Minimum status, matching and domination of graphs

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Abstract

Given a graph, the status of a vertex is the sum of the distances between the vertex and all other vertices. The minimum status of a graph is the minimum of statuses of all vertices of this graph. We give a sharp upper bound for the minimum status of a connected graph with fixed order and matching number (domination number, respectively), and characterize the unique trees achieving the bound. We also determine the unique tree such that its minimum status is as small as possible when order and matching number (domination number, respectively) are fixed.

Keywords: minimum status, proximity, matching number, domination number, minimum branch-weight

1 Introduction

All graphs considered in this paper are finite, simple and connected. Let G be a graph with vertex set V(G) and edge set E(G). For $u \in V(G)$, the status of u in G is defined as the sum of the distances from u to all other vertices in G, denoted by $s_G(u)$ or s(u), see [9]. That is, $s_G(u) = \sum_{v \in V(G)} d_G(u, v)$, where $d_G(u, v)$ is the distance between vertices u and v in G, i.e., the length (number of edges) of a shortest path connecting u and v in G. The status of a vertex is also known as its transmission [12, 15] or its total distance [6]. The minimum status of G, denoted by s(G), is defined as

 $s(G) = \min\{s_G(u) : u \in V(G)\}.$

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The study of this concept is essentially the study of the the proximity: If the order of G is n, then the proximity of G is defined as $\pi(G) = \frac{1}{n-1}s(G)$ [1, 2, 3]. The set $\{u \in V(G) : s_G(u) = s(G)\}$ is called the median of G.

For a graph G with $u \in V(G)$, G - u denotes the graph obtained from G by deleting u (and its incident edges). Let T be a tree. For $u \in V(T)$, the components of T - u are known as the branches of T at u. The branch-weight of a vertex u in T is the maximum number of vertices of branches of T at u, denoted by $w_T(u)$. The minimum branch-weight of T is defined as

$$w(T) = \min\{w_T(u) : u \in V(T)\}.$$

The set $\{u \in V(T) : w_T(u) = w(T)\}$ is called the centroid of T. Zelinka [21] showed that in a tree a vertex is in its median if and only if it is in its centroid, i.e., for any tree, its median is equal to its centroid, see also [14].

Aouchiche and Hansen [3] gave sharp lower and upper bounds for the proximity and hence the minimum status of a graph as a function of its order, and characterized the extremal graphs, and they also give a sharp lower bound for the proximity and hence the minimum status of a graph with fixed order and diameter. Lin et al. [16] found sharp lower and upper bounds for the minimum status of a graph with fixed order and maximum degree, characterized extremal graphs attaining the lower bound and found a necessary condition for graphs attaining the upper bound. Alternate proofs were given by Rissner and Burkard [19]. Lin et al. [17] extended the minimum status to weighted graphs, and they gave a formula for the minimum status of the Cartesian product of two weighted graphs. Rissner and Burkard [19] also showed that both radius and status obtain their minimum and maximum values at the same type of trees when order and maximum degree are fixed. Related work may be found in [5, 10, 11, 20].

A matching of a graph G is a set of edges in which any two distinct edges are not adjacent. The matching number of G, denoted by m(G), is the maximum of the cardinality of its matchings. For a graph G of order $n, m(G) \leq \lfloor \frac{n}{2} \rfloor$.

A dominating set of a graph G is a set S of vertices such that each vertex of G outside S is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of dominating sets of G. If G is a graph on n vertices without isolated vertices, then $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$, see [18]. A dominating set of G of cardinality $\gamma(G)$ is called $\gamma(G)$ -set. Bollobás and Cockayne [7] showed that a graph G without isolated vertices has a $\gamma(G)$ -set S such that for each vertex $u \in S$, there is a vertex of G outside S that is adjacent to u but no other vertices in S, and thus $m(G) \geq \gamma(G)$.

Following the above work, in this paper, we find sharp lower and upper bounds for the minimum status of trees in term of order and matching number (domination number, respectively), and we also characterize all extremal cases. The upper bounds may be extended to graphs.

2 Transformations

In this section, we give two types of transformations such that the minimum status is decreased or increased.

Lemma 2.1. [14, 16] Let T be a tree on n vertices and x a vertex of T. Then x is in the median of T if and only if $w_T(x) \leq \frac{n}{2}$.

For a graph G with $uv \in E(G)$, G - uv denotes the graph obtained from G by deleting the edge uv. If zw is an edge of the complement of a graph G, then G + zw denotes the graph obtained from G by adding the edge zw. For $v \in V(G)$, by $N_G(v)$ we denote the set of vertices of G that are adjacent to v, and by $d_G(v)$ we denote the degree of v in G, i.e., $d_G(v) = |N_G(v)|$.

A pendant vertex is a vertex of degree one. The vertex adjacent to a pendant vertex is said to be a quasi-pendant vertex. A pendant edge is an edge incident to a pendant vertex. A cut edge is an edge whose removal disconnect the graph. A non-pendant cut edge in a graph is a cut edge that is not a pendant edge. For a graph G with $u \in V(G)$ and $v \notin V(G)$, if G' is the graph with $V(G') = V(G) \cup \{v\}$ and $E(G') = E(G) \cup \{uv\}$, then we say that G' is obtained from G by attaching a pendant vertex at u.

Proposition 2.1. Let G be a connected graph and uv be a non-pendant cut edge of G. Let G_{uv} be the graph obtained from G by contracting uv to a vertex u and attaching a pendant vertex v to u. Then $s(G) > s(G_{uv})$.

Proof. Let x be a vertex in the median of G. Then $s(G) = s_G(x)$ and $s(G_{uv}) \leq s_{G_{uv}}(x)$. Let G_1 be the component of G - uv contains u and G_2 be the component of G - uv contains v. Suppose without loss of generality that $x \in V(G_1)$. As we pass from G to G_{uv} , the distance between x and a vertex of $V(G_1) \cup \{v\}$ remains unchanged and the distance between x and a vertex of $V(G_2) \setminus \{v\}$ is decreased by 1. It follows that

$$s(G) - s(G_{uv}) \geq s_G(x) - s_{G_{uv}}(x) = \sum_{w \in V(G)} (d_G(x, w) - d_{G_{uv}}(x, w)) = \sum_{w \in V(G_2) \setminus \{v\}} 1 = |V(G_2)| - 1 > 0,$$

i.e., $s(G) > s(G_{uv})$.

A pendant path at v in a graph G is a path connecting v and some vertex, say w in G, such that $d_G(w) = 1$, the degree of each internal vertex (if any exists) is two, and $d_G(v) \ge 2$.

 \Box

Proposition 2.2. Let T be a tree of order n with $u \in T$ and $N_T(u) = \{u_1, \ldots, u_k\}$, where $k \geq 3$. Let B_i be the branch of T at u containing u_i for $1 \leq i \leq k$. For $w \in V(B_2)$, let $T' = T - \{uu_i : 3 \leq i \leq t\} + \{wu_i : 3 \leq i \leq t\}$, where $3 \leq t \leq k$. Suppose that $|V(B_1)| \geq |V(B_2)|$. Then s(T') > s(T).

Proof. Since $|V(B_1)| + |V(B_2)| + 2 \le n$ and $|V(B_1)| \ge |V(B_2)|$, we have $|V(B_2)| \le \frac{n}{2} - 1$. Let x be a vertex in the median of T. For any $v \in V(B_2)$, we have $w_T(v) \ge n - |B_2| \ge \frac{n}{2} + 1$, and thus $x \notin V(B_2)$ by Lemma 2.1. Case 1. x = u.

In this case, $w_{T'}(x) = \max \{ w_T(x), \sum_{i=2}^t |V(B_i)| \}$. Suppose first that $\sum_{i=2}^t |V(B_i)| \leq \frac{n}{2}$. As $w_{T'}(u) \leq \frac{n}{2}$, u is in the median of T' by Lemma 2.1. As we pass from T to T', the distance between u and a vertex of $V(B_3) \cup \cdots \cup V(B_t)$ is increased by $d_T(u, w)$, and the distance between u and any other vertex remains unchanged. Thus

$$s(T') - s(T) = s_{T'}(u) - s_T(u) = d_T(u, w) \sum_{i=3}^t |V(B_i)| > 0,$$

implying s(T') > s(T). Thus, we may suppose that $\sum_{i=2}^{t} |V(B_i)| > \frac{n}{2}$. Then there must exist a vertex $v \in V(B_2)$ such that $w_{T'}(v) \leq \frac{n}{2}$. By Lemma 2.1, vis in the median of T'. Note that

$$d_{T'}(v, z) - d_T(u, z) = d_T(u, v) \text{ for } z \in V(B_1) \cup (V(B_{t+1}) \cdots \cup V(B_k)) \cup \{u\},$$
$$d_{T'}(v, z) - d_T(u, z) = d_T(v, z) - d_T(u, z) \ge -d_T(u, v) \text{ for } z \in V(B_2),$$

and

$$d_{T'}(v,z) - d_T(u,z) = d_T(v,w) \text{ for } z \in V(B_3) \cup \cdots \cup V(B_t).$$

Thus

$$s(T') - s(T) = s_{T'}(v) - s_T(u)$$

= $\sum_{z \in V(T')} d_{T'}(v, z) - \sum_{z \in V(T)} d_T(u, z)$
 $\geq d_T(u, v) \left(|V(B_1)| + \sum_{i=t+1}^k |V(B_i)| + 1 - |V(B_2)| \right)$
 $+ d_T(v, w) \sum_{i=3}^t |V(B_i)|$
 $\geq d_T(v, w)$
 $> 0,$

implying s(T') > s(T). **Case 2.** $x \in \bigcup_{i=1}^{k} V(B_i) \setminus V(B_2)$. In this case $w_{T'}(x) = w_T(x)$. By Lemma 2.1, x is also in the median of T'. Suppose first that $x \in V(B_i)$ for i = 1 or $t + 1 \leq i \leq k$. As we pass from T to T', the distance between x and a vertex of $V(B_3) \cup \cdots \cup V(B_t)$ is increased by $d_T(u, w)$, and the distance between x and any other vertex remains unchanged. It follows that

$$s(T') - s(T) = s_{T'}(x) - s_T(x) = d_T(u, w) \sum_{i=3}^t |V(B_i)| > 0,$$

and thus s(T') > s(T).

Next, suppose that $x \in V(B_i)$ for $3 \leq i \leq t$, As we pass from T to T', the distance between x and a vertex of $V(B_1) \cup (V(B_{t+1}) \cdots \cup V(B_k)) \cup \{u\}$ is increased by $d_T(u, w)$, the distance between x and a vertex z of $V(B_2)$ is decreased by $d_T(u, z) - d_T(w, z)$, which is less than or equal to $d_T(u, w)$, and the distance between x and any other vertex remains unchanged. Thus

$$s(T') - s(T) = s_{T'}(x) - s_T(x)$$

$$\geq d_T(u, w) \left(|V(B_1)| + \sum_{i=t+1}^k |V(B_i)| + 1 - |V(B_2)| \right)$$

$$\geq d_T(u, w) \left(\sum_{i=t+1}^k |V(B_i)| + 1 \right)$$

$$> 0,$$

and thus s(T') > s(T).

The result follows by combining the above cases.

3 Minimum status and matching number

In this section, we find sharp lower and upper bounds for the minimum status of a tree with fixed order and matching number and characterize the trees attaining these bounds. We note the upper bound may be extended for connected graphs.

By $A_{n,m}$ we denote the tree obtained from the star S_{n-m+1} by attaching a pendant edge to each of certain m-1 non-central vertices of S_{n-m+1} . The center of $A_{n,m}$ is the center of the star S_{n-m+1} .

Theorem 3.1. Let T be a tree of order n with matching number m, where $1 \le m \le \lfloor \frac{n}{2} \rfloor$. Then $s(T) \ge n + m - 2$ with equality if and only if $T \cong A_{n,m}$.

Proof. We show the result by induction on n.

First, we show the result for n = 2m by induction on m. It is obvious that $T \cong P_{2m} \cong A_{2m,m}$ for m = 1, 2 with s(T) = 3m - 2. Suppose that $m \ge 3$ and the result follows for trees of order 2(m-1) with matching number m - 1. Obviously, the diameter of T is at least 4. Choose a diametrical path $v_0v_1v_2\ldots v_d$ of T. As n = 2m and the matching number of T is m, we have $d_T(v_0) = 1$ and $d_T(v_1) = 2$. By induction hypothesis, $s(T - v_0 - v_1) \ge 3(m-1) - 2 = 3m - 5$ with equality if and only if $T - v_0 - v_1 \cong A_{2(m-1),m-1}$. Let x be a vertex of T in its median. By Lemma 2.1, $w_T(x) \le m$, and x is in the median of $T - v_0 - v_1$ if $w_T(x) \le m - 1$ or if $w_T(x) = m$ and v_0 belongs to the branch at x with m vertices. Suppose that $w_T(x) = m$ and v_0 does not belong to the branch B at x with m vertices. Let x^* be the neighbor of x in B. Then $w_T(x^*) = m$ and v_0 belongs to the branch at x^* is in the median of T and $T - v_0 - v_1$. Thus, we may assume that x is in the median of T and $T - v_0 - v_1$. It follows that

$$s(T) - s(T - v_0 - v_1) = s_T(x) - s_{T - v_0 - v_1}(x) = 2d_T(x, v_2) + 3 \ge 3$$

with equality if and only if $x = v_2$. Thus

$$s(T) \ge s(T - v_0 - v_1) + 3 \ge 3m - 5 + 3 = 3m - 2$$

with equalities if and only if $T - v_0 - v_1 \cong A_{2(m-1),m-1}$ and $x = v_2$, i.e., $T \cong A_{2m,m}$. This proves the result for n = 2m.

Now, suppose that n > 2m and the result follows for trees of order n - 1 with matching number m. Let T be a tree of order n with matching number m. As n > 2m, we have by [8, Lemma 2.7] that there is an matching M with |M| = m and a pendant vertex z such that M does not meet z. Let $yz \in E(T)$. It is evident that M is a matching of T - z, and thus the matching number of T - z is m. By induction hypothesis, $s(G-z) \ge n-1+m-2 = n+m-3$ with equality if and only if $T - z \cong A_{n-1,m}$. Let x' be in the median of T. Then by Lemma 2.1, $w_T(x') \le \frac{n}{2}$, and x' is in the median of T - z if $w_T(x') \le \frac{n-1}{2}$ or if $w_T(x') = \frac{n}{2}$ and z belongs to the branch of T with $w_T(x') = \frac{n}{2}$ vertices. Suppose that $w_T(x') = \frac{n}{2}$ and z does not belongs to the branch B' of T with $w_T(x') = \frac{n}{2}$ vertices. Then, as above, the neighbor of x' in B' is in the media of T and T'. Therefore

$$s(T) - s(T - z) = s_T(x') - s_{T-z}(x') = d_T(x', z) \ge 1$$

with equality if and only if x' = y. Thus

$$s(T) \ge s(T-z) + 1 \ge n + m - 2$$

with equalities if and only if $T - z \cong A_{n-1,m}$ and x' = y, i.e., $T \cong A_{n,m}$. \Box

For integers n, p and q with $p \ge q \ge 0$ and $p+q+2 \le n$, we define a graph called a dumbbell, denoted by $D_n(p,q)$, as the graph formed by attaching ppendant edges to one terminal vertex and q pendant edges to the other terminal vertex of a path P_{n-p-q} . If p+q = n-1, then we define $D(n, p, q) = S_n$. Obviously, $D_n(1,1) \cong P_n \cong D_n(1,0)$.

Lemma 3.1. If $p + q + 2 \le n$ and $p \ge q \ge 2$, then $s(D_n(p,q)) > s(D_n(p+1,q-1))$.

Proof. Let $T = D_n(p,q)$. Let $P = x_1 \dots x_{n-p-q}$ be the path of T connecting the two quasi-pendant vertices. Let u be a pendant vertex adjacent to x_{n-p-q} . Let $T^* = T - x_{n-p-q}u + x_1u$. Obviously, $T^* \cong D_n(p+1,q-1)$. Let x be a vertex in the median of T. Obviously, x lies on the path P. If $x = x_1$, then x_1 is also in the median of T^* as $w_{T^*}(x_1) < w_T(x_1) \leq \frac{n}{2}$. As we pass from T to T^* , the distance between x_1 and u is decreased by n - (p+q+1), and the distance between x_1 and any other vertex remains unchanged. Thus $s(T) - s(T^*) = n - (p+q+1) > 0$, implying $s(T) > s(T^*)$. If $x = x_i$ with $i \geq 2$, then x_{i-1} is a median of T^* as $w_{T^*}(x_{i-1}) \leq w_T(x_i) \leq \frac{n}{2}$. Let B_1 (B_2 , respectively) be the component of $T - x_{i-1}x_i$ contains x_i (x_{i-1} , respectively). Obviously, $|V(B_1)| = n - p - i + 1$ and $|V(B_2)| = p + i - 1$. Note that

$$d_T(x_i, z) - d_{T^*}(x_{i-1}, z) = -1 \text{ for } z \in V(B_1) \setminus \{u\},$$
$$d_T(x_i, z) - d_{T^*}(x_{i-1}, z) = 1 \text{ for } z \in V(B_2),$$

and

$$d_T(x_i, u) - d_{T^*}(x_{i-1}, u) = n - p - q + 1 - i - (i - 1) = n - p - q + 2 - 2i.$$

Thus

$$s(T^*) - s(T) = s_{T^*}(x_{i-1}) - s_T(x_i)$$

= $|V(B_1)| - 1 - |V(B_2)| - (n - p - q + 2 - 2i)$
= $q - p - 1$
< 0,

i.e., $s(T) > s(T^*)$.

A quasi-pendant vertex is a vertex that is adjacent to a pendant vertex.

Theorem 3.2. Let T be a tree on $n \ge 4$ vertices with matching number m, where $1 \le m \le \lfloor \frac{n}{2} \rfloor$. Then

$$s(T) \le m(n-m)$$

with equality if and only if $T \cong D_n(\lceil \frac{n+1}{2} \rceil - m, \lfloor \frac{n+1}{2} \rfloor - m)$.

Proof. Note that the diameter $D_n\left(\left\lceil \frac{n+1}{2}\right\rceil - m, \left\lfloor \frac{n+1}{2}\right\rfloor - m\right)$ is 2m and by Lemma 2.1, its center (the vertex of distance m from a pendant vertex) is in its median. Thus, we have

$$s\left(D_n\left(\left\lceil\frac{n+1}{2}\right\rceil-m,\left\lfloor\frac{n+1}{2}\right\rfloor-m\right)\right) = 2\sum_{i=1}^{m-1}i+(n-2m+1)m$$
$$= m(n-m).$$

Let T be a tree with order n and matching number m such that its minimum status is as large as possible. From the value of the minimum status of $D_n(\lceil \frac{n+1}{2} \rceil - m, \lfloor \frac{n+1}{2} \rfloor - m)$, it suffices to show that $T \cong D_n(\lceil \frac{n+1}{2} \rceil - m, \lfloor \frac{n+1}{2} \rfloor - m)$.

It is trivial for m = 1. Suppose that $m \ge 2$. Let α be the number of quasi-pendant vertices in T. As $m \ge 2$, the diameter of T is at least 3, and thus $\alpha \ge 2$.

We claim that $\alpha = 2$. Otherwise, $\alpha \geq 3$, there are at least three branches, say B_v , B_w and B_z , at some vertex u, and at least two of them are nontrivial, where $v, w, z \in N_T(u)$. Assume that $|V(B_v)| \geq |V(B_w)| \geq |V(B_z)|$. Obviously, B_v and B_w are non-trivial. Let M be a matching of T with |M| = m. Suppose that $uz \notin M$. Let u_1 be a quasi-pendant vertex of B_w . Then $T' = T - uz + u_1 z$ is a tree on n vertices with matching number m. By Proposition 2.2, s(T') > s(T), a contradiction. Thus, $uz \in M$, implying $uv, uw \notin M$. If B_z is nontrivial, then by reversing the roles of w and z as above, we have a contradiction. So B_z is trivial. Let T'' = T - uv + zv. Then m(T'') = m, and by Proposition 2.1 or 2.2, we have s(T'') > s(T), a contradiction. Therefore, we have $\alpha = 2$, as claimed. It follows that $T \cong D_n(p,q)$ for some p and q with $p \ge q \ge 1$. If p = 1, then $T \cong P_n$, $m = \lfloor \frac{n}{2} \rfloor$, and thus $T \cong D_n(1, 1) \cong D_n(1, 0)$.

Suppose p > 1. Assume that $T = D_n(p,q)$. Let $P_d = v_1 \dots v_d$ be a diametrical path of T with $d_T(v_2) = p + 1$, where d = n - p - q + 2. Note that $m = \lfloor \frac{d}{2} \rfloor$, we have d = 2m, 2m + 1. If d = 2m, then the matching number of $D_n(p-1,q)$ is m, and by Proposition 2.1, we have $s(D_n(p-1,q)) > s(D_n(p,q)) = s(T)$, a contradiction. Thus, d = 2m + 1. Then 2m + 1 = n - p - q + 2, i.e., p + q = n + 1 - 2m. By Lemma 3.1, we have $T \cong D_n(\lceil \frac{n+1}{2} \rceil - m, \lfloor \frac{n+1}{2} \rfloor - m)$.

Let G be a connected graph on n vertices with matching number m, where $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Obviously, $s(G) \leq s(T)$ for a spanning tree T of G. Thus, by Theorem 3.2,

 $s(G) \le m(n-m)$

with equality if $G \cong D_n(\lceil \frac{n+1}{2} \rceil - m, \lfloor \frac{n+1}{2} \rfloor - m).$

4 Minimum status and domination number

In this section, we find sharp lower and upper bounds for the minimum status of a tree with fixed order and domination number, and characterize the trees attaining these bounds. The upper bounds may be extended for connected graphs.

Theorem 4.1. Let T be a tree with order n and domination number γ , where $1 \leq \gamma \leq \lfloor \frac{n}{2} \rfloor$. Then

$$s(T) \ge n + \gamma - 2$$

with equality if and only if $T \cong A_{n,\gamma}$.

Proof. Note that $m(T) \ge \gamma$ and $\gamma(A_{n,\gamma}) = \gamma$. By Theorem 3.1 and Proposition 2.1, we have $s(T) \ge s(A_{n,m(T)}) \ge s(A_{n,\gamma}) = n + \gamma - 2$ with equalities if and only if $T \cong A_{n,m(T)}$ and $m(T) = \gamma$, i.e., $T \cong A_{n,\gamma}$.

The proximity $\pi = \pi(G)$ of a connected graph G = (V, E) is the minimum, over all vertices, of the average distance from a vertex to all others. Obviously, $s(G) = (n-1)\pi$.

In [3], an upper bound for the proximity was given, which is restated as below: Let G be a graph of order $n \ge 3$. Then $s(G) \le \lfloor \frac{n^2}{4} \rfloor$ with equality if and only if G is either the cycle C_n or the path P_n . Observe that $\gamma(C_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.

In the rest of this section, for trees with order n and domination number γ , we consider $1 \leq \gamma < \lceil \frac{n}{3} \rceil$ and $\lceil \frac{n}{3} \rceil < \gamma \leq \lfloor \frac{n}{2} \rfloor$, separately.

Theorem 4.2. Let T be a tree with order n and domination number γ , where $1 \leq \gamma < \lceil \frac{n}{3} \rceil$. Then

$$s(T) \leq \begin{cases} \frac{3\gamma-1}{2} \left(n - \frac{3\gamma-1}{2}\right) & \text{if } \gamma \text{ is odd} \\ \frac{3\gamma}{2} \left(n + 1 - \frac{3\gamma}{2}\right) - \left\lceil \frac{n}{2} \right\rceil & \text{if } \gamma \text{ is even} \end{cases}$$

with equality if and only if $T \cong D_n(\lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor).$

Proof. Let $H = D_n(\lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor)$. Obviously, the diameter of H is $3\gamma - 1$. Let uv be a pendant edge with $d_H(v) = \lceil \frac{n-3\gamma+2}{2} \rceil + 1$. Let x be a vertex of H with $d_H(x, u) = \lfloor \frac{3\gamma-1}{2} \rfloor$. Obviously, x is in the median of H by Lemma 2.1. If γ is odd, then

$$s(H) = s_H(x) = s(P_{3\gamma}) + \frac{3\gamma - 1}{2} \cdot (n - 3\gamma) = \frac{(3\gamma)^2 - 1}{4} + \frac{3\gamma - 1}{2} \cdot (n - 3\gamma) = \frac{3\gamma - 1}{2} \left(n - \frac{3\gamma - 1}{2}\right).$$

If γ is even, then

$$s(H) = s(P_{3\gamma}) + \left(\frac{3\gamma}{2} - 1\right) \left\lceil \frac{n - 3\gamma}{2} \right\rceil + \frac{3\gamma}{2} \cdot \left\lfloor \frac{n - 3\gamma}{2} \right\rfloor$$
$$= \frac{(3\gamma)^2}{4} + \frac{3\gamma}{2} \cdot (n - 3\gamma) - \left\lceil \frac{n - 3\gamma}{2} \right\rceil$$
$$= \frac{3\gamma}{2} \left(n + 1 - \frac{3\gamma}{2} \right) - \left\lceil \frac{n}{2} \right\rceil.$$

That is, we have

$$s\left(D_n\left(\left\lceil\frac{n-3\gamma+2}{2}\right\rceil, \left\lfloor\frac{n-3\gamma+2}{2}\right\rfloor\right)\right)$$

$$= \begin{cases} \frac{3\gamma-1}{2} \left(n - \frac{3\gamma-1}{2}\right) & \text{if } \gamma \text{ is odd,} \\ \frac{3\gamma}{2} \left(n + 1 - \frac{3\gamma}{2}\right) - \left\lceil \frac{n}{2} \right\rceil & \text{if } \gamma \text{ is even.} \end{cases}$$

Let T be a tree with order n and domination number at most γ such that its minimum status is as large as possible. By the value of $s\left(D_n\left(\lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor\right)\right)$, we only need to show that $T \cong D_n\left(\lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor\right)$.

It is trivial if $\gamma = 1$. Suppose that $\gamma \ge 2$. Claim 1. T has exactly two quasi-pendant vertices.

Otherwise, there are at least three branches at some vertex u of T, and at least two of them, say B_x and B_y , are nontrivial, where $x, y \in N_T(u)$. Suppose without loss of generality that $|V(B_x)| \ge |V(B_y)|$. Let S be a $\gamma(T)$ set. Suppose that $u \in S$. Let $w \in V(B_y) \cap S$. For a vertex $z \in N_T(u)$ with $z \ne x, y$, let T' = T - uz + wz. Obviously, T' is a tree. As $w \in S$, S is a dominating set of T', implying $\gamma(T') \le |S| = \gamma(T) \le \gamma$. By Proposition 2.2, s(T'') > s(T), a contradiction. It follows that $u \notin S$. Then there is a vertex $x' \in N_T(u) \cap S$. Let $\{y', z'\} \subseteq N_T(u) \setminus \{x'\}$. Let T'' = T - uz' + y'z' if $|V(B_{x'})| \ge |V(B_{y'})|$, and T'' = T - uz' + x'z' if $|V(B_{x'})| < |V(B_{y'})|$. Then T'' is a tree with $\gamma(T') \le \gamma(T) \le \gamma$. By Proposition 2.2, s(T'') > s(T), also a contradiction. Therefore, T has exactly two quasi-pendant vertices, as claimed.

By Claim 1, $T \cong D_n(p,q)$ for some p and q with $p \ge q \ge 1$. As $\gamma(T) \le \gamma < \lceil \frac{n}{3} \rceil$, we have $p \ge 2$.

Claim 2.
$$\gamma(T) = \gamma$$

Otherwise, we have $\gamma(T) < \gamma$. Let $T^* = D_n(p-1,q)$ if $p-1 \ge q$ and $T^* = D_n(p,q-1)$ if p = q. Evidently, $\gamma(T^*) \le \gamma(T) + 1 \le \gamma$. By Proposition 2.1, we have $s(T) < s(T^*)$, a contradiction. Hence $\gamma(T) = \gamma$, as claimed.

As $T \cong D_n(p,q)$ and $\gamma(T) = \gamma$ by Claim 2, we have $\lceil \frac{n-p-q+2}{3} \rceil = \gamma$, i.e., $n-p-q = 3\gamma - 2, 3\gamma - 3, 3\gamma - 4.$

Claim 3. $n - p - q = 3\gamma - 2$.

Otherwise, we have $n - p - q = 3\gamma - 3$, $3\gamma - 4$. Let $\widehat{T} = D_n(p - 1, q)$ if $p - 1 \ge q$ and $\widehat{T} = D_n(p, q - 1)$ for p = q. Obviously, $\gamma(\widehat{T}) = \lceil \frac{n - p - q + 3}{3} \rceil = \gamma$. By Proposition 2.1, $s(T) < s(\widehat{T})$, a contradiction.

By Claim 3, we have $p + q = n - 3\gamma + 2$. Now by Lemma 3.1, we have $T \cong D_n\left(\left\lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor\right)$.

Let G be a connected graph with order n and domination number γ , where $1 \leq \gamma < \lceil \frac{n}{3} \rceil$. Then

$$s(G) \le \begin{cases} \frac{3\gamma - 1}{2} \left(n - \frac{3\gamma - 1}{2} \right) & \text{if } \gamma \text{ is odd} \\ \frac{3\gamma}{2} \left(n + 1 - \frac{3\gamma}{2} \right) - \left\lceil \frac{n}{2} \right\rceil & \text{if } \gamma \text{ is even} \end{cases}$$

with equality if $G \cong D_n(\lceil \frac{n-3\gamma+2}{2} \rceil, \lfloor \frac{n-3\gamma+2}{2} \rfloor).$

A caterpillar is a tree in which removal of all pendant vertices gives a path. Let $C_n(p,q)$ be the caterpillar obtained by attaching a pendant vertex v'_i to v_i of the path $P_{n-p-q} = v_1 v_2 \dots v_{n-p-q}$ for $i = 1, \dots, p, n-p-2q+1, \dots, n-p-q$, where $p \ge q \ge 1$ and $2(p+q) \le n$.

Lemma 4.1. If $p \ge q+2$ and 2(p+q) < n, then $s(C_n(p-1,q+1)) > s(C_n(p,q))$.

Proof. Let $T = C_n(p-1,q+1)$, as labelled above. Let $T' = T - v_{n-p-2q}v'_{n-p-2q} + v_p v'_{n-p-2q}$. Obviously, $T' \cong C_n(p,q)$. Case 1. $2(p-1) < \lceil \frac{n}{2} \rceil$.

Let $x = v_{\lceil \frac{n}{2} \rceil - p + 1}$, then by Lemma 2.1, x is in the median of T as $w_T(x) \leq \frac{n}{2}$. As we pass T to T', the distance between x and v'_{n-p-2q} is decreased by $[n-p-2q-(\lceil \frac{n}{2} \rceil - p+1)+1]-(\lceil \frac{n}{2} \rceil - p+1-p+1) = 2(p-q-1)+(n-2\lceil \frac{n}{2} \rceil)$, and the distance between the x and any other vertex remains unchanged. Thus $s(T) - s(T') \geq s_T(x) - s_{T'}(x) = 2(p-q-1) + (n-2\lceil \frac{n}{2} \rceil) > 0$, implying s(T) > s(T').

Case 2. $2(p-1) \ge \lfloor \frac{n}{2} \rfloor$.

Let $x = v_{\lceil \frac{n}{4} \rceil}$, then by Lemma 2.1, x is in the median of T as $w_T(x) \leq \frac{n}{2}$. As we pass T to T', the distance between x and v'_{n-p-2q} is decreased by $n-p-2q+1-\lceil \frac{n}{4} \rceil - (p+1-\lceil \frac{n}{4} \rceil) = n-2(p+q)$, and the distance between x and any other vertex remains unchanged. Thus $s(T)-s(T') \geq s_T(x)-s_{T'}(x) = n-2(p+q) > 0$, implying s(T) > s(T').

Lemma 4.2. Let T be a caterpillar on n vertices with r pendant vertices, where $2 \le r \le \lfloor \frac{n}{2} \rfloor$ and each vertex of T is adjacent to at most one pendant vertex. Then $s(T) \le s(C_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor))$ with equality if and only if $T \cong C_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)$.

Proof. It is trivial for $r = 2, \frac{n}{2}$ as then $T \cong C_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)$. Suppose that $2 < r < \frac{n}{2}$. Obviously, the diameter of T is n - r + 1. Let T be a caterpillar satisfying the conditions of the lemma such that its minimum status is as large as possible. Let $v_1 \ldots v_{n-r+2}$ be a diametrical path of T and let U = $\{v \in V(T) : d_T(v) = 2\} \setminus \{v_2, v_{n-r+1}\}$. We claim that T - U has exactly two nontrivial components. Otherwise, there are three vertices v_i , v_j , and v_k in T such that $d(v_k) = 3$ and $\{v_i, v_j\} \subseteq U$, where i < k < j. Let v'_k be the unique pendant vertex adjacent to v_k . Let B_1 , B_2 be two nontrivial branches of T at v_k with $v_i \in V(B_1)$ and $v_j \in V(B_2)$. Suppose without loss of generality that $|V(B_1)| \geq |V(B_2)|$. Let $T' = T - v_k v'_k + v_j v'_k$. Evidently, T' is a caterpillar on n vertices with r pendant vertices, and each vertex of T'is adjacent to at most one pendant vertex. By Proposition 2.2, s(T') > s(T), which is a contradiction. Thus T - U has exactly two nontrivial components, as claimed. That is, $T \cong C_n(p,q)$ for some p,q with p+q=r. By Lemma 4.1, $T \cong C_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor).$

Lemma 4.3. [11] Let T be a tree on n vertices with domination γ , where $\gamma > \lfloor \frac{n}{3} \rfloor$. Then the diameter of T is at most $2n - 3\gamma + 1$.

Theorem 4.3. Let T be a tree on n vertices with domination number γ , where $\left\lceil \frac{n}{3} \right\rceil < \gamma \leq \lfloor \frac{n}{2} \rfloor$. Then

$$s(T) \le 3n\gamma + 3\gamma - n - \left\lceil \frac{n^2 + 18\gamma^2}{4} \right\rceil$$

with equality if and only if $T \cong C_n\left(\left\lceil \frac{3\gamma-n}{2} \right\rceil, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor\right)$. Proof. Let $a = \left\lceil \frac{3\gamma-n}{2} \right\rceil, b = \left\lfloor \frac{3\gamma-n}{2} \right\rfloor$ and $c = \left\lceil \frac{2n-3\gamma}{2} \right\rceil$. Obviously, $a+b=3\gamma-n$, and a = b, b+1. Let $H = C_n(a, b)$, as labelled as before. Let $x = v_c$. As $w_H(x) \leq \frac{n}{2}$, x is in the median of H by Lemma 2.1. Let $A = \{v_i : i = 1, \ldots, 2n - 3\gamma\}$ and $B = \{v'_i : i = 1, \ldots, a, 2n - 3\gamma - b + 1, \ldots, 2n - 3\gamma\}$. By direct calculation, we have

$$\sum_{u \in A} d_H(x, u) = s(P_{2n-3\gamma}) = \left\lfloor \frac{(2n-3\gamma)^2}{4} \right\rfloor$$

and

$$\sum_{u \in B} d_H(x, u) = \sum_{i=1}^a (c - a + i) + \sum_{i=1}^b (2n - 3\gamma - b + 1 - c + i)$$

=
$$\begin{cases} a(3n - 6\gamma + 1) + a(a + 1) & \text{if } a = b \\ a(3n - 6\gamma + 1) + a^2 - (2n - 3\gamma - b + 1 - c) & \text{if } a = b + 1 \end{cases}$$

=
$$\begin{cases} \frac{(3\gamma - n)(5n - 9\gamma + 4)}{4} & \text{if } 3\gamma - n \text{ is even,} \\ \frac{(3\gamma - n + 1)(5n - 9\gamma + 3)}{4} - \left(\lfloor \frac{2n - 3\gamma}{2} \rfloor + 1 - \frac{3\gamma - n - 1}{2}\right) & \text{if } 3\gamma - n \text{ is odd.} \end{cases}$$

Thus

$$\begin{split} s(H) &= \sum_{u \in A} d_H(x, u) + \sum_{u \in B} d_H(x, u) \\ &= \left\lfloor \frac{(2n - 3\gamma)^2}{4} \right\rfloor \\ &+ \begin{cases} \frac{(3\gamma - n)(5n - 9\gamma + 4)}{4} & \text{if } 3\gamma - n \text{ is even} \\ \frac{(3\gamma - n + 1)(5n - 9\gamma + 3)}{4} - \left(\lfloor \frac{2n - 3\gamma}{2} \rfloor + 1 - \frac{3\gamma - n - 1}{2}\right) & \text{if } 3\gamma - n \text{ is odd} \end{cases} \\ &= 3n\gamma + 3\gamma - n - \left\lceil \frac{n^2 + 18\gamma^2}{4} \right\rceil. \end{split}$$

So

$$s\left(C_n\left(\left\lceil\frac{3\gamma-n}{2}\right\rceil, \left\lfloor\frac{3\gamma-n}{2}\right\rfloor\right)\right) = 3n\gamma + 3\gamma - n - \left\lceil\frac{n^2 + 18\gamma^2}{4}\right\rceil.$$

Let T be a tree with order n and domination number at least γ such that its minimum status is as large as possible. By the value of $s\left(C_n\left(n, \left\lceil \frac{3\gamma-n}{2} \right\rceil, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor\right)\right)$, it suffices to show that $T \cong C_n\left(\left\lceil \frac{3\gamma-n}{2} \right\rceil, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor\right)$.

Claim 1. Each vertex of T is adjacent to at most one pendant vertex.

Otherwise, there is a vertex u adjacent to two pendant vertices, say v and w. Let T' = T - uv + vw. For any $\gamma(T')$ -set S, it contains one of v or w, and thus $S \cup \{u\} \setminus \{x\}$ with x = v, w is a dominating set of T, implying

 $\gamma \leq \gamma(T) \leq |S \cup \{u\} \setminus \{x\}| \leq |S| = \gamma(T')$. By Proposition 2.1, s(T') > s(T), a contradiction. Thus, each vertex of T is adjacent to at most one pendant vertex, as claimed.

Claim 2. If $d_T(u) \ge 3$, then $\gamma(T - uz) = \gamma(T)$ for $uz \in E(T)$.

Otherwise, $\gamma(T - uz) = \gamma(T) + 1$ as $\gamma(T - e) - \gamma(T) = 0, 1$ for $e \in E(T)$. Let B_v be the branch of T at u containing v for $v \in N_T(u)$. Let $\{x, y\} \subseteq N_T(u) \setminus \{z\}$. Suppose without loss of generality that $|V(B_x)| \geq |V(B_y)|$. Then T' = T - uz + yz is a tree with $\gamma(T') \geq \gamma(T' - yz) - 1 = \gamma(T - uz) - 1 = (\gamma(T) + 1) - 1 = \gamma(T) \geq \gamma$. By Proposition 2.2, s(T') > s(T), a contradiction. It follows that $\gamma(T - uz) = \gamma(T)$ for $uz \in E(T)$ if $d_T(u) \geq 3$, as claimed. Claim 3. T is a caterpillar.

Otherwise, there are at least three nontrivial branches B_x , B_y , B_z of T at u containing x, y, z, respectively, where $\{x, y, z\} \subseteq N_T(u)$. Since $d_T(u) \geq 3$, we have by Claim 2 that $\gamma(T - ux) = \gamma(T - uy) = \gamma(T - uz) = \gamma(T)$. Note that $\gamma(T-w) - \gamma(T) \ge -1$ for $w \in V(T)$. Suppose that $\gamma(B_x - x) - \gamma(B_x) =$ $\gamma(B_y - y) - \gamma(B_y) = \gamma(B_z - z) - \gamma(B_z) = -1$. Then, for each i = x, y, z, there is a $\gamma(B_i)$ -set S_i such that $i \in S_i$ and $S_i \setminus \{i\}$ is a $\gamma(B_i - i)$ -set. Let T^* be the tree obtained from T by deleting the vertices in $V(B_x)$ and let S be a $\gamma(T^*)$ -set. If $u \notin S$, then $S' = (S \setminus V(B_y) \cup V(B_z)) \cup (S_y \setminus \{y\}) \cup (S_z \setminus \{z\}) \cup \{u\}$ is a dominating set of T^* with cardinality $|S'| = |S| - \gamma(B_y) - \gamma(B_z) + \gamma(B_y - \gamma(B_y)) - \gamma(B_y) - \gamma(B_y$ $y + \gamma(B_z - z) + 1 = |S| - 1 = \gamma(T^*) - 1 < \gamma(T^*)$, which is impossible. It follows that $u \in S$. Then $S \cup (S_x \setminus \{x\})$ is a dominating set of T with cardinality $\gamma(T^*) + \gamma(B_x - x)$. Therefore $\gamma(T) \leq \gamma(T^*) + \gamma(B_x - x)$. As $\gamma(T-ux) = \gamma(T^*) + \gamma(B_x)$, we have $(T-ux) - \gamma(T) \ge \gamma(B_x) - \gamma(B_x - x) = 1$, which implies that $\gamma(T - ux) - \gamma(T) = 1$, a contradiction to the fact that $\gamma(T-ux) = \gamma(T)$. Therefore there is a vertex in $\{x, y, z\}$, say z, such that $\gamma(B_z - z) - \gamma(B_z) > -1$. Assume that $|V(B_x)| \geq |V(B_y)|$. Let $w \in V(B_y)$ be a quasi-pendant vertex of T. Let T' = T - uz + wz. As w is adjacent to a unique pendant vertex, there is a $\gamma(T')$ -set R containing w. If $z \in R$, then R is also a dominating set of T' - wz, and so $\gamma(T' - wz) \leq \gamma(T')$. If $z \notin R$, then $R \cap V(B_z - z)$ is a $\gamma(B_z - z)$ -set, and so for a $\gamma(B_z)$ -set R_1 , $(R \setminus R \cap V(B_z - z)) \cup R_1$ is a dominating set of T' - wz with cardinality $|R| - \gamma(B_z - z) + \gamma(B_z) \leq |R|$, implying that $\gamma(T' - wz) \leq \gamma(T')$. Therefore $\gamma(T') = \gamma(T'-wz) = \gamma(T-uz) = \gamma(T) \ge \gamma$. By Proposition 2.2, s(T') > s(T), a contradiction. This proves Claim 3.

Let r be the number of pendant vertices of T. By Claims 1 and 3 and Lemma 4.2, we have $T \cong C_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)$. Let d the diameter of T. Then r = n - d + 1. As $\gamma(T) > \lceil \frac{n}{3} \rceil$, we have by Lemma 4.3 that $d \leq 2n - 3\gamma(T) + 1$. Thus $r \geq 3\gamma(T) - n \geq 3\gamma - n$. Note that $C_n(\lceil \frac{3\gamma - n}{2} \rceil, \lfloor \frac{3\gamma - n}{2} \rfloor)$ has exactly $3\gamma - n$ pendant vertices and its domination number is γ . By Proposition 2.1, if $r > 3\gamma - n$, then $s(C_n(\lceil \frac{r}{2} \rceil, \lfloor \frac{r}{2} \rfloor)) < s(C_n(\lceil \frac{3\gamma - n}{2} \rceil, \lfloor \frac{3\gamma - n}{2} \rfloor))$, a contradiction. Therefore $r = 3\gamma - n$, i.e., $G \cong C_n(\lceil \frac{3\gamma - n}{2} \rceil, \lfloor \frac{3\gamma - n}{2} \rfloor)$.

Let G be a connected graph on n vertices with domination γ , where $\left\lceil \frac{n}{3} \right\rceil < 1$

 $\gamma \leq \lfloor \frac{n}{2} \rfloor$. Then

$$s(G) \le 3n\gamma + 3\gamma - n - \left\lceil \frac{n^2 + 18\gamma^2}{4} \right\rceil$$

with equality if $G \cong C_n\left(\left\lceil \frac{3\gamma-n}{2} \right\rceil, \left\lfloor \frac{3\gamma-n}{2} \right\rfloor\right)$.

5 Concluding remarks

We present sharp lower and upper bounds on the minimum status of a tree using order and matching number (domination number respectively). The trees that attain these bounds are determined. The minimum status is a fundamental graph parameter to measure the centrality of a graph or network [3, 9, 19]. The notion of centrality has been widely used in many different areas. Some other parameters, like radius [9, 19], average distance [12], and distance spectral radius [4], also play roles in the measurement of centrality of a graph. For example, Rissner and Burkard [19] also established analogous results on trees with minimum and maximum radius. It is of interest to investigate the relationship among these parameters.

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