# Two Reflected Gray Code based orders on some restricted growth sequences 

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#### Abstract

We consider two order relations: that induced by the $m$-ary reflected Gray code and a suffix partitioned variation of it. We show that both of them when applied to some sets of restricted growth sequences still yield Gray codes. These sets of sequences are: subexcedant or ascent sequences, restricted growth functions, and staircase words. In each case we give efficient exhaustive generating algorithms and compare the obtained results.


## 1 Introduction and motivations

The term 'Gray code' was taken from Frank Gray, who patented Binary Reflected Gray Code (BRGC) in 1953 [1]. The concept of BRGC is extended to Reflected Gray Code (RGC), to accommodate $m$-tuples (sequence), with $m>2[2]$. In these Gray codes, successive sequence differ in a single position, and by +1 or -1 in this position. More generally, if a list of sequences is such that the Hamming distance between successive sequences is upper bounded by a constant $d$, then the list is said a $d$-Gray code. So in particular, BRGC and RGC are 1-Gray codes. In addition, if the positions where the successive sequences differ are adjacent, then we say that the list is a $d$-adjacent Gray code.

For long time, the design of Gray codes for combinatorial classes and their corresponding generating algorithms was an ad-hoc task, that is, done case by case according to the class under consideration. Recently, general techniques which fit to large classes of combinatorial objects were developed and used. Among them are, for example, the ECO-method (initiated in Gray code context in [3], see also (4), prefix rotations (yielding bubble languages, see [5, 6] and references therein), or Reflected Gray Code based order relations; this last technique was used implicitly, for example in [7,8, and developed systematically as a general method in 4, 9, 13]. The results presented in this paper are in the light of this last direction. More precisely, we show that two order relations induced by Reflected Gray Code and its variation also give Gray codes for some classes of restricted growth sequences defined by means of statistics. These classes are: subexcedant or ascent sequences, restricted growth functions, and staircase words. We give efficient (CAT) generating algorithm for each obtained Gray code.

## 2 Preliminaries

### 2.1 Gray code orders

Let $G_{n}(m)$ be the set of length $n m$-ary sequences $s_{1} s_{2} \ldots s_{n}$ with $s_{i} \in\{0,1, \ldots, m-1\}$; clearly, $G_{n}(m)$ is the product set $\{0,1, \ldots, m-1\}^{n}$. The Reflected Gray Code ( $R G C$ for short) for the set $G_{n}(m)$, denoted by $\mathcal{G}_{n}(m)$, is the natural extension of the Binary Reflected Gray Code to this set. The list $\mathcal{G}_{n}(m)$ is defined recursively by the following relation [2]:

$$
\mathcal{G}_{n}(m)=\left\{\begin{array}{cc}
\epsilon & \text { if } \quad n=0,  \tag{1}\\
0 \mathcal{G}_{n-1}(m), 1 \overline{\mathcal{G}_{n-1}(m)}, 2 \mathcal{G}_{n-1}(m), \ldots,(m-1) \mathcal{G}_{n-1}^{\prime}(m) & \text { if } \quad n>0,
\end{array}\right.
$$

where $\epsilon$ is the empty sequence, $\overline{\mathcal{G}_{n-1}(m)}$ is the reverse of $\mathcal{G}_{n-1}(m)$, and $\mathcal{G}_{n-1}^{\prime}(m)$ is $\mathcal{G}_{n-1}(m)$ or $\overline{\mathcal{G}_{n-1}(m)}$ according to $m$ is odd or even.

In $\mathcal{G}_{n}(m)$, two successive sequences differ in a single position and by +1 or -1 in this position. A list for a set of sequences induces an order relation to this set, and we give two order relations induced by the RGC and its variation, namely RGC order [10] and Co-RGC order.

We adopt the convention that lower case bold letters represent tuples, for example: $s=$ $s_{1} s_{2} \ldots s_{n}, \boldsymbol{a}=a_{1} a_{2} \ldots a_{k}, \boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$.

Definition 1. The Reflected Gray Code order $\prec$ on $G_{n}(m)$ is defined as: $s=s_{1} s_{2} \ldots s_{n}$ is less than $\boldsymbol{t}=t_{1} t_{2} \ldots t_{n}$, denoted by $\boldsymbol{s} \prec \boldsymbol{t}$, if either

- $\sum_{i=1}^{k-1} s_{i}$ is even and $s_{k}<t_{k}$, or
- $\sum_{i=1}^{k-1} s_{i}$ is odd and $s_{k}>t_{k}$,
where $k$ is the leftmost position where $\boldsymbol{s}$ and $\boldsymbol{t}$ differ.
It is easy to see that $\mathcal{G}_{n}(m)$ defined in relation (1) lists sequences in $G_{n}(m)$ in $\prec$ order.
Now we give a variation of $\mathcal{G}_{n}(m)$. Let $s_{1} s_{2} \ldots s_{n}$ be a sequence in $G_{n}(m)$. The complement of $s_{i}, 1 \leq i \leq n$, is

$$
\left(m-1-s_{i}\right),
$$

and the reverse of $s_{1} s_{2} \ldots s_{n}$ is

$$
s_{n} s_{n-1} \ldots s_{1} .
$$

Let $\widetilde{\mathcal{G}}_{n}(m)$ be the list obtained by transforming each sequence $\boldsymbol{s}$ in $\mathcal{G}_{n}(m)$ as follows:

- complementing each digit in $s$ if $m$ is even, or complementing only digits in odd positions if $m$ is odd, then
- reversing the obtained sequence.

Clearly, $\widetilde{\mathcal{G}}_{n}(m)$ is also a Gray code for $G_{n}(m)$, and the sequences therein are listed in Co-Reflected Gray Code order, as defined formally below.

Definition 2. The Co-Reflected Gray Code order $\prec_{c}$ on $G_{n}(m)$ is defined as: $\boldsymbol{s}=s_{1} s_{2} \ldots s_{n}$ is less than $\boldsymbol{t}=t_{1} t_{2} \ldots t_{n}$, denoted by $\boldsymbol{s} \prec_{c} \boldsymbol{t}$, if either

- $\sum_{i=k+1}^{n} s_{i}+(n-k)$ is even and $s_{k}>t_{k}$, or
- $\sum_{i=k+1}^{n} s_{i}+(n-k)$ is odd and $s_{k}<t_{k}$,
where $k$ is the rightmost position where $\boldsymbol{s}$ and $\boldsymbol{t}$ differ.
Although this definition sounds somewhat arbitrary, as we will see in Section 4 it turns out that $\prec_{c}$ order gives suffix partitioned Gray codes for some sets of restricted growth sequences. Obviously, the restriction of $\mathcal{G}_{n}(m)$ (resp. $\widetilde{\mathcal{G}}_{n}(m)$ ) to a set of sequences is simply the list of sequences in the set listed in $\prec$ (resp. $\prec_{c}$ ) order.


### 2.2 Restricted growth sequences defined by means of statistics

Through this paper we consider sequences over non-negative integers. A statistic on a set of sequences is an association of an integer to each sequence in the set. For a sequence $s_{1} s_{2} \ldots s_{n}$, its length minus one, numbers of ascents/levels/descents, maximal value, and last value are classical examples of statistics. They are defined as follows, see also [14]:

- $\operatorname{len}\left(s_{1} s_{2} \ldots s_{n}\right)=n-1$;
- $\operatorname{asc}\left(s_{1} s_{2} \ldots s_{n}\right)=\operatorname{card}\left\{i \mid 1 \leq i<n\right.$ and $\left.s_{i}<s_{i+1}\right\} ;$
- $\operatorname{lev}\left(s_{1} s_{2} \ldots s_{n}\right)=\operatorname{card}\left\{i \mid 1 \leq i<n\right.$ and $\left.s_{i}=s_{i+1}\right\} ;$
- $\operatorname{des}\left(s_{1} s_{2} \ldots s_{n}\right)=\operatorname{card}\left\{i \mid 1 \leq i<n\right.$ and $\left.s_{i}>s_{i+1}\right\} ;$
- $\mathrm{m}\left(s_{1} s_{2} \ldots s_{n}\right)=\max \left\{s_{1}, s_{2}, \ldots, s_{n}\right\} ;$
- $\operatorname{lv}\left(s_{1} s_{2} \ldots s_{n}\right)=s_{n}$.

If st is one of the statistics len, asc, $m$, and Iv , then st satisfy the following:

$$
\begin{equation*}
\operatorname{st}\left(s_{1} s_{2} \ldots s_{n}\right) \leq n-1, \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\text { if } s_{n}=\operatorname{st}\left(s_{1} s_{2} \ldots s_{n-1}\right)+1 \text {, then } s_{n}=\operatorname{st}\left(s_{1} s_{2} \ldots s_{n-1} s_{n}\right) \text {. } \tag{3}
\end{equation*}
$$

On the contrary, the statistics lev and des do not satisfy relation (3). Accordingly, through this paper we will consider only the four statistics above. However, as we will point out, some of the results presented here are also true for arbitrary statistics satisfying relations (2) and (3).

Definition 3. For a given statistic st, an st-restricted growth sequence $s_{1} s_{2} \ldots s_{n}$ is a sequence with $s_{1}=0$ and

$$
\begin{equation*}
0 \leq s_{k+1} \leq \operatorname{st}\left(s_{1} s_{2} \ldots s_{k}\right)+1 \text { for } 1 \leq k<n \tag{4}
\end{equation*}
$$

and the set of st-restricted growth sequences is the set of all sequences $s_{1} s_{2} \ldots s_{n}$ satisfying relation (4).

From this definition, it follows that any prefix of an st-restricted growth sequence is also (a shorter) st-restricted growth sequence.

Remark 1. If st is a statistic satisfying relations (2) and (3) above, then

1. $\max \left\{\operatorname{st}\left(s_{1} s_{2} \ldots s_{n}\right) \mid s_{1} s_{2} \ldots s_{n}\right.$ is an st-restricted growth sequence $\}=n-1$;
2. if $s_{1} s_{2} \ldots s_{n}$ is an st-restricted growth sequence, then for any $k, 1 \leq k<n, s_{k+1}=$ $\operatorname{st}\left(s_{1} s_{2} \ldots s_{k}\right)+1$ implies $s_{k+1}=\operatorname{st}\left(s_{1} s_{2} \ldots s_{k} s_{k+1}\right)$.

The sets of st-restricted growth sequences, where st is one of the statistics len, asc, m, and Iv , are defined below.

## Definition 4.

- The set $S E_{n}$ of subexcedant sequences of length $n$ is defined as:

$$
S E_{n}=\left\{s_{1} s_{2} \ldots s_{n} \mid s_{1}=0 \text { and } 0 \leq s_{k+1} \leq \operatorname{len}\left(s_{1} s_{2} \ldots s_{k}\right)+1 \text { for } 1 \leq k<n\right\}
$$

- The set $A_{n}$ of ascent sequences of length $n$ is defined as:

$$
A_{n}=\left\{s_{1} s_{2} \ldots s_{n} \mid s_{1}=0 \text { and } 0 \leq s_{k+1} \leq \operatorname{asc}\left(s_{1} s_{2} \ldots s_{k}\right)+1 \text { for } 1 \leq k<n\right\}
$$

- The set $R_{n}$ of restricted growth functions of length $n$ is defined as:

$$
R_{n}=\left\{s_{1} s_{2} \ldots s_{n} \mid s_{1}=0 \text { and } 0 \leq s_{k+1} \leq \mathrm{m}\left(s_{1} s_{2} \ldots s_{k}\right)+1 \text { for } 1 \leq k<n\right\}
$$

- The set $S_{n}$ of staircase words of length $n$ is defined as:

$$
S_{n}=\left\{s_{1} s_{2} \ldots s_{n} \mid s_{1}=0 \text { and } 0 \leq s_{k+1} \leq \operatorname{lv}\left(s_{1} s_{2} \ldots s_{k}\right)+1 \text { for } 1 \leq k<n\right\}
$$

Notice that alternatively, $S E_{n}=\{0\} \times\{0,1\} \times \ldots \times\{0,1, \ldots, n-1\}$.
Remark 2. $S_{n} \subset R_{n} \subset A_{n} \subset S E_{n} \subset G_{n}(n)$.
Below we give examples to illustrate Remark 2.

## Example 1.

- If $\boldsymbol{s}=010145$, then $\boldsymbol{s} \in S E_{6}$, but $\boldsymbol{s} \notin A_{6}, \boldsymbol{s} \notin R_{6}$ and $\boldsymbol{s} \notin S_{6}$.
- If $\boldsymbol{s}=010103$, then $s \in S E_{6}, s \in A_{6}$, but $s \notin R_{6}$ and $s \notin S_{6}$.
- If $\boldsymbol{s}=010102$, then $s \in S E_{6}, s \in A_{6}$, and $s \in R_{6}$, but $s \notin S_{6}$.
- If $s=010101$, then $s \in S E_{6}, s \in A_{6}, s \in R_{6}$ and $s \in S_{6}$.

See Table 3 for the sets $S_{5}, R_{5}$, and $A_{5}$ listed in $\prec$ order, and Table 4 for the same sets listed in $\prec_{c}$ order.
We denote by

- $\mathcal{X}_{n}$ the list for the set $X_{n}$ in $\prec$ order, and by $\widetilde{\mathcal{X}}_{n}$ that in $\prec_{c}$ order;
- $\operatorname{succ}_{X}(s), s \in X_{n}$, the successor of $s$ in the set $X_{n}$ listed in $\prec$ order; that is, the smallest sequence in $X_{n}$ larger than $s$ with respect to $\prec$ order;
- $\widetilde{\operatorname{succ}}_{X}(\boldsymbol{s})$ the counterpart of $\operatorname{succ}_{X}(\boldsymbol{s})$ with respect to $\prec_{c}$ order;
- $\operatorname{first}(\mathcal{L})$ the first sequence in the list $\mathcal{L}$;
- $\operatorname{last}(\mathcal{L})$ the last sequence in the list $\mathcal{L}$.


### 2.3 Constant Amortized Time algorithms and principle

An exhaustive generating algorithm is said to run in constant amortized time (CAT for short) if the total amount of computation is proportional to the number of generated objects. And so, a CAT algorithm can be considered an efficient algorithm.

Ruskey and van Baronaigien 15 introduced three CAT properties, and proved that if a recursive generating procedure satisfies them, then it runs in constant amortized time (see also [16]). They called this general technique to prove the efficiency of a generating algorithm as CAT principle, and the involved properties are:

1. Every call of the procedure results in the output of at least one object;
2. Excluding the computation done by the recursive calls, the amount of computation of any call is proportional to the degree of the call, that is, the number of call initiated by the current call;
3. The number of calls of degree one, if any, is $O(N)$, where $N$ is the number of generated objects.

All the generating algorithms we present in this paper satisfy these three desiderata, and so they are efficient.

## 3 The Reflected Gray Code order for the sets $S E_{n}, A_{n}, R_{n}$, and $S_{n}$

### 3.1 The bound of Hamming distance between successive sequences in the lists $\mathcal{S E} \mathcal{E}_{n}, \mathcal{A}_{n}, \mathcal{R}_{n}$, and $\mathcal{S}_{n}$

Here we will show that the Hamming distance between two successive sequences in each of the mentioned lists is upper bounded by a constant, and so the lists are Gray codes.

Without another specification, $X_{n}$ generically denotes one of the sets $S E_{n}, A_{n}, R_{n}$, or $S_{n}$; and $\mathcal{X}_{n}$ denotes its corresponding list in $\prec$ order, that is, one of the lists $\mathcal{S} \mathcal{E}_{n}, \mathcal{A}_{n}, \mathcal{R}_{n}$, or $\mathcal{S}_{n}$. Later in this section, Theorem 1 and Proposition 2 state that, in each case, the set $X_{n}$ listed in $\prec$ order yields a Gray code.

Lemma 1. If $\boldsymbol{s}=s_{1} s_{2} \ldots s_{n}$ and $\boldsymbol{t}=t_{1} t_{2} \ldots t_{n}$ are two sequences in $X_{n}$ with $\boldsymbol{t}=\operatorname{succ}_{X}(\boldsymbol{s})$ and $k$ is the leftmost position where they differ, then $s_{k}=t_{k}+1$ or $s_{k}=t_{k}-1$.

Proof. Let $\boldsymbol{t}=\operatorname{succ}_{X}(s)$ and $k$ be the leftmost position where they differ. Let us suppose that $s_{k}<t_{k}$ and $s_{k} \neq t_{k}-1$ (the case $s_{k}>t_{k}$ and $s_{k} \neq t_{k}+1$ being similar).
It is easy to check that

$$
\boldsymbol{u}=s_{1} s_{2} \ldots s_{k-1}\left(s_{k}+1\right) 0 \ldots 0
$$

belongs to $X_{n}$, and considering the definition of $\prec$ order relation, it follows that $s \prec \boldsymbol{u} \prec \boldsymbol{t}$, which is in contradiction with $t=\operatorname{succ}_{X}(s)$, and the statement holds.

If $\boldsymbol{a}=a_{1} a_{2} \ldots a_{k} \in X_{k}$, then for any $n>k, \boldsymbol{a}$ is the prefix of at least one sequence in $X_{n}$, and we denote by $\boldsymbol{a} \mid \mathcal{X}_{n}$ the sublist of $\mathcal{X}_{n}$ of all sequences having the prefix $\boldsymbol{a}$. Clearly, a list in $\prec$ order for a set of sequences is a prefix partitioned list (all sequences with same prefix are contiguous), and for any $\boldsymbol{a} \in X_{k}$ and $n>k$, it follows that $\boldsymbol{a} \mid \mathcal{X}_{n}$ is a contiguous sublist of $\mathcal{X}_{n}$.

For a given $\boldsymbol{a} \in X_{k}$, the set of all $x$ such that $\boldsymbol{a} x \in X_{k+1}$ is called the defining set of the prefix $\boldsymbol{a}$, and obviously $\boldsymbol{a} x$ is also a prefix of some sequences in $X_{n}$, for any $n>k$. We denote by

$$
\begin{equation*}
\omega_{X}(\boldsymbol{a})=\max \left\{x \mid \boldsymbol{a} x \in X_{k+1}\right\} \tag{5}
\end{equation*}
$$

the largest value in the defining set of $\boldsymbol{a}$. And if we denote $\omega_{X}(\boldsymbol{a})$ by $M$, then by Remark 1 we have

$$
\begin{aligned}
M & =\operatorname{st}(\boldsymbol{a})+1 \\
& =\operatorname{st}(\boldsymbol{a} M)
\end{aligned}
$$

And consequently,

$$
\begin{align*}
\omega_{X}(\boldsymbol{a} M) & =\operatorname{st}(\boldsymbol{a} M)+1 \\
& =M+1 \tag{6}
\end{align*}
$$

The next proposition gives the pattern of $\boldsymbol{s} \in X_{n}$, if $\boldsymbol{s}=\operatorname{last}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$ or $\boldsymbol{s}=\operatorname{first}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$.
Proposition 1. Let $k<n$ and $\boldsymbol{a}=a_{1} a_{2} \ldots a_{k} \in X_{k}$. If $\boldsymbol{s}=\operatorname{last}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$, then the pattern of $\boldsymbol{s}$ is given by:

- if $\sum_{i=1}^{k} a_{i}$ is odd, then $\boldsymbol{s}=\boldsymbol{a} 0 \ldots 0$;
- if $\sum_{i=1}^{k} a_{i}$ is even and $M$ is odd, then $\boldsymbol{s}=\boldsymbol{a} M 0 \ldots 0$;
- if $\sum_{i=1}^{k} a_{i}$ is even and $M$ is even, then $\boldsymbol{s}=\boldsymbol{a} M(M+1) 0 \ldots 0$;
where $M$ denotes $\omega_{X}(\boldsymbol{a})$.
Similar results hold for $\boldsymbol{s}=\operatorname{first}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$ by replacing 'odd' by 'even', and vice versa, for the parity of $\sum_{i=1}^{k} a_{i}$.

Proof. Let $\boldsymbol{s}=a_{1} a_{2} \ldots a_{k} s_{k+1} \ldots s_{n}=\operatorname{last}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$.
If $\sum_{i=1}^{k} a_{i}$ is odd, then by considering the definition of $\prec$ order, it follows that $s_{k+1}$ is the smallest value in the defining set of $\boldsymbol{a}$, and so $s_{k+1}=0$, and finally $\boldsymbol{s}=\boldsymbol{a} 0 \ldots 0$, and the first point holds. Now let us suppose that $\sum_{i=1}^{k} a_{i}$ is even. In this case $s_{k+1}$ equals $\omega_{X}(\boldsymbol{a})=M$, the largest value in the defining set of $\boldsymbol{a}$. When in addition $M$ is odd, so is the summation of $\boldsymbol{a} M$, the length $k+1$ prefix of $\boldsymbol{s}$, and thus $\boldsymbol{s}=\boldsymbol{a} M 0 \ldots 0$, and the second point holds.
Finally, when $M$ is even, then $s_{k+2}$ is the largest value in the defining set of $\boldsymbol{a} M$, which by relation (6) is $M+1$. In this case $M+1$ is odd, and thus $\boldsymbol{s}=\boldsymbol{a} M(M+1) 0 \ldots 0$, and the last point holds.
The proof for the case $s=\operatorname{first}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$ is similar.
By Proposition 1 above, we have the following:
Theorem 1. The lists $\mathcal{A}_{n}, \mathcal{R}_{n}$ and $\mathcal{S}_{n}$ are 3-adjacent Gray codes.
Proof. Let $\mathcal{X}_{n}$ be one of the lists $\mathcal{A}_{n}, \mathcal{R}_{n}$ or $\mathcal{S}_{n}$, and $\boldsymbol{t}=\operatorname{succ}_{X}(\boldsymbol{s})$. Let $k$ be the leftmost position where $\boldsymbol{s}$ and $\boldsymbol{t}$ differ, and let us denote by $\boldsymbol{a}$ the length $k$ prefix of $\boldsymbol{s}$ and $\boldsymbol{a}^{\prime}$ that of $\boldsymbol{t}$; so, $\boldsymbol{s}=\operatorname{last}\left(\boldsymbol{a} \mid \mathcal{X}_{n}\right)$ and $\boldsymbol{t}=\operatorname{first}\left(\boldsymbol{a}^{\prime} \mid \mathcal{X}_{n}\right)$. If $k+3 \leq n$, then by Proposition 11, it follows that $s_{k+3}=s_{k+4}=\cdots=s_{n}=0$ and $t_{k+3}=t_{k+4}=\cdots=t_{n}=0$. So $s$ and $\boldsymbol{t}$ differ only in position $k$, and possibly in position $k+1$ and in position $k+2$.
Now we show the adjacency, that is, if $k+2 \leq n$ and $s_{k+1}=t_{k+1}$ implies $s_{k+2}=t_{k+2}$. If $s_{k+1}=t_{k+1}$, by Lemma 1, it follows that the summation of the length $k$ prefix of $s$ and that of $\boldsymbol{t}$ have different parity, and two cases can occur:

- $s_{k+1}=t_{k+1}=0$, and by Proposition 1, it follows that $s_{k+2}=t_{k+2}=0$; or
- $s_{k+1}=t_{k+1} \neq 0$, and thus $s_{k+1}=t_{k+1}=\omega(\boldsymbol{a})=\omega\left(\boldsymbol{a}^{\prime}\right)$. In this case, $\omega(\boldsymbol{a})$ either is odd and
so $s_{k+2}=t_{k+2}=0$, or is even and so $s_{k+2}=t_{k+2}=\omega(\boldsymbol{a})+1$.
In both cases, $s_{k+2}=t_{k+2}$.
It is well known that the restriction of $\mathcal{G}_{n}(m)$ defined in relation (1) to any product space remains a 1-Gray code, see for example [13]. In particular, for $S E_{n}=\{0\} \times\{0,1\} \times \ldots \times$ $\{0,1, \ldots, n-1\}$ we have the next proposition. Its proof is simply based on Lemma $\mathbb{1}$ Proposition $\mathbb{1}$, and on the additional remark: for any $\boldsymbol{a} \in S E_{k}, k<n$, it follows that $\omega_{S E}(\boldsymbol{a})=k$.
Proposition 2. The list $\mathcal{S E}_{n}$ is 1-Gray code.
It is worth to mention that for any statistic st satisfying relations (2) and (3), the list in $\prec$ order for the set of st-restricted growth sequences of length $n$ is an at most 3-Gray code.
Actually, the lists $\mathcal{S E}_{n}, \mathcal{A}_{n}, \mathcal{R}_{n}$, and $\mathcal{S}_{n}$ are circular Gray codes, that is, the last and the first sequences in the list differ in the same way. Indeed, by the definition of $\prec$ order, it follows that:
- $\operatorname{first}\left(\mathcal{X}_{n}\right)=000 \ldots 0$;
- $\operatorname{last}\left(\mathcal{X}_{n}\right)=010 \ldots 0$;
where $\mathcal{X}_{n}$ is one of the list $\mathcal{S} \mathcal{E}_{n}, \mathcal{A}_{n}, \mathcal{R}_{n}$, or $\mathcal{S}_{n}$.


### 3.2 Generating algorithms for the lists $\mathcal{S E}, \mathcal{A}_{n}, \mathcal{R}_{n}$, and $\mathcal{S}_{n}$

Procedure Gen1 in Figure 1 is a general procedure generating exhaustively the list of st-restricted growth sequences, where st is a statistic satisfying relations (2) and (3). According to particular instances of the function Omega_X called by it (and so, of the statistic st), Gen1 produces specific st-restricted growth sequences, and in particular the lists $\mathcal{S} \mathcal{E}_{n}, \mathcal{A}_{n}, \mathcal{R}_{n}$, and $\mathcal{S}_{n}$. From the length one sequence 0 , Gen 1 constructs recursively increasing length st-restricted growth sequences: for a given prefix $s_{1} s_{2} \ldots s_{k}$ it produces all prefixes $s_{1} s_{2} \ldots s_{k} i$, with $i$ covering (in increasing or decreasing order) the defining set of $s_{1} s_{2} \ldots s_{k}$; and eventually all length $n$ st-restricted growth sequences. It has the following parameters:

- $k$, the position in the sequence $s$ which is updated by the current call;
- $x$, belongs to the defining set of $s_{1} s_{2} \ldots s_{k-1}$, and is the value to be assigned to $s_{k}$;
- dir, the direction (ascending for $\operatorname{dir} \bmod 2=0$ and descending for $\operatorname{dir} \bmod 2=1$ ) to cover the defining set of $s_{1} s_{2} \ldots s_{k-1}$;
- $v$, the value of the statistic of the prefix $s_{1} s_{2} \ldots s_{k-1}$ from which the value of the statistic of the current prefix $s_{1} s_{2} \ldots s_{k}$ is computed. Remark that $v=\omega_{X}\left(s_{1} s_{2} \ldots s_{k-1}\right)-1$.

Function Omega_ $X$ computes $\omega_{X}\left(s_{1} s_{2} \ldots s_{k}\right)$ (see relation (5) ), and the main call is Gen1(1, 0, 0, 0).
In Gen1 the amount of computation of each call is proportional with the degree of the call, and there are no degree one calls, and so it satisfies the CAT principle stated at the end of Section 2. and so it is an efficient generating algorithm. The computational tree of Gen1 producing the list $\mathcal{A}_{4}$ is given in Figure 22, Each node at level $k, 1 \leq k \leq 4$, represents prefixes $s_{1} s_{2} \ldots s_{k}$, and leaves sequences in $\mathcal{A}_{4}$.

```
procedure Gen1(k,x,dir,v)
global: n, s;
sk}:=x
if k=n
then Print s;
else u:= Omega_X (v,k);
    if dir mod 2=0
    then for }i:=0\mathrm{ to }u\mathrm{ do
            Gen1(k+1,i,i,u-1);
            enddo
    else for i:= u downto 0 do
            Gen1(k+1,i,i+1,u-1);
        enddo
endif
end.
```

(a)
(i) function Omega_SE $(w, p)$ return $p$; end.
(ii) function Omega_A $(w, p)$
if $p>1$ and $s_{p}>s_{p-1}$
then return $w+2$;
else return $w+1$;
endif
end.
(iii) function Omega $\_\mathrm{R}(w, p)$ if $p>1$ and $s_{p}>w$
then return $w+2$;
else return $w+1$;
endif
end.
(iv)function Omega_S $(w, p)$
return $s_{p}+1$;
end.
(b)

Figure 1: (a) Algorithm Gen1, generating the list $\mathcal{X}_{n}$; (b) Particular function Omega_ $X$ called by Gen1, and returning the value for $\omega_{X}\left(s_{1} s_{2} \ldots s_{k}\right)$, if $X_{n}$ is one of the sets: (i) $S E_{n}$, (ii) $A_{n}$, (iii) $R_{n}$, and (iv) $S_{n}$.


Figure 2: The tree induced by the initial call $\operatorname{Gen} 1(1,0,0,0)$ for $n=4$ and generating the list $\mathcal{A}_{4}$.

## 4 The Co-Reflected Gray Code order for the sets $S E_{n}, A_{n}, R_{n}$, and $S_{n}$

In this section we will consider, as in the previous one, the sets $S E_{n}, A_{n}, R_{n}$ and $S_{n}$, but listed in $\prec_{c}$ order. Our main goal is to prove that the obtained lists are Gray codes as well, and to develop generating algorithms for these lists. Recall that $X_{n}$ generically denotes one of the sets $S E_{n}, A_{n}, R_{n}$, or $S_{n}$; and let $\widetilde{\mathcal{X}}_{n}$ denote their corresponding list in $\prec_{c}$ order, that are, $\widetilde{\mathcal{S E}}_{n}, \widetilde{\mathcal{A}}_{n}$, $\widetilde{\mathcal{R}}_{n}$, or $\widetilde{\mathcal{S}}_{n}$. Clearly, a set of sequences listed in $\prec_{c}$ order is a suffix partitioned list, that is, all sequences with same suffix are contiguous, and such are the lists we consider here.

For a set $X_{n}$ and a sequence $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$, we call $\boldsymbol{b}$ an admissible suffix in $X_{n}$ if there exists at least a sequence in $X_{n}$ having suffix $\boldsymbol{b}$. For example, 124 is an admissible suffix in $A_{6}$, because there are sequences in $A_{6}$ ending with 124, namely 012124 and 010124 . On the other hand, 224 is not an admissible suffix in $A_{6}$; indeed, there is no length 6 ascent sequence ending with 224.

We denote by $\widetilde{\mathcal{X}}_{n} \mid \boldsymbol{b}$ the sublist of $\widetilde{\mathcal{X}}_{n}$ of all sequences having suffix $\boldsymbol{b}$, and clearly, $\widetilde{\mathcal{X}}_{n} \mid \boldsymbol{b}$ is a contiguous sublist of $\widetilde{\mathcal{X}}_{n}$. The set of all $x$ such that $x \boldsymbol{b}$ is also an admissible suffix in $X_{n}$ is called the defining set of the suffix $\boldsymbol{b}$.

For $\prec$ order discussed in Section 3, the characterization of prefixes is straightforward: $a_{1} a_{2} \ldots a_{k}$ is the prefix of some sequences in $X_{n}, n>k$, if and only if $a_{1} a_{2} \ldots a_{k}$ is in $X_{k}$. And the defining set of the prefix $a_{1} a_{2} \ldots a_{k}$ is $\left\{0,1, \ldots, \operatorname{st}\left(a_{1} a_{2} \ldots a_{k}\right)+1\right\}$. In the case of $\prec_{c}$ order, it turns out that similar notions are more complicated: for example, 13 is an admissible suffix in $A_{5}$, but 13 is not in $A_{2}$; and the defining set of the suffix 13 is $\{0,2\}$, because 013 and 213 are both admissible suffixes in $A_{5}$, but 113 is not. See Table 4 for the set $A_{5}$ listed in $\prec_{c}$ order.

### 4.1 Suffix expansion of sequences in the sets $S E_{n}, A_{n}, R_{n}$, and $S_{n}$

For a suffix partitioned list, we need to build st-restricted growth sequences under consideration from right to left, i.e., by expanding their suffix. For this purpose, we need the notions defined below.

Definition 5. Let $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}, 1 \leq k<n$, an admissible suffix in $X_{n}$.

- $\alpha_{X}(\boldsymbol{b})$ is the set of all elements in the defining set of the suffix $\boldsymbol{b}$. Formally:

$$
\alpha_{X}(\boldsymbol{b})=\left\{x \mid x \boldsymbol{b} \text { is an admissible suffix in } X_{n}\right\},
$$

and for the empty suffix $\epsilon, \alpha_{X}(\epsilon)=\{0,1, \ldots, n-1\}$.

- $\mu_{X}(\boldsymbol{b})$ is the minimum required value of the statistic defining the set $X_{n}$, and provided by a length $(k+1)$ prefix of a sequence in $X_{n}$ having suffix $\boldsymbol{b}$. Formally:

$$
\mu_{X}(\boldsymbol{b})=\min \left\{\operatorname{st}\left(s_{1} s_{2} \ldots s_{k} b_{k+1}\right) \mid s_{1} s_{2} \ldots s_{k} \boldsymbol{b} \in X_{n}\right\} .
$$

Notice that $\mu_{X}(x \boldsymbol{b}) \in\left\{\mu_{X}(\boldsymbol{b})-1, \mu_{X}(\boldsymbol{b}), x\right\}$ for $x \in \alpha_{X}(\boldsymbol{b})$.
Remark 3. Let st be one of statistics asc, m or lv , and $\boldsymbol{s}=s_{1} s_{2} \ldots s_{n}$ be an st-restricted growth sequence. If there is a $k<n$ such that $s_{k+1}=k$, then $s_{i}=i-1$ for all $i, 1 \leq i \leq k$.

Proof. If $s_{k}<k-1$, then in each case for st , $\operatorname{st}\left(s_{1} s_{2} \ldots s_{k}\right)<k-1$, which is in contradiction with $s_{k+1}=k$, and so $s_{k}=k-1$. Similarly, $s_{k-1}=k-2, \ldots, s_{2}=1$, and $s_{1}=0$.

Under the conditions in the previous remark, $s_{k+1}=k$ imposes that all values at the left of $k+1$ in $s$ are uniquely determined. As we will see later, in the induced tree of the generating algorithm, all descendants of a node with $s_{k+1}=k$ have degree one, and we will eliminate the obtained degree-one path in order not to alter the algorithm efficiency.

It is routine to check the following propositions. (Actually, Proposition 3 is a consequence of Remark (1)
Proposition 3. Let $X_{n}$ be one of the sets $S E_{n}, A_{n}, R_{n}$, or $S_{n}$. If $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}, 1 \leq k<n$, is an admissible suffix in $X_{n}$, then $b_{k+1} \leq \mu_{X}(\boldsymbol{b})$.
Proposition 4. Let $Y_{n}$ be one of the sets $A_{n}, R_{n}$, or $S_{n}$. If $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}, 1 \leq k<n$, is an admissible suffix in $Y_{n}$, then

1 if $\boldsymbol{b}=b_{n}$, that is, a length one admissible suffix, then $\mu_{Y}(\boldsymbol{b})=b_{n}$;
$2 \mu_{Y}(\boldsymbol{b})=k$ if and only if $b_{k+1}=k$;
3 if $x \boldsymbol{b}$ is also an admissible suffix in $Y_{n}$ (i.e., $x \in \alpha_{Y}(\boldsymbol{b})$ ) and $x \geq b_{k+1}$, then

$$
\mu_{Y}(x \boldsymbol{b})=\max \left\{x, \mu_{Y}(\boldsymbol{b})\right\} .
$$

The following propositions give the values for $\alpha_{X}(\boldsymbol{b})$ and $\mu_{X}(x \boldsymbol{b})$, if $X_{n}$ is one of the sets $S E_{n}, A_{n}, R_{n}$, or $S_{n}$. We do not provide the proofs for Propositions [5, 6, 11, and 12, because they are obviously based on the definition of the corresponding sequences.
Proposition 5. Let $\boldsymbol{b}=\epsilon$ or $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $S E_{n}$. Then

$$
\alpha_{S E}(\boldsymbol{b})= \begin{cases}\{0,1, \ldots, n-1\} & \text { if } \boldsymbol{b}=\epsilon, \\ \{0,1, \ldots, k-1\} & \text { otherwise. }\end{cases}
$$

Proposition 6. Let $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $S E_{n}$ and $x \in \alpha_{S E}(\boldsymbol{b})$. Then

$$
\mu_{S E}(x \boldsymbol{b})=\mu_{S E}(\boldsymbol{b})-1 .
$$

Obviously, for a length one suffix $\boldsymbol{b}=b_{n}$, it follows that $\mu_{S E}(\boldsymbol{b})=n-1$.
Example 2. If $\boldsymbol{b}=\epsilon$, and $n=10$, then $\alpha_{S E}(\boldsymbol{b})=\{0,1, \ldots, 9\}$; and for $\boldsymbol{b}=b_{10}, b_{10} \in\{0,1, \ldots, 9\}$, it follows that $\mu_{S E}(x \boldsymbol{b})=9-1=8$, for all $x \in \alpha_{S E}(\boldsymbol{b})$.
Proposition 7. Let $\boldsymbol{b}=\epsilon$ or $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $A_{n}$. Then

$$
\alpha_{A}(\boldsymbol{b})= \begin{cases}\{0,1, \ldots, n-1\} & \text { if } \boldsymbol{b}=\epsilon, \\ \{k-1\} & \text { if } \mu_{A}(\boldsymbol{b})=k, \text { or } \mu_{A}(\boldsymbol{b})=k-1 \text { and } b_{k+1}=0, \\ \left\{0,1, \ldots, b_{k+1}-1\right\} \cup\{k-1\} & \text { if } \mu_{A}(\boldsymbol{b})=k-1 \text { and } 0<b_{k+1}<k, \\ \{0,1, \ldots, k-1\} & \text { if } \mu_{A}(\boldsymbol{b})<k-1 .\end{cases}
$$

Proof. If $\boldsymbol{b}=\epsilon$, the result is obvious.
For $\boldsymbol{b} \neq \epsilon$, let $x \in \alpha_{A}(\boldsymbol{b})$.
If $\mu_{A}(\boldsymbol{b})=k$, by Proposition 4 point $2, b_{k+1}=k$ and by Remark 3 we have $x=k-1$.
If $\mu_{A}(\boldsymbol{b})=k-1$ and $b_{k+1}=0$, then $\operatorname{asc}\left(x b_{k+1}\right)=0$, and so $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})=k-1$, and again by Proposition 4 point 2 we have $x=k-1$.
If $\mu_{A}(\boldsymbol{b})=k-1$ and $0<b_{k+1}<k$, then there are two possibilities for $\mu_{A}(x \boldsymbol{b})$ :

- $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})=k-1$, if $\operatorname{asc}\left(x b_{k+1}\right)=0$, and as above $x=k-1 ;$
- $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})-1=k-2$, if $\operatorname{asc}\left(x b_{k+1}\right)=1$. In this case $x \in\left\{0,1, \ldots, b_{k+1}-1\right\}$.

If $\mu_{A}(\boldsymbol{b})<k-1$ (and consequently $0 \leq b_{k+1}<k$ ), then there are two possibilities for $\mu_{A}(x \boldsymbol{b})$ :

- $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})$, if $\operatorname{asc}\left(x b_{k+1}\right)=0$, and we have $x \in\left\{b_{k+1}, b_{k+1}+1, \ldots, k-1\right\}$;
- $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})-1$, if $\operatorname{asc}\left(x b_{k+1}\right)=1$, and we have $x \in\left\{0,1, \ldots, b_{k+1}-1\right\}$.

Proposition 8. Let $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $A_{n}$ and $x \in \alpha_{A}(\boldsymbol{b})$. Then

$$
\mu_{A}(x \boldsymbol{b})= \begin{cases}x & \text { if } \quad x \geq \mu_{A}(\boldsymbol{b}) \\ \mu_{A}(\boldsymbol{b}) & \text { if } \quad b_{k+1} \leq x<\mu_{A}(\boldsymbol{b}) \\ \mu_{A}(\boldsymbol{b})-1 & \text { if } x<b_{k+1}\end{cases}
$$

Proof. If $x \geq \mu_{A}(\boldsymbol{b})$, by Proposition 3 it follows that $x \geq b_{k+1}$, and by Proposition 4 point 3, that $\mu_{A}(x \boldsymbol{b})=\max \left\{x, \mu_{A}(\boldsymbol{b})\right\}=x$.
If $b_{k+1} \leq x<\mu_{A}(\boldsymbol{b})$, then, again by Proposition 4 point 3 , it follows that $\mu_{A}(x \boldsymbol{b})=\max \left\{x, \mu_{A}(\boldsymbol{b})\right\}=$ $\mu_{A}(\boldsymbol{b})$.
If $x<b_{k+1}$, then $\operatorname{asc}\left(x b_{k+1}\right)=1$, so $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})-1$.
Example 3. Let $k=5, n=9$, and $\boldsymbol{b}=b_{6} b_{7} b_{8} b_{9}=2050$ be an admissible suffix in $A_{9}$. Clearly, $\mu_{A}(\boldsymbol{b})$, the minimum number of ascents in a prefix $s_{1} s_{2} \ldots s_{5} b_{6}$ such that $s_{1} s_{2} \ldots s_{5} \boldsymbol{b} \in A_{9}$, is 4 . In this case, denoting $s_{5}$ by $x$, we have

- the set $\alpha_{A}(\boldsymbol{b})$ of all possible values for $x$ is $\left\{0,1,, \ldots, b_{k+1}-1\right\} \cup\{k-1\}=\{0,1\} \cup\{4\}$.
- $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})-1=4-1=3$, if $x \in\{0,1\}$; or $\mu_{A}(x \boldsymbol{b})=\mu_{A}(\boldsymbol{b})=4$, if $x=4$.

Proposition 9. Let $\boldsymbol{b}=\epsilon$ or $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $R_{n}$. Then

$$
\alpha_{R}(\boldsymbol{b})= \begin{cases}\{0,1, \ldots, n-1\} & \text { if } \quad \boldsymbol{b}=\epsilon, \\ \{k-1\} & \text { if } \mu_{R}(\boldsymbol{b})=k, \text { or } \mu_{R}(\boldsymbol{b})=k-1 \text { and } b_{k+1}<k-1 \\ \{0,1, \ldots, k-1\} & \text { if } \mu_{R}(\boldsymbol{b})=k-1 \text { and } b_{k+1}=k-1, \text { or } \mu_{R}(\boldsymbol{b})<k-1 .\end{cases}
$$

Proof. If $\boldsymbol{b}=\epsilon$, the result is obvious.
For $\boldsymbol{b} \neq \epsilon$, let $x \in \alpha_{R}(\boldsymbol{b})$.
If $\mu_{R}(\boldsymbol{b})=k$, by Proposition 4 point $2, b_{k+1}=k$ and by Remark 3 we have $x=k-1$.
If $\mu_{R}(\boldsymbol{b})=k-1$ and $b_{k+1}<k-1$, then the maximal value of the statistic m (defining the set $R_{n}$ ) of a length $k+1$ prefix ending with $b_{k+1}<k-1$ is $k-1$, and it is reached when $x=k-1$. If $\mu_{R}(\boldsymbol{b})=k-1$ and $b_{k+1}=k-1$, then there are two possibilities for $\mu_{R}(x \boldsymbol{b})$ :

- $\mu_{R}(x \boldsymbol{b})=\mu_{R}(\boldsymbol{b})=k-1$, and as above, this implies $x=k-1$;
- $\mu_{R}(x \boldsymbol{b})=\mu_{R}(\boldsymbol{b})-1=k-2$, which implies $x \in\{0,1, \ldots, k-2\}$.

Finally, if $\mu_{R}(\boldsymbol{b})<k-1$, then $x$ can be any value in $\{0,1, \ldots, k-1\}$.
Proposition 10. Let $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $R_{n}$ and $x \in \alpha_{R}(\boldsymbol{b})$. Then

$$
\mu_{R}(x \boldsymbol{b})=\left\{\begin{array}{lll}
x & \text { if } \quad x \geq \mu_{R}(\boldsymbol{b}), \\
\mu_{R}(\boldsymbol{b}) & \text { if } \quad b_{k+1} \leq x<\mu_{R}(\boldsymbol{b}) \text { or } x<b_{k+1}<\mu_{R}(\boldsymbol{b}), \\
\mu_{R}(\boldsymbol{b})-1 & \text { if } \quad x<b_{k+1}=\mu_{R}(\boldsymbol{b}) .
\end{array}\right.
$$

Proof. The case $x \geq \mu_{R}(\boldsymbol{b})$ is analogous with the similar case in Proposition 8 ,
The next case is equivalent with $x<\mu_{R}(\boldsymbol{b})$ and $b_{k+1}<\mu_{R}(\boldsymbol{b})$, and since $R_{n}$ corresponds to the statistic m , the result holds.
Finally, if $x<b_{k+1}=\mu_{R}(\boldsymbol{b})$, then $\mu_{R}(\boldsymbol{b})=\mu_{R}(x \boldsymbol{b})+1$, and so $\mu_{R}(x \boldsymbol{b})=\mu_{R}(\boldsymbol{b})-1$.
Example 4. Let $k=4, n=7$, and $\boldsymbol{b}=b_{5} b_{6} b_{7}=241$ be an admissible suffix in $R_{7}$. It follows that $b_{k+1}=2, \mu_{R}(\boldsymbol{b})=3$ and

- $\alpha_{R}(\boldsymbol{b})=\{k-1\}=\{3\} ;$
- $\mu_{R}(x \boldsymbol{b})=x=3$.

Proposition 11. Let $\boldsymbol{b}=\epsilon$ or $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $S_{n}$. Then

$$
\alpha_{S}(\boldsymbol{b})= \begin{cases}\{0,1, \ldots, n-1\} & \text { if } \boldsymbol{b}=\epsilon, \\ \{k-1\} & \text { if } b_{k+1}=k, \\ \{C, C+1, \ldots, k-1\} & \text { if } 0 \leq b_{k+1} \leq k-1,\end{cases}
$$

where $C=\max \left\{0, b_{k+1}-1\right\}$.
Since $\mu_{S}(\boldsymbol{b})=b_{k+1}$, the next result follows:
Proposition 12. Let $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $S_{n}$ and $x \in \alpha_{S}(\boldsymbol{b})$. Then

$$
\mu_{S}(x \boldsymbol{b})=x .
$$

Example 5. Let $k=6, n=9$, and $\boldsymbol{b}=b_{7} b_{8} b_{9}=457$ be an admissible suffix in $S_{9}$. So we have $\mu_{S}(\boldsymbol{b})=4, C=\max \left\{0, b_{k+1}-1\right\}=\max \{0,3\}=3$, and

- $\alpha_{S}(\boldsymbol{b})=\{C, C+1, \ldots, k-1\}=\{3,4,5\} ;$
- $\mu_{S}(x \boldsymbol{b})=x$, where $x \in\{3,4,5\}$.


### 4.2 The bound of Hamming distance between successive sequences in the lists $\widetilde{\mathcal{S E}}_{n}, \widetilde{\mathcal{A}}_{n}, \widetilde{\mathcal{R}}_{n}$, and $\widetilde{\mathcal{S}}_{n}$

Now we show that the Hamming distance between two successive sequences in the mentioned lists is upper bounded by a constant, which implies that the lists are Gray codes. These results are embodied in Theorems (2) (3) and 4.

### 4.2.1 The list $\widetilde{\mathcal{S E}}_{n}$

Theorem 2. The list $\widetilde{\mathcal{S E}}_{n}$ is 1-Gray code.
Proof. The result follows from the fact that the restriction of the 1-Gray code list $\mathcal{G}_{n}(n)$ to any product space remains a 1-Gray code (see [13]), in particular to the set

$$
V_{n}=\vartheta_{1} \times \vartheta_{2} \times \ldots \times \vartheta_{n},
$$

where

- $\vartheta_{i}=\{0,1, \ldots, n-i\}$, if $n$ is odd and $i$ is even, or
- $\vartheta_{i}=\{i-1, i, \ldots, n-1\}$, if $n$ is even, or $n$ and $i$ are both odd.

Then by applying to each sequence $s$ in the list $\mathcal{V}_{n}$ the two transforms mentioned before Definition 2, namely:

- complementing each digit in $s$ if $n$ is even, or only digits in odd positions if $n$ is odd, then
- reversing the obtained sequence,
the desired 1-Gray code for the set $S E_{n}$ in $\prec_{c}$ order is obtained.


### 4.2.2 The lists $\widetilde{\mathcal{A}}_{n}$ and $\widetilde{\mathcal{R}}_{n}$

The next proposition describes the pattern of $s=\operatorname{last}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$ and $s=\operatorname{first}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$, where $\widetilde{\mathcal{Y}}_{n}$ is one of the lists $\widetilde{\mathcal{A}}_{n}$ or $\widetilde{\mathcal{R}}_{n}$.

Proposition 13. Let $Y_{n}$ be one of the sets ${\underset{\widetilde{\nu}}{n}}^{\widetilde{\nu}_{n}}$ or $R_{n}$, and $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $Y_{n}$. If $\boldsymbol{s}=\operatorname{last}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$ or $\boldsymbol{s}=\operatorname{first}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$, then $\boldsymbol{s}$ has one of the following patterns:

- $s=012 \ldots(k-2)(k-1) \boldsymbol{b}$, or
- $s=012 \ldots(k-2) 0 b$.

Proof. Let $\boldsymbol{s}=s_{1} s_{2} \ldots s_{k} b_{k+1} b_{k+2} \ldots b_{n}$. Since $\boldsymbol{s}=\operatorname{last}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$ or $s=$ first $\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$, according to $\alpha_{Y}(\boldsymbol{b})$ given in Propositions 7 and 9, it follows that $s_{k} \in\{0, k-1\}$. In other words, $s_{k}$ is either the smallest or the largest value in $\alpha_{Y}(\boldsymbol{b})$.
If $s_{k}=k-1$, then by Remark 3 we have $s=012 \ldots(k-2)(k-1) \boldsymbol{b}$.
If $s_{k}=0$, then considering the definition of $\prec_{c}$ order we have either

- $s=\operatorname{first}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$ and $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is odd, or
- $s=\operatorname{last}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$ and $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is even.

For the first case, again by the definition of $\prec_{c}$ order, it follows that $s_{k-1}$ must be the largest value in $\alpha_{Y}(0 \boldsymbol{b})$, and so $s_{k-1}=k-2$, and by Remark 3, $s=012 \ldots(k-2) 0 \boldsymbol{b}$. Similarly, the same result is obtained for the second case.

A direct consequence of the previous proposition is the next theorem.
Theorem 3. The lists $\widetilde{\mathcal{A}}_{n}$ and $\widetilde{\mathcal{R}}_{n}$ are 2-adjacent Gray codes.

Proof. Let $s, \boldsymbol{t} \in Y_{n}$, with $\boldsymbol{t}=\widetilde{\operatorname{succ}}_{Y}(\boldsymbol{s})$. If $k+1$ is the rightmost position where $s$ and $\boldsymbol{t}$ differ, then there are admissible suffixes $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ and $\boldsymbol{b}^{\prime}=b_{k+1}^{\prime} b_{k+2} \ldots b_{n}$ in $Y_{n}$ such that $\boldsymbol{s}=\operatorname{last}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}\right)$ and $\boldsymbol{t}=\operatorname{first}\left(\widetilde{\mathcal{Y}}_{n} \mid \boldsymbol{b}^{\prime}\right)$.
By Proposition [13, $s$ has pattern

$$
\begin{aligned}
& 012 \ldots(k-2)(k-1) \boldsymbol{b}, \text { or } \\
& 012 \ldots(k-2) 0 \boldsymbol{b}
\end{aligned}
$$

and $\boldsymbol{t}$ has pattern

$$
\begin{aligned}
& 012 \ldots(k-2)(k-1) \boldsymbol{b}^{\prime}, \text { or } \\
& 012 \ldots(k-2) 0 \boldsymbol{b}^{\prime} .
\end{aligned}
$$

And in any case, $\boldsymbol{s}$ and $\boldsymbol{t}$ differ in position $k+1$ and possibly in position $k$.

### 4.2.3 The list $\widetilde{\mathcal{S}}_{n}$

The next proposition gives the pattern of last $\left(\widetilde{\mathcal{S}}_{n} \mid \boldsymbol{b}\right)$ and first $\left(\widetilde{\mathcal{S}}_{n} \mid \boldsymbol{b}\right)$ for an admissible suffix $\boldsymbol{b}$ in $S_{n}$.
Proposition 14. Let $\boldsymbol{b}=b_{k+1} b_{k+2} \ldots b_{n}$ be an admissible suffix in $S_{n}$. If $\boldsymbol{s}=\operatorname{last}\left(\widetilde{\mathcal{S}}_{n} \mid \boldsymbol{b}\right)$, then the pattern of $s$ is given by:

- if $b_{k+1}=k$ or $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is odd, then

$$
s=012 \ldots(k-2)(k-1) \boldsymbol{b}
$$

- if $b_{k+1}<k$ and $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is even, and either $b_{k+1}=0$ or $b_{k+1}$ is odd, then

$$
s=012 \ldots(k-2)\left(\max \left\{0, b_{k+1}-1\right\}\right) \boldsymbol{b}
$$

- if $b_{k+1}<k$ and $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is even, and $b_{k+1}>0$ is even, then

$$
s=012 \ldots(k-3)\left(b_{k+1}-2\right)\left(b_{k+1}-1\right) \boldsymbol{b}
$$

Similar results hold for $\boldsymbol{s}=\operatorname{first}\left(\widetilde{\mathcal{S}}_{n} \mid \boldsymbol{b}\right)$ by replacing 'odd' by 'even', and vice versa, for the parity of $\sum_{i=k+1}^{n} b_{i}+(n-k)$.
Proof. Let $\boldsymbol{s}=s_{1} s_{2} \ldots s_{k} b_{k+1} \ldots b_{n}=\operatorname{last}\left(\widetilde{\mathcal{S}}_{n} \mid \boldsymbol{b}\right)$.
If $b_{k+1}=k$ or $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is odd, then $s_{k}$ is the largest value in $\alpha_{S}(\boldsymbol{b})$, so $s_{k}=k-1$, and by Remark 3, $s_{i}=i-1$ for $1 \leq i \leq k$. So the first case holds.
If $\sum_{i=k+1}^{n} b_{i}+(n-k)$ is even and $b_{k+1}<k$, then $s_{k}$ is the smallest value in $\alpha_{S}(\boldsymbol{b})$, namely $\max \left\{0, b_{k+1}-1\right\}$, which is even if $b_{k+1}=0$ or $b_{k+1}$ is odd. Thus, by the definition of $\prec_{c}$ order, $s_{k-1}$ is the largest value in $\alpha_{S}\left(s_{k} \boldsymbol{b}\right)$, which is $k-2$, and by Remark 3, the second case holds.
For the last case, as above, $s_{k}=\max \left\{0, b_{k+1}-1\right\}$, and considering $b_{k+1}>0$ and even, it follows that $s_{k}=b_{k+1}-1$ is odd. Thus $s_{k-1}$ is the minimal value in $\alpha_{S}\left(s_{k} \boldsymbol{b}\right)$, that is $b_{k+1}-2$, which in turns is even, and the last case holds.
The proof for the case $s=\operatorname{first}\left(\widetilde{\mathcal{S}}_{n} \mid \boldsymbol{b}\right)$ is similar.
Theorem 4. The list $\widetilde{\mathcal{S}}_{n}$ is 3-adjacent Gray codes.

Proof. Let $\boldsymbol{t}=\widetilde{\operatorname{succ}}_{S}(\boldsymbol{s})$, and $k+1$ be the rightmost position where $\boldsymbol{s}$ and $\boldsymbol{t}$ differ. Let us denote by $\boldsymbol{b}$ the length $(n-k)$ suffix of $\boldsymbol{s}$ and $\boldsymbol{b}^{\prime}$ that of $\boldsymbol{t}$; so, $\boldsymbol{s}=\operatorname{last}\left(\boldsymbol{b} \mid \mathcal{S}_{n}\right)$ and $\boldsymbol{t}=\mathrm{first}\left(\boldsymbol{b}^{\prime} \mid \mathcal{S}_{n}\right)$. It follows by Proposition 14, that $s_{i}=t_{i}=i-1$ for all $i \leq k-2$, and so the other differences possibly occur in position $k$ and in position $k-1$.

Considering all valid combinations for $s$ and $t$ as given in Proposition 14, the proof of the adjacency is routine, and based on the following: $s_{k} \neq t_{k}$ if and only if $s_{k-1} \neq t_{k-1}$. It follows that $\boldsymbol{s}$ and $\boldsymbol{t}$ differ in one position, or three positions which are adjacent.

In addition, the lists $\widetilde{\mathcal{A}}_{n}, \widetilde{\mathcal{R}}_{n}$, and $\widetilde{\mathcal{S}}_{n}$, are circular Gray codes. This is a consequence of the following remarks based on Propositions 13 and 14

- $\operatorname{first}\left(\widetilde{\mathcal{Y}}_{n}\right)=012 \ldots(n-2)(n-1)$;
- $\operatorname{last}\left(\widetilde{\mathcal{Y}}_{n}\right)=012 \ldots(n-2) 0$;
where $\widetilde{\mathcal{Y}}_{n}$ is one of the lists $\widetilde{\mathcal{A}}_{n}, \widetilde{\mathcal{R}}_{n}$, or $\widetilde{\mathcal{S}}_{n}$.


### 4.3 Generating algorithm for $\widetilde{\mathcal{S E}}_{n}, \widetilde{\mathcal{A}}_{n}, \widetilde{\mathcal{R}}_{n}$, and $\widetilde{\mathcal{S}}_{n}$

Here we explain algorithm Gen2 in Figure 3 which generates suffix partitioned Gray codes for restricted growth sequences; according to particular instances of the functions called by it, Gen2 produces the list $\widetilde{\mathcal{S E}}_{n}, \widetilde{\mathcal{A}}_{n}, \widetilde{\mathcal{R}}_{n}$, or $\widetilde{\mathcal{S}}_{n}$. Actually, for convenience, Gen2 produces length $(n+1)$ sequences $s=s_{1} s_{2} \ldots s_{n+1}$ with $s_{n+1}=0$, and so, neglecting the last value in each sequence $s$ the desired list is obtained. Notice that with this dummy value for $s_{n+1}$ we have $\mu_{X}\left(s_{k} s_{k+1} \ldots s_{n}\right)=\mu_{X}\left(s_{k} s_{k+1} \ldots s_{n} 0\right)$, for $k \leq n$, and similarly for $\alpha_{X}$.

In Gen2, the sequence $s$ is a global variable, and initialized by $01 \ldots(n-1) 0$, which is the first length $n$ sequence in $\prec_{c}$ order, followed by a 0 ; and the main call is $\operatorname{Gen} 2(n+1,0,0,0)$. Procedure Gen2 has the following parameters (the first three of them are similar with those of procedure Gen1):

- $k$, the position in the sequence $s$ which is updated by the current call;
- $x$, the value to be assigned to $s_{k}$;
- dir, gives the direction in which $s_{k-1}$ covers $\alpha_{X}\left(s_{k} s_{k+1} \ldots s_{n} 0\right)$, the defining set of the current suffix;
- $v$, the value of $\mu_{X}\left(s_{k+1} \ldots s_{n} 0\right)$.

The functions called by Gen2 are given in Figures 4 and 5. They are principally based on the evaluation of $\alpha_{X}$ and $\mu_{X}$ for the current suffix of $s$, and are:

- Mu_ $X(k, x, v)$ returns the value of $\mu_{X}\left(s_{k} s_{k+1} \ldots s_{n} 0\right)$, with $x=s_{k}$.
- IsDegreeOne_ $X(k)$ stops the recursive calls when $\alpha_{X}\left(s_{k} s_{k+1} \ldots s_{n} 0\right)$ has only one element, namely $k-2$. In this case, by Remark 3 the sequence is uniquely determined by the current suffix, and this prevents Gen2 to produce degree one calls. In addition, IsDegreeOne_ $X(k)$ sets appropriatelly $d-1$ values at the left of $s_{k}$, where $d$ is the upper bound of the Hamming distance in the list (changes at the left of $s_{1}$ are considered with no effect). This can be considered as a Path Elimination Technique [16].
- Lowest_ $X(k)$ is called when IsDegreeOne_ $X$ returns false, and gives the lowest value in $\alpha_{X}\left(s_{k} s_{k+1} \ldots s_{n} 0\right)$.
- SecLargest_ $X(k, u)$, is called when IsDegreeOne_ $X$ returns false, and gives the second largest value in $\alpha_{X}\left(s_{k} s_{k+1} \ldots s_{n} 0\right)$ (the largest value being always $k-2$ ).

By this construction, algorithm Gen2 has no degree one calls and it satisfies the CAT principle. Figure 6 shows the tree induced by the algorithm when generates $\widetilde{\mathcal{A}}_{4}$.

```
procedure \(\operatorname{Gen} 2(k, x, \operatorname{dir}, v)\)
global \(n, s\);
\(s_{k}:=x\);
\(u:=\mathrm{Mu} \_X(k, x, v)\);
if IsDegreeOne_ \(X(k, u)\)
then Print \(s\);
else \(c:=\) Lowest_ \(X(k)\);
    \(d:=\) SecLargest_ \(X(k, u)\);
    if \(\operatorname{dir} \bmod 2=1\)
    then for \(i:=c\) to \(d\) do
                \(\operatorname{Gen} 2(k-1, i, i, u)\);
            enddo
    endif
    \(\operatorname{Gen} 2(k-1, k-2,(k-1-d i r) \bmod 2, u)\);
    if \(\operatorname{dir} \bmod 2=0\)
    then for \(i:=d\) downto \(c\) do
                \(\operatorname{Gen} 2(k-1, i, i+1, u)\);
        enddo
    endif
endif
end.
```

Figure 3: Algorithm Gen2, generating the list $\widetilde{\mathcal{X}}_{n}$.

```
function Mu_A(m,i,w)
    if i\geqw
    then return i;
    else if i\geqsm+1
        then return w;
        else return w-1;
        endif
    endif
end.
function IsDegreeOne_A(m,v)
    if v=m-1 or
        (v=m-2 and sm}=0
    then }\mp@subsup{s}{m-1}{}:=m-2\mathrm{ ;
        return true;
    else return false;
    endif
end.
function Lowest_A(m)
    return 0;
end.
function SecLargest_A(m,w)
    if w=m-2 and sm}>
        and }\mp@subsup{s}{m}{}<m-
    then return }\mp@subsup{s}{m}{}-1\mathrm{ ;
    else return m-3;
    endif
end.
```

(a)
function Mu_R( $m, i, w)$
if $i \geq w$
then return $i$;
else if $s_{m+1}<w$ then return $w$; else return $w-1$; endif
endif
end.
function IsDegreeOne $\mathcal{R}(m, v)$
if $v=m-1$ or
$\left(v=m-2\right.$ and $\left.s_{m}<m-2\right)$
then $s_{m-1}:=m-2$;
return true;
else return false;
endif
end.
function Lowest_R( $m$ )
return 0 ;
end.
function SecLargest_R( $m, w$ )
return $m-3$;
end.

Figure 4: Particular functions called by Gen2, generating the lists: (a) $\widetilde{\mathcal{A}}_{n}$, and (b) $\widetilde{\mathcal{R}}_{n}$.

```
function Mu_S(m,i,w)
    return i;
end.
function IsDegreeOne_S(m,v)
    if }\mp@subsup{s}{m}{}=m-
    then }\mp@subsup{s}{m-1}{}:=m-2
                    sm-2 :=m-3;
                    return true;
    else return false;
    endif
end.
function Lowest_S(m)
    if }\mp@subsup{s}{m}{}>1\mathrm{ and }\mp@subsup{s}{m}{}\leqm-
    then return }\mp@subsup{s}{m}{}-1\mathrm{ ;
    else return 0;
    endif
end.
function SecLargest_S(m,w)
    return m-3;
end.
```

(a)

```
function Mu_SE \((m, i, w)\)
    if \(m=n\)
    then return \(n-1\);
    else return \(w-1\);
end.
function IsDegreeOne_SE \((m, v)\)
    if \(m=2\)
    then return true;
    else return false;
end.
function Lowest_SE \((m)\)
    return 0;
end.
function SecLargest_SE \((m, w)\)
    return \(m-3\);
end.
```

(b)

Figure 5: Particular functions called by Gen2, generating the lists: (a) $\widetilde{\mathcal{S}}_{n}$, and (b) $\widetilde{\mathcal{S E}}_{n}$.


Figure 6: The tree induced by the call $\operatorname{Gen} 2(5,0,0,0)$, generating the list $\widetilde{\mathcal{A}}_{4}$.

## 5 Final remarks

We conclude this paper by comparing for each of the sets $S E_{n}, A_{n}, R_{n}$ and $S_{n}$ the prefix partitioned Gray codes induced by $\prec$ order and the suffix partitioned one induced by $\prec_{c}$ order. This can be done by comparing the Hamming distance between all pairs of successive sequences either in the worst case or in average.

Table 11summarizes Theorems 1, 3, and 4, and Proposition 2, and gives the upper bound of the Hamming distance (that is, the worst case Hamming distance) for the two order relations. It shows that for the sets $S E_{n}$ and $S_{n}$ these relations have same performances, and for the sets $A_{n}$ and $R_{n}, \prec_{c}$ order induces more restrictive Gray codes.

| Set | $\prec($ RGC $)$ <br> order | $\prec_{c}($ Co-RGC $)$ <br> order |
| :---: | :---: | :---: |
| $S E_{n}$ | 1 | 1 |
| $A_{n}$ | 3 | 2 |
| $R_{n}$ | 3 | 2 |
| $S_{n}$ | 3 | 3 |

Table 1: The bound of the Hamming distance between two successive sequences in $\prec$ and $\prec_{c}$ orders.

For a list $\mathcal{L}$ of sequences, the average Hamming distance is defined as

$$
\frac{\sum d(\boldsymbol{s}, \boldsymbol{t})}{N-1}
$$

where the summation is taken over all $\boldsymbol{s}$ in $\mathcal{L}$, except its last element, $\boldsymbol{t}$ is the successor of $\boldsymbol{s}$ in $\mathcal{L}, d$ is the Hamming distance, and $N$ the number of sequences in $\mathcal{L}$.

Surprisingly, despite $\prec_{c}$ order has same or better performances in terms of worst case Hamming distance, if we consider the average Hamming distance, numerical evidences show that $\prec$ order is 'more optimal' than $\prec_{c}$ order on $A_{n}, R_{n}(n \geq 5)$, and $S_{n}(n \geq 6)$. And this phenomenon strengthens for large $n$; see Table 2,

| $n$ | $\prec($ RGC ) order |  |  |  | $\prec_{c}$ (Co-RGC) order |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\mathcal{S E}_{n}$ | $\mathcal{A}_{n}$ | $\mathcal{R}_{n}$ | $\mathcal{S}_{n}$ | $\widetilde{\mathcal{S E}}_{n}$ | $\widetilde{\mathcal{A}}_{n}$ | $\widetilde{\mathcal{R}}_{n}$ | $\widetilde{\mathcal{S}}_{n}$ |
|  | 1 | 1.21 | 1.21 | 1.31 | 1 | 1.14 | 1.14 | 1.15 |
| 5 | 1 | 1.13 | 1.12 | 1.29 | 1 | 1.19 | 1.18 | 1.24 |
| 6 | 1 | 1.09 | 1.07 | 1.27 | 1 | 1.23 | 1.20 | 1.31 |
| 7 | 1 | 1.06 | 1.06 | 1.26 | 1 | 1.25 | 1.22 | 1.35 |
| 8 | 1 | 1.04 | 1.04 | 1.25 | 1 | 1.26 | 1.23 | 1.37 |
| 9 | 1 | 1.03 | 1.03 | 1.24 | 1 | 1.28 | 1.24 | 1.39 |
| 10 | 1 | 1.02 | 1.03 | 1.23 | 1 | 1.28 | 1.24 | 1.41 |

Table 2: The average Hamming distance for $\prec$ order and $\prec_{c}$ order.
Algorithmically, $\prec_{c}$ order has the advantage that its corresponding generating algorithm, Gen2, is more appropriate to be parallelized than its $\prec$ order counterpart, Gen1. Indeed, the main call of Gen2 produces $n$ recursive calls (compare to two recursive calls produced by the
main call of Gen1), and so we can have more parallelized computations; and this is more suitable for large $n$. See Figure 2 and 6 for examples of computational trees.

Finally, it will be of interest to explore order relation based Gray codes for restricted growth sequences defined by statistics other than those considered in this paper. In this vein we suggest the following conjecture, checked by computer for $n \leq 10$, and concerning descent sequences (defined similarly with ascent sequences in Section 2).

Conjecture 1. The set of length $n$ descent sequences listed in $\prec_{c}$ order is a 4-adjacent Gray code.

## Appendix

| Sequence | $S_{5}$ | $R_{5}$ | $A_{5}$ | Sequence | $S_{5}$ | $R_{5}$ | $A_{5}$ | Sequence | $S_{5}$ | $R_{5}$ | $A_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 00000 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01233 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01111 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00001 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01234 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01110 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00012 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01223 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01120 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00011 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01222 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01121 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00010 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01221 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01122 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00123 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01220 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01123 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00122 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01210 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01023 |  | $\checkmark$ | $\checkmark$ |
| 00121 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01211 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01022 |  | $\checkmark$ | $\checkmark$ |
| 00120 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01212 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01021 |  | $\checkmark$ | $\checkmark$ |
| 00110 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01213 |  | $\checkmark$ | $\checkmark$ | 01020 |  | $\checkmark$ | $\checkmark$ |
| 00111 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01203 |  | $\checkmark$ | $\checkmark$ | 01010 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00112 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01202 |  | $\checkmark$ | $\checkmark$ | 01011 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00102 |  | $\checkmark$ | $\checkmark$ | 01201 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01012 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00101 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01200 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01013 |  |  | $\checkmark$ |
| 00100 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01100 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01002 |  | $\checkmark$ | $\checkmark$ |
| 01230 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01101 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01001 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01231 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01102 |  | $\checkmark$ | $\checkmark$ | 01000 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01232 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01112 | $\checkmark$ | $\checkmark$ | $\checkmark$ |  |  |  |  |

Table 3: The sets $S_{5}, R_{5}$, and $A_{5}$ listed in $\prec$ order.

| Sequence | $S_{5}$ | $R_{5}$ | $A_{5}$ | Sequence | $S_{5}$ | $R_{5}$ | $A_{5}$ | Sequence | $S_{5}$ | $R_{5}$ | $A_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 01234 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01222 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01101 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01233 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01122 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01201 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01023 |  | $\checkmark$ | $\checkmark$ | 00122 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01200 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00123 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01022 |  | $\checkmark$ | $\checkmark$ | 01100 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01123 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01232 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00100 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01223 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01231 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00000 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01213 |  | $\checkmark$ | $\checkmark$ | 01021 |  | $\checkmark$ | $\checkmark$ | 01000 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01013 |  |  | $\checkmark$ | 00121 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01010 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01203 |  | $\checkmark$ | $\checkmark$ | 01121 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00010 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01202 |  | $\checkmark$ | $\checkmark$ | 01221 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00110 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01102 |  | $\checkmark$ | $\checkmark$ | 01211 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01110 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00102 |  | $\checkmark$ | $\checkmark$ | 01111 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01210 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01002 |  | $\checkmark$ | $\checkmark$ | 00111 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01220 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01012 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00011 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01120 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00012 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01011 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00120 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 00112 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01001 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01020 |  | $\checkmark$ | $\checkmark$ |
| 01112 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00001 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 01230 | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| 01212 | $\checkmark$ | $\checkmark$ | $\checkmark$ | 00101 | $\checkmark$ | $\checkmark$ | $\checkmark$ |  |  |  |  |

Table 4: The sets $S_{5}, R_{5}$, and $A_{5}$ listed in $\prec_{c}$ order.

## References

[1] Gray, F. (1953) Pulse code communication, U.S. Patent 2632058 .
[2] Er, M.C. (1984) On generating the $N$-ary reflected Gray code, IEEE Transaction on Computers, 33(8), 739-741.
[3] Bernini, A., Grazzini, E., Pergola, E. and Pinzani, R. (2007) A general exhaustive generation algorithm for Gray structures, Acta Informatica, 44(5), 361-376.
[4] Vajnovszki, V. (2010) Generating involutions, derangements, and relatives by ECO, DMTCS, 12 (1), 109-122.
[5] Ruskey, F. and Williams, A. (2009) The coolest way to generate combinations, Discrete Mathematics, 309, 5305-5320.
[6] Ruskey, F., Sawada, J. and Williams, A. (2012) Binary bubble languages, Journal of Combinatorial Theory, Ser. A 119(1), 155-169.
[7] Klingsberg, P. (1981) A Gray code for compositions, Journal of Algorithms, 3 (1), 4144.
[8] Walsh, T. (2000) Loop-free sequencing of bounded integer compositions, Journal of Combinatorial Mathematics and Combinatorial Computing, 33, 323-345.
[9] Baril, J.-L. and Vajnovszki, V. (2005) Minimal change list for Lucas strings and some graph theoretic consequences, Theoretical Computer Science, 346, 189-199.
[10] Vajnovszki, V. (2001) A loopless generation of bitstrings without $p$ consecutive ones, Discrete Mathematics and Theoretical Computer Science - Springer, 227-240.
[11] Vajnovszki, V. (2007) Gray code order for Lyndon words, Discrete Mathematics and Theoretical Computer Science, 9 (2), 145-152.
[12] Vajnovszki, V. (2008) More restrictive Gray codes for necklaces and Lyndon words, Information Processing Letters, 106, 96-99.
[13] Vajnovszki, V. and Vernay, R. (2011) Restricted compositions and permutations: from old to new Gray codes, Information Processin Letters, 111, 650-655.
[14] Mansour, T. and Vajnovszki, V. (2013) Efficient generation of restricted growth words, Information Processing Letters, 113, 613-616.
[15] van Baronaigien, D.R. and Ruskey, F. (1993) Efficient generation of subsets with a given sum, JCMCC, 14, 87-96.
[16] Ruskey, F. Combinatorial Generation. Book in preparation.

