

Spectra, hitting times, and resistance distances of q -subdivision graphs

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Abstract

Graph operations or products play an important role in complex networks. In this paper, we study the properties of q -subdivision graphs, which have been applied to model complex networks. For a simple connected graph G , its q -subdivision graph $S_q(G)$ is obtained from G through replacing every edge uv in G by q disjoint paths of length 2, with each path having u and v as its ends. We derive explicit formulas for many quantities of $S_q(G)$ in terms of those corresponding to G , including the eigenvalues and eigenvectors of normalized adjacency matrix, two-node hitting time, Kemeny constant, two-node resistance distance, Kirchhoff index, additive degree-Kirchhoff index, and multiplicative degree-Kirchhoff index. We also study the properties of the iterated q -subdivision graphs, based on which we obtain the closed-form expressions for a family of hierarchical lattices, which has been used to describe scale-free fractal networks.

Keywords: Normalized Laplacian spectrum, Subdivision graph, Random walk, Hitting time, Kirchhoff index, Effective resistance

1. Introduction

As powerful tools of network science, graph operations and products have been widely used to construct complex networks with the remarkable scale-free [1], small-world [2], and fractal [3] characteristics observed in realistic networks [4]. A clear advantage for generating complex networks by graph operations and products lies in the allowance of rigorous analysis for

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structural and dynamical properties of the resulting networks. In addition, various real massive networks comprise of smaller pieces, such as communities [5], motifs [6], and cliques [7]. Graph operations and products represent a natural way to create a huge graph out of small ones. Due to the great relevance, diverse graph operations and products have been introduced or developed for practical purposes, e.g. designing models for complex networks. Frequently used graph operations and products include edge iteration [8, 9], planar triangulation [10, 11, 12], Kronecker product [13, 14, 15], hierarchical product [16, 17, 18], and corona product [19, 20, 21].

Among various graph operations and products, subdivision is one of the most popular ones. For a simple graph G , its subdivision graph is the graph obtained from G by inserting a new node into every edge of G . The properties of subdivision graphs have been extensively studied [22, 23, 24]. Moreover, many extended subdivision graphs were proposed, such as q -full subdivision graph [25, 26] and q -subdivision graphs [27]. The q -full subdivision graph of G is obtained from G by replacing each of its edges with pairwise internally disjoint paths of length $q + 1$, while the q -subdivision graph $S_q(G)$ of G is obtained from G by replacing each edge uv with q disjoint paths of length 2: $ux_1v, ux_2v, \dots, ux_qv$. The q -subdivision operation was iteratively applied to the particular graph consisting of an edge, generating the hierarchical lattices—a model of complex networks with the striking scale-free fractal topologies [28], which has received much recent attention [29, 30, 31]. However, in contrast to the traditional subdivision, the properties of $S_q(G)$ for a general graph G are still not well understood, despite the wide ranges of applications for this graph operation.

In this paper, we present an extensive study of the properties for q -subdivision graph $S_q(G)$ of a simple connected graph G . We provide explicit formulas for eigenvalues and eigenvectors of normalized adjacency matrix for $S_q(G)$ in terms of those associated with G , based on which we determine two-node hitting time and the Kemeny constant for random walks on $S_q(G)$ in terms of those corresponding to G . Also, we derive the expressions of two-node resistance distance, Kirchhoff index, additive degree-Kirchhoff index for $S_q(G)$, and multiplicative degree-Kirchhoff index, in terms of the quantities for G . Finally, we obtain closed-form solutions to related quantities for iterated q -subdivisions of a graph G , and apply those obtained results to the scale-free fractal hierarchical lattices, leading to explicit expressions for some quantities.

2. Preliminaries

In this section, we introduce some basic concepts for a graph, random walks and electrical networks.

2.1. Graph and Matrix Notation

Let $G(V, E)$ be a simple connected graph with n nodes and m edges. The n nodes constitute node set $V(G) = \{1, 2, \dots, n\}$, and m edges form edge set $E(G) = \{e_1, e_2, \dots, e_m\}$.

Let A denote the adjacency matrix of G , the entry $A(i, j)$ of which is 1 (or 0) if nodes i and j are (not) directly connected in G . Let $\Gamma(i)$ denote the set of neighbors of node i in graph G . Then the degree of node i is $d_i = \sum_{j \in \Gamma(i)} A(i, j)$, which constitutes the i th entry of the diagonal degree matrix D of G . The incidence matrix of G is an $n \times m$ matrix B , where $B(i, j) = 1$ (or 0) if i is (not) incident with e_j .

Lemma 2.1 [32] *Let G be a simple connected graph with n nodes. Then the rank of its incidence matrix B is $\text{rank}(B) = n - 1$ if G is bipartite, and $\text{rank}(B) = n$ otherwise.*

2.2. Random Walks on Graphs

For a graph G , we can define a discrete-time unbiased random walk taking place on it. For any time step, the walker jumps from its current location, node i , to another node j with probability $A(i, j)/d_i$. Such a random walk on G is in fact a Markov chain characterized by the transition probability matrix $T = D^{-1}A$, with the entry $T(i, j)$ equal to $A(i, j)/d_i$. For a random walk on graph G , the stationary distribution is an n -dimension vector $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ satisfying $\pi T = \pi$ and $\sum_{i=1}^n \pi_i = 1$. It is easy to verify that $\pi = (d_1/2m, d_2/2m, \dots, d_n/2m)$ for unbiased random walks on G .

The transition probability matrix T of graph G is not symmetric. However, T is similar to the normalized adjacency matrix P of G , which is defined by

$$P = D^{-\frac{1}{2}}AD^{-\frac{1}{2}} = D^{\frac{1}{2}}TD^{-\frac{1}{2}}.$$

Obviously, P is symmetric, with the (i, j) th entry being $P(i, j) = \frac{A(i, j)}{\sqrt{d_i d_j}}$.

Lemma 2.2 [33] *Let G be a simple connected graph with n nodes, and let $1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \geq -1$ be the eigenvalues of its normalized adjacency matrix P . Then $\lambda_n = -1$ if and only if G is bipartite.*

Let v_1, v_2, \dots, v_n be the normalized mutually orthonormal eigenvectors corresponding to the n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, where $v_i = (v_{i1}, v_{i2}, \dots, v_{in})^\top$. Then,

$$v_1 = \left(\sqrt{d_1/2m}, \sqrt{d_2/2m}, \dots, \sqrt{d_n/2m} \right)^\top \quad (1)$$

and

$$\sum_{k=1}^n v_{ik}v_{jk} = \sum_{k=1}^n v_{ki}v_{kj} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

As for a bipartite graph G , whose node set $V(G)$ can be divided into two disjoint subsets V_1 and V_2 , i.e., $V(G) = V_1 \cup V_2$, we have

$$v_{ni} = \sqrt{d_i/2m}, i \in V_1; \quad v_{nj} = -\sqrt{d_j/2m}, j \in V_2. \quad (3)$$

A fundamental quantity related to random walks is the hitting time. The hitting time T_{ij} from one node i to another node j is the expected time taken by a walker to first reach node j starting from node i , which is relevant in various scenarios [34]. Many interesting quantities of graph G can be defined or derived from hitting times. For example, for a graph G , its Kemeny's constant $K(G)$ is defined as the expected number of steps required for a walker starting from node i to a destination node, which is chosen randomly according to a stationary distribution of random walks on G [35]. The Kemeny's constant $K(G)$ is independent of the selection of starting node i [36].

The hitting time T_{ij} for random walks on graph G is encoded in the eigenvalues and eigenvectors of its normalized adjacency matrix P .

Theorem 2.3 [37] *For random walks on a simple connected graph G , the hitting time T_{ij} from one node i to another node j is*

$$T_{ij} = 2m \sum_{k=2}^n \frac{1}{1 - \lambda_k} \left(\frac{v_{kj}^2}{d_j} - \frac{v_{ki}v_{kj}}{\sqrt{d_i d_j}} \right).$$

In particular, when G is a bipartite graph with $V(G) = V_1 \cup V_2$, then

$$T_{ij} = 2m \sum_{k=2}^{n-1} \frac{1}{1 - \lambda_k} \left(\frac{v_{kj}^2}{d_j} - \frac{v_{ki}v_{kj}}{\sqrt{d_i d_j}} \right),$$

if i and j are both in V_1 or V_2 ;

$$T_{ij} = 2m \sum_{k=2}^{n-1} \frac{1}{1 - \lambda_k} \left(\frac{v_{kj}^2}{d_j} - \frac{v_{ki}v_{kj}}{\sqrt{d_i d_j}} \right) + 1,$$

otherwise.

In contrast, the Kemeny's constant of G is only dependent on the eigenvalues of P .

Lemma 2.4 [38] *Let G be a simple connected graph with n nodes. Then*

$$K(G) = \sum_{j=1}^n \pi_j T_{ij} = \sum_{i=2}^n \frac{1}{1 - \lambda_i},$$

where $1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \geq -1$ are eigenvalues of matrix P .

2.3. Electrical Networks

For a simple connected graph G , we can define a corresponding electrical network G^* , which is obtained from G by replacing each edge in G with a unit resistor [39]. The resistance distance r_{ij} between a pair of nodes i and j in G is equal to the effective resistance between i and j in G^* . Similar to the hitting time T_{ij} , resistance distance r_{ij} can also be expressed in terms of the eigenvalues and eigenvectors of normalized adjacency matrix P .

Lemma 2.5 [40] *Let G be a simple connected graph. Then resistance distance r_{ij} between nodes i and j is*

$$r_{ij} = \sum_{k=2}^n \frac{1}{1 - \lambda_k} \left(\frac{v_{ki}}{\sqrt{d_i}} - \frac{v_{kj}}{\sqrt{d_j}} \right)^2.$$

Lemma 2.6 [41] *Let G be a simple connected graph with n nodes. Then the sum of resistance distances between all pairs of adjacent nodes in G is equivalent to $n - 1$, i.e.*

$$\sum_{ij \in E(G)} r_{ij} = n - 1.$$

where the summation is taken over all the edges in G .

There are some intimate relationships between random walks on graphs and electrical networks. For example, the resistance distance r_{ij} is closely related to hitting times T_{ij} and T_{ji} of G , as stated in the following lemma.

Lemma 2.7 [42] *For any pair of nodes i and j in a graph G with m edges, the following relation holds true:*

$$2mr_{ij} = T_{ij} + T_{ji}.$$

The resistance distance is an important quantity [43]. Various graph invariants based on resistance distances have been defined and studied. Among these invariants, the Kirchhoff index [44] is of vital importance.

Definition 2.8 [44] *The Kirchhoff index of a graph G is defined as*

$$\mathcal{K}(G) = \frac{1}{2} \sum_{i,j=1}^n r_{ij} = \sum_{\{i,j\} \subseteq V(G)} r_{ij}.$$

Kirchhoff index has found wide applications. For example, it can be used as measures of the overall connectedness of a network [45], the robustness of first-order consensus algorithm in noisy networks [46], as well as the edge centrality of complex networks [47]. In recent years, several modifications for Kirchhoff index have been proposed, including additive degree-Kirchhoff index [48] and multiplicative degree-Kirchhoff index [40]. For a graph G , its additive degree-Kirchhoff index $\bar{\mathcal{K}}(G)$ and multiplicative degree-Kirchhoff index $\tilde{\mathcal{K}}(G)$ are defined as

$$\bar{\mathcal{K}}(G) = \frac{1}{2} \sum_{i,j=1}^n (d_i + d_j)r_{ij} = \sum_{\{i,j\} \subseteq V(G)} (d_i + d_j)r_{ij}$$

and

$$\tilde{\mathcal{K}}(G) = \frac{1}{2} \sum_{i,j=1}^n d_i d_j r_{ij} = \sum_{\{i,j\} \subseteq V(G)} d_i d_j r_{ij},$$

respectively.

It has been proved that $\tilde{\mathcal{K}}(G)$ can be represented in terms of the eigenvalues of the matrix P .

Lemma 2.9 [40] *Let G be a graph with n nodes and m edges. Then*

$$\tilde{\mathcal{K}}(G) = 2m \sum_{i=2}^n \frac{1}{1 - \lambda_i}.$$

3. q -subdivision Graphs and Their Matrices

In this section, we introduce the q -subdivision graph of a graph G , which is an extension of the traditional subdivision graph, since 1-subdivision graph is exactly the subdivision graph. The subdivision of G , denoted by $S(G)$, is the graph obtained from G by inserting a new node into each edge in G . The subdivision graph can be easily extended to a general case.

Definition 3.1 Let G be a simple connected graph. For a positive integer q , the q -subdivision graph of G , denoted by $S_q(G)$, is the graph obtained from G by replacing each edge uv in G with q disjoint paths of length 2: ux_1v , ux_2v, \dots, ux_qv .

In what follows, for a quantity Z of G , we use \hat{Z} to denote the corresponding quantity associated with $S_q(G)$. Then it is easy to verify that in the q -subdivision graph $S_q(G)$, there are $\hat{n} = n + mq$ nodes and $\hat{m} = 2mq$ edges.

By definition, $S_q(G)$ is a bipartite graph, irrespective of G . Then, the node set $\hat{V} := V(S_q(G))$ of $S_q(G)$ can be divided into two disjoint parts V and V' , where V is the set of old nodes inherited from G , while V' is the set of new nodes generated in the process of performing q -subdivision operation on G . Moreover, V' can be further classified into q parts as $V' = V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(q)}$, where each $V^{(i)}$ ($i = 1, 2, \dots, q$) contains m new nodes produced by m different edges in G . Namely,

$$\hat{V} = V \cup V^{(1)} \cup V^{(2)} \cup \dots \cup V^{(q)}. \quad (4)$$

By construction, for each old edge uv , there exists one and only one node x in each $V^{(i)}$ ($i = 1, 2, \dots, q$), satisfying $\hat{\Gamma}(x) = \{u, v\}$. Thus, for two different sets $V^{(i)}$ and $V^{(j)}$, the structural and dynamical properties of nodes belonging them are equivalent to each other.

For $S_q(G)$, its adjacency matrix \hat{A} , diagonal degree matrix \hat{D} , and normalized adjacency matrix \hat{P} , can be expressed in terms of related matrices of G as

$$\hat{A} = \begin{pmatrix} O & B & \dots & B \\ B^\top & O & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ B^\top & O & \dots & O \end{pmatrix},$$

$$\hat{D} = \text{diag}\{qD, \underbrace{2I_m, \dots, 2I_m}_q\},$$

$$\hat{P} = \hat{D}^{-\frac{1}{2}} \hat{A} \hat{D}^{-\frac{1}{2}} = \frac{1}{\sqrt{2q}} \begin{pmatrix} O & D^{-\frac{1}{2}}B & \dots & D^{-\frac{1}{2}}B \\ B^\top D^{-\frac{1}{2}} & O & \dots & O \\ \vdots & \vdots & \ddots & O \\ B^\top D^{-\frac{1}{2}} & O & \dots & O \end{pmatrix}, \quad (5)$$

where I_m is the $m \times m$ identity matrix.

4. Eigenvalues and Eigenvectors of Normalized Adjacency Matrix for q -subdivision Graphs

In this section, we study the eigenvalues and eigenvectors of normalized adjacency matrix \hat{P} for q -subdivision graphs $S_q(G)$. We will show that both eigenvalues and eigenvectors for \hat{P} can be expressed in terms of those related quantities associated with graph G .

Lemma 4.1 *Let $\hat{\lambda}$ be any non-zero eigenvalue of \hat{P} . Then, after appropriate labeling of nodes, the eigenvector ψ corresponding to $\hat{\lambda}$ can be rewritten in the form $\psi^\top = (\psi_1, \psi_2, \dots, \psi_{n+qm})^\top = (\psi'^\top, \psi^{(1)\top}, \psi^{(2)\top}, \dots, \psi^{(q)\top})$, where ψ' is an n dimensional vector, $\psi^{(i)}$ ($i = 1, 2, \dots, q$) is an m dimensional vector satisfying $\psi^{(1)} = \psi^{(2)} = \dots = \psi^{(q)}$.*

Proof. By definition of eigenvalues and eigenvectors, we have $\hat{\lambda}\psi = \hat{P}\psi$. Considering Eq. (5), we obtain $\hat{\lambda}\psi^{(i)} = \frac{1}{\sqrt{2q}}B^\top D^{-\frac{1}{2}}\psi'$, $i = 1, 2, \dots, q$. When $\hat{\lambda} \neq 0$, $\psi^{(i)} = \frac{1}{\lambda\sqrt{2q}}B^\top D^{-\frac{1}{2}}\psi'$ holds for all $i = 1, 2, \dots, q$. \square

Now we are ready to evaluate the full eigenvalues and their multiplicities of \hat{P} .

Lemma 4.2 *Let $\hat{\lambda}$ be a non-zero eigenvalue of \hat{P} . Then, $2\hat{\lambda}^2 - 1$ is an eigenvalue of P and its multiplicity, denoted by $m_P(2\hat{\lambda}^2 - 1)$ is identical to the multiplicity $m_{\hat{P}}(\hat{\lambda})$ for eigenvalue $\hat{\lambda}$ of \hat{P} .*

Proof. Let $\hat{\lambda}$ be an eigenvalue of \hat{P} , and let $\psi^\top = (\psi_1, \psi_2, \dots, \psi_{n+mq})^\top$ be its corresponding eigenvector. In addition, let ψ' be an n -dimensional vector obtained from ψ by restricting its components to the node set V . Then ψ' is an eigenvector of P for G , as we will show below. By definition,

$$\hat{\lambda}\psi = \hat{P}\psi. \quad (6)$$

Our goal is to express $\hat{\lambda}$ in terms of eigenvalues of P . For this purpose, we consider an old node x in \hat{P} . Let d_x and \hat{d}_x denote the degree of node x in graphs G and $S_q(G)$, respectively. Then, by construction, we have the following relation $\hat{d}_x = q d_x$. From Eq. (4), the neighbors of x can be divided into q classes, which are, respectively, in $V^{(1)}, V^{(2)}, \dots, V^{(q)}$. Moreover, the properties of nodes in these q classes are identical. We can appropriately label the nodes in V such that $\{1, 2, \dots, d_x\}$ is the set of neighbors for x in G . Then in $S_q(G)$ we assume that nodes with labelling $n + km + i$ ($0 \leq k \leq q - 1$ and $1 \leq i \leq d_x$) are neighbors of x , which belong to $V^{(k+1)}$.

Note that in $S_q(G)$ each new neighbor of x is simultaneously connected to an old neighbor of x . For the sake of convenience, we assume that the neighbors of the newly-added node $n + km + i$ ($0 \leq k \leq q - 1$ and $1 \leq i \leq d_x$) are x and i .

From Eq. (6), we obtain the equation corresponding to node x , which reads

$$\begin{aligned}\hat{\lambda}\psi_x &= \sum_{j=1}^{\hat{d}_x} \hat{P}(x, j)\psi_j = \sum_{k=0}^{q-1} \sum_{i=1}^{d_x} \hat{P}(x, n + km + i)\psi_{n+km+i} \\ &= q \sum_{i=1}^{d_x} \hat{P}(x, n + i)\psi_{n+i},\end{aligned}\tag{7}$$

where the next-to-last equality is obtained according to Lemma 4.1. Analogously, for a newly-added node s with neighboring set $\hat{\Gamma}(s) = \{u, v\}$, we obtain

$$\hat{\lambda}\psi_s = \hat{P}(s, u)\psi_u + \hat{P}(s, v)\psi_v,\tag{8}$$

which implies

$$\psi_s = \frac{\hat{P}(s, u)}{\hat{\lambda}}\psi_u + \frac{\hat{P}(s, v)}{\hat{\lambda}}\psi_v.\tag{9}$$

Thus, ψ_{n+i} in Eq. (6) can be written as

$$\psi_{n+i} = \frac{\hat{P}(n + i, x)}{\hat{\lambda}}\psi_x + \frac{\hat{P}(n + i, i)}{\hat{\lambda}}\psi_i.\tag{10}$$

Inserting Eq. (10) into Eq. (7) leads to

$$\hat{\lambda}\psi_x = q \sum_{i=1}^{d_x} \frac{\left(\hat{P}(x, n + i)\right)^2}{\hat{\lambda}}\psi_x + q \sum_{i=1}^{d_x} \frac{\hat{P}(n + i, x)\hat{P}(n + i, i)}{\hat{\lambda}}\psi_i.\tag{11}$$

By definition of \hat{P} , for each neighbour $n + i$ of node x , one has

$$\hat{P}(x, n + i) = \frac{1}{\sqrt{2\hat{d}_x}} = \frac{1}{\sqrt{2q\hat{d}_x}}.\tag{12}$$

In addition, for an old neighbor i of node x , the following relation holds:

$$\begin{aligned}\hat{P}(n + i, x)\hat{P}(n + i, i) &= \frac{1}{\sqrt{2\hat{d}_i}} \frac{1}{\sqrt{2\hat{d}_x}} \\ &= \frac{1}{2} \frac{1}{\sqrt{\hat{d}_i\hat{d}_x}} = \frac{1}{2} \hat{P}(x, i).\end{aligned}\tag{13}$$

Substituting Eqs. (12) and (13) back into Eq. (11), we arrive at

$$\left(\hat{\lambda}^2 - \frac{1}{2}\right)\psi_x = \frac{q}{2} \sum_{i=1}^{d_x} \hat{P}(x, i)\psi_i, \quad (14)$$

which only involves old nodes in V . Thus, according to the following relation

$$\hat{P}(x, i) = \frac{1}{\sqrt{\hat{d}_x \hat{d}_i}} = \frac{1}{q\sqrt{d_x d_i}} = \frac{P(x, i)}{q}, \quad (15)$$

we have

$$(2\hat{\lambda}^2 - 1)\psi_x = \sum_{i=1}^{d_x} P(x, i)\psi_i, \quad \forall x \in V, \quad (16)$$

implying that $2\hat{\lambda}^2 - 1$ is an eigenvalue of P , and ψ' , the n -dimensional restricted vector defined above, is one associated eigenvector. Furthermore, ψ can be totally determined by ψ' using Eq. (9). Thus $m_P(2\hat{\lambda}^2 - 1) \geq m_{\hat{P}}(\hat{\lambda})$.

Suppose that $m_P(2\hat{\lambda}^2 - 1) > m_{\hat{P}}(\hat{\lambda})$. This means that there should exist an extra eigenvector ψ_e associated to $2\hat{\lambda}^2 - 1$ without a corresponding eigenvector in \hat{P} . But Eq. (9) provides ψ_e with an associated eigenvector of \hat{P} since $\lambda \neq 0$. This contradicts our assumption. Therefore, $m_P(2\hat{\lambda}^2 - 1) = m_{\hat{P}}(\hat{\lambda})$. \square

Lemma 4.3 *Let λ be any eigenvalue of P such that $\lambda \neq -1$. Then $\sqrt{\frac{1+\lambda}{2}}$ and $-\sqrt{\frac{1+\lambda}{2}}$ are eigenvalues of \hat{P} and $m_{\hat{P}}\left(\sqrt{\frac{1+\lambda}{2}}\right) = m_{\hat{P}}\left(-\sqrt{\frac{1+\lambda}{2}}\right) = m_P(\lambda)$.*

Proof. This is a direct consequence of Lemma 4.2. \square

Lemmas 4.2 and 4.3 show that all nonzero eigenvalues and their corresponding eigenvectors of \hat{P} can be obtained from those of P . For those zero eigenvalues and their associated eigenvectors of \hat{P} , we can characterize them easily.

Theorem 4.4 *Let G be a simple connected graph with n nodes and m edges. Let $1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \geq -1$ be the eigenvalues of P , and let v_1, v_2, \dots, v_n be their corresponding orthonormal eigenvectors. Then*

- (i) *if G is non-bipartite, then $\pm\sqrt{\frac{1+\lambda_i}{2}}$, $i = 1, 2, \dots, n$, are eigenvalues of \hat{P} , and the element of their orthonormal eigenvectors corresponding to*

node j is

$$\begin{cases} \frac{1}{\sqrt{2}}v_{ij}, & j \in V, \\ \pm\sqrt{\frac{1}{2q(1+\lambda_i)}}\left(\frac{v_{is}}{\sqrt{d_s}} + \frac{v_{it}}{\sqrt{d_t}}\right), & j \in V', \hat{\Gamma}(j) = \{s, t\}; \end{cases}$$

and 0's are eigenvalues of \hat{P} with multiplicity $mq - n$, with their corresponding orthonormal eigenvectors being

$$\begin{pmatrix} 0 \\ Y_z \end{pmatrix}, z = 1, 2, \dots, mq - n,$$

where $Y_1, Y_2, \dots, Y_{mq-n}$ are an orthonormal basis of the kernel space of matrix

$$C := \underbrace{\begin{pmatrix} B & B & \cdots & B \end{pmatrix}}_q.$$

- (ii) if G is non-bipartite, then $\pm\sqrt{\frac{1+\lambda_i}{2}}$, $i = 1, 2, \dots, n-1$, are eigenvalues of \hat{P} , and the element of their orthonormal eigenvectors corresponding to node j is

$$\begin{cases} \frac{1}{\sqrt{2}}v_{ij}, & j \in V, \\ \pm\sqrt{\frac{1}{2q(1+\lambda_i)}}\left(\frac{v_{is}}{\sqrt{d_s}} + \frac{v_{it}}{\sqrt{d_t}}\right), & j \in V', \hat{\Gamma}(j) = \{s, t\}; \end{cases}$$

and 0's are eigenvalues of \hat{P} with multiplicity $mq - n + 2$, with their corresponding orthonormal eigenvectors being

$$\begin{pmatrix} v_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Y_z \end{pmatrix}, z = 1, 2, \dots, mq - n + 1;$$

where $Y_1, Y_2, \dots, Y_{mq-n+1}$ are an orthonormal basis of the kernel space of matrix

$$C := \underbrace{\begin{pmatrix} B & B & \cdots & B \end{pmatrix}}_q.$$

Proof. We first prove (i). Since G is non-bipartite, by Lemma 2.2, every eigenvalue λ_i of P is not equal to -1 . According to Lemma 4.3, we can obtain all nonzero eigenvalues $\pm\sqrt{\frac{1+\lambda_i}{2}}$ of \hat{P} and their multiplicity. Moreover, using Eq. (9), the element of their associated eigenvectors corresponding to node j is

$$\begin{cases} v_{ij}, & j \in V, \\ \pm\sqrt{\frac{1}{q(1+\lambda_i)}}\left(\frac{v_{is}}{\sqrt{d_s}} + \frac{v_{it}}{\sqrt{d_t}}\right), & j \in V', \hat{\Gamma}(j) = \{s, t\}, \end{cases}$$

which can be orthonormalized to obtain the orthonormal eigenvectors.

For the zero eigenvalues, from Lemma 2.1, $\text{rank}(B) = n$ since G is non-bipartite. Thus, $\text{rank}(C) = n$ and $\dim(\text{Ker}(C)) = mq - n$. Let $Y_1, Y_2, \dots, Y_{mq-n}$ be an orthonormal basis of the kernel space of matrix C . It is easy to confirm that $\begin{pmatrix} 0 \\ Y_z \end{pmatrix}$, $z = 1, 2, \dots, mq - n$, are eigenvectors for eigenvalues 0 of matrix \hat{P} .

For (ii), our proof is similar. We just need to verify that

$$\hat{P} \begin{pmatrix} v_n \\ 0 \end{pmatrix} = O_{(n+mq) \times 1},$$

which is trivial according to Eq. (5). \square

Note that when $q = 1$, Theorem 4.4 coincides with result in [49].

5. Hitting Times for Random Walks on q -subdivision Graphs

Theorem 4.4 provides complete information about the eigenvalues and eigenvectors of \hat{P} in terms of those P . In this section, we use this information to determine two-node hitting time and Kemeny constant for unbiased random walks on $S_q(G)$.

5.1. Two-Node Hitting Time

We first compute the hitting time from one node to another in $S_q(G)$. To this end, we express the orthonormal eigenvectors of $S_q(G)$ in more explicit forms. By Eqs. (1) (3) and Theorem 4.4, we can directly derive the following results.

- (i) The eigenvectors corresponding to eigenvalues $\pm\sqrt{\frac{1+\lambda_1}{2}} = \pm 1$ for matrix \hat{P} are

$$\left(\sqrt{\frac{d_1}{4m}}, \dots, \sqrt{\frac{d_n}{4m}}, \sqrt{\frac{1}{2mq}}, \dots, \sqrt{\frac{1}{2mq}} \right)^\top \quad (17)$$

and

$$\left(\sqrt{\frac{d_1}{4m}}, \dots, \sqrt{\frac{d_n}{4m}}, -\sqrt{\frac{1}{2mq}}, \dots, -\sqrt{\frac{1}{2mq}} \right)^\top, \quad (18)$$

respectively.

(ii) If G is non-bipartite, for each $j \in V'$ with $\hat{\Gamma}(j) = \{s, t\}$,

$$\sum_{z=1}^{mq-n} Y_{zj}^2 = 1 - \frac{1}{mq} - \sum_{k=2}^n \frac{1}{(1 + \lambda_k)q} \left(\frac{v_{ks}}{\sqrt{d_s}} + \frac{v_{kt}}{\sqrt{d_t}} \right)^2. \quad (19)$$

(iii) If G is bipartite, for each $j \in V'$ with $\hat{\Gamma}(j) = \{s, t\}$,

$$\sum_{z=1}^{mq-n+1} Y_{zj}^2 = 1 - \frac{1}{mq} - \sum_{k=2}^{n-1} \frac{1}{(1 + \lambda_k)q} \left(\frac{v_{ks}}{\sqrt{d_s}} + \frac{v_{kt}}{\sqrt{d_t}} \right)^2. \quad (20)$$

Now we present our results for hitting times of random walks on $S_q(G)$.

Theorem 5.1 *Let G be a simple connected graph with n nodes and m edges. $S_q(G)$ is the q -subdivision graph of G with $\hat{V} = V \cup V'$. Then*

- (i) if $i, j \in V$, then $\hat{T}_{ij} = 4T_{ij}$;
- (ii) if $i \in V', j \in V$, $\hat{\Gamma}(i) = \{s, t\}$, then

$$\begin{aligned} \hat{T}_{ij} &= 1 + 2(T_{sj} + T_{tj}); \\ \hat{T}_{ji} &= 2mq - 1 + 2(T_{js} + T_{jt}) - (T_{ts} + T_{st}); \end{aligned}$$

- (iii) if $i, j \in V'$, $\hat{\Gamma}(i) = \{s, t\}$, $\hat{\Gamma}(j) = \{u, v\}$, then

$$\hat{T}_{ij} = 2mq + T_{su} + T_{tu} + T_{sv} + T_{tv} - (T_{uv} + T_{vu}).$$

Proof. Note that $\hat{m} = 2qm$, $\hat{d}_i = qd_i$ if $i \in V$, and $\hat{d}_i = 2$ if $i \in V'$.

We first prove (i). We distinguish two cases: (a) G is a non-bipartite graph, and (b) G is a bipartite graph. When G is a non-bipartite graph, by Theorems 2.3 and 4.4, we have

$$\begin{aligned} \hat{T}_{ij} &= 2\hat{m} \left(\sum_{k=2}^n \left(\frac{1}{1 - \sqrt{\frac{1+\lambda_k}{2}}} + \frac{1}{1 + \sqrt{\frac{1+\lambda_k}{2}}} \right) \left(\frac{v_{kj}^2}{2qd_j} - \frac{v_{ki}v_{kj}}{2q\sqrt{d_i d_j}} \right) \right) \\ &= 8m \sum_{k=2}^n \frac{1}{1 - \lambda_k} \left(\frac{v_{kj}^2}{d_j} - \frac{v_{ki}v_{kj}}{\sqrt{d_i d_j}} \right) = 4T_{ij}. \end{aligned}$$

When G is a bipartite graph, the proof is similar. Thus (i) is proved.

We continue to prove (ii). Since $\hat{\Gamma}(i) = \{s, t\}$,

$$\hat{T}_{ij} = 1 + \frac{1}{2} \left(\hat{T}_{sj} + \hat{T}_{tj} \right) = 1 + 2(T_{sj} + T_{tj}).$$

While for \hat{T}_{ji} , we also divide it into two cases: (a) G is a non-bipartite graph, and (b) G is a bipartite graph. For the first case that G is non-bipartite, by Theorem 2.3 and Eqs. (17) and (19), we have

$$\begin{aligned}
\hat{T}_{ji} &= 1 + 2\hat{m} \left(\sum_{k=2}^n \left(\frac{1}{1 - \sqrt{\frac{1+\lambda_k}{2}}} + \frac{1}{1 + \sqrt{\frac{1+\lambda_k}{2}}} \right) \frac{1}{4q(1+\lambda_k)} \left(\frac{v_{ks}}{\sqrt{d_s}} + \frac{v_{kt}}{\sqrt{d_t}} \right)^2 \right. \\
&\quad \left. - \sum_{k=2}^n \left(\frac{1}{1 - \sqrt{\frac{1+\lambda_k}{2}}} - \frac{1}{1 + \sqrt{\frac{1+\lambda_k}{2}}} \right) \right. \\
&\quad \left. \times \frac{v_{kj}}{2q\sqrt{2(1+\lambda_k)d_j}} \left(\frac{v_{ks}}{\sqrt{d_s}} + \frac{v_{kt}}{\sqrt{d_t}} \right) + \sum_{z=1}^{mq-n} \frac{Y_{zi}^2}{2} \right) \\
&= 1 + 4mq \left(\sum_{k=2}^n \frac{1}{q(1-\lambda_k)(1+\lambda_k)} \left(\frac{v_{ks}}{\sqrt{d_s}} + \frac{v_{kt}}{\sqrt{d_t}} \right)^2 - \sum_{k=2}^n \frac{1}{q(1-\lambda_k)} \right. \\
&\quad \left. \left(\frac{v_{ks}v_{kj}}{\sqrt{d_s}d_j} + \frac{v_{kt}v_{kj}}{\sqrt{d_t}d_j} \right) + \frac{1}{2} \left(1 - \frac{1}{mq} - \sum_{k=2}^n \frac{1}{q(1+\lambda_k)} \left(\frac{v_{ks}}{\sqrt{d_s}} + \frac{v_{kt}}{\sqrt{d_t}} \right)^2 \right) \right) \\
&= 2mq - 1 + 4m \sum_{k=2}^n \frac{1}{1-\lambda_k} \left(\left(\frac{v_{ks}^2}{d_s} - \frac{v_{ks}v_{kj}}{\sqrt{d_s}d_j} \right) + \left(\frac{v_{kt}^2}{d_t} - \frac{v_{kt}v_{kj}}{\sqrt{d_t}d_j} \right) \right. \\
&\quad \left. - \frac{1}{2} \left(\frac{v_{ks}}{\sqrt{d_s}} - \frac{v_{kt}}{\sqrt{d_t}} \right)^2 \right) = 2mq - 1 + 2(T_{js} + T_{jt}) - (T_{ts} + T_{st}).
\end{aligned}$$

If G is bipartite, our proof is similar.

We finally prove (iii). Considering $\hat{\Gamma}(i) = \{s, t\}$ and $\hat{\Gamma}(j) = \{u, v\}$, we obtain

$$\begin{aligned}
\hat{T}_{ij} &= 1 + \frac{1}{2} (\hat{T}_{sj} + \hat{T}_{tj}) \\
&= 1 + \frac{1}{2} (2mq - 1 + 2(T_{su} + T_{sv}) - (T_{uv} + T_{vu}) \\
&\quad + 2mq - 1 + 2(T_{tu} + T_{tv}) - (T_{uv} + T_{vu})) \\
&= 2mq + T_{su} + T_{tu} + T_{sv} + T_{tv} - (T_{uv} + T_{vu}).
\end{aligned}$$

This completes the proof. \square

5.2. Kemeny's Constant

In addition to the two-node hitting time, the Kemeny's constant of $S_q(G)$ can also be expressed in terms of that of G .

Theorem 5.2 *Let G be a simple connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph. Then*

$$K(S_q(G)) = 4K(G) + \frac{2mq - 2n + 1}{2}.$$

Proof. Suppose that $1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n \geq -1$ are the eigenvalues of the matrix P . We first consider the case that G is a non-bipartite graph. For this case, by Lemma 2.4 and Theorem 4.4, we have

$$\begin{aligned} K(S_q(G)) &= \sum_{k=2}^n \left(\frac{1}{1 - \sqrt{\frac{1+\lambda_k}{2}}} + \frac{1}{1 + \sqrt{\frac{1+\lambda_k}{2}}} \right) + \frac{1}{2} + mq - n \\ &= 4K(G) + \frac{2mq - 2n + 1}{2}. \end{aligned}$$

For the other case that G is bipartite, we can prove similarly. \square

6. Resistance Distances of q -subdivision Graphs

In this section, we determine the two-node resistance distance, multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index, and Kirchhoff index of $S_q(G)$, in terms of those of G .

6.1. Two-Node Resistance Distance

We first determine the resistance distance between any pair of nodes in $S_q(G)$.

Theorem 6.1 *Let G be a simple connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph of G with node set $\hat{V} = V \cup V'$. Then*

(1) *for $i, j \in V$,*

$$\hat{r}_{ij} = \frac{2}{q}r_{ij};$$

(2) *for $i \in V', j \in V$ and $\hat{\Gamma}(i) = \{s, t\}$,*

$$\hat{r}_{ij} = \frac{1}{2} + \frac{2r_{sj} + 2r_{tj} - r_{st}}{2q};$$

(3) *for $i, j \in V', \hat{\Gamma}(i) = \{s, t\}$ and $\hat{\Gamma}(j) = \{u, v\}$,*

$$\hat{r}_{ij} = 1 + \frac{r_{su} + r_{tu} + r_{sv} + r_{tv} - r_{st} - r_{uv}}{2q}.$$

Proof. The results follow directly from Lemma 2.7 and Theorem 5.1. \square

6.2. Some Intermediary Results

In the next subsections, we will derive the Kirchhoff index, the additive degree-Kirchhoff index and the multiplicative degree-Kirchhoff index for $S_q(G)$. In the computation of the first two graph invariants, we need the following two properties for resistance distance in $S_q(G)$.

Lemma 6.2 *Let G be a simple connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph of G with node set $\hat{V} = V \cup V'$. Then*

$$\sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij} = \bar{\mathcal{K}}(G) + \frac{mnq - n^2 + n}{2}.$$

Proof. Note that, $\sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij}$ can be divided into two sum terms as

$$\sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij} = \sum_{i \in V'} \sum_{j \in \hat{\Gamma}(i)} \hat{r}_{ij} + \sum_{i \in V'} \sum_{j \in V \setminus \hat{\Gamma}(i)} \hat{r}_{ij}. \quad (21)$$

We next compute the above two sum terms separately.

(i) As for the first term, by Lemma 2.6, we have

$$\sum_{i \in V'} \sum_{j \in \hat{\Gamma}(i)} \hat{r}_{ij} = \sum_{ij \in \hat{E}} \hat{r}_{ij} = |\hat{V}| - 1 = mq + n - 1. \quad (22)$$

(ii) As for the second term, suppose that $\hat{\Gamma}(i) = \{s, t\}$. According to Eq. (4), Lemma 2.6 and Theorem 6.1, we have

$$\begin{aligned} \sum_{i \in V'} \sum_{j \in V \setminus \hat{\Gamma}(i)} \hat{r}_{ij} &= \sum_{f=1}^q \sum_{i \in V^{(f)}} \sum_{j \in V \setminus \hat{\Gamma}(i)} \left(\frac{1}{2} + \frac{2r_{sj} + 2r_{tj} - r_{st}}{2q} \right) \\ &= q \sum_{i \in V^{(1)}} \sum_{j \in V \setminus \hat{\Gamma}(i)} \left(\frac{1}{2} + \frac{2r_{sj} + 2r_{tj} - r_{st}}{2q} \right) \\ &= \sum_{i \in V^{(1)}} \left(\frac{(n-2)q}{2} + \sum_{j \in V \setminus \hat{\Gamma}(i)} (r_{sj} + r_{tj}) - \frac{n-2}{2} r_{st} \right). \end{aligned} \quad (23)$$

For convenience, let r_s be the sum of resistance distances between s and all other nodes in graph G , that is,

$$r_s = \sum_{\substack{j \in V \\ j \neq s}} r_{sj}.$$

Thus, Eq. (23) can be rewritten as

$$\begin{aligned} \sum_{i \in V'} \sum_{j \in V \setminus \hat{\Gamma}(i)} \hat{r}_{ij} &= \sum_{i \in V^{(1)}} \left(\frac{(n-2)q}{2} + r_s + r_t - \frac{n+2}{2} r_{st} \right) \\ &= \frac{m(n-2)q}{2} + \sum_{i \in V^{(1)}} (r_s + r_t) - \frac{n+2}{2} \sum_{i \in V^{(1)}} r_{st}. \end{aligned} \quad (24)$$

The term $\sum_{i \in V^{(1)}} (r_s + r_t)$ can be computed as

$$\begin{aligned} \sum_{i \in V^{(1)}} (r_s + r_t) &= \sum_{st \in E} (r_s + r_t) = \sum_{s \in V} d_s r_s \\ &= \sum_{\{i,j\} \subseteq V} (d_i + d_j) r_{ij} = \bar{\mathcal{K}}(G). \end{aligned} \quad (25)$$

By Lemma 2.6, the term $\frac{n+2}{2} \sum_{i \in V^{(1)}} r_{st}$ can be evaluated as

$$\frac{n+2}{2} \sum_{i \in V^{(1)}} r_{st} = \frac{n+2}{2} \sum_{st \in E} r_{st} = \frac{(n+2)(n-1)}{2}. \quad (26)$$

Plugging Eqs. (25) and (26) into Eq. (24) gives

$$\sum_{i \in V'} \sum_{j \in V \setminus \hat{\Gamma}(i)} \hat{r}_{ij} = \frac{m(n-2)q}{2} + \bar{\mathcal{K}}(G) - \frac{(n+2)(n-1)}{2}. \quad (27)$$

Combining Eqs. (22) and (27) gives the desired result. \square

Lemma 6.3 *Let G be a connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph of G with node set $\hat{V} = V \cup V'$. Then*

$$\sum_{\{i,j\} \subseteq V'} \hat{r}_{ij} = \frac{q}{2} \tilde{\mathcal{K}}(G) + \frac{mq(mq-1)}{2} - \frac{m(n-1)q}{2}.$$

Proof. Suppose that $\hat{\Gamma}(i) = \{s, t\}$ and $\hat{\Gamma}(j) = \{u, v\}$. Then by Theorem 6.1,

$$\begin{aligned} \sum_{\{i,j\} \subseteq V'} \hat{r}_{ij} &= \sum_{\{i,j\} \subseteq V'} \left(1 + \frac{r_{su} + r_{tu} + r_{sv} + r_{tv}}{2q} - \frac{r_{st} + r_{uv}}{2q} \right) \\ &= \frac{mq(mq-1)}{2} + \sum_{\{i,j\} \subseteq V'} \frac{r_{su} + r_{tu} + r_{sv} + r_{tv}}{2q} - \sum_{\{i,j\} \subseteq V'} \frac{r_{st} + r_{uv}}{2q}. \end{aligned} \quad (28)$$

We now compute the second term in Eq. (28). It is not easy to evaluate it directly. We will compute it in an alternative way. For any pair of nodes $\{k, l\} \subseteq V$, we consider how many times r_{kl} appears in the summation. Observe that r_{kl} is summed once if and only if there exists a unique subset $\{i, j\} \subseteq V'$ such that $k \in \hat{\Gamma}(i)$ and $l \in \hat{\Gamma}(j)$. Thus, our problem could be simplified and converted to the following one: how many pairwise different aforementioned subsets exist? It is not difficult to see that if k is not adjacent to l in G there exist $q^2 d_k d_l$ subsets, and that if $kl \in E(G)$, there exist $q^2 d_k d_l - q$ such subsets. Then, once again by Lemma 2.6, we have

$$\begin{aligned}
& \sum_{\{i,j\} \subseteq V'} \frac{r_{su} + r_{tu} + r_{sv} + r_{tv}}{2q} \\
&= \sum_{\substack{\{k,l\} \subseteq V \\ kl \notin E}} \frac{q^2 d_k d_l}{2q} r_{kl} + \sum_{\substack{\{k,l\} \subseteq V \\ kl \in E}} \frac{q^2 d_k d_l - q}{2q} r_{kl} \\
&= \sum_{\{k,l\} \subseteq V} \frac{q d_k d_l}{2} r_{kl} - \frac{1}{2} \sum_{\substack{\{k,l\} \subseteq V \\ kl \in E}} r_{kl} = \frac{q}{2} \sum_{\{k,l\} \subseteq V} d_k d_l r_{kl} - \frac{n-1}{2} \\
&= \frac{q}{2} \tilde{\mathcal{K}}(G) - \frac{n-1}{2}.
\end{aligned} \tag{29}$$

We proceed to evaluate the third term in Eq. (28). Note that for any two different nodes i and j in V' , if their neighbours are the same, i.e., $\hat{\Gamma}(i) = \hat{\Gamma}(j) = \{s, t\}$, we use $i \sim j$ to denote this relation. Otherwise, the sets of their neighbours are different, we call $i \not\sim j$. According to these two relations and Eq. (4), it follows that

$$\begin{aligned}
& \sum_{\{i,j\} \subseteq V'} \frac{r_{st} + r_{uv}}{2q} = \frac{1}{4q} \sum_{i \in V'} \sum_{j \in V'} (r_{st} + r_{uv}) \\
&= \frac{1}{4q} \sum_{f=1}^q \sum_{i \in V(f)} \left(\sum_{i \not\sim j} (r_{st} + r_{uv}) + \sum_{i \sim j} (r_{st} + r_{st}) \right) \\
&= \frac{1}{4q} q \sum_{st \in E} \left(q \sum_{\substack{uv \in E \\ uv \neq st}} (r_{st} + r_{uv}) + 2(q-1)r_{st} \right) \\
&= \frac{1}{4} \sum_{st \in E} \left(q \sum_{uv \in E} r_{uv} + (mq-2)r_{st} \right).
\end{aligned} \tag{30}$$

By Lemma 2.6, Eq. (30) can be recast as

$$\begin{aligned} \sum_{\{i,j\} \subseteq V'} \frac{r_{st} + r_{uv}}{2q} &= \frac{1}{4} \sum_{st \in E} \left((n-1)q + (mq-2)r_{st} \right) \\ &= \frac{m(n-1)q}{4} + \frac{mq-2}{4}(n-1). \end{aligned} \quad (31)$$

Plugging Eqs. (29) and (31) into Eq. (28) gives the result. \square

6.3. Multiplicative Degree-Kirchhoff Index

We first determine the multiplicative degree-Kirchhoff index for $S_q(G)$.

Theorem 6.4 *Let G be a connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph of G . Then*

$$\tilde{\mathcal{K}}(S_q(G)) = 8q\tilde{\mathcal{K}}(G) + 2mq(2mq - 2n + 1).$$

Proof. According to Lemma 2.4 and 2.9, Theorem 6.4 is an obvious consequence of Theorem 5.2. \square

6.4. Additive Degree-Kirchhoff Index

We continue to determine the additive degree-Kirchhoff index for $S_q(G)$.

Theorem 6.5 *Let G be a simple connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph of G . Then*

$$\bar{\mathcal{K}}(S_q(G)) = 4\bar{\mathcal{K}}(G) + 4q\tilde{\mathcal{K}}(G) + mq(3mq - 2n + 1) - n(n-1).$$

Proof. By definition of the additive degree-Kirchhoff index, we have

$$\begin{aligned} \bar{\mathcal{K}}(S_q(G)) &= \sum_{\{i,j\} \subseteq V \cup V'} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} \\ &= \sum_{\{i,j\} \subseteq V} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} + \sum_{i \in V'} \sum_{j \in V} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} + \sum_{\{i,j\} \subseteq V'} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij}. \end{aligned} \quad (32)$$

We now compute the three sum terms on the last row of Eq. (32) one by one.

For the first sum term, by Theorem 6.1, we have

$$\sum_{\{i,j\} \subseteq V} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} = \sum_{\{i,j\} \subseteq V} (qd_i + qd_j) \frac{2}{q} r_{ij} = 2\bar{\mathcal{K}}(G). \quad (33)$$

For the second sum term, it can be evaluated as

$$\begin{aligned} \sum_{i \in V'} \sum_{j \in V} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} &= \sum_{i \in V'} \sum_{j \in V} (2 + qd_j) \hat{r}_{ij} \\ &= 2 \sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij} + q \sum_{i \in V'} \sum_{j \in V} d_j \hat{r}_{ij}. \end{aligned} \quad (34)$$

By Lemma 6.2, we have

$$2 \sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij} = 2\bar{\mathcal{K}}(G) + mnq - n^2 + n. \quad (35)$$

On the other hand, by Lemma 2.6 and Theorem 6.1,

$$\begin{aligned} q \sum_{i \in V'} \sum_{j \in V} d_j \hat{r}_{ij} &= \sum_{i \in V'} \sum_{j \in V} qd_j \left(\frac{1}{2} + \frac{2r_{sj} + 2r_{tj} - r_{st}}{2q} \right) \\ &= \frac{q}{2} \sum_{i \in V'} \sum_{j \in V} d_j + \sum_{i \in V'} \sum_{j \in V} d_j (r_{sj} + r_{tj}) - \frac{1}{2} \sum_{i \in V'} \sum_{j \in V} d_j r_{st} \\ &= \frac{q}{2} \sum_{i \in V'} 2m + \sum_{i \in V'} \sum_{j \in V} d_j (r_{sj} + r_{tj}) - \frac{1}{2} \sum_{i \in V'} 2mr_{st} \\ &= m^2 q^2 + \sum_{i \in V'} \sum_{j \in V} d_j (r_{sj} + r_{tj}) - mq(n-1). \end{aligned} \quad (36)$$

For the middle term in Eq. (36), we have

$$\begin{aligned} \sum_{i \in V'} \sum_{j \in V} d_j (r_{sj} + r_{tj}) &= q \sum_{i \in V^{(1)}} \sum_{j \in V} d_j (r_{sj} + r_{tj}) = q \sum_{j \in V} \sum_{i \in V^{(1)}} d_j (r_{sj} + r_{tj}) \\ &= q \sum_{j \in V} \sum_{k \in V} d_j d_k r_{kj} = 2q\tilde{\mathcal{K}}(G). \end{aligned} \quad (37)$$

Combining Eqs. (34)-(37) yields

$$\sum_{i \in V'} \sum_{j \in V} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} = 2\bar{\mathcal{K}}(G) + 2q\tilde{\mathcal{K}}(G) + m^2 q^2 + mq - n^2 + n. \quad (38)$$

For the third sum term in Eq. (32), by Lemma 6.3, we have

$$\sum_{\{i,j\} \subseteq V'} (\hat{d}_i + \hat{d}_j) \hat{r}_{ij} = 4 \sum_{\{i,j\} \subseteq V'} \hat{r}_{ij} = 2q\tilde{\mathcal{K}}(G) + 2m^2 q^2 - 2mnq. \quad (39)$$

Substituting Eqs. (33) (38) and (39) back into Eq. (32), our proof is completed after simple calculations. \square

6.5. Kirchhoff Index

We finally determine the Kirchhoff index for $S_q(G)$.

Theorem 6.6 *Let G be a simple connected graph with n nodes and m edges, and let $S_q(G)$ be the q -subdivision graph of G . Then*

$$\mathcal{K}(S_q(G)) = \frac{2}{q}\mathcal{K}(G) + \bar{\mathcal{K}}(G) + \frac{q}{2}\tilde{\mathcal{K}}(G) + \frac{m^2q^2 - n(n-1)}{2}.$$

Proof. According to Definition 2.8 and Eq. (4), we have

$$\begin{aligned} \mathcal{K}(S_q(G)) &= \sum_{\{i,j\} \subseteq \hat{V}} \hat{r}_{ij} = \sum_{\{i,j\} \subseteq V \cup V'} \hat{r}_{ij} \\ &= \sum_{\{i,j\} \subseteq V} \hat{r}_{ij} + \sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij} + \sum_{\{i,j\} \subseteq V'} \hat{r}_{ij}. \end{aligned} \quad (40)$$

We shall compute the three sum terms in Eq. (40) one by one.

For the first sum term, by Theorem 6.1,

$$\sum_{\{i,j\} \subseteq V} \hat{r}_{ij} = \sum_{\{i,j\} \subseteq V} \frac{2}{q}r_{ij} = \frac{2}{q}\mathcal{K}(G). \quad (41)$$

For the second sum term, by Lemma 6.2 we obtain

$$\sum_{i \in V'} \sum_{j \in V} \hat{r}_{ij} = \bar{\mathcal{K}}(G) + \frac{mnq - n^2 + n}{2}. \quad (42)$$

For the third sum term, by Lemma 6.3, we have

$$\sum_{\{i,j\} \subseteq V'} \hat{r}_{ij} = \frac{q}{2}\tilde{\mathcal{K}}(G) + \frac{mq(mq-1)}{2} - \frac{m(n-1)q}{2}. \quad (43)$$

Plugging Eqs. (41)-(43) into Eq. (40) leads to the desired result. \square

7. Properties of Iterated q -subdivision graphs and Their Applications

The q -subdivision graphs have found many applications in physics and network science. For example, by iteratively applying q -subdivision operation on an edge we can obtain the hierarchical lattices, which can be used to mimic complex networks with the striking scale-free fractal topologies [28]. In this section, we study the properties of iterated q -subdivision graphs, based on which we further obtain exact expressions for some interesting quantities for the hierarchical lattices.

7.1. Definition of Iterated q -subdivision Graphs

The family of iterated q -subdivision graphs $S_{q,k}(G)$ of a graph G is defined as follows. For $k = 0$, $S_{q,0}(G) = G$. For $k \geq 1$, $S_{q,k}(G)$ is obtained from $S_{q,k-1}(G)$ by performing the q -subdivision operation on $S_{q,k-1}(G)$. In other words, $S_{q,k}(G) = S_q(S_{q,k-1}(G))$. For a quantity Z of G , we use $Z_{q,k}$ to denote the corresponding quantity associated with $S_{q,k}(G)$. Then, in $S_{q,k}(G)$, the number of edges is

$$m_{q,k} = 2qm_{q,k-1} = (2q)^k m, \quad (44)$$

and the number of nodes is

$$n_{q,k} = n_{q,k-1} + qm_{q,k-1} = \frac{mq[(2q)^k - 1]}{2q - 1} + n. \quad (45)$$

7.2. Formulas of Quantities of Iterated q -subdivision Graphs

We here present expressions for some interesting quantities for iterated q -subdivision graphs $S_{q,k}(G)$.

7.2.1. Kemeny's Constant

Theorem 7.1 *Let G be a simple connected graph with n nodes and m edges. Then*

(i) if $q = 2$,

$$K_{2,k} = 4^k K_{2,0} + \frac{mk4^k}{3} + \left(\frac{4m+3}{6} - n\right) \frac{4^k - 1}{3};$$

(ii) if $q \neq 2$,

$$\begin{aligned} K_{q,k} = & 4^k K_{q,0} + \frac{mq(q-1)}{(q-2)(2q-1)} \left[(2q)^k - 4^k \right] \\ & + \left(\frac{2mq+2q-1}{2(2q-1)} - n \right) \frac{4^k - 1}{3}. \end{aligned}$$

Proof. According to Theorem 5.2 and Eqs. (44) and (45), we obtain

$$\begin{aligned} K_{q,k} = & 4K_{q,k-1} + \frac{1}{2} (2m_{q,k-1}q - 2n_{q,k-1} + 1) \\ = & 4K_{q,k-1} + mq(2q)^{k-1} - \frac{mq[(2q)^{k-1} - 1]}{2q-1} - n + \frac{1}{2}. \end{aligned}$$

Divided by 4^k on both sides, we obtain

$$\frac{K_{q,k}}{4^k} - \frac{K_{q,k-1}}{4^{k-1}} = \frac{m(q-1)}{2q-1} \left(\frac{q}{2}\right)^k + \left(\frac{2mq+2q-1}{2(2q-1)} - n\right) \frac{1}{4^k}.$$

If $q \neq 2$, we derive the following relation

$$\frac{K_{q,k}}{4^k} - \frac{K_{q,0}}{4^0} = \frac{mq(q-1)}{2(2q-1)} \frac{1 - (q/2)^k}{1 - q/2} + \left(\frac{2mq+2q-1}{2(2q-1)} - n\right) \frac{1/4[1 - (1/4)^k]}{1 - 1/4},$$

which leads to the result through simple calculations.

For the case $q = 2$, the proof is similar. \square

7.2.2. Multiplicative Degree-Kirchhoff Index

Theorem 7.2 *Let G be a simple connected graph with n nodes and m edges. Then*

(i) if $q = 2$,

$$\tilde{\mathcal{K}}_{2,k} = 16^k \tilde{\mathcal{K}}_{2,0} + \frac{2m^2 k 16^k}{3} + \left(\frac{4m^2 + 3m}{3} - 2mn\right) \frac{16^k - 4^k}{3};$$

(ii) if $q \neq 2$,

$$\begin{aligned} \tilde{\mathcal{K}}_{q,k} = & (8q)^k \tilde{\mathcal{K}}_{q,0} + \frac{2m^2 q (q-1)}{(q-2)(2q-1)} \left[(2q)^{2k} - (8q)^k \right] \\ & + \left(\frac{2m^2 q + 2mq - m}{2q-1} - 2mn \right) \frac{(8q)^k - (2q)^k}{3}. \end{aligned}$$

Proof. By Lemmas 2.4 and 2.9, the result follows directly from Theorem 7.1. \square

7.2.3. Additive Degree-Kirchhoff Index

Theorem 7.3 *Let G be a simple connected graph with n nodes and m edges. Then*

(i) if $q = 2$,

$$\begin{aligned} \bar{\mathcal{K}}_{2,k} = & 4^k \bar{\mathcal{K}}_{2,0} + \frac{2(16^k - 4^k) \tilde{\mathcal{K}}_{2,0}}{3} + \frac{16^k - 4^k}{9} 2m(2m - 2n + 1) \\ & + \frac{16^k}{9} 4m^2 k - \frac{4^k - 1}{27} (2m - 3n)(2m - 3n + 3); \end{aligned}$$

(ii) if $q \neq 2$,

$$\begin{aligned}
\bar{\mathcal{K}}_{q,k} &= 4^k \bar{\mathcal{K}}_{q,0} + \frac{3m^2 q^3 (q-1) [(2q)^{2k} - 4^k]}{(q-2)(q+1)(2q-1)^2} \\
&\quad + \frac{[(8q)^k - 4^k]}{2q-1} \left(q \tilde{\mathcal{K}}_{q,0} - \frac{2m^2 q^2}{3(q-2)} - \frac{m(2n-1)q}{3} \right) \\
&\quad - \frac{mq [(2q)^k - 4^k]}{3(2q-1)} \left(\frac{2mq}{2q-1} - 2n+1 \right) \\
&\quad - \frac{4^k - 1}{3} \left(\left(\frac{mq}{2q-1} - n \right) \left(\frac{mq}{2q-1} - n+1 \right) \right).
\end{aligned}$$

Proof. By Theorem 6.5 and Eqs. (44) and (45), we have

$$\begin{aligned}
\bar{\mathcal{K}}_{q,k} &= 4\bar{\mathcal{K}}_{q,k-1} + 4q\tilde{\mathcal{K}}_{q,k-1} + m_{q,k-1}q(3m_{q,k-1}q - 2n_{q,k-1} + 1) \\
&\quad - n_{q,k-1}(n_{q,k-1} - 1) \\
&= 4\bar{\mathcal{K}}_{q,k-1} + 4q\tilde{\mathcal{K}}_{q,k-1} + (2q)^{2k-2} \frac{4m^2 q^2 (3q^2 - 4q + 1)}{(2q-1)^2} \\
&\quad + (2q)^{k-1} \left(\frac{4m^2 q^3}{(2q-1)^2} - \frac{2m(2n-1)q^2}{2q-1} \right) \\
&\quad - \left(n - \frac{mq}{2q-1} \right)^2 + n - \frac{mq}{2q-1}.
\end{aligned} \tag{46}$$

For $q = 2$, inserting Theorem 7.2 into Eq. (46) yields

$$\begin{aligned}
\bar{\mathcal{K}}_{2,k} &= 4\bar{\mathcal{K}}_{2,k-1} + 8\tilde{\mathcal{K}}_{2,k-1} + \frac{80m^2 \cdot 16^{k-1}}{9} + 4^{k-1} \left(\frac{32m^2}{9} - \frac{8m(2n-1)}{3} \right) \\
&\quad - \frac{(2m-3n)(2m-3n+3)}{9} \\
&= 4\bar{\mathcal{K}}_{2,k-1} + 16^{k-1} \left(8\tilde{\mathcal{K}}_{2,0} + \frac{8m((6k+8)m - 6n+3)}{9} \right) \\
&\quad - \frac{(2m-3n)(2m-3n+3)}{9}.
\end{aligned} \tag{47}$$

By dividing both sides by 4^k , we could derive the result through simple calculations.

Analogously, if $q \neq 2$, using Theorem 7.2, we rewrite Eq. (46) as

$$\begin{aligned}\bar{\mathcal{K}}_{q,k} = & 4\bar{\mathcal{K}}_{q,k-1} + (2q)^{2k-2} \frac{12m^2q^3(q-1)^2}{(q-2)(2q-1)^2} + (8q)^{k-1} \left(4q\tilde{\mathcal{K}}_{q,0} - \frac{8m^2q^2}{3(q-2)} \right. \\ & \left. - \frac{4m(2n-1)q}{3} \right) - (2q)^{k-1} \left(\frac{4m^2q^2(q-2)}{3(2q-1)^2} - \frac{2m(2n-1)q(q-2)}{3(2q-1)} \right) \\ & - \left(n - \frac{mq}{2q-1} \right)^2 + n - \frac{mq}{2q-1}.\end{aligned}\tag{48}$$

Once again, by dividing both sides by 4^k , we obtain a geometric sequence, which is solved to yield the result. \square

7.2.4. Kirchhoff Index

Theorem 7.4 *Let G be a simple connected graph with n nodes and m edges. Then*

(1) if $q = 2$,

$$\begin{aligned}\mathcal{K}_{2,k} = & \mathcal{K}_{2,0} + \frac{4^k - 1}{3} \bar{\mathcal{K}}_{2,0} + \frac{(4^k - 1)^2}{9} \tilde{\mathcal{K}}_{2,0} + \frac{16^k - 1}{135} m((10k + 14)m - 10n + 5) \\ & + \frac{4^k - 1}{81} (-16m^2 + 24mn - 12m - 9n(n-1)) \\ & - \frac{k(2m - 3n)(2m - 3n + 3)}{54};\end{aligned}$$

(2) if $q \neq 2$,

$$\begin{aligned}\mathcal{K}_{q,k} = & \left(\frac{2}{q}\right)^k \mathcal{K}_{q,0} + \frac{q[4^k - (2/q)^k]}{2(2q-1)} \bar{\mathcal{K}}_{q,0} + \frac{q^2[(8q)^k - 2 \cdot 4^k + (2/q)^k]}{4(2q-1)^2} \tilde{\mathcal{K}}_{q,0} \\ & + \frac{m^2q^3(q-1)[(2q)^{2k} - (2/q)^k]}{2(q-2)(q+1)(2q-1)^2} - \frac{mq^2[(8q)^k - (2/q)^k]}{6(2q-1)^2} \left(\frac{mq}{q-2} + n - \frac{1}{2} \right) \\ & + \frac{mq^2[(2q)^k - (2/q)^k]}{3(q+1)(2q-1)} \left(-\frac{mq}{2q-1} + n - \frac{1}{2} \right) - \frac{q[4^k - (2/q)^k]}{2q-1} \\ & \left(\frac{m^2q^2(q-1)}{2(2q-1)^2(q+1)} + \frac{1}{6} \left(\frac{mq}{2q-1} - n \right) \left(\frac{mq}{2q-1} - n + 1 \right) \right) \\ & + \frac{q[(2/q)^k - 1]}{6(q-2)} \left(\frac{mq}{2q-1} - n \right) \left(\frac{mq}{2q-1} - n + 1 \right).\end{aligned}$$

Proof. By Theorem 6.6, we have

$$\begin{aligned}
\mathcal{K}_{q,k} &= \frac{2}{q}\mathcal{K}_{q,k-1} + \bar{\mathcal{K}}_{q,k-1} + \frac{q}{2}\tilde{\mathcal{K}}_{q,k-1} + \frac{m_{q,k-1}^2q^2 - n_{q,k-1}(n_{q,k-1} - 1)}{2} \\
&= \frac{2}{q}\mathcal{K}_{q,k-1} + \bar{\mathcal{K}}_{q,k-1} + \frac{q}{2}\tilde{\mathcal{K}}_{q,k-1} + (2q)^{2k}\frac{m^2q(q-1)}{2(2q-1)^2} \\
&\quad - \frac{(2q)^k m}{4(2q-1)}\left(2n - \frac{2mq}{2q-1} - 1\right) - \frac{1}{2}\left(n - \frac{mq}{2q-1}\right)\left(n - \frac{mq}{2q-1} - 1\right).
\end{aligned} \tag{49}$$

We first consider the case $q = 2$. Inserting Theorems 7.2 and 7.3 into Eq. (49) yields

$$\begin{aligned}
\mathcal{K}_{2,k} &= \mathcal{K}_{2,k-1} + \bar{\mathcal{K}}_{2,k-1} + \tilde{\mathcal{K}}_{2,k-1} + \frac{16^k}{9}m^2 + \frac{4^k}{36}m(4m - 6n + 3) \\
&\quad - \frac{(2m - 3n)(2m - 3n + 3)}{18} \\
&= \mathcal{K}_{2,k-1} + 4^{k-1}\bar{\mathcal{K}}_{2,0} + \frac{5 \cdot 16^{k-1} - 2 \cdot 4^{k-1}}{3}\tilde{\mathcal{K}}_{2,0} \\
&\quad + \frac{16^{k-1}}{9}m\left((10k + 14)m - 10n + 5\right) \\
&\quad + \frac{4^{k-1}}{27}\left(-16m^2 + 24mn - 12m - 9n(n-1)\right) \\
&\quad - \frac{(2m - 3n)(2m - 3n + 3)}{54},
\end{aligned} \tag{50}$$

which could lead to our result through simple calculations.

For the other case $q \neq 2$, once again by Theorems 7.2 and 7.3, Eq. (49) is rewritten as

$$\begin{aligned}
\mathcal{K}_{q,k} &= \frac{2}{q}\mathcal{K}_{q,k-1} + (2q)^{2k}\frac{m^2(q-1)(2q^3-1)}{4(q-2)(q+1)(2q-1)^2} + (8q)^k\left(\frac{2q+1}{16(2q-1)}\tilde{\mathcal{K}}_{q,0}\right. \\
&\quad \left. - \frac{m^2q(2q+1)}{24(q-2)(2q-1)} - \frac{m(2n-1)(2q+1)}{48(2q-1)}\right) \\
&\quad + (2q)^k\left(\frac{m(2n-1)(q-1)}{6(2q-1)} - \frac{m^2q(q-1)}{3(2q-1)^2}\right) + 4^k\left(\frac{\bar{\mathcal{K}}_{q,0}}{4} - \frac{q\tilde{\mathcal{K}}_{q,0}}{4(2q-1)}\right. \\
&\quad \left. - \frac{m^2q^2(q-1)}{4(2q-1)^2(q+1)} - \frac{1}{12}\left(n - \frac{mq}{2q-1}\right)\left(n - \frac{mq}{2q-1} - 1\right)\right) \\
&\quad - \frac{1}{6}\left(n - \frac{mq}{2q-1}\right)\left(n - \frac{mq}{2q-1} - 1\right).
\end{aligned} \tag{51}$$

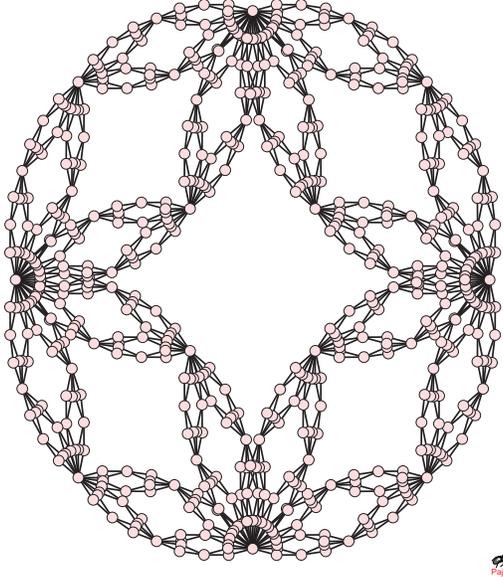


Figure 1: The hierarchical lattice $H_{2,5}$.

Dividing both sides by $(\frac{q}{2})^k$, we obtain a geometric sequence, which is summed to yield the result.

For $q = 2$, we could derive the result similarly. \square

Our results in this section generalize those previously obtained for subdivision graphs [50, 51], but our computation method is much simpler.

7.3. Applications to the Hierarchical Lattices

The hierarchical lattices [27] are a particular example of iterated q -subdivision graphs. They are constructed in an iterative way. Let $H_{q,k}$, $q \geq 2$ and $k \geq 0$, denote the hierarchical lattices after k iterations. For $k = 0$, $H_{q,0}$ is an edge connecting two nodes. For $k \geq 1$, $H_{q,k}$ is obtained from $H_{q,k-1}$ by performing the q -subdivision operation on $H_{q,k-1}$. Thus, the hierarchical lattices are actually iterated q -subdivision graphs $S_{q,k}(G)$ when G is a graph consisting of two nodes linked by an edge. They have been recently introduced as a model of complex networks with scale-free fractal properties [28]. Fig. 1 illustrates a particular hierarchical lattice $H_{2,5}$. Below, we present some properties of the hierarchical lattices, by using the results derived in last subsections.

For $H_{q,0}$, its adjacency matrix and normalized adjacency matrix are both $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Their eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$, with corresponding orthonormal eigenvectors being $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. In addition, for $H_{q,0}$ the Kemeny constant, multiplicative degree-Kirchhoff index, additive degree-Kirchhoff index, and Kirchhoff index are $K(H_{q,0}) = \frac{1}{2}$, $\tilde{\mathcal{K}}(H_{q,0}) = 1$, $\bar{\mathcal{K}}(H_{q,0}) = 2$, and $\mathcal{K}(H_{q,0}) = 1$, respectively. Then, by Theorems 7.1, 7.2, 7.3, and 7.4, we obtain the following exact solutions to the Kemeny constant $K(H_{q,k})$, multiplicative degree-Kirchhoff index $\tilde{\mathcal{K}}(H_{q,k})$, additive degree-Kirchhoff index $\bar{\mathcal{K}}(H_{q,k})$, and Kirchhoff index $\mathcal{K}(H_{q,k})$ for $H_{q,k}$.

$$K(H_{q,k}) = \begin{cases} \frac{(6k+4)4^k+5}{18}, & \text{if } q = 2; \\ \frac{q(q-1)(2q)^k}{(q-2)(2q-1)} - \frac{q \cdot 4^k}{3(q-2)} + \frac{4q-3}{6(2q-1)}, & \text{if } q \neq 2. \end{cases} \quad (52)$$

$$\tilde{\mathcal{K}}(H_{q,k}) = \begin{cases} \frac{(6k+4)16^k+5 \cdot 4^k}{9}, & \text{if } q = 2; \\ \frac{2q(q-1)(2q)^{2k}}{(q-2)(2q-1)} - \frac{2q(8q)^k}{3(q-2)} + \frac{(4q-3)(2q)^k}{3(2q-1)}, & \text{if } q \neq 2. \end{cases} \quad (53)$$

$$\bar{\mathcal{K}}(H_{q,k}) = \begin{cases} \frac{4(k+1)16^k}{9} + \frac{38 \cdot 4^k}{27} + \frac{4}{27}, & \text{if } q = 2; \\ \frac{3q^3(q-1)(2q)^{2k}}{(q-2)(q+1)(2q-1)^2} - \frac{2q^2(8q)^k}{3(q-2)(2q-1)} + \frac{q(4q-3)(2q)^k}{3(2q-1)^2} \\ + \frac{2(3q^2+2q-2)4^k}{3(q+1)(2q-1)} + \frac{(q-1)(3q-2)}{3(2q-1)^2}, & \text{if } q \neq 2. \end{cases} \quad (54)$$

$$\mathcal{K}(H_{q,k}) = \begin{cases} \frac{2(5k+7)16^k}{135} + \frac{38 \cdot 4^k}{81} + \frac{173-60k}{405}, & \text{if } q = 2; \\ \frac{q^3(q-1)(2q)^{2k}}{2(q-2)(q+1)(2q-1)^2} - \frac{q^3(8q)^k}{6(q-2)(2q-1)^2} + \frac{q^2(4q-3)(2q)^k}{6(q+1)(2q-1)^2} \\ + \frac{q(3q^2+2q-2)4^k}{3(q+1)(2q-1)^2} + \frac{5q^4-9q^3-5q^2+12q-4}{2(q-2)(q+1)(2q-1)^2} \left(\frac{2}{q}\right)^k - \frac{(q-1)q(3q-2)}{6(q-2)(2q-1)^2}, & \text{if } q \neq 2. \end{cases} \quad (55)$$

We note that Eq. (52) is in complete agreement with the result obtained in [30].

8. Conclusions

The q -subdivision operation is an extension of traditional subdivision operation on a graph, which has been applied to construct complex networks. In this paper, we studied various properties of q -subdivision graph $S_q(G)$ of a simple connected graph G , and expressed some quantities of $S_q(G)$ in terms of associated with G . We first derived formulas for eigenvalues and eigenvectors of normalized adjacency matrix for $S_q(G)$. We then determined two-node hitting time and resistance distance for any pair of

nodes in $S_q(G)$, using the connection between eigenvalues and eigenvectors of normalized adjacency matrix and hitting time and resistance distance. Moreover, we obtained the Kemeny constant, Kirchhoff index, multiplicative degree-Kirchhoff index, and additive degree-Kirchhoff index for $S_q(G)$. Finally, we derived explicit formulas for some interesting quantities of iterated q -subdivisions for any graph G , using which we obtained closed-form expressions for those corresponding quantities of the scale-free fractal hierarchical lattices.

It deserves to mention that our computation method and process also apply to other graph operations, such as q -triangulation. For a graph G , its q -triangulation is a obtained from G : For each edge e in G we create q new nodes, and connect them to both end nodes of e . The q -triangulation is a generalization of traditional triangulation operation [52], which has been used to generate scale-free small-world networks [53].

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