n\| + \|\sigma^n\| \le \widetilde{C}\big(\|\varepsilon^n\| + ((1+H^n)e^n, e^n)^{1/2}\big), \tag{5.95}$$

for $0 \le n \le n^*$. Now

$$\|\varepsilon^{n+1}\|^2 + ((1+H^{n+1})e^{n+1}, e^{n+1}) = \|\varepsilon^{n+1}\|^2 + ((1+H^n)e^{n+1}, e^{n+1}) + \beta_1^n, \tag{5.96}$$

where $\beta_1^n = ((H^{n+1} - H^n)e^{n+1}, e^{n+1})$, while, from (5.75), (5.76) we see that

$$\begin{aligned} &\|\varepsilon^{n+1}\|^2 + ((1+H^n)e^{n+1}, e^{n+1}) \\ &\qquad = \|\gamma^n\|^2 + ((1+H^n)\sigma^n, \sigma^n) + \tfrac{1}{3}\beta_2^n + 2\beta_3^n + \beta_4^n + \tfrac{1}{36}\beta_5^n, \end{aligned} \tag{5.97}$$

where the terms $\beta_2^n = k[(\gamma^n, \omega^n) + ((1+H^n)\sigma^n, w^n) + (\omega^n, \delta_1^n) + ((1+H^n)w^n, \delta_2^n)]$, $\beta_3^n = (\gamma^n, \delta_1^n) + ((1+H^n)\sigma^n, \delta_2^n)$, $\beta_4^n = \|\delta_1^n\|^2 + ((1+H^n)\delta_2^n, \delta_2^n)$, $\beta_5^n = k^2(\|\omega^n\|^2 + \|w^n\|^2)$ will be estimated in the sequel. For β_2^n we have $|\beta_2^n| \le Ck(\|\gamma^n\|\|\omega^n\| + \|\sigma^n\|\|w^n\| + \|\omega^n\|\|\delta_1^n\| + \|w^n\|\|\delta_2^n\|)$. Hence, from (5.95), (5.70), (5.74), Lemma 5.3 and Lemma 5.2 we obtain $|\beta_2^n| \le Ck\big(\|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n)\big) + Ck^2\Big(\|\varepsilon^n\| + \big((1+H^n)e^n, e^n\big)^{1/2}\Big)(k^3 + h^{r-1})$, and therefore

$$|\beta_2^n| \le Ck\big(\|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n)\big) + k^3(k^3 + h^{r-1})^2. \tag{5.98}$$

Furthermore,

$$|\beta_3^n| \le C(\|\gamma^n\|\|\delta_1^n\| + \|\sigma^n\|\|\delta_2^n\|) \le Ck(k^3 + h^{r-1})\Big(\|\varepsilon^n\| + \big((1 + H^n)e^n, e^n\big)^{1/2}\Big),$$

from which

$$|\beta_3^n| \le Ck\big(\|\varepsilon^n\|^2 + ((1 + H^n)e^n, e^n)\big) + Ck(k^3 + h^{r-1})^2. \tag{5.99}$$

Also,

$$|\beta_4^n| \le Ck(k^3 + h^{r-1})^2 \tag{5.100}$$

and

$$|\beta_5^n| \le Ck^2(\|\varepsilon^n\|^2 + ((1 + H^n)e^n, e^n)). \tag{5.101}$$

Now, from (5.76) we see that $\|e^{n+1}\|^2 = \|\sigma^n\|^2 + \frac{k}{3}[(\sigma^n, w^n) + (w^n, \delta_2^n)] + \frac{k^2}{36}\|w^n\|^2 + \|\delta_2^n\|^2 + (\sigma^n, \delta_2^n)$, and, therefore, from (5.95), (5.74), Lemma 5.3 and Lemma 5.2 $\|e^{n+1}\|^2 \le C\big(\|\varepsilon^n\|^2 + ((1 + H^n)e^n, e^n)\big) + Ck^2(k^3 + h^{r-1})^2$. Thus $|\beta_1^n| = |((H^{n+1} - H^n)e^{n+1}, e^{n+1})| \le Ck\big(\|\varepsilon^n\|^2 + ((1 + H^n)e^n, e^n)\big) + Ck(k^3 + h^{r-1})^2$. From this estimate and (5.97)-(5.101), we obtain in (5.96), taking into account (5.94) also, $\|\varepsilon^{n+1}\|^2 + ((1+H^{n+1})e^{n+1}, e^{n+1}) \le (1+Ck)\big(\|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n)\big)$ $+Ck(k^3 + h^{r-1})^2$. Hence, from Gronwall's lemma we obtain $\|\varepsilon^n\|^2 + ((1+H^n)e^n, e^n) \le C(\|\varepsilon^0\|^2 + ((1+H^0)e^0, e^0)) + C(k^3+h^{r-1})^2$, for $0 \le n \le n^*+1$, or according to Lemma 5.2, $\|\varepsilon^n\|^2 + \|e^n\|^2 \le C(\|\varepsilon^0\|^2 + \|e^0\|^2) + C(k^3 + h^{r-1})^2$. Therefore

$$\|\varepsilon^n\| + \|e^n\| \le C(k^3 + h^{r-1}),$$

for $0 \le n \le n^* + 1$. Using the inverse properties of the spaces S_h, $S_{h,0}$ and the fact that $r \ge 3$ we conclude that n^* is not maximal. Hence we may go up to $n^* = M - 1$, and the conclusion of the proposition follows. □

We close this section by presenting the results of a relevant numerical experiment. We solve the nonhomogeneous SW system with exact solutions given by the functions $\eta(x,t) = \exp(2t)(\cos(\pi x) + x + 2)$, $u(x,t) = \exp(xt)(\sin(\pi x) + 5x^2(x-1))$, for $0 \le x \le 1$, $t \ge 0$, using cubic splines on a uniform mesh on $[0,1]$ with $h = 1/N$ for the spatial discretization and the Shu-Osher scheme with $k = h/10$ for time stepping. (It was determined experimentally that the maximum value of the Courant-number for stability was about 0.115 for this problem.) Table 5.1 shows the L^2-, L^∞- and H^1-errors and associated rates of convergence for this problem at $T = 0.5$ as N is increased. The rate of convergence in L^2 stabilizes to about 3 for both components of the solution, which is the expected temporal rate, as the experimental spatial rate is 4 in view of the numerical results in Table 4.2. The L^∞-errors converge at a rate which appears to be equal to 3 again (we expect a $O(k^3 + h^4)$ behaviour), and so do the H^1-errors as well, for which the expected error is of $O(k^3 + h^3)$.

6. Remarks

6.1. **Periodic boundary conditions.** In this section we consider the *periodic* initial-value problem for the usual and the symmetric shallow-water systems, which we discretize using the standard Galerkin method with periodic splines of order $r \ge 2$ on a uniform mesh. Using suitable *quasiinterpolants* of smooth periodic functions in the space of periodic splines (cf. [20]), we prove optimal-order L^2-error estimates for the semidiscrete approximations of both systems. A similar error

TABLE 5.1. L^2-, L^∞-, and H^1-errors and orders of convergence, cubic splines on a uniform mesh with $h = 1/N$ and Shu-Osher time stepping with $k = h/10$, (SW).

L^2-errors

N	η	$order$	u	$order$
40	0.1578(−6)		0.7319(−7)	
80	0.1202(−7)	3.7146	0.5452(−8)	3.7468
160	0.1123(−8)	3.4200	0.5015(−9)	3.4425
320	0.1255(−9)	3.1616	0.5562(−10)	3.1726
480	0.3626(−10)	3.0621	0.1606(−10)	3.0637
640	0.1519(−10)	3.0244	0.6732(−11)	3.0223

L^∞-errors

N	η	$order$	u	$order$
40	0.3656(−6)		0.1664(−6)	
80	0.2886(−7)	3.6631	0.1273(−7)	3.7084
160	0.2541(−8)	3.5056	0.1143(−8)	3.4773
320	0.2498(−9)	3.3466	0.1175(−9)	3.2821
480	0.6788(−10)	3.2134	0.3239(−10)	3.1781
640	0.2764(−10)	3.1232	0.1320(−10)	3.1202

H^1-errors

N	η	$order$	u	$order$
40	0.2502(−4)		0.1708(−4)	
80	0.3068(−5)	3.0277	0.2143(−5)	2.9946
160	0.3797(−6)	3.0144	0.2685(−6)	2.9966
320	0.4719(−7)	3.0083	0.3361(−7)	2.9980
480	0.1396(−7)	3.0039	0.9965(−8)	2.9984
640	0.5888(−8)	3.0008	0.4206(−8)	2.9983

analysis in the case of Boussinesq (i.e. dispersive) systems was done in [4]. For the purposes of the present subsection we shall denote, for integer $k \geq 0$, by H^k_{per} the usual, L^2-based, real Sobolev space of periodic functions on $[0,1]$ with associated norm $\|\cdot\|_k$, and by C^k_{per} the space of periodic functions in $C^k[0,1]$.

We consider the periodic initial-value problem for the shallow water systems. In the case of the usual system we seek $\eta = \eta(x,t)$, $u = u(x,t)$, 1-periodic in x for all $t \in [0,T]$, such that

$$(\mathrm{SW}_{per}) \qquad \begin{aligned} &\eta_t + u_x + (\eta u)_x = 0, \\ &u_t + \eta_x + u u_x = 0, \end{aligned} \quad x \in [0,1], \ \ t \in [0,T],$$
$$\eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x), \quad x \in [0,1],$$

where η_0, u_0 are given smooth 1-periodic functions. The analogous problem for the symmetric system is

$$(\mathrm{SSW}_{per}) \qquad \begin{aligned} &\eta_t + u_x + \tfrac{1}{2}(\eta u)_x = 0, \\ &u_t + \eta_x + \tfrac{3}{2} u u_x + \tfrac{1}{2}\eta\eta_x = 0, \end{aligned} \quad x \in [0,1], \ \ t \in [0,T],$$
$$\eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x), \quad x \in [0,1],$$

where again $\eta(\cdot,t)$, $u(\cdot,t)$ are 1-periodic for $0 \leq t \leq T$ and η_0, u_0 are given smooth 1-periodic functions. We shall assume that (SW_{per}) has a unique smooth enough solution on $[0,T]$ and that there exists a positive α such that $1 + \eta(x,t) \geq \alpha > 0$ for $x \in [0,1]$, $t \in [0,T]$. Similarly, it will be assumed that (SSW_{per}) has a unique smooth enough solution for $0 \le t \le T$. For a theory of local existence-uniqueness of solutions of (SW$_{per}$) we refer the reader to [15].

Let N be a positive integer, $h = 1/N$, and $x_j = jh$, $0 \le j \le N$. For integer $r \ge 2$ let S_h be the N-dimensional space of smooth 1-periodic splines, i.e. $S_h = \{\phi \in C^{r-2}_{per}[0,1] : \phi\big|_{[x_j,x_{j+1}]} \in \mathbb{P}_{r-1}\,, 1 \le j \le N-1\}$. It is well known that S_h has the approximation property that given $w \in H^s_{per}$, where $1 \le s \le r$, there exists a $\chi \in S_h$ such that

$$\sum_{j=0}^{s-1} h^j \|w-\chi\|_j \le Ch^s\|w\|_s, \quad 1 \le s \le r, \tag{6.1}$$

where C is a constant independent of h and w. In addition, the inverse inequalities (2.3) and (2.4) hold in the present framework as well. Following Thomée and Wendroff, [20], one may construct a basis $\{\phi\}_{j=1}^N$ of S_h, with $\mathrm{supp}(\phi_j) = O(h)$, such that for a sufficiently smooth 1-periodic function w, the associated *quasiinterpolant* $Q_h w = \sum_{j=1}^N w(x_j)\phi_j$ satisfies

$$\|w - Q_h w\| \le Ch^r\|w^{(r)}\|. \tag{6.2}$$

In addition, it follows from [20] that the basis $\{\phi\}_{j=1}^N$ may be chosen so that the following properties hold:

(i) If $\psi \in S_h$, then

$$\|\psi\| \le Ch^{-1} \max_{1\le i\le N} |(\psi,\phi_i)|. \tag{6.3}$$

(ii) Let w be a sufficiently smooth 1-periodic function and ν, κ integers such that $0 \le \nu,\kappa \le r-1$. Then

$$\big((Q_h w)^{(\nu)}, \phi_i^{(\kappa)}\big) = (-1)^\kappa h w^{(\nu+\kappa)}(x_i) + O(h^{2r+j-\nu-\kappa}), \quad 1 \le i \le N, \tag{6.4}$$

where $j=1$ if $\nu+\kappa$ is even, and $j=2$ if $\nu+\kappa$ is odd.

(iii) Let f, g be sufficiently smooth 1-periodic functions and ν and κ as in (ii) above. Let

$$\beta_i = \big(f(Q_h g)^{(\nu)}, \phi_i^{(\kappa)}\big) - (-1)^\kappa\big(Q_h[(fg^{(\nu)})^{(\kappa)}], \phi_i\big), \quad 1 \le i \le N.$$

Then

$$\max_{1\le i\le N} |\beta_i| = O(h^{2r+j-\nu-\kappa}), \tag{6.5}$$

where j is as in (ii).

The semidiscretizations of the two systems are defined as follows. In the case of (SW$_{per}$) we seek η_h, $u_h : [0,T] \to S_h$ satisfying

$$\begin{aligned} &(\eta_{ht},\phi) + (u_{hx},\phi) + ((\eta_h u_h)_x,\phi) = 0, \quad \forall \phi \in S_h,\ 0 \le t \le T,\\ &(u_{ht},\chi) + (\eta_{hx},\chi) + (u_h u_{hx},\chi) = 0, \quad \forall \chi \in S_h,\ 0 \le t \le T, \end{aligned} \tag{6.6}$$

$$\eta_h(0) = \eta_{0,h}, \quad u_h(0) = u_{0,h},$$

where $\eta_{0,h}$, $u_{0,h} \in S_h$ are any approximations of η_0, u_0 in S_h satisfying $\|\eta_{0,h}-\eta_0\| + \|u_{0,h}-u_0\| = O(h^r)$. The analogous semidiscrete ivp for (SSW$_{per}$) is

$$\begin{aligned} &(\eta_{ht},\phi) + (u_{hx},\phi) + \tfrac{1}{2}((\eta_h u_h)_x,\phi) = 0, \quad \forall \phi \in S_h,\ 0 \le t \le T,\\ &(u_{ht},\chi) + (\eta_{hx},\chi) + \tfrac{1}{2}(\eta_h\eta_{hx},\chi) + \tfrac{3}{2}(u_h u_{hx},\chi) = 0, \quad \forall \chi \in S_h,\ 0 \le t \le T, \end{aligned} \tag{6.7}$$

$$\eta_h(0) = \eta_{0,h}, \quad u_h(0) = u_{0,h},$$

with $\eta_{0,h}$, $u_{0,h}$ as above. It is clear that (6.6) has a unique solution locally in time; and due to the conservation property (2.10), which holds for solutions of (6.7) as well, (6.7) has a unique solution in any temporal interval $[0, T]$.

The error analysis in the case of (SSW$_{per}$) is straightforward due to the symmetry of the system. We first estimate a truncation error for the system (6.7) defined for all $t \in [0, T]$ in terms of the quasiinterpolants of η and u.

Lemma 6.1. *Let (η, u) be the solution of* (SSW$_{per}$) *and $H = Q_h\eta$, $U = Q_hu$. Define ψ and $\zeta \in S_h$ so that for $0 \le t \le T$,*

$$(H_t, \phi) + (U_t, \phi) + \tfrac{1}{2}((HU)_x, \phi) = (\psi, \phi), \quad \forall \phi \in S_h, \tag{6.8}$$

$$(U_t, \chi) + (H_t, \chi) + \tfrac{1}{2}(HH_x, \chi) + \tfrac{3}{2}(UU_x, \chi) = (\zeta, \chi), \quad \forall \chi \in S_h. \tag{6.9}$$

Then, there is a constant C independent of h such that

$$\|\psi(t)\| + \|\zeta(t)\| \le Ch^r, \quad 0 \le t \le T. \tag{6.10}$$

Proof. Applying (6.4) and (6.8) and using the first p.d.e. of (SSW$_{per}$) yields for $1 \le i \le N$, $t \in [0, T]$ $(\psi, \phi_i) = h(\eta_t + u_x)(x_i, t) + \frac{1}{2}((HU)_x, \phi_i) + O(h^{2r+1}) = \frac{1}{2}\big([(HU) - Q_h(\eta u)]_x, \phi_i\big) + O(h^{2r+1})$. Since $HU - Q_h(\eta u) = \eta u - \varepsilon u - e\eta + \varepsilon e - Q_h(\eta u)$, where $\varepsilon := \eta - H$, $e := u - U$, we have, using (6.5), for $1 \le i \le N$, $(\psi, \phi_i) = \frac{1}{2}\big((\varepsilon e)_x, \phi_i\big) - \frac{1}{2}\big((\eta u)_x - Q_h[(\eta u)_x], \phi_i\big) + O(h^{2r+1})$. Therefore, by (6.3) we obtain, using (6.1) and (6.2), $\|\psi\| \le C\|\varepsilon\|_1\|e\|_1 + O(h^r) \le Ch^r$. The analogous estimate for $\|\zeta\|$ follows along similar lines. $\square$

We now proceed to prove an optimal-order L^2-error estimate for the solution of (6.7).

Proposition 6.1. *Let (η, u), (η_h, u_h) be the solutions of* (SSW$_{per}$), *(6.7), respectively. Then*

$$\max_{0 \le t \le T} (\|\eta - \eta_h\| + \|u - u_h\|) \le Ch^r. \tag{6.11}$$

Proof. Let $\theta := H - \eta_h = Q_h\eta - \eta_h$ and $\xi := U - u_h = Q_hu - u_h$. Then, from (6.7) and (6.8), (6.9) we have for $t \in [0, T]$,

(6.12)
$$(\theta_t, \phi) + (\xi_x, \phi) + \tfrac{1}{2}\big((H\xi + U\theta - \theta\xi)_x, \phi\big) = (\psi, \phi), \quad \forall \phi \in S_h,$$
(6.13)
$$(\xi_t, \chi) + (\theta_x, \chi) + \tfrac{1}{2}\big((H\theta)_x - \theta\theta_x, \chi\big) + \tfrac{3}{2}\big((U\xi)_x - \xi\xi_x, \chi\big) = (\zeta, \chi), \quad \forall \chi \in S_h.$$

Taking $\phi = \theta$ in (6.12), $\chi = \xi$ in (6.13), adding the resulting equations, and using periodicity we obtain for $0 \le t \le T$,

$$\tfrac{1}{2}\tfrac{d}{dt}(\|\theta\|^2 + \|\xi\|^2) + \tfrac{1}{2}(H_x\theta, \xi) + \tfrac{1}{4}(U_x\theta, \theta) + \tfrac{3}{4}(U_x\xi, \xi) = (\psi, \theta) + (\zeta, \chi). \tag{6.14}$$

From (6.2) and the inverse inequalities we have for $0 \le t \le T$, $\|H_x\|_\infty \le C$, $\|U_x\|_\infty \le C$, where C is independent of h. Therefore it follows from (6.10) and (6.14) that for $0 \le t \le T$, $\frac{1}{2}\frac{d}{dt}(\|\theta\|^2 + \|\xi\|^2) \le C(\|\theta\|^2 + \|\xi\|^2 + h^{2r})$. An application of Gronwall's lemma, (6.2), and our choice of $\eta_{0,h}$ and $u_{0,h}$ now yield the desired estimate (6.11). $\square$

We now estimate the errors of the semidiscrerization of (SW_{per}). As before we may prove

Lemma 6.2. *Let* (η, u) *be the solution of* (SW_{per}) *and* $H = Q_h\eta$, $U = Q_h u$. *Define* ψ, $\zeta \in S_h$ *so that for* $t \in [0, T]$,

$$(H_t, \phi) + (U_x, \phi) + ((HU)_x, \phi) = (\psi, \phi), \quad \forall \phi \in S_h, \tag{6.15}$$

$$(U_t, \chi) + (H_x, \chi) + (UU_x, \chi) = (\zeta, \chi), \quad \forall \chi \in S_h. \tag{6.16}$$

Then, for some constant C *independent of* h, *we have*

$$\|\psi(t)\| + \|\zeta(t)\| \le Ch^r, \quad 0 \le t \le T. \tag{6.17}$$

The proof of the main error estimate for (SW_{per}) is not as straightforward as that of the symmetric system but goes through if we use ideas from the proof of Proposition 2.2.

Proposition 6.2. *Let* (η, u) *be the solution of* (SW_{per}). *Then, for* h *sufficiently small,* (6.6) *has a unique solution* (η_h, u_h) *for* $0 \le t \le T$, *satisfying*

$$\max_{0 \le t \le T} (\|\eta - \eta_h\| + \|u - u_h\|) \le Ch^r. \tag{6.18}$$

Proof. We let again $\theta := H - \eta_h = Q_h\eta - \eta_h$ and $\xi := U - u_h = Q_h u - u_h$. Then, from (6.6) and (6.15)-(6.16), we have, while the solution of (6.6) exists,

$$(\theta_t, \phi) + (\xi_x, \phi) + ((H\xi + U\theta - \theta\xi)_x, \phi) = (\psi, \phi), \quad \forall \phi \in S_h, \tag{6.19}$$

$$(\xi_t, \chi) + (\theta_x, \chi) + ((U\xi)_x - \xi\xi_x, \chi) = (\zeta, \chi), \quad \forall \chi \in S_h. \tag{6.20}$$

Putting $\phi = \theta$ in (6.19) and using periodicity we have

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 + (\xi_x, \theta) + ((H\xi)_x, \theta) + \tfrac{1}{2}(U_x\theta, \theta) - \tfrac{1}{2}(\xi_x\theta, \theta) = (\psi, \theta). \tag{6.21}$$

Now, using the inverse inequalities and (6.2) we see that

$$((H\xi)_x, \theta) = ((\eta\xi)_x, \theta) + ((H - \eta)_x\xi, \theta) + ((H - \eta)\xi_x, \theta) \le ((\eta\xi)_x, \theta) + C\|\xi\|\|\theta\|. \tag{6.22}$$

Let $t_h > 0$ denote a maximal value of t such that (η_h, u_h) exists and $\|\xi_x\|_\infty \le 1$ for $0 \le t \le t_h$, and suppose that $t_h < T$. From (6.21), (6.22) and (6.17) we conclude then that

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 - (\theta_x, \gamma) \le C(h^r\|\theta\| + \|\xi\|\|\theta\|), \quad 0 \le t \le t_h, \tag{6.23}$$

where $\gamma := (1 + \eta)\xi$. We now put in (6.20) $\chi = P\gamma = P[(1 + \eta)\xi]$, where P is the L^2-projection on S_h, and obtain for $0 \le t \le t_h$,

$$(\xi_t, (1 + \eta)\xi) + (\theta_x, P\gamma) = -((U\xi)_x - \xi\xi_x, P\gamma) + (\zeta, P\gamma). \tag{6.24}$$

Now, using periodicity, we have $((U\xi)_x, P\gamma) = (U\xi_x, P\gamma - \gamma) + (U_x\xi, P\gamma - \gamma) + (U_x\xi, (1 + \eta)\xi) - \big((U(1 + \eta))_x, \xi^2\big)$. Using again the superapproximation property (2.23), which holds in the space of periodic splines as well, the fact that $\|U\|_{1,\infty} \le C$, and inverse properties, we obtain from the above

$$((U\xi)_x, P\gamma) \le C\|\xi\|^2. \tag{6.25}$$

Using, in addition, the fact that $\|\xi_x\|_\infty \le 1$ in $[0, t_h]$, we also have

$$(\xi\xi_x, P\gamma) = (\xi\xi_x, P\gamma - \gamma) + (\xi\xi_x, \gamma) \le C\|\xi\|^2. \tag{6.26}$$

Therefore, by (6.16), (6.25), (6.26), and (6.24) we have

$$(\xi_t, (1 + \eta)\xi) + (\theta_x, P\gamma) \le C(h^r\|\xi\| + \|\xi\|^2), \quad 0 \le t \le t_h. \tag{6.27}$$

Adding (6.23) and (6.27) we see that $\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + (\xi_t, (1+\eta)\xi) + (\theta_x, P\gamma - \gamma) \leq C\big(h^r(\|\xi\| + \|\theta\|) + \|\xi\|^2 + \|\theta\|^2\big)$. As in the proof of Proposition 2.2 we have, mutatis mutandis, that $\|\theta(t)\| + \|\xi(t)\| \leq Ch^r$, $0 \leq t \leq t_h$, for a constant C independent of h. It follows that $\|\xi_x\|_\infty \leq Ch^{r-3/2}$, i.e. that t_h is not maximal if h is chosen sufficiently small. The result of the proposition now follows in the standard manner. □

6.2. **Comparison of SW and SSW for small-amplitude solutions.** As is well known, the system of shallow water equations (which has been written thus far in terms of nondimensional, unscaled variables) is derived from the 2D Euler equations for surface water waves in the long-wave regime, i.e. when $\sigma := \frac{h_0}{\lambda} \ll 1$, where h_0 is the depth of the horizontal channel and λ is a typical wavelength. Under the additional assumption that the wave amplitude is small, i.e. when $\varepsilon := \frac{\alpha}{h_0} \ll 1$, one may formally derive (cf. [13], [5]) from the Euler equations one of the original versions of a Boussinesq system written in nondimensional, scaled variables in the form

$$\begin{aligned} &\eta_t + u_x + \varepsilon(\eta u)_x + \tfrac{\sigma^2}{3}u_{xxx} = O(\varepsilon\sigma^2, \sigma^4),\\ &u_t + \eta_x + \varepsilon u u_x = O(\varepsilon\sigma^2, \sigma^4), \end{aligned}$$

where u denotes the horizontal velocity at the free surface and η is the displacement of the free surface from its rest position. (Here $x \in \mathbb{R}$ is proportional to length along the channel and $t \geq 0$ is proportional to time.) If we assume that the dispersive effects are small, in the sense that $\varepsilon \sim \sigma$, we obtain $\eta_t + u_x + \varepsilon(\eta u)_x = O(\varepsilon^2)$, $u_t + \eta_x + \varepsilon u u_x = O(\varepsilon^3)$, from which, replacing the right-hand side by zero, we get the system

$$\eta_t + u_x + \varepsilon(\eta u)_x = 0, \tag{6.28}$$

$$u_t + \eta_x + \varepsilon u u_x = 0, \tag{6.29}$$

a scaled version of the shallow water equations valid for small-amplitude waves in the regime $\varepsilon \sim \sigma \ll 1$.

Making in (6.28)-(6.29) the nonlinear change of variable $v = u(1 + \frac{\varepsilon}{2}\eta)$ used in [6] in the context of dispersive waves, we obtain that $\eta_t + v_x + \frac{\varepsilon}{2}(\eta v)_x = O(\varepsilon^2)$, $v_t + \eta_x + \frac{\varepsilon}{2}\eta\eta_x + \frac{3\varepsilon}{2}vv_x = O(\varepsilon^2)$, i.e. that (η, v) satisfies a scaled version of the symmetric shallow water equations which is formally equivalent as a model up to $O(\varepsilon^2)$ terms to the scaled shallow water system.

Now let (η^s, u^s) denote the solution of the Cauchy problem for the symmetric system

$$\eta^s_t + u^s_x + \tfrac{\varepsilon}{2}(\eta^s u^s)_x = 0, \tag{6.30}$$

$$u^s_t + \eta^s_x + \tfrac{\varepsilon}{2}\eta^s\eta^s_x + \tfrac{3\varepsilon}{2}u^s u^s_x = 0, \tag{6.31}$$

for $x \in \mathbb{R}$, $t \geq 0$, with initial data

$$\eta^s(x,0) = \eta^s_0(x), \quad u^s(x,0) = u^s_0(x), \quad x \in \mathbb{R}, \tag{6.32}$$

and consider the Cauchy problem for the system (6.28)-(6.29) with initial conditions

$$\eta(x,0) = \eta_0(x), \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{6.33}$$

Using the theory of local existence for initial-value problems for quasilinear hyperbolic systems, [12], [18], and examining the proofs of Proposition 4 and Corollary 2 of [6], we may conclude that the results of [6] hold also in the nondispersive case, and specifically for the initial-value problems (6.28), (6.29), (6.33) and (6.30)-(6.32). In particular, if $(\eta_0^s, u_0^s) \in \big(H^\ell(\mathbb{R})\big)^2$ for some $\ell > 3/2$, there exists $T_0 > 0$ independent of ε and a unique solution $(\eta^s, u^s) \in C\big([0, \frac{T_0}{\varepsilon}]; (H^\ell(\mathbb{R}))^2\big)$ of (6.30)-(6.32). In addition, $\|(\eta^s, u^s)\|_{W^{k,\infty}\left(0,\frac{T_0}{\varepsilon};(H^{\ell-k}(\mathbb{R}))^2\right)} \leq C_0$ for some constant C_0 independent of ε and for all k such that $\ell - k > 3/2$. An entirely analogous result (with different constants T_0' and C_0') holds for the solutions (η, u) of the initial-value problem for the shallow water system (6.28), (6.29), (6.33). Under these hypotheses and if

$$\eta_0^s = \eta_0, \quad u_0^s = u_0(1 + \tfrac{\varepsilon}{2}\eta_0), \tag{6.34}$$

and $T = \min(T_0, T_0')$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,

$$\|\eta - \eta^s\|_{L^\infty(0,t;H^\ell(\mathbb{R}))} + \|u - (1 - \tfrac{\varepsilon}{2}\eta^s)u^s\|_{L^\infty(0,t;H^\ell(\mathbb{R}))} \leq C\varepsilon^2 t, \tag{6.35}$$

for all $t \in [0, \frac{T}{\varepsilon}]$ and some constant C independent of ε. If therefore the initial data in (6.32) and (6.33) are related by (6.34), the solutions (η, u) and (η^s, u^s) of the two systems (transformed as in (6.35)) differ by an amount of at most $O(\varepsilon^2 t)$ for t up to $O(T/\varepsilon)$. (Note that initially smooth solutions of both systems are expected in general to develop singularities after times of $O(1/\varepsilon)$.)

We will now investigate by computational means whether an estimate of the form (6.35) holds also in the case of initial-boundary-value problems for the two systems when they are posed on a finite interval, say on $[0, 1]$, with the velocity variable equal to zero at the endpoints. We consider therefore the ibvp's $(\mathrm{SW}_\varepsilon)$ consisting of (6.28) and (6.29) for $x \in [0, 1]$, $t \geq 0$, initial conditions of the form (6.33) for $x \in [0, 1]$ and boundary conditions $u(0, t) = u(1, t) = 0$ for $t \geq 0$, and the analogous problem $(\mathrm{SSW}_\varepsilon)$ consisting of (6.30)-(6.32) for $x \in [0, 1]$, $t \geq 0$, and boundary conditions $u^s(0, t) = u^s(1, t) = 0$, $t \geq 0$. (Note that the change of variables $u^s = u(1 + \frac{\varepsilon}{2}\eta)$ preserves the homogeneous boundary conditions on the velocity.) We solve both problems numerically using cubic splines on a uniform mesh in space coupled with the third-order Shu-Osher temporal discretization with $h = 10^{-3}$, $k = 10^{-3}$, taking as initial conditions for $(\mathrm{SW}_\varepsilon)$ the functions $\eta_0(x) = 1$, $u_0(x) = x(x - 1)$, $x \in [0, 1]$, and for $(\mathrm{SSW}_\varepsilon)$ $\eta_0^s = \eta_0$, $u_0^s = u_0(1 + \frac{\varepsilon}{2}\eta_0)$. In Figure 6.1 we plot as functions of t the quantities

$$L_2 - error := \|\eta - \eta^s\| + \|u - u^s(1 - \tfrac{\varepsilon}{2}\eta^s)\|,$$
$$H^1 - error := \|\eta - \eta^s\|_1 + \|u - u^s(1 - \tfrac{\varepsilon}{2}\eta^s)\|_1,$$

where (η, u) and (η^s, u^s) are the numerical approximations of the solutions of $(\mathrm{SW}_\varepsilon)$ and $(\mathrm{SSW}_\varepsilon)$, respectively, evolving from the stated initial conditions for various values of ε. For values of ε up to 10^{-3} the temporal profile is practically linear up to about $t = 300$, and the same is observed for $\varepsilon = 10^{-2}$ up to about $t = 100$ for the L^2-error. In the case $\varepsilon = 10^{-2}$, note the change of scale in the t-axis in the figure: a singularity starts developing after about $t = 120$ (when $t\varepsilon = O(1)$). In Table 6.1 we present the values of the L^2- and H^1-errors from the same computations at $t = 50$, 100, 200, 300 as functions of diminishing ε in the range where the models are valid, i.e. before singularities emerge. The computed numerical orders of convergence in ε for each fixed t are practically equal to 2.

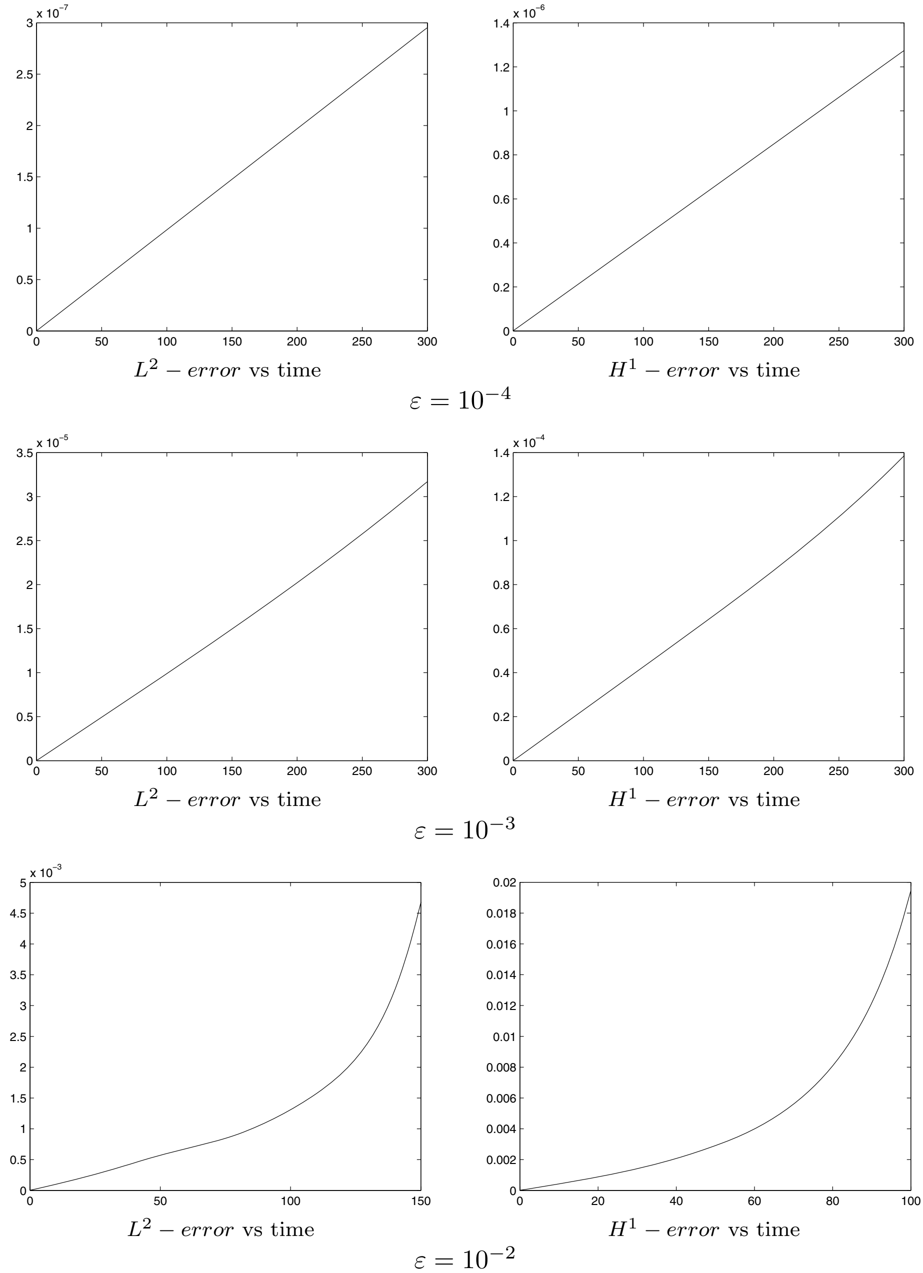

FIGURE 6.1. L^2 and H^1 norms of the differences $\big(\eta - \eta^s, u - u^s(1 - \frac{\varepsilon}{2}\eta^s)\big)$, ("$L^2$-, H^1-errors") as functions of t for $\varepsilon = 10^{-4}$, 10^{-3}, 10^{-2}.

TABLE 6.1. Data of Figure 6.1. L^2- and H^1-errors at $t = 50$, 100, 200, 300 as functions of ε, and order of convergence as $\varepsilon \to 0$.

	time = 50		time = 100		time = 200		time = 300	
ε	L^2-*error*	*order*	L^2-*error*	*order*	L^2-*error*	*order*	L^2-*error*	*order*
10^{-2}	5.7064(−04)		1.3102(−03)					
10^{-3}	4.9362(−06)	2.0630	9.8992(−06)	2.1217	2.0206(−05)		3.1706(−05)	
10^{-4}	4.9224(−08)	2.0012	9.8437(−08)	2.0024	1.9688(−07)	2.0113	2.9536(−07)	2.0308
10^{-5}	4.9214(−10)	2.0001	9.8415(−10)	2.0001	1.9682(−09)	2.0001	2.9523(−09)	2.0002

	time = 50		time = 100		time = 200		time = 300	
ε	H^1-*error*	*order*	H^1-*error*	*order*	H^1-*error*	*order*	H^1-*error*	*order*
10^{-2}	2.9087(−03)		1.9463(−02)					
10^{-3}	2.1336(−05)	2.1346	4.2709(−05)	2.6587	8.6497(−05)		1.3853(−04)	
10^{-4}	2.1254(−07)	2.0017	4.2482(−07)	2.0023	8.4936(−07)	2.0079	1.2740(−06)	2.0364
10^{-5}	2.1305(−09)	2.0000	4.2485(−09)	2.0000	8.4888(−09)	2.0003	1.2725(−08)	2.0005

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