

A graphical approach to measurement-based quantum computing

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Abstract

Quantum computations are easily represented in the graphical notation known as the ZX-calculus, a.k.a. the red-green calculus. We demonstrate its use in reasoning about measurement-based quantum computing, where the graphical syntax directly captures the structure of the entangled states used to represent computations, and show that the notion of information flow within the entangled states gives rise to rewriting strategies for proving the correctness of quantum programs.

Quantum computation, at least for the finite dimensional systems usually considered, lives in the setting of finite dimensional Hilbert spaces. Even ignoring the possibly enormous dimension of the spaces involved, a Hilbert space is a very rich mathematical environment which often hides the structure of the states and conceals the behaviour of their maps, making it difficult to analyse quantum programs. Can this difficulty be circumvented?

In this chapter, we will present an abstract formulation of quantum theory, based on algebraic features present in the Hilbert space theory, but making no reference to Hilbert spaces themselves. The reader will perhaps be unsurprised to learn that the tool of choice for this reformulation of quantum mechanics is category theory, and in particular the theory of symmetric monoidal categories (SMCs).

In a seminal paper [1], Abramsky and Coecke introduced the notions of \dagger -symmetric monoidal category (\dagger -SMC) and \dagger -compact category, and, exploiting the fact that the category of finite dimensional Hilbert spaces and linear maps (henceforth called **fdHilb**) forms a \dagger -compact category, gave a high-level proof of correctness of the quantum teleportation protocol [4]. In so doing, they showed that quantum protocols do not necessarily rely upon the full apparatus of Hilbert spaces: a more abstract presentation of quantum mechanics can suffice. We will use such a high level presentation to analyse measurement-based quantum programs.

As discussed earlier in this volume, \dagger -compact categories admit a graphical notation where the morphisms of the category are represented by diagrams. Sequential composition of morphisms is represented by plugging together diagrams, and parallel composition (i.e. the tensor product) is represented by juxtaposition

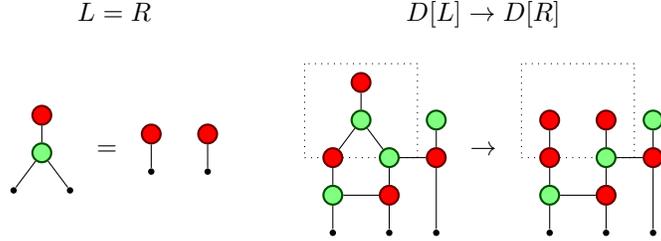


Figure 1: Rewriting as substitution

of diagrams. The crucial point, fully elaborated by Selinger [43], is that the equations of the category are fully captured by homotopic transformation of diagrams. In other words, two morphisms are equal, according to the axioms of \dagger -SMCs, if and only if the corresponding diagrams can be continuously deformed into each other. Hence, by transcribing a morphism into the graphical notation a large amount of equational structure is incorporated directly into the syntax of the formal system.

The axioms of \dagger -compact categories will not be sufficient, however, to study the quantum systems of interest in this chapter. We will introduce more structure by choosing specific generators for the category of diagrams, and imposing certain equations between diagrams involving those generators. These equations induce an equivalence relation between diagrams based on substitution. More concretely we view each equation, say $L = R$, as a rewrite rule, and whenever we find L occurring as a subdiagram in some larger diagram D , we may perform the rewrite $D[L] \rightarrow D[R]$, as shown in Figure 1. Since all diagrams are typed this substitution is always possible. We will treat diagrams and rewriting informally here, but the interested reader can find a detailed account in the paper of Dixon and Kissinger [22].

The additional equations imposed on diagrams are those corresponding to two different algebra structures found on the underlying Hilbert space. Thanks to the theorem of Coecke, Pavlovic and Vicary [11], there is a bijective correspondence between orthonormal bases for the Hilbert space — which from our point of view represent quantum observables — and special commutative \dagger -Frobenius algebras. Therefore we encode each observable by an algebra, and its associated equations give the first collection of rewrite rules. For the purpose of analysing measurement-based quantum programs, we'll only need to consider two different algebras, namely those corresponding to the X and Z spin observables. These observables have, in addition, a further property: they are complementary, and in a particularly strong sense. Intuitively, complementarity means that perfect knowledge of one observable implies complete ignorance of the other. In previous work, Coecke and the author [13] showed that this kind of complementarity can also be formalised in terms of algebras. Strongly complementary observables form a bialgebra, in fact a Hopf algebra. We also impose the defining equations

of these structures onto the diagrammatic language to get another family of rewrite rules.

In summary, the graphical calculus consists of diagrams generated by two Frobenius algebras for the X and Z observables; the equations imposed by the monoidal structure may be effectively forgotten because, thanks to the context of \dagger -compact categories, the notion of equality of diagrams already contains all of them. We then impose rewrite rules corresponding to equations for the Frobenius structure, and then further equations stating that these generators, when combined appropriately, form a Hopf algebra. This setup, combined with additional elements to be introduced later, forms the ZX-calculus, also known as the red-green calculus.

The ZX-calculus is known to be weaker than the full theory of Hilbert spaces, but it is sound, meaning that any equation derived in it will also hold when translated back to the Hilbert space formalism. It replaces matrices with a structured and discrete notation which exposes the relation between different parts of a quantum system. Furthermore, its graphical nature allows quantum circuits to be represented very easily, and more importantly, the ad hoc notation for graph states used in measurement-based quantum computation can be derived from algebraic considerations alone. Hence we can see the beautiful interplay between the structure of an entangled state and the algebraic objects which would represent this state. The rewrite rules then allow transformations between, e.g., the circuit model and the measurement-based model, and expose how information flows within the entangled state during the process of executing a measurement-based program. This last fact will be at the heart of our analysis: we will demonstrate how the non-determinism induced by quantum measurements may be tracked through the graph structure of an entangled state and verify that a given computation is in fact deterministic.

Background and related work We assume that the reader is familiar with the basics of monoidal category theory; aside from other chapters in this volume, the articles of Abramsky and Tzevelekos [2], and Coecke and Paquette [16] provide suitable introductions, while [34] is the standard text. Compact closed categories were introduced by Kelly and Laplaza [33], and the notion of dagger-compactness first arose in [1]. Diagrammatic notation for monoidal categories has a long history going back to work of Kelly [31, 32] and Penrose [39]; one can also view the proof-net syntax of linear logic in this light [27, 5]. The essential reference on this subject is Selinger’s survey [43], which pulls together a great deal of material scattered throughout the literature.

The study of quantum mechanics through categorical eyes was initiated by the paper of Abramsky and Coecke [1], and the explicit use of diagrams was emphasised by Coecke [12]. The notion of classical structure, here referred to as *observable structure* was introduced by Coecke and Pavlovic [10], and further developed by those authors in collaboration with Paquette [9, 17]. The key theorem, that any classical structure in finite dimensional Hilbert space is equivalent to a basis for that space, was shown by Coecke, Pavlovic and

Vicary [11]. The that complementarity could also be formalised in terms of interacting algebras was introduced by Coecke and the author [8, 13]: these papers introduced several fundamental ideas which have since found application in areas as diverse as quantum foundations [26, 14, 15], topological quantum computation [29], and measurement-based quantum computing, which is our main concern here.

One can view the quantum teleportation protocol [4] as the first measurement-based quantum computation; albeit the program computes the identity function. Gottesman and Chuang showed later that this idea could be generalised to a universal computation model [28]. The model of interest here is not the teleportation model, but rather the one-way model introduced by Raussendorf and Briegel [40, 41, 42]. Our work here is based on the work carried out by Danos, Kashefi and Panangaden [20] to provide this model with a formal syntax, the measurement calculus. Danos and Kashefi [18] introduced the concept of *flow*, later renamed *causal flow* to study the problem of determinism in the one-way model. Mhalla and Perdrix demonstrated an efficient algorithm for finding optimal flows [36], while Browne, Kashefi, Mhalla, and Perdrix introduced the notion of *generalised flow* [7].

This chapter mainly draws on the author’s joint work with Perdrix [24], although work relating the graphical/categorical approach to MBQC goes back rather further [23].

Outline of the chapter The next section is a primer on the basics of quantum mechanics. Section 2 introduces the algebraic framework of interacting observables (as presented in [13]) in full generality; Section 3 presents the ZX-calculus, which is the specific instance of that framework as a formal graphical calculus based on qubits with the spin observables X and Z . Section 4 introduces the one-way model and the measurement calculus, and shows how to represent them in the ZX-calculus. Finally, Section 5 examines determinism in the one-way model, and shows how to use the property of flow to generate rewrite sequences to prove the correctness of the measurement calculus programs.

Notation We will use the Dirac notation throughout: vectors are denoted by *kets* $|\psi\rangle$, and their duals by *bras* $\langle\phi|$. The inner product is written $\langle\phi|\psi\rangle$, and is taken to be linear in the second component and anti-linear in the first. We will usually denote \mathbb{C}^2 by Q , since it is the state space of *qubits*. The vectors comprising the standard basis for Q , sometime called the or Z -basis, are written $|0\rangle$ and $|1\rangle$; we denote the X -basis by

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

When writing tensor products of qubits we will usually suppress the tensor symbol, and write e.g. $|00\rangle$ in place of $|0\rangle \otimes |0\rangle$. The base field \mathbb{C} will often be written simply as I since it is the unit of the monoidal category structure.

If u and v are vertices of an undirected graph, then $v \sim u$ means that they are adjacent.

When drawing diagrams, we use the pessimistic convention: diagrams should be read from top to bottom.

1 The rudiments of quantum computing

This section is necessarily rather brief; for a more complete treatment we suggest the excellent books by Mermin [35], and Kaye, Laflamme and Mosca [30].

Whereas a classical bit has only two values, its quantum analogue — the qubit — is a unit vector in a two-dimensional Hilbert space. It is impossible to distinguish two states which differ only by a global phase, so we quotient the state space by the relation $|\psi\rangle \sim e^{i\alpha} |\psi\rangle$. The state space of a compound quantum system, i.e. one formed by combining individual systems, is given by the tensor product of the constituent state spaces, so a state consisting of n qubits is a vector in a Hilbert space of dimension 2^n . Such a state space necessarily contains states that cannot be decomposed into a product of n individual qubits. For example, the Bell state,

$$|\Phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} ,$$

is a perfectly valid state of two qubits, but there are no single qubit states such that $|\phi\rangle \otimes |\psi\rangle = |\Phi_+\rangle$. This simple mathematical fact underlies the phenomenon of *entanglement*. These indecomposable states reflect non-local correlations between the subsystems, and form the main building block of the paradigm called *measurement-based quantum computation*.

The evolution of an undisturbed quantum system is given by a unitary operator:

$$|\psi(t)\rangle = U_t |\psi(t_0)\rangle$$

A quantum computation is typically described as a quantum circuit: this is just a sequence of unitary operations acting on some number of qubits. Unitarity implies that quantum computations are reversible, since the unitaries form a group. The exception is quantum measurement.

Quantum measurements have two properties which run contrary to classical intuition. Firstly their outcomes are *probabilistic*: in almost all quantum states the outcome of a given measurement cannot be known with certainty. Secondly, they have *side-effects*, so that state after a measurement will usually be different to that before it. Mathematically speaking we identify a quantum measurement with a self-adjoint operator,

$$M = \sum_i \lambda_i |v_i\rangle\langle v_i| .$$

The possible observed values are the eigenvalues λ_i . We are only concerned with non-degenerate measurements here, so we assume that all the λ_i are distinct and non-zero. Given a quantum state $|\psi\rangle$, the probability of observing λ_i is given by the inner product

$$p(\lambda_i) = |\langle v_i | \psi \rangle|^2 .$$

Most importantly, the new state of the system after the measurement is the corresponding eigenvector $|v_i\rangle$. In other words, observing λ_i is effectively the same as acting on the state with the projection operator $|v_i\rangle\langle v_i|$, or, if the measured system is destroyed by the measurement—which will be the case for the systems of interest here—simply $\langle v_i|$. The actual values of the measurements are not important here, so we will regard them simply as labels for the outcomes.

If one part of an entangled quantum state is measured the effect of that measurement can be observed in other parts of the state. For example, consider again the Bell state $|\Phi_+\rangle$. If its first qubit is measured in the $|0\rangle, |1\rangle$ basis then the two outcomes are equally likely; suppose that $|0\rangle$ is observed. The resulting effect is to act on the joint state with the operator $|0\rangle\langle 0| \otimes \text{id}$. Ignoring normalisation, we have

$$\begin{aligned} (|0\rangle\langle 0| \otimes \text{id}) |\Phi_+\rangle &= (|0\rangle\langle 0| \otimes \text{id}) |00\rangle + (|0\rangle\langle 0| \otimes \text{id}) |11\rangle \\ &= \langle 0|0\rangle |00\rangle + \langle 0|1\rangle |11\rangle = |00\rangle . \end{aligned}$$

Hence, now performing the same measurement on the second qubit will produce outcome $|0\rangle$ with probability one. Despite acting on only one part of the system, we have produced a global change. (Notice also that the new joint state is no longer entangled.)

This, in a nutshell, is the concept behind measurement-based quantum computation: we begin with a large entangled state, and by performing carefully chosen measurements upon it, the unmeasured parts are driven toward the desired result. As a first approximation, we could say that the structure of entangled quantum states defines the desired computation, while the measurements themselves function more to “push” information through this structure. From this point of view, the measurements play a role similar to the evaluation maps in functional programming, effectively “applying” their outcomes to the function defined by the rest of the state. (This is not entirely accurate; as we shall see, the choice of measurements does play a role in defining the computation.)

It is frequently useful to generalise the notion of quantum state to admit probabilistic mixtures of states. In this setting, states comprising a single state vector as described above are called *pure states*, while the others are called *mixed states*. These more general states are represented by *density operators*: that is, trace one Hermitian matrices of the form

$$\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$$

where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. Note that the components $|\psi_i\rangle$ need not be orthogonal. For pure states we have $p_1 = 1$ and $p_i = 0$ for $i > 1$. The decomposition of a mixed state is not unique. For example, the maximally mixed qubit can arise by preparing either $|0\rangle$ or $|1\rangle$ with equal probability, or equivalent by preparing either $|+\rangle$ and $|-\rangle$:

$$\frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{|+\rangle\langle +| + |-\rangle\langle -|}{2}.$$

The most general class of operations that are possible in the density matrix formalism are *completely positive maps* also called *superoperators*. The action of such a map \mathcal{E} on a state ρ is given by:

$$\rho' = \mathcal{E}\rho\mathcal{E}^\dagger.$$

Since \mathcal{E} is positive ρ' is again a density matrix. The most basic examples of superoperators are unitary maps and quantum measurements. For example, given a measurement $M = \sum_i \lambda_i |v_i\rangle\langle v_i| = \sum_i \lambda_i P_i$, the effect of performing the measurement on state ρ is

$$\rho' = \sum_i P_i \rho P_i = \sum_{i,j} \text{Tr}[\rho P_i] |v_i\rangle\langle v_j|$$

where $\text{Tr}[\rho P_i]$ gives the probability of observing outcome i . While mixed states arise for a variety of reasons in quantum computation, in this chapter the randomness introduced by measurement will be the only source of uncertainty.

2 Observables and strong complementarity

2.1 Observables and observable structures

Given a quantum system whose state space is the Hilbert space A , we will assume that any orthonormal basis $\{|a_i\rangle\}_i$ for A defines an observable on that system. Furthermore, these will be the only observables of interest.

The no-cloning [44] and no-deleting [38] theorems state that it is impossible to perfectly copy or erase an unknown quantum state. However, it is possible to perform both of these operations if the state is guaranteed to be an outcome of a known observable; that is, if it is a member of some given basis. We can therefore view each quantum observable as determining a classical data type, whose elements are possible outcomes, and whose operations are copying and deleting. For example, the copying and deleting operations for the standard basis are given by the linear maps

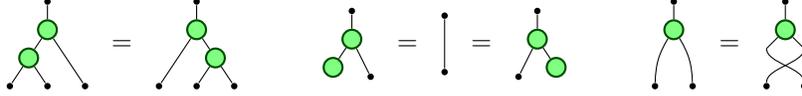
$$\begin{aligned} \delta_Z : Q &\rightarrow Q \otimes Q & \epsilon_Z : Q &\rightarrow I \\ \delta_Z : |i\rangle &\mapsto |ii\rangle & \epsilon_Z : |i\rangle &\mapsto 1 \end{aligned}$$

Note that ϵ_Z is an unnormalised bra, namely $\sqrt{2}\langle +|$. Graphically we will denote these operations by

$$\delta_Z = \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \quad \epsilon_Z = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

What axioms should such operations obey? Informally we may say that if we copy something, and then copy one of the copies, it should not matter which

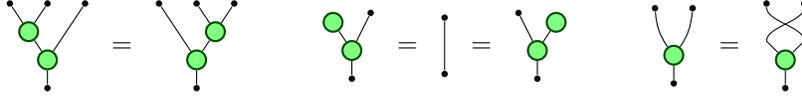
copy we copied; that if we copy something and immediately erase one of the copies, the combined operation should have no effect; and, if we copy something, the two copies may be exchanged without making any difference. The same thing stated formally is that (δ_Z, ϵ_Z) should form a cocommutative comonoid on Q . Presented graphically we have the following:



Since we operate in a \dagger -category we may also consider the adjoint operations

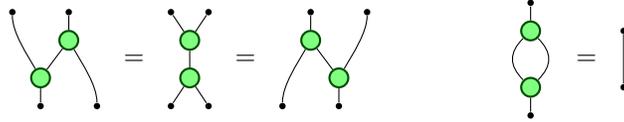


which automatically form a commutative monoid, and obey the same pictorial equations as (δ_Z, ϵ_Z) but flipped upside down:



Notice that ϵ_Z^\dagger is an unnormalised ket, $\sqrt{2}|+\rangle$.

Taken together, the 4-tuple $(\delta_Z, \epsilon_Z, \delta_Z^\dagger, \epsilon_Z^\dagger)$ forms a special commutative \dagger -Frobenius algebra; this amounts to saying that in addition to the above, the following equations also hold:



These equations, the Frobenius law and the special condition respectively, will not be motivated here; see [9, 10].

The preceding discussion of algebras is motivated by the following theorem:

Theorem 2.1 ([11]). *There is a bijective correspondence between orthonormal bases for a finite-dimensional Hilbert space A , and special commutative Frobenius algebras on A .*

Since we have assumed that all observables are non-degenerate, Theorem 2.1 permits us to treat Frobenius algebras of the above type as an abstract version of quantum observables. This definition moreover makes no reference to the fact that the underlying object is a Hilbert space, and hence it can be used in any \dagger -SMC. For this reason, we will henceforward refer to special commutative \dagger -Frobenius algebras by the term *observable structure*¹.

Remark 2.2. Note that this representation of observables by algebras is not a representation of the *measurement* of an observable. The additional formal apparatus required to account for the non-deterministic aspect of the measurement will be introduced in Section 3.4.

¹The same object has also been called a *classical structure* [17] and a *basis structure* [26]

2.2 Unbiasedness and the phase group

Given an orthonormal basis $\{|a_i\rangle\}_i$ for a d -dimensional Hilbert space A , a vector $|\psi\rangle$ is unbiased for $\{|a_i\rangle\}_i$ if for all i , we have $|\langle a_i|\psi\rangle| = \frac{1}{\sqrt{d}}$. For example, $|+\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle)$ is unbiased for the standard basis on Q , for all values of α ; indeed these are the only unbiased states for the standard basis. Incorporating this concept into our diagrammatic language yields a surprising amount of power.

Recall that in \dagger -category a morphism $f : A \rightarrow B$ is unitary if $f^\dagger \circ f = \text{id}_A$ and $f \circ f^\dagger = \text{id}_B$; diagrammatically this is written,

$$\begin{array}{c} \bullet \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \bullet \end{array}$$

where the picture for f^\dagger is obtained by flipping the picture for f upside down. This notion of unitarity agrees with usual one in **fdHilb**. Now we can make:

Definition 2.3. Let (δ, ϵ) be an observable structure on A , and let $\alpha : I \rightarrow A$ be a point of A . Define a map $\Lambda(\alpha) : A \rightarrow A$ by

$$\Lambda(\alpha) := \delta^\dagger \circ (\alpha \otimes \text{id}_A) = \begin{array}{c} \triangle \alpha \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \end{array}$$

Definition 2.4. Let $\alpha : I \rightarrow A$ be a point of A , and $\Lambda(\cdot)$ as in Definition 2.3. We say α is unbiased for (δ, ϵ) if $\Lambda(\alpha)$ is unitary.

$$\begin{array}{c} \triangle \alpha \\ \downarrow \\ \bullet \\ \downarrow \\ \bullet \end{array} \text{ unbiased} \iff \begin{array}{c} \triangle \alpha \\ \downarrow \\ \bullet \\ \downarrow \\ \triangle \alpha \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

When α is unbiased, the map $\Lambda(\alpha)$ is called a *phase map* for (δ, ϵ) . According to the unit law for the monoid structure, $\Lambda(\epsilon^\dagger)$ yields the identity map, which is unitary. Therefore every observable structure has at least one unbiased point, namely ϵ^\dagger . Further since they are unitary, the phase maps form a group, indeed an abelian group as the following calculation shows:

$$\begin{array}{c} \triangle \alpha \\ \downarrow \\ \bullet \\ \downarrow \\ \triangle \beta \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \triangle \beta \\ \downarrow \\ \bullet \\ \downarrow \\ \triangle \alpha \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \triangle \alpha \\ \downarrow \\ \bullet \\ \downarrow \\ \triangle \beta \\ \downarrow \\ \bullet \end{array} = \begin{array}{c} \triangle \beta \\ \downarrow \\ \bullet \\ \downarrow \\ \triangle \alpha \\ \downarrow \\ \bullet \end{array}$$

Since all the phase maps commute, we make the following notational convention:

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \alpha \\ \square \end{array} := \begin{array}{c} \bullet \\ \diagup \alpha \\ \bullet \\ \diagdown \end{array},$$

for which we have the equations

$$\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \alpha \\ \square \end{array} \right)^\dagger = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} -\alpha \\ \square \end{array} \quad \text{and} \quad \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \alpha \\ \square \\ \beta \\ \square \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \alpha + \beta \\ \square \end{array}.$$

The alert reader will have noted that the unbiased points themselves form an abelian group, isomorphic to the phase group, under the multiplication δ^\dagger ; one can equivalently define the phase group via this route.

Remark 2.5. While we will be exclusively interested in the case where $\alpha : I \rightarrow A$ is an unbiased point for some observable, most of the above still holds when α is an arbitrary point of A . In that case we get a commutative monoid rather than a group. Note especially that Theorem 2.6, below, still applies.

Returning to the example of (δ_Z, ϵ_Z) on the qubit, the vectors $|+\alpha\rangle$, when multiplied by $\sqrt{2}$, yield the phase maps

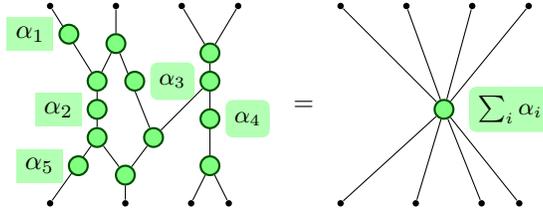
$$\Lambda_Z(\sqrt{2}|+\alpha\rangle) = Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix},$$

comprising rotations around the Z axis of the Bloch sphere.

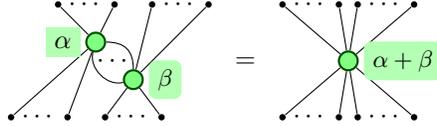
We can now state the key theorem for the diagrammatic treatment of observable structures and their phase groups:

Theorem 2.6 ([13]). *Let D be a connected diagram generated by an observable structure (δ, ϵ) and its phase group; then D is determined completely by its number of inputs, its number of outputs, and the sum $\sum_i \alpha_i$ of phase group elements occurring in it.*

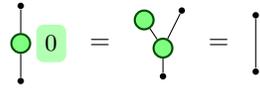
The above result is effectively a normal form theorem for observable structures, however we will use it instead to justify a new notational convention, and simply collapse any connected diagram down to a single vertex, which we refer to as a *spider*:



The label will be omitted when $\alpha = 0$. We can therefore adopt spiders as the generators of the diagrammatic language, governed by a single equational scheme, the spider rule:



Example 2.7. What is the spider with one input, one output, and $\alpha = 0$? The answer is provided by the unit law of the observable structure: it must be the identity, as shown below.

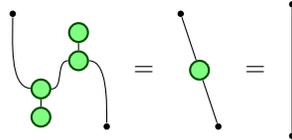


Once we have made the above convention with respect to the identity, all the earlier equations are included in the spider rule, hence this formulation is equivalent to the definition in terms of δ , ϵ and $\Lambda(\alpha)$.

Example 2.8. For any given observable structure (δ, ϵ) we can produce a bipartite state $d : I \rightarrow A \otimes A$ by $d = \delta \circ \epsilon$:



If we partially compose this state with its adjoint we obtain, via the spider rule, the identity:



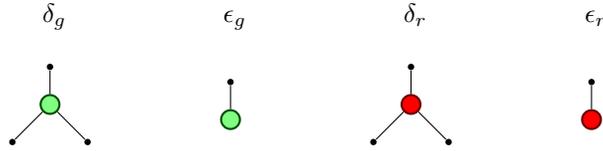
Hence, the object A bears a self-dual compact structure. It is straight forward to construct observable structures for $A \otimes A$ given one on A , so the monoidal category generated by A is compact closed.

2.3 Strong complementarity

In quantum theory, two observables are said to be *complementary* if measuring one of them reveals no information about the other, for example the X and Z spins. Notice that both elements of X basis, $|+\rangle$ and $|-\rangle$, are unbiased with respect to the Z basis, and vice versa; these bases are said to be *mutually unbiased*. Mutually unbiased bases correspond to complementary observables: given an eigenstate of Z , the inner product with either eigenstate of X has the same absolute value, and hence both outcomes are equiprobable when an X measurement is performed. In this section we will present, though not justify, a

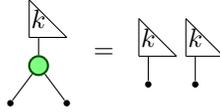
characterisation of complementarity in terms of observable structures rather than bases. In fact, we will present the axioms for observable structures which are *strongly complementary*, a property enjoyed by well-behaved pairs of observables. While the observables we are most interested in—the X and Z spins—are strongly complementary, the material of this section is completely general; the special features of the X and Z observables are treated in the next section.

Since we are now dealing with two observables, we will have two observable structures, (δ_g, ϵ_g) and (δ_r, ϵ_r) , which are represented by green and red coloured spiders. (For those reading without the benefit of colour, green will appear as light grey, red as dark grey.)



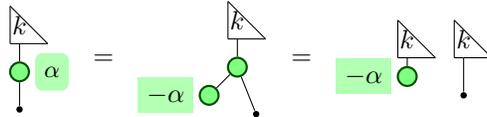
The map δ was originally introduced as copying operation, however it was axiomatised without any reference to the objects it copies. We now correct this.

Definition 2.9. A point $k : I \rightarrow A$ is called *classical* for an observable structure (δ, ϵ) if it satisfies $\delta \circ k = k \otimes k$.



In the language of coalgebras, classical points are called *set-like elements*.

Lemma 2.10. Let k denote a classical point for (δ_g, ϵ_g) , and let α be any unbiased point for (δ_g, ϵ_g) ; k is an eigenpoint of the corresponding phase map $\Lambda_g(\alpha)$.



Note the appearance of a scalar element here – the eigenvalue of k .

Remark 2.11. Any diagram with no inputs or outputs represents an arrow of type $I \rightarrow I$. When interpreted in **fdHilb** these are simply complex numbers. Since quantum mechanics does not distinguish states that differ by a scalar factor we will ignore these whenever they appear. Further, *many of the equations presented below hold only up to scalar normalising factor*. We omit these in order to simplify the presentation; if needed they can easily be reconstructed.

Definition 2.12. Two observable structures (δ_g, ϵ_g) and (δ_r, ϵ_r) on A are called *strongly complementary* if

1. For every point $k : I \rightarrow A$, if k is classical for (δ_g, ϵ_g) then it is unbiased for (δ_r, ϵ_r) , and vice versa.
2. ϵ_r^\dagger is classical for (δ_g, ϵ_g) and ϵ_g^\dagger is classical for (δ_r, ϵ_r) , i.e.:



3. The equation $(\delta_r^\dagger \otimes \delta_r^\dagger) \circ (\text{id}_A \otimes \sigma \otimes \text{id}_A) \circ (\delta_g \otimes \delta_g) = \delta_g \circ \delta_r^\dagger$ holds, i.e.:

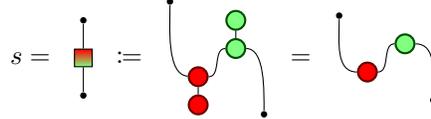


where σ denotes the symmetry of the monoidal structure.

Since we are operating in \dagger -SMC, conditions 2 and 3 also imply flipped versions of the same equations. Given this, these two conditions could be replaced by a unified condition:

- The 4-tuple $(\delta_g, \epsilon_g, \delta_r^\dagger, \epsilon_r^\dagger)$ forms a *bialgebra* on A .

In fact, as well being a bialgebra, a pair of strongly complementary observable structures is in addition a Hopf algebra. Recall $d = \delta \circ \epsilon^\dagger$, and define $s : A \rightarrow A$ by $s = (d_r^\dagger \otimes \text{id}_A) \circ (\text{id}_A \otimes d_g)$. We introduce a new element of the graphical notation for s :



Now we have:

Lemma 2.13. *The 5-tuple $(\delta_g, \epsilon_g, \delta_r^\dagger, \epsilon_r^\dagger, s)$ forms a Hopf algebra on A , i.e.*

$$\delta_r^\dagger \circ (s \otimes \text{id}_A) \circ \delta_g = \epsilon_r^\dagger \circ \epsilon_g.$$
(2)

Remark 2.14. We have stated Lemma 2.13 as a consequence of the bialgebra structure; in fact, under a mild side condition, equation (2) can be shown to be equivalent to condition 1 of Definition 2.12. See [13] for full details.

The classical points have some useful additional properties, which we will now state; the reader can find proofs in [13].

Thanks to Definition 2.12, if k is classical for (δ_g, ϵ_g) then $\Lambda_r(k)$ is an element of the phase group for the strongly complementary observable (δ_r, ϵ_r) . We draw the classical points in the colour of the observable with respect to which they are unbiased, and rely on the label to indicate that it is in fact a classical point: Latin letters will indicate classical points, while Greek letters will denote arbitrary unbiased points.

Proposition 2.15. *Let k, k' be classical points for (δ_g, ϵ_g) , and let h be classical for (δ_r, ϵ_r) ; then:*

1. *The phase map $\Lambda_r(k)$ is a comonoid homomorphism of (δ_g, ϵ_g) :*

2. *The phase maps $\Lambda_g(h)$ and $\Lambda_r(k)$ commute, up to a scalar factor:*

3. *The point $\delta_r^\dagger(k \otimes k')$ is also classical for (δ_g, ϵ_g) :*

Corollary 2.16. *If (δ_g, ϵ_g) has finitely many classical points, then they form a subgroup among the group of unbiased points of (δ_r, ϵ_r) .*

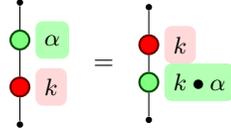
When we consider observable structures over Hilbert spaces, having finitely many classical points is just the statement the underlying space is finite dimensional. Since this is the case for all the situations of interest for this chapter we will henceforth assume that the *classical phases* always form a subgroup.

There is also an important interaction between the classical points and the phase group.

Proposition 2.17. *Let k be a classical point for (δ_g, ϵ_g) , let α be an unbiased point for (δ_g, ϵ_g) , and define $k \bullet \alpha := \Lambda_r(k) \circ \alpha$; then:*

1. *$k \bullet \alpha$ is again unbiased for (δ_g, ϵ_g) ;*

$$2. \Lambda_r(k) \circ \Lambda_g(\alpha) = \Lambda_g(k \bullet \alpha) \circ \Lambda_r(k).$$



3. $\Lambda_r(k)$ is a group automorphism of the unbiased points of (δ_g, ϵ_g) , and conjugation by $\Lambda_r(k)$ is an automorphism of the corresponding phase group.

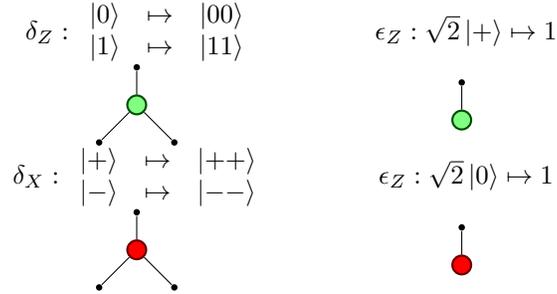
To restate some of the preceding: the classical points of one observable structure always form a subgroup among the unbiased points of the strongly complementary observable; this subgroup in turn acts as an automorphism group upon the unbiased points of the first observable structure.

While all the preceding results will be (sometimes implicitly) used in the subsequent sections, we will be able to make this all rather more concrete by focusing on the specific case of the Z and X spin observables.

3 The zx-calculus

3.1 The Z and X observables

In this section, and in the rest of the chapter, we'll represent the Z and X spin observables by following two strongly complementary observable structures on \mathbb{C}^2 :



One of the most significant simplifications that occurs when working with the Z and X observables is that they both generate the same compact structure; that is, we have the equation

$$d_Z = d_X = |00\rangle + |11\rangle$$

Since there is no need to distinguish between a green or red cup (or cap), we will drop the dot from diagrammatic notation whenever possible. Since the category bears a single compact structure, we can treat the internal structure of any diagram as an undirected graph and appeal to the principle of diagrammatic

equivalence described earlier: if two diagrams are isomorphic as labelled graphs, they are equal.

In direct consequence, the antipode map of the Hopf algebra structure is trivial:

$$s = (d_Z^\dagger \otimes \text{id}_A) \circ (\text{id}_A \otimes d_X) = \text{id}_Q$$

The defining equation of the Hopf algebra structure can therefore be simplified:

Recall that a point $|\alpha_Z\rangle$ is unbiased for the standard basis $|0\rangle, |1\rangle$ if and only if it has the form $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle)$. Hence the phase group for the Z observable is just the circle group, i.e., the interval $[0, 2\pi)$ under addition modulo 2π . The X observable's phase group is isomorphic, so we represent the unbiased points and phase maps as shown below.

The phase maps are unitary, so the dagger sends each element to its inverse, i.e., it negates the angle:

Since we are operating in dimension 2, there are two classical points, corresponding to the angles 0 and π . The action of the non-trivial classical map is to negate the phase:

Given any two bases for a Hilbert space there is a unitary isomorphism that maps one basis to the other; in the case of the Z and X bases this map is the familiar Hadamard matrix. We'll introduce an extra element into the diagrammatic language to represent this map:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \text{H}$$

The Hadamard is a self-adjoint unitary, hence we have the equations:

$$\left(\begin{array}{c} \bullet \\ | \\ \boxed{H} \\ | \\ \bullet \end{array} \right)^\dagger = \begin{array}{c} \bullet \\ | \\ \boxed{H} \\ | \\ \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \boxed{H} \\ | \\ \boxed{H} \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

Since the Hadamard maps one basis to the other, we could use it to *define* one observable structure in terms of the other:

$$\begin{array}{c} \bullet \\ | \\ \boxed{H} \\ | \\ \bullet \\ / \quad \backslash \\ \boxed{H} \quad \boxed{H} \\ | \quad | \\ \bullet \quad \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \quad \begin{array}{c} \bullet \\ | \\ \boxed{H} \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

Strictly speaking, it is redundant to continue with both the Z and X observable structures: we could eliminate, for example, the X vertices. However, for some purposes it is more convenient to use both Z and X , while for other purposes it is easier to work with just Z and H , so we maintain all three elements in the syntax, and endorse the following *colour duality principle*.

Proposition 3.1. *Every statement made in the diagrammatic language also holds with the colours reversed*

We will switch freely between the two-coloured presentation, and the one-colour and Hadamard view depending on which is most convenient at any given time.

3.2 Syntax and semantics I

The ZX-calculus is a formal graphical notation, based on the notion of an *open graph*.

Definition 3.2. An *open graph* is a triple (G, I, O) consisting of an undirected graph $G = (V, E)$ and distinguished subsets $I, O \subseteq V$ of *input* and *output* vertices I and O . The set of vertices $I \cup O$ is called the *boundary* of G , and $V \setminus (I \cup O)$ is the *interior* of G .

A term of the ZX-calculus is called a *diagram*; this is an open graph with some additional properties and structure.

Definition 3.3. A *diagram* is an open graph (G, I, O) , where (i) all the boundary vertices are of degree one; (ii) the set of inputs I and the set of outputs O are both totally ordered; and (iii) whose interior vertices are restricted to the following types:

- Z vertices with m inputs and n outputs, labelled by an angle $\alpha \in [0, 2\pi)$; these are denoted $Z_n^m(\alpha)$, and shown graphically as (light) green circles,

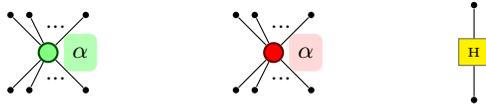


Figure 2: Interior vertices of diagrams

- X vertices with m inputs and n outputs, labelled by an angle $\alpha \in [0, 2\pi)$; these are these are denoted $X_n^m(\alpha)$, and shown graphically as (dark) red circles,
- H (or Hadamard) vertices, restricted to degree 2; shown as squares.

If a X or Z vertex has $\alpha = 0$ then the label is entirely omitted. The allowed vertices are shown in Figure 2.

Since the inputs and outputs of of a diagram are totally ordered, we can identify them with natural numbers and speak of the k th input, etc.

Remark 3.4. When a vertex occurs inside the graph, the distinction between inputs and outputs is purely conventional: one can view them simply as vertices of degree $n + m$; however, this distinction allows the semantics to be stated more directly, see below.

The collection of diagrams forms a compact category in the obvious way: the objects are natural numbers and the arrows $m \rightarrow n$ are those diagrams with m inputs and n outputs; composition $g \circ f$ is formed by identifying the inputs of g with the outputs of f and erasing the corresponding vertices; $f \otimes g$ is the diagram formed by the disjoint union of f and g with I_f ordered before I_g , and similarly for the outputs. This is basically the free (self-dual) compact category generated by the arrows shown in Figure 2.

We can make this category \dagger -compact by specifying that f^\dagger is the same diagram as f , but with the inputs and outputs exchanged, and all the angles negated.

This construction yields a category that does not incorporate the algebraic structure of strongly complementary observables. To obtain the desired category we must quotient by the equations shown in Figure 3. We denote the category so-obtained by \mathbb{D} .

Remark 3.5. The equations shown in Figure 3 are not exactly those described in Sections 2 and 3.1, however they are equivalent to them. We shall therefore, on occasion, use properties discussed earlier as derived rules in computations.

Since \mathbb{D} is a monoidal category we can assign an interpretation to any diagram by providing a monoidal functor from \mathbb{D} to any other monoidal category. Since we are interested in quantum mechanics, the obvious target category is \mathbf{fdHilb} .

Definition 3.6. Let $[[\cdot]] : \mathbb{D} \rightarrow \mathbf{fdHilb}$ be a symmetric monoidal functor defined on objects by

$$[[1]] = \mathbb{C}^2$$

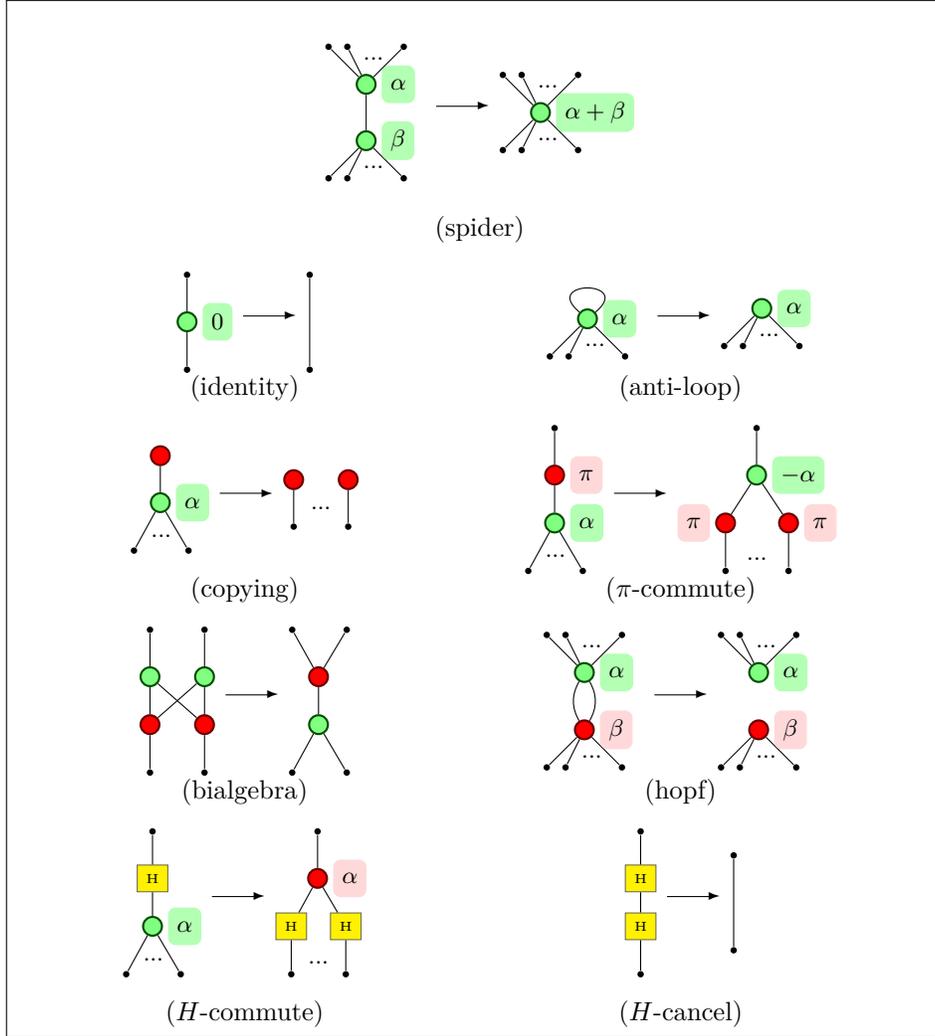


Figure 3: Rewrite rules for the ZX-calculus. We present the rules for the Z subsystem; to obtain the complete set of rules exchange the colours in the rules shown above.

and on the generators by:

$$\begin{aligned}
 \llbracket Z_n^m(\alpha) \rrbracket &= \llbracket \begin{array}{c} \dots \\ \dots \\ \text{green } \alpha \\ \dots \\ \dots \end{array} \rrbracket = \begin{cases} |0\rangle^{\otimes m} \mapsto |0\rangle^{\otimes n} \\ |1\rangle^{\otimes m} \mapsto e^{i\alpha} |1\rangle^{\otimes n} \end{cases}, \\
 \llbracket X_n^m(\alpha) \rrbracket &= \llbracket \begin{array}{c} \dots \\ \dots \\ \text{red } \alpha \\ \dots \\ \dots \end{array} \rrbracket = \begin{cases} |+\rangle^{\otimes m} \mapsto |+\rangle^{\otimes n} \\ |-\rangle^{\otimes m} \mapsto e^{i\alpha} |-\rangle^{\otimes n} \end{cases}, \\
 \llbracket H \rrbracket &= \llbracket \begin{array}{c} \text{yellow } H \\ \vdots \end{array} \rrbracket = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
 \end{aligned}$$

The value of $\llbracket \cdot \rrbracket$ on all other objects and arrows is then fixed by the requirement that it be a symmetric monoidal functor².

Theorem 3.7 (Soundness). *For any diagrams D and D' in \mathbb{D} , if $D = D'$ then $\llbracket D \rrbracket = \llbracket D' \rrbracket$ in \mathbf{fdHilb} .*

Proof. Notice that the compact closed structure is preserved automatically because $\llbracket \cdot \rrbracket$ is a monoidal functor. It just remains to check that all the equations of Figure 3 hold in the image of $\llbracket \cdot \rrbracket$. \square

Remark 3.8. While Theorem 3.7 shows that every equation provable in the ZX-calculus is true in Hilbert spaces, the converse does not hold: there are diagrams D and D' such that $\llbracket D \rrbracket = \llbracket D' \rrbracket$ but the equation $D = D'$ cannot be derived from the rules of the calculus. See [25] for details.

Proposition 3.9. *For any diagram D in \mathbb{D} , we have $\llbracket D^\dagger \rrbracket = \llbracket D \rrbracket^\dagger$.*

3.3 Quantum circuits

The quantum circuit model is simple and intuitive quantum computational model. Analogous to traditional Boolean circuits, a quantum circuit consists of a register of qubits, to which quantum logic gates—that is one- or two-qubit unitary operations—are applied, in sequence and in parallel³. A fairly typical set of logic gates is shown in Figure 4, however these are not all necessary, as the following theorem states.

$$\begin{array}{cc}
 Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 Z_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} & H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
 \wedge X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \wedge Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
 \end{array}$$

Figure 4: Quantum logic gates

Theorem 3.10 ([3]). *The set $\{Z_\alpha, H, \wedge X\}$ suffices to generate all unitary matrices on Q^n .*

Corollary 3.11. *The ZX-calculus can represent all unitary matrices on Q^n .*

²The full details of this construction regarding cyclic graphs and traces can be found in [23].

³The circuit model usually incorporates measurements too, but this will not be necessary here. See, e.g., chapter 4 of [37].

Proof. It suffices to show that there are ZX-calculus terms for the matrices Z_α , H and $\wedge X$. We have

$$\llbracket \begin{array}{c} \bullet \\ | \\ \boxed{H} \\ | \\ \bullet \end{array} \rrbracket = H, \quad \llbracket \begin{array}{c} \bullet \\ | \\ \boxed{\alpha} \\ | \\ \bullet \end{array} \rrbracket = Z_\alpha \quad \text{and} \quad \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \wedge X$$

which can be verified by direct calculation. Note that

$$\llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket$$

so the presentation of $\wedge X$ is unambiguous. \square

Example 3.12 (The $\wedge Z$ -gate). The $\wedge Z$ -gate can be obtained by using a Hadamard (H) gate to transform the second qubit of a $\wedge X$ gate. We obtain a simpler representation using the colour-change rule

$$\llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket$$

From the presentation of $\wedge Z$ in the ZX-calculus, we can immediately read off that it is symmetric in its inputs. Furthermore, we can prove one of the basic properties of the $\wedge Z$ gate, namely that it is self-inverse.

$$\llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket$$

Example 3.13 (Bell state). The following is a ZX-calculus term representing a quantum circuit which produces a Bell state, $|00\rangle + |11\rangle$. We can verify this fact by the equations of the calculus.

$$\llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket = \llbracket \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \rrbracket$$

The corresponding ZX-calculus derivation is a proof of the correctness of this circuit.

The ZX-calculus can represent many things which do not correspond to quantum circuits. We now present a criterion to recognise which diagrams do correspond to quantum circuits.

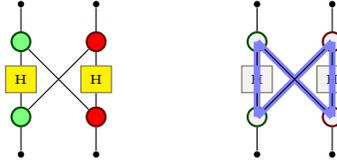
Definition 3.14. A diagram is called *circuit-like* if:

- (C1). all of its vertices can be covered by a set \mathcal{P} of disjoint directed paths, each of which ends in an output;
- (C2). for every oriented cycle γ in the diagram, if γ contains at least 2 edges from different paths in \mathcal{P} , then it traverses at least one of them in the direction opposite to that induced by the path; and,
- (C3). it is a simple graph, and is 3-coloured.

Intuitively, the paths of \mathcal{P} represent the trajectories of the individual qubits through the circuit, whereas those edges not included in any path represent two-qubit gates. Condition (C2) guarantees that the diagram has a causally consistent temporal order, while condition (C3) forces the diagram to be minimal with respect to the spider, anti-loop, and Hopf rules.

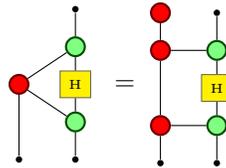
Remark 3.15. Definition 3.14 requires each path to end in an output vertex, but does not demand that the initial vertex is an input. This allows the representation of quantum circuits with some or all inputs fixed; see Example 3.13 above.

Example 3.16. The following diagram is not circuit-like, since condition (C2) fails.



There is only one possible path covering of this diagram; the indicated cycle runs contrary to the path.

Example 3.17. The following diagram is circuit-like, and, as shown, is equivalent to something which clearly *looks* like a circuit.



Of course, the right-hand diagram is *not* circuit-like because it does not satisfy condition (C3); indeed, it reduces to the left-hand diagram by two applications of the spider rule.

As the preceding example shows, the definition of circuit-like—in particular the technical third condition—is rather stronger than strictly necessary to ensure that a diagram corresponds to a valid circuit. For example, it forces the two-qubit gates to be $\wedge X$ rather than $\wedge Z$. However it is not hard to prove the following result:

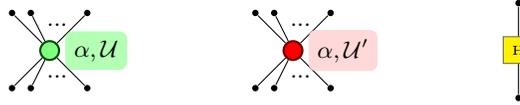


Figure 5: Interior vertices of \mathcal{V} -labelled diagrams

Proposition 3.18. *If $D : n \rightarrow m$ is a circuit-like diagram, then $\llbracket D \rrbracket$ is a unitary embedding $Q^n \rightarrow Q^m$; conversely, if $\llbracket D \rrbracket$ is a unitary embedding, then there exists some circuit-like D' such that $D = D'$ by the rules of the ZX-calculus.*

3.4 Syntax and semantics II: measurements

While the version of the ZX-calculus we have presented so far can represent the projection onto some measurement outcome, for example we have $\llbracket \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \rrbracket = \langle + |$, the physical process of measurement, including its non-deterministic aspect, cannot be represented. To address this we now introduce the notion of a \mathcal{V} -labelled diagram. This will require a modification to the syntax, and a new interpretation based on superoperators, rather than **fdHilb**.

Definition 3.19. Let \mathcal{V} be some set of variables. A *conditional diagram* is a diagram (cf Definition 3.3) where each Z or X vertex with $\alpha \neq 0$ is additionally labelled by a set $\mathcal{U} \subseteq \mathcal{V}$.

If a vertex is labelled by a $\mathcal{U} \neq \emptyset$ then it is called *conditional*, otherwise it is *unconditional*. A diagram with no conditional vertices is called unconditional. The allowed vertices of \mathcal{V} -labelled diagrams are shown in Figure 5.

The equational rules must also be modified to take account the labels: certain rewrites are only allowed when the variable sets agree, in a sense that will be made clear below. The updated rules are shown in Figure 6.

For any given \mathcal{V} , the \mathcal{V} -labelled diagrams, quotiented by the equations of Figure 6 form a \dagger -compact category denoted $\mathbb{D}(\mathcal{V})$; $\mathbb{D}(\emptyset)$ is exactly the category \mathbb{D} defined earlier.

Definition 3.20. A function $v : \mathcal{V} \rightarrow \{0, 1\}$ is called a *valuation* of \mathcal{V} ; for each valuation v , we define a functor $\hat{v} : \mathbb{D}(\mathcal{V}) \rightarrow \mathbb{D}$ which produces a new diagram by relabelling the Z and X vertices. If a vertex z is labelled by α and \mathcal{U} , then $\hat{v}(z)$ is labelled by 0 if $\sum_{s \in \mathcal{U}} v(s) = 0$ and α otherwise.

The modified rewrite rules for \mathcal{V} -labelled diagrams are simply the original equations, with the constraint that they should be true in all valuations.

Definition 3.21. Let D be a diagram in $\mathbb{D}(\mathcal{V})$ such that every variable of \mathcal{V} occurs in D . Define a symmetric monoidal functor $\llbracket \cdot \rrbracket_{\mathcal{V}} : \mathbb{D}(\mathcal{V}) \rightarrow \mathbf{SuperOp}$ by setting $\llbracket 1 \rrbracket_{\mathcal{V}} = \mathbb{C}^2 \times \mathbb{C}^2$ and, for every diagram D , defining $\llbracket D \rrbracket_{\mathcal{V}}$ as the superoperator constructed by summing over all the valuations of \mathcal{V} :

$$\rho \mapsto \sum_{v \in 2^{\mathcal{V}}} \llbracket \hat{v}(D) \rrbracket \rho \llbracket \hat{v}(D) \rrbracket^{\dagger}.$$

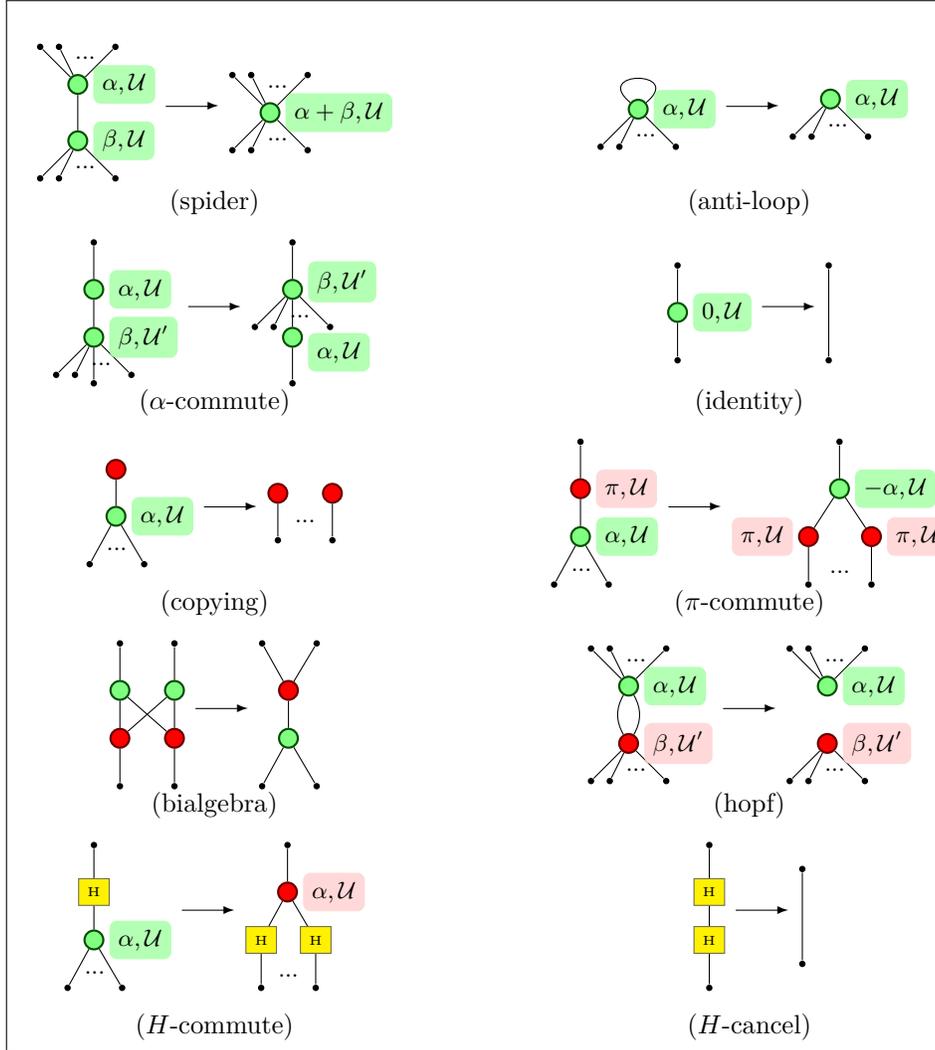
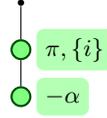


Figure 6: Rewrite rules for the ZX-calculus with conditional vertices. We present the rules for the Z subsystem; to obtain the complete set of rules exchange the colours in the rules shown above.

Proposition 3.22. For any diagram D in $\mathbb{D}(\mathcal{V})$, we have $\llbracket D^\dagger \rrbracket_{\mathcal{V}} = \llbracket D \rrbracket_{\mathcal{V}}^\dagger$.

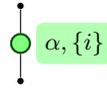
Example 3.23. The following diagram represents the measurement of a single qubit in the basis $|\pm_\alpha\rangle = |0\rangle \pm e^{i\alpha}|1\rangle$.



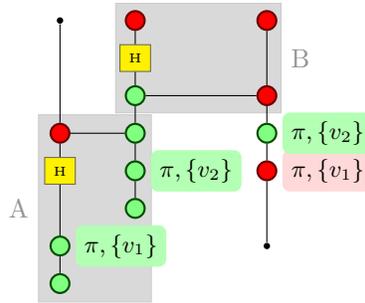
The variable i encodes which of the two possible outcomes occurred, as can be seen by computing the denotation:

$$\begin{aligned} \rho \mapsto \sum_{v=0,1} \llbracket \hat{v}(PIC) \rrbracket \rho \llbracket \hat{v}(PIC) \rrbracket^\dagger &= \langle +_\alpha | \rho | +_\alpha \rangle + \langle +_\alpha | Z \rho Z | +_\alpha \rangle \\ &= \langle +_\alpha | \rho | +_\alpha \rangle + \langle -_\alpha | \rho | -_\alpha \rangle \end{aligned}$$

Example 3.24. A classically controlled Pauli- Z operation is represented by the following diagram:

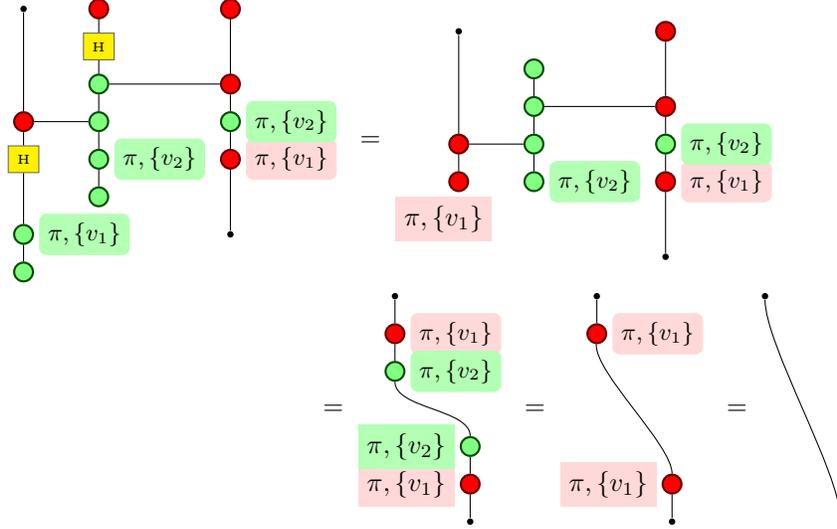


Example 3.25. Combining the two previous examples, we present the teleportation protocol [4].



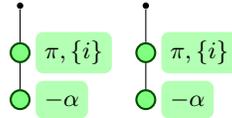
The block labelled “B” is the circuit to prepare a Bell state, while “A” represents the Bell basis measurement of two qubits. Notice that the same variables label the measurements as the corresponding correction operators, indicating that these operations must be correlated. We can rewrite this diagram to prove the

correctness of the protocol.



Since all the conditional vertices are removed by the final step, we can conclude that the teleportation protocol is *deterministic*.

Example 3.26. It is also possible to write down diagrams which correspond to quite unphysical operations. For example the diagram



represents two one-qubit measurements whose outcomes are always perfectly correlated, regardless of the input. This is of course plainly impossible in quantum mechanics.

To avoid such situations, each single qubit measurement must be labelled by a fresh variable; any other vertex labelled by the same variable must be interpreted as an operation which is classically controlled by the outcome of that measurement.

4 The measurement calculus

Now we turn our attention to the details of *measurement-based quantum computation* (MBQC). While there are several approaches to MBQC, we will be concerned only with the *one-way model* (1WQC) introduced by Raussendorf and Briegel [40, 41, 6].

Whereas the quantum circuit model consists of reversible unitary gates, the 1WQC carries out computation via the irreversible state changes induced

by quantum measurements. The computation begins with some input qubits coupled to a large entangled resource state, called a cluster state or *graph state*. Single qubit measurements are performed upon the state. Since the measured qubits are no longer entangled with the rest of the resource we can view the measurement process as consuming the resource, hence the name *one-way*. The computation proceeds by a number of rounds of measurement, where the choice of measurement performed in later rounds may depend on the observed outcomes of earlier measurements, until finally only the output qubits remain unmeasured. It may then be necessary to apply some single-qubit unitary corrections to obtain the desired result. Aside from the initial creation of the resource state, all the operations of the 1WQC act locally on a single qubit, so it is perhaps surprising that the 1WQC is universal for quantum computing: any unitary operation on n qubits may be computed by the 1WQC.

We shall formally describe the 1WQC using the syntax of the *measurement calculus* of Danos, Kashefi and Panangaden [20]. Measurement calculus programs, called *patterns*, consist of a (finite) set of qubits, a (finite) set of Boolean variables called *signals*, and a sequence of *commands* $C_n \dots C_1 C_0$, read from right to left. The possible commands are:

- N_i : initialise qubit i in the state $|+\rangle$.
- E_{ij} : entangle qubits i and j by apply applying a $\wedge Z$ operation.
- M_i^α : measure qubit i in the basis $|\pm_\alpha\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\alpha}|1\rangle)$. We assume the measurements are destructive, so qubit i will play no further role in the computation.
- X_i and Z_i : apply a 1-qubit Pauli X (resp. Z) operator to qubit i . These are called *corrections*.

The measurement and correction commands may be classically controlled by the value of one or more signals.

- $^s[M_i^\alpha]^t$: perform the measurement $M_i^{(-1)^s\alpha+t\pi}$.
- X_i^s and Z_i^s : if $s = 1$ perform the command X_i (resp. Z_i), otherwise do nothing.

In principle the signals could obtain their values from any source, however they will always be associated to the outcome of a measurement already performed in the same pattern⁴. If the $|+\alpha\rangle$ outcome is obtained, the corresponding signal is set to zero; otherwise it set to one. This introduces the third of three determinacy conditions:

1. The initialisation of a qubit is the first command acting on it.

⁴ It is easy to generalise the conditional commands to allow classical control by some arithmetic expression of signals, however this is not required here, and indeed does not increase the expressiveness of the measurement calculus.

2. The measurement of a qubit is the last command acting on it.
3. No command depends upon a measurement not already performed.

Any qubits not initialised are *inputs*; any qubit not measured is an *output*.

Example 4.1. Consider the 2-qubit pattern

$$\mathfrak{P}_H := X_2^{s_1} M_1^0 E_{12} N_2 .$$

Since qubit 1 is not initialised, it must be an input; similarly qubit 2 is an output. The only signal is associated to the measurement of qubit 1. Suppose that the input qubit is in state $|\psi\rangle = a|0\rangle + b|1\rangle$. The execution proceeds as follows:

$$\begin{aligned} |\psi\rangle &\xrightarrow{N_2} |\psi\rangle \otimes |+\rangle = a|00\rangle + a|01\rangle + b|10\rangle + b|11\rangle \\ &\xrightarrow{E_{12}} a|00\rangle + a|01\rangle + b|10\rangle - b|11\rangle \\ &= |+\rangle (a|+\rangle + b|-\rangle) + |-\rangle (a|+\rangle - b|-\rangle) \end{aligned}$$

So far all we have done is construct the initial entanglement. The next step is the measurement M_1^0 . Suppose that the result of the measurement is 0, i.e. the projection onto $|+\rangle$; then qubit 1 is eliminated and we have the new state $a|+\rangle + b|-\rangle$. Since the signal s_1 is zero, there is no need to perform the final correction. On the other hand, should the outcome of the measurement be 1, the resulting state will be $a|+\rangle - b|-\rangle$, and since $s_1 = 1$ the X_2 correction must be applied, again producing a final state of $a|+\rangle + b|-\rangle$. Hence the overall effect of \mathfrak{P}_H is independent of the outcome of the measurement: in either case the computer applies a Hadamard gate upon its input.

This example illustrates a key feature of the 1WQC. The measurement introduces a branch in the execution where one outcome corresponds to the ‘correct’ behaviour, in the sense that the projection achieves the desired computational effect, and one branch contains an ‘error’ which must be corrected later in the pattern. More generally, a pattern with n measurements potentially performs 2^n different linear maps on its input, which are called the *branch maps*; the branch where all the measurements output 0 is called the *positive branch*, and we make the convention that this branch is the computation that we intend to carry out. The pattern is deterministic if all the branch maps have the same effect on the input as the positive branch.

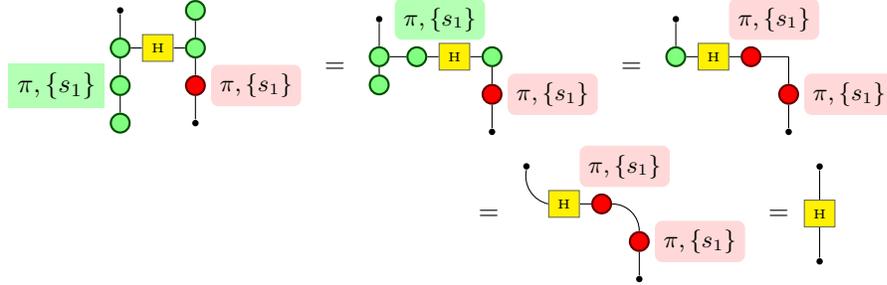
Remark 4.2. The concept of branch map is used in [20] to define the semantics of the measurement calculus. We omit this, because we shortly provide a semantics via a translation into the ZX-calculus. The interpretation presented here is equivalent to that of Danos, Kashefi, and Panangaden [20].

Theorem 4.3. *For any pattern \mathfrak{P} , there exists an equivalent pattern—in the sense of having the same semantics—whose command sequence has the form*

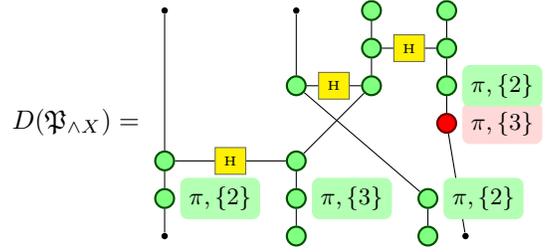
$$\mathfrak{P}^* = CMEN$$

where C , M , E , and N are sequences of commands consisting exclusively of corrections, measurements, entangling operations, and initialisations respectively.

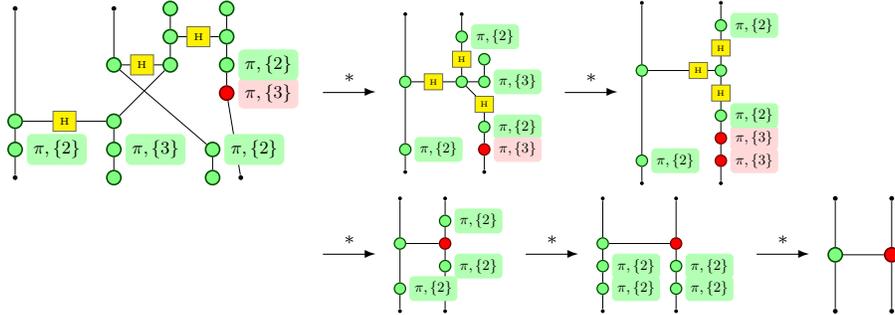
the Hadamard gate using purely diagrammatic reasoning:



Example 4.6. The ubiquitous CNOT operation can be computed by the pattern $\mathfrak{P}_{\wedge X} = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4$ [20]. This yields the diagram



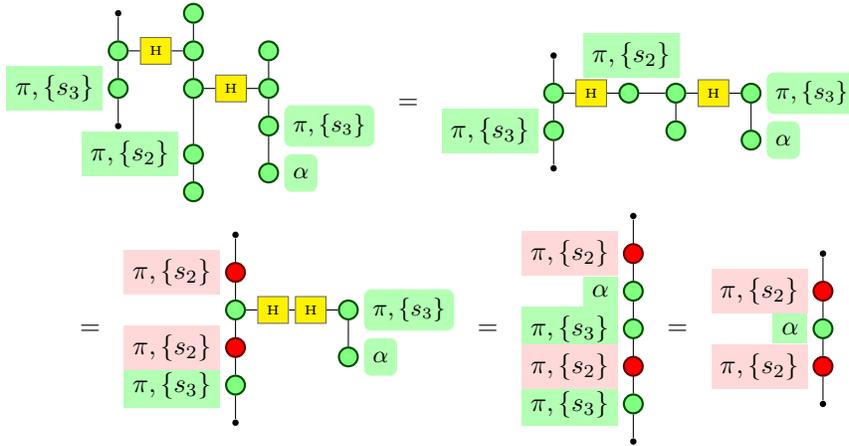
with qubit 1 the leftmost, and qubit 4 is the rightmost. Now, we can prove the correctness of the pattern by rewriting:



One can clearly see in this example how the non-determinism introduced by measurements is corrected by conditional operations later in the pattern. The possibility of performing such corrections depends on the *geometry* of the pattern, the entanglement graph implicitly defined by the pattern. This will be the main concern of the next section.

5 Determinism and flow

Consider the pattern $\mathfrak{N} = Z_1^{s_2} M_2^0 M_3^\alpha E_{12} E_{23} N_2 N_3$. Working in the ZX-calculus, it can be rewritten as follows:



Hence we have a pattern which is non-deterministic, acting as a Z -rotation by either α or $-\alpha$ depending on the outcome of measurement 2. Furthermore, unlike the previous examples, there is no way to remove the dependence on s_2 by correction at qubit 1, or by conditional measurement at qubit 3.

As this example shows, not every pattern performs a deterministic computation, and this possibility depends not just upon the conditional operations introduced by the programmer, but also the structure of the entangled resource used in the computation. This structure is called the *geometry* of the pattern. A pattern can perform a deterministic computation if its geometry has a graph theoretic property called *flow*. Examples 4.1 and 4.6 have flow, while the pattern \mathfrak{N} above does not.

Flow is a sufficient property for determinism, but not a necessary one: it guarantees *strong* and *uniform* determinism. Strong determinism means that all the branch maps of the pattern are equal, while uniformity means that the pattern is deterministic for all possible choices of its measurement angles. Before moving on, we make the following obvious observation:

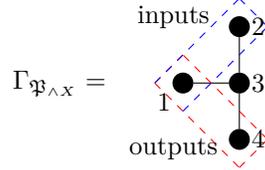
Theorem 5.1. *If $D(\mathfrak{P})$ can be rewritten to an unconditional diagram then \mathfrak{P} is strongly deterministic.*

Hence, to show that a pattern is deterministic, it suffices to find some rewrite sequence which removes all the conditional vertices. Typically this is done by ‘pushing’ the conditional vertex introduced by measurement through the diagram until it meets a matching corrector. The two conditional vertices then cancel each other out. The rest of this section will explore when this is possible.

Definition 5.2. The *geometry* of a pattern \mathfrak{P} , denoted $\Gamma_{\mathfrak{P}}$, is the open graph $((V, E), I, O)$ defined by taking the qubits of \mathfrak{P} as vertices V , the input and

output qubits as the sets I and O , and defining the edge relation by $v \sim u$ if and only if the command E_{vu} occurs in \mathfrak{P} .

Example 5.3. Consider $\mathfrak{P}_{\wedge X} = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3$ as in Example 4.6. We then have



Definition 5.4. Let $G = ((V, E), I, O)$ be an open graph; a *flow* on G is a pair (f, \prec) , where f is a function $V \setminus O \rightarrow V \setminus I$ and \prec is a partial order on V , satisfying

- (F1). $f(u) \sim u$;
- (F2). $u < f(u)$;
- (F3). If $f(u) \sim v$ and $u \neq v$ then $u < v$.

Intuitively, the function f specifies a causal successor for every measured qubit. Should the measurement at qubit i give the ‘wrong’ answer, then a conditional operation at qubit $f(i)$ can be used to correct the resulting error. The partial order ensures that no causal loops can form: that is, it is not necessary to apply a correction to a qubit that has already been measured. We have the following:

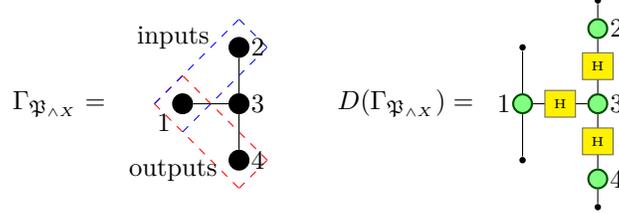
Theorem 5.5 ([19]). *If G is an open graph with flow then there exists a strongly and uniformly deterministic pattern \mathfrak{P} such that $G = \Gamma_{\mathfrak{P}}$.*

It must be emphasised that flow is a property of the geometry *not* the pattern itself. In order for a pattern to be deterministic, correct placement of the conditional operations is still required. Danos and Kashefi give the pattern \mathfrak{P} explicitly in [19]; we will reconstruct this later.

Definition 5.6. Let $\Gamma = ((V, E), I, O)$ be an open graph; we define an unconditional diagram $D(\Gamma)$ as follows:

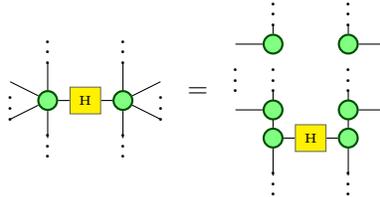
- The vertices of $D(\Gamma)$ are given by the disjoint union $V + E + I + O$. Should Γ have vertex v contained in both I and O then $D(\Gamma)$ contains *three* corresponding vertices in $D(\Gamma)$; we use subscripts v_V, v_I, v_O to disambiguate.
- The vertices are typed depending which disjoint subset they originate in: those from V (the original vertices of Γ) have type Z , without any label; those from E have type H ; and those from $I + O$ are boundary vertices, with I providing the inputs and O the outputs.
- If e is an edge in Γ connecting vertices u and v , then we have $u_V \sim e$ and $e \sim v_V$ in $D(\Gamma)$. For the boundary vertices we have $v_I \sim v_V$ and $v_O \sim v_V$.

Example 5.7. Consider again the $\wedge X$ pattern, or rather its geometry.

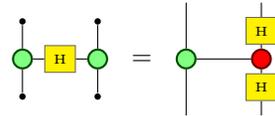


Theorem 5.8. Let \mathfrak{P} be a pattern; if $\Gamma_{\mathfrak{P}}$ has flow then $D(\Gamma_{\mathfrak{P}})$ is equivalent to a circuit-like diagram.

Proof. Suppose that $\Gamma_{\mathfrak{P}}$ has a flow (f, \prec) . Let J be the vertices of $\Gamma_{\mathfrak{P}}$ which are minimal with respect to \prec . For each vertex $j \in J$ we can define a finite sequence $p_j = j, f(j), f^2(j), \dots, f^n(j)$ where the last element of the sequence is an output qubit. By definition of f , the collection $\cup_{j \in J} p_j$ contains all the vertices of $\Gamma_{\mathfrak{P}}$. Each p_j defines a path in $D(\Gamma_{\mathfrak{P}})$, and the collection of these paths covers all the Z vertices of $D(\Gamma_{\mathfrak{P}})$; we can trivially extend these paths to include the boundary vertices adjacent to their end points. The collection $\{p_j\}_{j \in J}$ provides the path covering required by the definition of circuit-like (cf. Definition 3.14). $D(\Gamma_{\mathfrak{P}})$ satisfies condition (C2), because of the partial order structure of the flow, and condition (C3) by construction, however it does not satisfy condition (C1), since some vertices are not covered by the path. Specifically, those H vertices e where $v \sim e \sim u$ and $f(u) \neq v$ are not covered by the path. At each such vertex we perform the following rewrite:



Now the H vertices can be removed via the rewrite shown below:



After which any pairs of adjacent H vertices may be cancelled, and the spider rule can be used to guarantee that the diagram is three coloured. \square

The converse to Theorem 5.8 does not hold; in [24] it is shown that geometries which have *generalised flow*, discussed below, can be rewritten to circuit-like diagrams. However, we can give a weaker result which holds for flow.

Definition 5.9. Let D be a diagram, and let $U \subseteq V$ be a set of its vertices satisfying

- $u \in U$ implies that u has type H ;
- $v_1 \sim u \sim v_2$ implies that v_1 and v_2 are either both of type Z or both of type X .

Then D is called *weakly circuit-like* if the following conditions hold.

- (W1). The vertices $V \setminus U$ can be covered by a set \mathcal{P} of disjoint directed paths, each of which ends in an output;
- (W2). for every oriented cycle γ in the diagram, if γ contains at least 2 edges from different paths in \mathcal{P} , then it traverses at least one of them in the direction opposite to that induced by the path; and,
- (W3). it is a simple graph, and is 3-coloured.

Weakly circuit-like diagrams correspond to circuits where the 2-qubit gates may be $\wedge Z$ gates as well as $\wedge X$ gates.

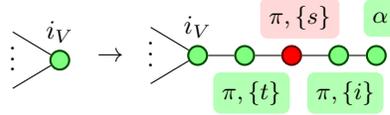
Theorem 5.10. *Let \mathfrak{P} be a pattern; if $D(\Gamma_{\mathfrak{P}})$ is weakly circuit-like then $\Gamma_{\mathfrak{P}}$ has flow.*

Proof. By construction, the Z vertices are in bijective correspondence with the qubits of \mathfrak{P} ; hence we need only define a flow (f, \prec) over the Z vertices. Let p be one of the paths of the path covering of $D(\Gamma_{\mathfrak{P}})$; p then defines a linear order over the Z vertices it covers. Define f by $f(u) = v$ whenever v is the successor of u in this order. Since the paths are disjoint, the same procedure can be carried out for every path, and since the paths cover all the Z vertices this defines the required function f . The union of these linear orders gives a partial order over the Z vertices; to obtain the required \prec this order can be completed by imposing condition (F3). The acyclicity condition (W2) guarantees that this is possible. \square

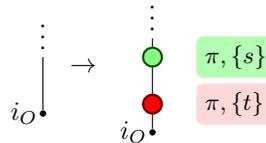
Evidently, the circuit-like diagram produced from $D(\Gamma_{\mathfrak{P}})$ is not equivalent to $D(\mathfrak{P})$, however they are closely related.

Definition 5.11. Let \mathfrak{P} be pattern; construct a new diagram $D(\Gamma_{\mathfrak{P}})^*$ from $D(\Gamma_{\mathfrak{P}})$ as follows:

- If ${}^s[M_i^\alpha]^t$ occurs in \mathfrak{P} then modify $D(\Gamma_{\mathfrak{P}})$ by adjoining the subdiagram corresponding to the measurement, as shown below:

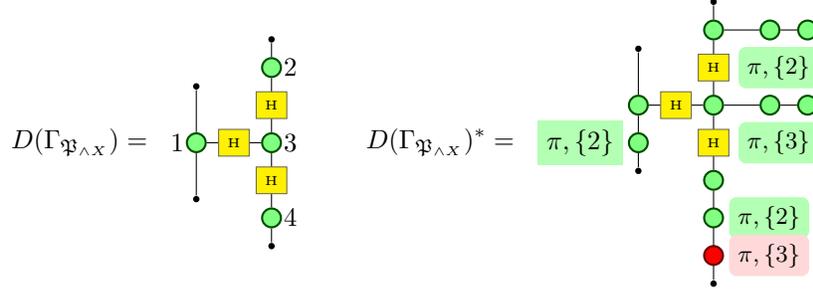


- If X_i^s or Z_i^s occurs in \mathfrak{P} then adjoin the subdiagram corresponding to the correction as shown below:



Note that since the \mathfrak{P} is in standard form, corrections can only appear at an output qubit.

Example 5.12. Recall the pattern $\mathfrak{P}_{\wedge X} = X_4^3 Z_4^2 Z_1^2 M_3^0 M_2^0 E_{13} E_{23} E_{34} N_3 N_4$. We have:

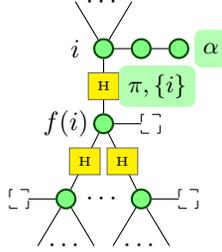


Lemma 5.13. For any pattern \mathfrak{P} we have $D(\mathfrak{P}) \rightarrow D(\Gamma_{\mathfrak{P}})^*$

Proof. All the Z vertices in $D(\mathfrak{P})$ which are introduced by N_i and E_{ij} commands form a connected subgraph, hence they can all be contracted together via the spider rule; this gives $D(\Gamma_{\mathfrak{P}})^*$. \square

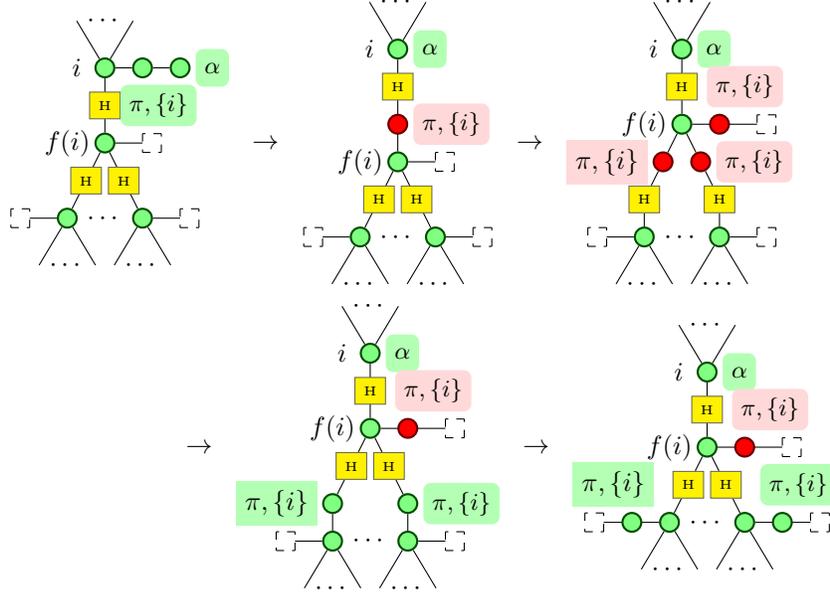
Corollary 5.14. If \mathfrak{P} has flow then the subgraph of $D(\mathfrak{P})$ excluding the measurements rewrites to a circuit-like diagram.

If \mathfrak{P} has a circuit-like geometry then Theorem 5.5 shows that it could be deterministic, if there are corrections in the appropriate places. The ZX-calculus can be used to determine where the corrections must be placed. Suppose that we have the configuration shown below. (The dotted boxes represent either measurements or outputs.)



The measurement at qubit i introduces an error term which must be cancelled

at a later qubit. We can perform the following rewrite sequence:

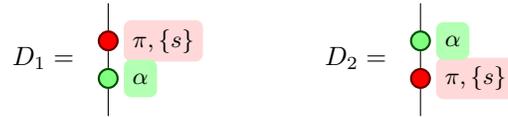


Hence in order to correct for the measurement at qubit i , we must perform a conditional X at qubit $f(i)$, and a condition Z at all its neighbours except i itself. Since the geometry is circuit-like all of these qubits are later in the execution of the pattern than the measurement of i itself. Hence we can conclude:

Theorem 5.15. *Let \mathfrak{P} be a pattern such that $\Gamma_{\mathfrak{P}}$ has flow and, for every measured qubit i , the command sequence contains correctors $X_{f(i)}^i$ and Z_j^i for all $i \neq j \sim f(i)$; then $D(\mathfrak{P})$ rewrites to an unconditional circuit-like diagram.*

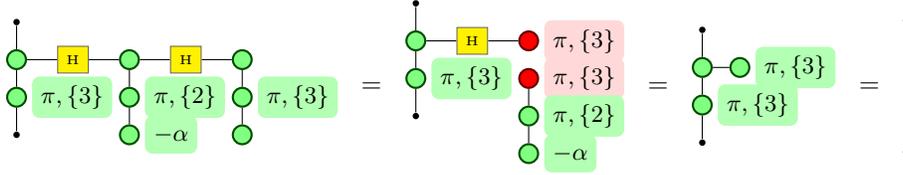
Of course, if j is a measured qubit the corrections can be absorbed into the measurements.

The Z correction at qubit j is effectively the same as an error introduced by measuring j , hence this operation can be deferred by another step, and the correction performed at $f(j)$ and its neighbours. However the same is not true for the corrector X . Consider the following diagrams:



In this case we have $\llbracket D_1 \rrbracket \neq \llbracket D_2 \rrbracket$ unless $\alpha = 0$ or $\alpha = \pi$, hence these diagrams are cannot be rewritten to one another. Therefore uniform determinism requires that the X correction be performed at $f(i)$ and not later in the computation. On the other hand, if $\alpha \in \{0, \pi\}$ then these diagrams are equal. This points to an important advantage of the ZX-calculus.

The diagram $D(\mathfrak{P})$ contains all the information of the pattern itself, not just its geometry. Therefore by rewriting as discussed above the correctness of \mathfrak{P} can be verified directly, and in particular should the MBQC programmer have made an error in the placement of the corrections this error will be revealed. Further, since the ZX-calculus is sensitive to the values of the angles, it can also be used to show that a pattern is deterministic even when it is not uniformly so. Consider the pattern $\mathfrak{P} = M_3^0 M_2^\alpha E_{23} E_{12} N_2 N_3$. This pattern does not have flow. However it is deterministic:



The disconnected component is just a scalar factor which we drop since it has no bearing on the computation.

In this chapter we have focused on flow but flow is not a necessary condition for strong and uniform determinism. With flow each measurement has a single successor where the correction must be performed. If each qubit has instead a set of correcting qubits this yields the notion of *generalised flow* [7]. A computation is called *stepwise* deterministic if after each measurement the non-determinism can be removed by correction. Generalised flow is both necessary and sufficient for strong, stepwise, uniform determinism. The ZX-calculus can also be used to handle generalised flow; in fact using the rewrite rules, especially the bialgebra rule, any geometry which has generalised flow can be rewritten to an equivalent pattern which has flow. The details can be found in [24].

6 Conclusions

Complementarity has long been recognised as one of the fundamental ingredients of quantum mechanics, although it is usually understood negatively: as the failure of certain classical properties. Here we have demonstrated a positive characterisation of complementarity, in terms of the existence of certain algebraic structures. The ZX-calculus provide a very rich language for reasoning about quantum systems that fully exploits this algebraic structure.

Due to its graphical nature, the ZX-calculus is extremely legible, exposing the close parallels between quantum circuits and 1WQC patterns with flow. In the preceding section we have seen how the almost metaphorical property of flow actually defines a trajectory within a diagram along which information—in this the outcome of some quantum measurement—must travel in order to produce a deterministic computation. This analysis can be taken further in the analysis of *generalised flow* [24], where the role of the bialgebra rule is crucial, functioning as a kind ‘interference’ principle, where different paths cancel each out.

The graphical syntax we have employed here is not simply a gimmick. Since the ZX-calculus is based on algebraic first principles, it can unify differing

computational paradigms such quantum circuits and the one-way model in a single setting. Further, since it is based on graphs, it is amenable to automation, opening the door to mechanised reasoning about quantum programs [21].

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