



# Stability Analysis of Hybrid Automata with Set-Valued Vector Fields Using Sums of Squares(Hybrid Dynamical Systems, <Special Section>Concurrent/Hybrid Systems: Theory and...

Masubuchi, Izumi  
Tsuji, Tokihisa

---

(Citation)

IEICE transactions on fundamentals of electronics, communications and computer sciences, E89-A(11):3185-3191

(Issue Date)

2006-11-01

(Resource Type)

journal article

(Version)

Version of Record

(URL)

<https://hdl.handle.net/20.500.14094/90001336>



# Stability Analysis of Hybrid Automata with Set-Valued Vector Fields Using Sums of Squares

Izumi MASUBUCHI<sup>†a)</sup>, Member and Tokihisa TSUJI<sup>††</sup>, Nonmember

**SUMMARY** Stability analysis is one of the most important problems in analysis of hybrid dynamical systems. In this paper, a computational method of Lyapunov functions is proposed for stability analysis of hybrid automata that have set-valued vector fields. For this purpose, a formulation of matrix-valued sums of squares is provided and applied to derive an LMI/LME problem whose solution yields a Lyapunov function.

**key words:** hybrid automata, robust stability,  $L_2$ -gain, sums of squares, LMIs

## 1. Introduction

One of the most important notion in theory of dynamical systems is stability, under which the variables of systems evolve in an adequate region. Though it is the most basic property of dynamical systems, analysis of stability is not very easy except for such as linear systems. A considerable amount of attention has been paid to stability analysis of hybrid dynamical systems (HDS) in the last decade. Various extensions of notions of stability for differential equations have been proposed for HDS [3], [9]–[11], where the theory of Lyapunov has been generalized to deal with executions of HDS, which consist of continuous and discrete evolutions. Computational methods have been shown for piecewise linear systems [4], [9] and more general classes of HDS [5], [7], where the latter utilize *sums of squares* [6], [8] with which semidefinite programming is applicable to problems of seeking positive functions.

In this paper, extending the results of [5], we propose a computational method of Lyapunov functions for stability analysis of hybrid automata [1], [2], [5], [12] that have set-valued vector fields for continuous evolution. Admitting set-valued vector fields enables to handle more general classes of HDS than those ever considered before such as in [5], [7]. Then stability analysis involves what corresponds to robustness analysis in continuous dynamical systems. We reduce a Lyapunov-like stability condition to a condition of matrix inequalities of functions and apply matrix-valued sums of squares to solve those inequalities. For this purpose we generalize the formulation of scalar-valued sums of squares [6] to matrix-valued and show linear matrix inequalities and

equations (LMIs and LMEs in short) whose solution yields a Lyapunov function of HDS. LMIs are convex inequalities and solving LMIs/LMEs enjoys fast and globally convergent numerical algorithms.

**Notation.** Denote by  $\mathbf{R}^p$ ,  $\mathbf{R}^{q \times r}$  and  $\mathbf{R}^{p \times q \times r}$  the sets of vectors, matrices and (3,0)-tensors of real numbers, respectively. Elements of  $v \in \mathbf{R}^p$ ,  $M \in \mathbf{R}^{q \times r}$ ,  $\mathbf{T} \in \mathbf{R}^{p \times q \times r}$  are represented as  $v^i$ ,  $M^{jk}$ ,  $\mathbf{T}^{ijk}$ , respectively. The elements of the tensor product  $\mathbf{T} = v \otimes M$  are  $\mathbf{T}^{ijk} = v^i M^{jk}$ . Let  $M^{i\bullet}$  and  $M^{\bullet j}$  stand for the  $i$ -th row vector and the  $j$ -th column vector of matrix  $M$ , respectively. We define inner products for  $\mathbf{R}^p$ ,  $\mathbf{R}^{q \times r}$ ,  $\mathbf{R}^{p \times q \times r}$  by

$$\begin{aligned} \langle v, w \rangle &= \sum_i v^i w^i, \quad v, w \in \mathbf{R}^p, \\ \langle M, N \rangle &= \sum_{j,k} M^{jk} N^{jk}, \quad M, N \in \mathbf{R}^{q \times r}, \\ \langle \mathbf{T}, \mathbf{U} \rangle &= \sum_{i,j,k} \mathbf{T}^{ijk} \mathbf{U}^{ijk}, \quad \mathbf{T}, \mathbf{U} \in \mathbf{R}^{p \times q \times r}. \end{aligned}$$

For  $v \in \mathbf{R}^p$  and  $\mathbf{T} \in \mathbf{R}^{p \times q \times r}$ , we denote by  $\langle v, \mathbf{T} \rangle$  the contraction of  $v \in \mathbf{R}^p$  and  $\mathbf{T} \in \mathbf{R}^{p \times q \times r}$ , where  $\langle v, \mathbf{T} \rangle$  is a  $q \times r$  matrix whose  $(j, k)$ -element is given as  $[\langle v, \mathbf{T} \rangle]^{jk} = \sum_i v^i \mathbf{T}^{ijk}$ . Let  $F(\cdot)$  be a linear map from  $\mathbf{R}^{p \times q \times r}$  to  $\mathbf{R}^{s \times t}$ . Then the adjoint of  $F(\cdot)$  is defined as the map  $F^*(\cdot)$  that satisfies

$$\langle F(\mathbf{T}), Z \rangle = \langle \mathbf{T}, F^*(Z) \rangle$$

for all  $\mathbf{T} \in \mathbf{R}^{p \times q \times r}$  and  $Z \in \mathbf{R}^{s \times t}$ . Let  $M^T$  denote the transpose of a matrix (or vector)  $M$ . For a symmetric matrix  $M$  the inequality  $M > (\geq) 0$  means that  $M$  is positive (semi-)definite. The standard Euclidean norm is denoted by  $\|\cdot\|$ .

## 2. Preliminaries

### 2.1 Hybrid Automata and Executions

Let us define a class of hybrid automata and their executions based on [12]. For a finite collection  $V$  of variables  $v$ , let  $\mathbf{V}$  (in boldface) denote the set of valuations of the variables. Variables whose set of valuation is finite or countable are referred to as *discrete* and to variables whose set of valuations is a subset of a Euclidean space or a manifold as *continuous*.

**Definition 1:** A hybrid automaton  $\mathbf{HA}$  is a collection

Manuscript received March 31, 2006.

Manuscript revised June 10, 2006.

Final manuscript received July 13, 2006.

<sup>†</sup>The author is with the Graduate School of Engineering, Hiroshima University, Higashi-hiroshima-shi, 739-8527 Japan.

<sup>††</sup>The author is with Daihatsu Motor Co., LTD., Ikeda-shi, 563-8561 Japan.

a) E-mail: msb@hiroshima-u.ac.jp

DOI: 10.1093/ietfec/e89-a.11.3185

$\mathbf{HA} = \{Q, X, \text{Init}, F, D, E, G, R\}$ , where

- $Q$  is a finite collection of discrete variables with  $\mathbf{Q}$  being a finite set. An element of  $\mathbf{Q}$  is referred to as *mode*;
- $X$  is a finite collection of continuous variables with  $\mathbf{X} = \mathbf{R}^{n_q}$  for  $q \in \mathbf{Q}$ , where  $n_q$  is the dimension of  $\mathbf{X}$  for  $q \in \mathbf{Q}$ ;
- $\text{Init} \subset \Xi$  is a set of initial values, where  $\Xi = \{(q, x) : q \in \mathbf{Q}, x \in \mathbf{R}^{n_q}\}$ ;
- $F : (q, x) \in \Xi \rightarrow F_q(x) \in 2^{T\mathbf{X}^\dagger}$  is a *set-valued vector field*;
- $D : q \in \mathbf{Q} \rightarrow D_q \in 2^{\mathbf{R}^{n_q}}$  is a map<sup>††</sup> assigning a subset ‘*mode-invariant set*’ of  $\mathbf{X}$  for each  $q \in \mathbf{Q}$ ;
- $E \subset \mathbf{Q} \times \mathbf{Q}$  is a set of edges, where  $e = (r, q) \in E$  means that the graph of the automaton has an edge from  $q$  to  $r$ ;
- $G : e = (r, q) \in E \rightarrow G_e \in 2^{\mathbf{R}^{n_q}}$  is a map assigning a subset ‘*guard*’ of  $\mathbf{R}^{n_q}$  for each  $e = (r, q)$ ;
- $R : (e, x) \in E \times G_e \rightarrow R_e(x) \in 2^{\mathbf{R}^{n_r}}$  with  $e = (r, q)$  is a map called *reset* that assigns a subset of  $\mathbf{R}^{n_r}$  for each  $e = (r, q) \in E$  and  $x \in G_e$ .  $\square$

**Example 1:** Let us consider a hybrid automaton of the class of Definition 1 with:

$$\begin{aligned} \mathbf{Q} &= \{+, -\}, \quad \mathbf{X}_+ = \mathbf{X}_- = \mathbf{R}^2, \\ F_\pm(x) &= \left\{ \begin{bmatrix} -3x_1 \mp x_1^2 + \Delta x_1 \\ -x_2 + \Delta x_1 \end{bmatrix} : |\Delta| \leq 1 \right\}, \\ D_+ &= \{x : x_1 \geq -1\}, \quad D_- = \{x : x_1 \leq 1\}, \\ E &= \{e_1 = (-, +), e_2 = (+, -)\}, \\ G_{e_1} &= \{x : x_1 = -1\}, \quad G_{e_2} = \{x : x_1 = 1\}, \\ R_{e_1}(x) &= R_{e_2}(x) = \{x\}, \\ \text{Init} &= \{(q, x) : q \in \mathbf{Q}, x \in D_q\}. \end{aligned}$$

Then the hybrid automaton  $\mathbf{HA}$  represents a hybrid dynamical system with two modes ‘+,’ ‘-’ and the continuous variable in  $\mathbf{R}^2$ . The continuous variable  $x$  evolves in  $\mathbf{R}^2$  with  $\dot{x}$  belonging to the set  $F_q(x)$  as far as  $x$  remains in the mode invariant set  $D_q$ . Suppose  $q = +$ . If  $x$  arrives at a point in  $G_{e_1}$ , i.e., if  $x_1 = -1$ , then the mode changes to ‘-,’ where  $e_1 = (-, +)$  corresponds to mode transition from ‘+’ to ‘-.’ The edge  $E$  is the set of all possible transitions of the modes and thus the mode can change also from ‘-’ to ‘+.’ According to  $R$  of this example, the continuous variable  $x$  does not jump at mode transitions.  $\square$

The descriptions of the evolution of the variable  $(q, x)$  in Example 1 are formalized below by defining *hybrid time trajectory* and *executions*, which follow those in [12]. The hybrid time trajectory is a sequence of time intervals in which the discrete variable is constant. The execution is defined as a function of a hybrid time trajectory that is accepted by the hybrid automaton  $\mathbf{HA}$ .

**Definition 2:** A *hybrid time trajectory* is a sequence of intervals  $\mathcal{I} = \{I_k\}_{k=0}^N$  with  $N < \infty$  or  $N = \infty$  such that

- $I_k = [t_k, t'_k]$  for all  $0 \leq k < N$ ,
- if  $N < \infty$  then  $I_N = [t_N, t'_N]$  or  $I_N = [t_N, t'_N)$ ,

- $t_k \leq t'_k$  and  $t'_k = t_{k+1}$  for all  $0 \leq k < N$ .

Denote by  $\langle \mathcal{I} \rangle$  the sequence of integers  $\{0, 1, \dots, N\}$  and let  $|I_k| = t'_k - t_k$  for a hybrid time trajectory  $\mathcal{I}$ .  $\square$

**Definition 3:** Consider a triplet  $\chi = \{\mathcal{I}, q, x\}$ , where  $\mathcal{I} = \{I_k\}_{k=0}^N$  is a hybrid time trajectory,  $q : k \in \langle \mathcal{I} \rangle \rightarrow q_k \in \mathbf{Q}$  and  $x = \{x_k(\cdot) : k \in \langle \mathcal{I} \rangle\}$  with  $x_k(t)$  being a  $C^1$  map from  $I_k$  to  $\mathbf{R}^{n_{q_k}}$ . If  $\chi$  satisfies the following conditions 1–3, we say that  $\chi$  is an *execution* of the hybrid automaton  $\mathbf{HA}$ , or  $\mathbf{HA}$  *accepts*  $\chi$ .

1. (*Initial condition*)  $(q_0, x_0(t_0)) \in \text{Init}$ .
2. (*Continuous evolution*) For each  $k \in \langle \mathcal{I} \rangle$  such that  $|I_k| > 0$ , it holds that

$$\begin{cases} \dot{x}_k(t) \in F_{q_k}(x_k(t)), & t \in I_k, \\ x_k(t) \in D_{q_k}, & t \in [t_k, t'_k]. \end{cases} \quad (1)$$

3. (*Discrete evolution*) For each  $k \in \langle \mathcal{I} \rangle$ , it holds that

$$\begin{cases} e_k := (q_{k+1}, q_k) \in E, \\ x_k(t'_k) \in G_{e_k}, \\ x_{k+1}(t'_k) \in R_{e_k}(x_k(t'_k)). \end{cases} \quad (2)$$

$\square$

## 2.2 Stability of Executions

We define the following equilibrium set to consider stability of executions.

**Definition 4:** Let  $\mathbf{Q}_0$  be a subset of  $\mathbf{Q}$ . A set  $\mathbf{Equi} = \{(q, x_q) : q \in \mathbf{Q}_0, x_q \in D_q\}$  is an *equilibrium set* if

- $F_q(x_q) = \{0\}$ ,  $q \in \mathbf{Q}_0$ ,
- if  $e = (r, q) \in E$  and  $q \in \mathbf{Q}_0$ , then  $r \in \mathbf{Q}_0$  and  $R_e(x_q) = \{x_r\}$ .  $\square$

Below we assume  $x_q = 0$  for all  $q \in \mathbf{Q}_0$  without loss of generality. Then  $\mathbf{Equi} = \{(q, 0) : q \in \mathbf{Q}_0\}$ . We define a notion of stability of the continuous variable as a generalization of Lyapunov stability of continuous dynamical systems.

**Definition 5:** The set  $\mathbf{Equi}$  is said to be *stable* if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\|x_0(t_0)\| < \delta \Rightarrow \|x_k(t)\| < \varepsilon, \quad \forall k \in \langle \mathcal{I} \rangle, \quad \forall t \in I_k$$

holds for every execution  $\chi = \{\mathcal{I}, q, x\}$  accepted by  $\mathbf{HA}$ .  $\square$

Based on [9], [10], we use a Lyapunov-like stability condition of executions shown below. Denote by  $\mathbf{R}^+$  the set of nonnegative real numbers. A continuous function  $\alpha : \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is said to be a *class- $\mathcal{K}$*  function if  $\alpha(0) = 0$  and  $\alpha(r_1) < \alpha(r_2)$  for any  $0 \leq r_1 < r_2$ .

<sup>†</sup> $T\mathbf{X}$  is the tangent space of  $\mathbf{X}$ . In this paper we can always identify it as  $\mathbf{R}^{n_q}$ .

<sup>††</sup>We sometimes use notation such as  $D_q, F_q(x)$  to represent maps with discrete variables such as  $q$ .

**Proposition 1:** The set **Equi** is stable if there exist  $V_q(x) : (q, x) \in \Xi \rightarrow \mathbf{R}$ , where  $V_q$  is  $\mathbf{C}^1$  for fixed  $q$ , and class- $\mathcal{K}$  functions  $\alpha(\cdot), \beta(\cdot)$  for which the following conditions hold:

$$\alpha(\|x\|) \leq V_q(x) \leq \beta(\|x\|), \quad \forall q \in \mathbf{Q}, \forall x \in D_q, \quad (3)$$

$$\frac{\partial V_q(x)}{\partial x} v \leq 0, \quad \forall v \in F_q(x), \forall q \in \mathbf{Q}, \forall x \in D_q, \quad (4)$$

$$V_r(R_e(x)) \leq V_q(x), \quad \forall e = (r, q) \in E, \quad \forall x \in G_e. \quad (5)$$

□

We call  $V_q(x)$  satisfying the conditions of the above proposition *Lyapunov function*.

**Example 2:** Consider the hybrid automaton given in Example 1. Let  $\mathbf{Q}_0 = \mathbf{Q}$ . Then **Equi** =  $\{(q, 0) : q \in \mathbf{Q}_0\}$  is stable as seen below. Set  $V_q(x) = (x_1^2 + x_2^2)/2$ . Then the conditions (3) and (5) are obvious. For  $v \in F_{\pm}(x)$ ,

$$\frac{\partial V_q(x)}{\partial x} v = -3x_1^2 \mp x_1^3 - x_2^2 + \Delta(x_1^2 + x_1x_2) \quad (6)$$

for some  $\Delta \in [-1, 1]$ , which implies

$$\begin{aligned} (6) &\leq -3x_1^2 \mp x_1^3 - x_2^2 + (x_1^2 + |x_1x_2|) \\ &= -\frac{7}{4}x_1^2 \left(1 \pm \frac{4}{7}x_1\right) - \left(\frac{1}{2}|x_1| - |x_2|\right)^2 \end{aligned} \quad (7)$$

From the definition of  $D_q$ ,  $1 + 4x_1/7 \geq 3/7$  if  $q = '+'$  since  $x \in D_+$  implies  $x_1 \geq -1$ . The same holds also for  $q = '-'$ . Therefore (7) is nonpositive. □

### 2.3 Matrix-Valued Sums of Squares

In Sect. 3, we consider stability analysis of a class of hybrid automata via numerical methods. This subsection is devoted to formulate *matrix-valued sums of squares* as a generalization of (scalar-valued) sums of squares shown in [6]. Sums of squares have been widely used to derive positive functions, such as Lyapunov functions (e.g., [5], [7]). We utilize matrix-valued sums of squares to compute positive semidefinite matrices that are functions of  $(x, q)$ , by which stability analysis is solved via convex optimization.

Let  $\omega^{(m)}(x) : x \in \mathbf{R}^n \rightarrow \mathbf{R}^p$ ,  $m = 1, \dots, M$  be linearly independent vector-valued functions. Denote the set of these functions by

$$\mathcal{S} = \{\omega^{(1)}(x), \dots, \omega^{(M)}(x)\}$$

and the span of  $\mathcal{S}$  by

$$\begin{aligned} \mathcal{F}(\mathcal{S}) &= \text{span}\{\mathcal{S}\} \\ &= \left\{ s(x) = \sum_{m=1}^M c_m \omega^{(m)}(x) : c_1, \dots, c_M \in \mathbf{R} \right\}. \end{aligned}$$

Let  $\mathcal{K}(\mathcal{S})$  be the set of finite linear combinations of dyads of elements of  $\mathcal{S}$ :

$$\mathcal{K}(\mathcal{S}) = \left\{ W(x) = \sum_{i=1}^{K_w} s^{(i)}(x) s^{(i)}(x)^\top : s^{(1)}, \dots, s^{(K_w)} \in \mathcal{S} \right\},$$

where the number of summand  $K_w$  depends on  $W(x)$ . This is a subset of  $p \times p$ -matrix-valued functions and we refer to elements of  $\mathcal{K}(\mathcal{S})$  as *matrix-valued sums of squares*. Denote  $\Omega(x) = \begin{bmatrix} \omega^{(1)}(x) & \dots & \omega^{(M)}(x) \end{bmatrix}$  and define a vector-valued function  $l(x) : x \in \mathbf{R}^n \rightarrow \mathbf{R}^N$  whose elements  $l^{(i)}(x)$ ,  $i = 1, \dots, N$  are linearly independent and a linear map  $\Lambda(\mathbf{T}) : \mathbf{T} \in \mathbf{R}^{N \times p \times p} \rightarrow \mathbf{R}^{M \times M}$  that satisfy

$$\Lambda(l(x) \otimes R) = \Omega(x)^\top R \Omega(x) \quad (8)$$

for all  $x \in \mathbf{R}^n$  and  $R \in \mathbf{R}^{p \times p}$ .

**Remark 1:** The map  $\Lambda(\cdot)$  in (8) is a generalization of that defined in [6] and they coincide with each other for  $p = 1$ . Since  $\Lambda(\cdot)$  is linear, it is represented in terms of elements as  $[\Lambda(\mathbf{T})]^{mn} = \sum_{i,j,k} \Lambda_{ijk}^{mn} \mathbf{T}^{ijk}$ , where  $\Lambda_{ijk}^{mn}$ 's are the coefficients of the linear map  $\Lambda(\cdot)$ . Then (8) is equivalent to  $\sum_i \Lambda_{ijk}^{mn} [l(x)]^i = \omega_j^{(m)}(x) \omega_k^{(n)}(x)$ . □

The following lemma gives a representation of  $\mathcal{K}(\mathcal{S})$ .

**Lemma 1:** A matrix-valued function  $W(x)$  belongs to  $\mathcal{K}(\mathcal{S})$  if and only if it is represented with a positive semidefinite matrix  $Y \in \mathbf{R}^{M \times M}$  as

$$W(x) = \Omega(x) Y \Omega(x)^\top = \langle l(x), \Lambda^*(Y) \rangle. \quad (9)$$

□

**Proof.** Suppose that  $W(x) \in \mathcal{K}(\mathcal{S})$ . Then for some  $s^{(i)} = \sum_{j=1}^M c_{ij} \omega^{(j)}(x)$  we have

$$\begin{aligned} W(x) &= \sum_{i=1}^{K_w} \left( \sum_{j=1}^M c_{ij} \omega^{(j)}(x) \right) \left( \sum_{k=1}^M c_{ik} \omega^{(k)}(x) \right)^\top \\ &= \sum_{j,k=1}^M \left( \sum_{i=1}^{K_w} c_{ij} c_{ik} \omega^{(j)}(x) \omega^{(k)}(x)^\top \right). \end{aligned}$$

Define  $Y^{jk} = \sum_{i=1}^{K_w} c_{ij} c_{ik}$ . Then  $Y \geq 0$  and

$$W(x) = \sum_{j,k=1}^M Y^{jk} \omega^{(j)}(x) \omega^{(k)}(x)^\top = \Omega(x) Y \Omega(x)^\top.$$

Following the above conversely, we see  $W(x) \in \mathcal{K}(\mathcal{S})$  if  $W(x) = \Omega(x)^\top Y \Omega(x)$ , where  $c_{ij}$ 's are defined as the elements of  $C$  that decomposes  $Y \geq 0$  as  $Y = CC^\top$ .

The second equality in (9) is shown as follows. For any  $Y \in \mathbf{R}^{M \times M}$  and  $R \in \mathbf{R}^{p \times p}$ , we have

$$\begin{aligned} \langle \Lambda(l(x) \otimes R), Y \rangle &= \langle l(x) \otimes R, \Lambda^*(Y) \rangle \\ &= \sum_{i,j,k} [l(x)]^i R^{jk} \sum_{m,n} \Lambda_{ijk}^{mn} Y^{mn} \\ &= \sum_{j,k} R^{jk} \left( \sum_i [l(x)]^i \sum_{m,n} \Lambda_{ijk}^{mn} Y^{mn} \right) \\ &= \langle R, \langle l(x), \Lambda^*(Y) \rangle \rangle, \end{aligned} \quad (10)$$

while (8) and the first equality of (9) imply

$$\begin{aligned} \langle \Lambda(l(x) \otimes R), Y \rangle &= \langle \Omega(x) R \Omega(x)^T, Y \rangle \\ &= \text{Tr} \Omega(x) R^T \Omega(x)^T Y = \text{Tr} R^T \Omega(x)^T Y \Omega(x) \\ &= \langle R, \Omega(x)^T Y \Omega(x) \rangle = \langle R, W(x) \rangle. \end{aligned} \quad (11)$$

Since  $R$  is arbitrary, we obtain  $W(x) = \langle l(x), \Lambda^*(Y) \rangle$  from (10) and (11).  $\square$

**Remark 2:** The representation of (9) in terms of  $l(x)$  is a generalization of [6] for scalar-valued sums of squares to matrix-valued. Below in this paper we utilize this expression.  $\square$

**Remark 3:** The element of the matrix  $W(x)$  in (9) is  $[W(x)]^{jk} = \sum_{m,n,i} \Lambda_{ijk}^{mn} Y^{mn} [l(x)]^i$ . The elementwise representation of  $[\Lambda^*(Y)]^{ij}$ , which is  $N$ -vector valued as  $l(x)$ , is given by

$$[\Lambda^*(Y)]^{ij} = \begin{bmatrix} \sum_{m,n} \Lambda_{ij1}^{mn} Y^{mn} \\ \vdots \\ \sum_{m,n} \Lambda_{ijN}^{mn} Y^{mn} \end{bmatrix}.$$

Thus we have  $[W(x)]^{jk} = \langle l(x), [\Lambda^*(Y)]^{ij} \rangle$ , where  $\langle \bullet, \bullet \rangle$  is the interior product of  $N$ -vectors.  $\square$

### 3. Stability Analysis via Matrix-Valued Sums of Squares

#### 3.1 Description of a Class of Hybrid Automata

We consider hybrid automata that have the following expressions, which we utilize to carry out stability analysis via a computational method based on matrix-valued sums of squares, as well as we make several assumptions.

1. The map  $F_q(x)$ , which assigns a set of vector field to  $(q, x)$ , is represented as

$$\begin{aligned} F_q(x) &= \{f_q^A(x) + f_q^B(x) \Delta f_q^C(x) : \\ &\quad \bar{\sigma}(\Delta) \leq k_q\}, \end{aligned} \quad (12)$$

where  $f_q^A(x) \in \mathbf{R}^{n_q}$ ,  $f_q^B(x) \in \mathbf{R}^{n_q \times n_q^B}$  and  $f_q^C(x) \in \mathbf{R}^{n_q^C \times n_q}$  with  $TX$  being identified as  $\mathbf{R}^{n_q}$ . The scalar  $k_q \geq 0$  gives the upper bound of the largest singular value, denoted by  $\bar{\sigma}(\cdot)$ , of  $\Delta \in \mathbf{R}^{n_q^B \times n_q^C}$ . For each execution,  $x_k(t)$  satisfies the following differential equation

$$\begin{aligned} \dot{x}_k(t) &= f_{q_k}^A(x(t)) \\ &\quad + f_{q_k}^B(x(t)) \Delta_{q_k}(t, x(t)) f_{q_k}^C(x(t)) \end{aligned} \quad (13)$$

for some  $\Delta_{q_k}(t, x)$  satisfying  $\bar{\sigma}(\Delta_{q_k}(t, x)) \leq k_{q_k}$ .

2. The mode-invariant sets are given as  $D_q = \{x : d_{qi}(x) \geq 0, i = 1, \dots, N_{Dq}\}$ , where  $d_{qi} : \mathbf{R}^{n_q} \rightarrow \mathbf{R}, i = 1, \dots, N_{Dq}$  are smooth functions on  $\mathbf{R}^{n_q}$ .
3. The reset map is a singleton:  $R_e(x) = \{R_e^N(x)\}$  with  $R_e^N(x) : (e, x) \in E \times G_e \rightarrow \mathbf{R}^{n_q}$  with  $e = (r, q)$ .

4. The guard  $G_e, e = (r, q)$  and the mode-invariant set satisfy  $\bigcup_r G_{(r,q)} = \partial D_q$ .
5. The guard for  $e = (r, q)$  is represented as  $G_e = \bigcup_{j=1}^{N_{Ge}} \{g_{e0}^{(j)}(\xi) : \xi \in G_e^{(j)}\}$  with  $G_e^{(j)} = \{\xi \in \mathbf{R}^{n_q-1} : g_{ei}^{(j)}(\xi) \geq 0, i = 1, \dots, N_{Ge}^{(j)}\}$ , where  $g_{e0}^{(j)} : \mathbf{R}^{n_q-1} \rightarrow \mathbf{R}^{n_q}$  and  $g_{ei}^{(j)} : \mathbf{R}^{n_q-1} \rightarrow \mathbf{R}, i = 1, \dots, N_{Ge}^{(j)}$  are smooth functions on  $\mathbf{R}^{n_q-1}$ .
6. **Init**  $\subset \{(q, x) : x \in D_q, q \in \mathbf{Q}\}$ .

The explicit expressions of  $F_q(x), D_q, R_e(x)$  and  $G_e$  are given in the items 1, 2, 3 and 5, respectively, while the assumptions in the items 1, 4, 6 ensure the existence of executions [5].

#### 3.2 Stability Condition

We invoke a widely-known technique to handle set-valued vector fields parameterized with norm bounded elements as (12).

**Lemma 2:** The set **Equi** is stable if there exist  $V_q(x) : (q, x) \in \Xi \rightarrow \mathbf{R}$ , where  $V_q$  is  $C^1$  for fixed  $q$ , a strictly positive function  $\sigma_q(x)$  and class- $\mathcal{K}$  functions  $\alpha(\cdot), \beta(\cdot)$  for which the following conditions hold:

$$\alpha(\|x\|) \leq V_q(x) \leq \beta(\|x\|), \quad \forall x \in D_q, q \in \mathbf{Q}, \quad (14)$$

$$\begin{aligned} &\frac{\partial V_q(x)}{\partial x} f_q^A(x) + \sigma_q(x) f_q^C(x)^T f_q^C(x) \\ &\quad + \frac{1}{4\sigma_q(x)} \frac{\partial V_q(x)}{\partial x} f_q^B(x) f_q^B(x)^T \frac{\partial V_q(x)}{\partial x^T} \leq 0, \\ &\quad x \in D_q, q \in \mathbf{Q}, \end{aligned} \quad (15)$$

$$V_r(R_e^N(x)) \leq V_q(x), \quad x \in G_e, e = (r, q) \in E. \quad (16)$$

$\square$

**Example 3:** The hybrid automaton in Example 1 can be represented as (12) with

$$\begin{aligned} f_{\pm}^A(x) &= \begin{bmatrix} -3x_1 \mp x_1^2 \\ -x_2 \end{bmatrix}, \quad f_{\pm}^B(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ f_{\pm}^C(x) &= x_1, \quad k_q = 1. \end{aligned}$$

This hybrid automaton satisfies (14)–(16) with  $V_q(x) = (x_1^2 + x_2^2)/2$  and  $\sigma_q(x) = 1$ . In fact, it holds that

$$\begin{aligned} &(\text{the left-hand side of (15)}) \\ &= -\frac{5}{3}x_1^2 \left(1 \pm \frac{3}{5}x_1\right) - \frac{3}{4} \left(\frac{1}{3}x_1 - x_2\right)^2 \\ &\leq -\frac{2}{3}x_1^2 - \frac{3}{4} \left(\frac{1}{3}x_1 - x_2\right)^2 \leq 0, \end{aligned}$$

since  $1 \pm (3/5)x_1 \geq 2/5$  due to  $x \in D_q$ .  $\square$

The proof of Lemma 2 follows easily as that of quadratic stability for linear uncertain systems, by observing that the inequalities (14)–(16) imply (3)–(5). We focus on obtaining functions  $V_q(x)$  and  $\sigma_q(x)$  that satisfy the conditions of Lemma 2, based on matrix-valued sums of squares

in Sect. 2.3.

First, in general, a symmetric-matrix-valued function  $A(x)$  is guaranteed to be positive on a given region  $X_g := \{x \in \mathbf{R}^n : g_i(x) \geq 0, i = 1, \dots, k\}$  whenever there exist functions  $\lambda_i(x)$  satisfying

$$A(x) - \sum_{i=1}^k \lambda_i(x) g_i(x) \geq 0, \quad \lambda_i(x) \geq 0, \quad i = 1, \dots, k \quad (17)$$

for all  $x \in \mathbf{R}^n$ . Now let  $\lambda_i(x)$ 's and the left-hand side of the first inequality of (17) be sums of squares. Then with appropriate basis matrices  $\Omega_i(x)$  and positive semidefinite matrices  $Y_i$  for  $i = 0, 1, \dots, k$  the inequality condition (17) is rewritten as

$$A(x) - \sum_{i=1}^k g_i(x) \Omega_i(x)^T Y_i \Omega_i(x) = \Omega_0(x)^T Y_0 \Omega_0(x),$$

which has the following alternative representation

$$A(x) = \Omega(x)^T Y \Omega(x),$$

where  $Y = \text{diag}\{Y_0, Y_1, \dots, Y_k\}$  and

$$\Omega(x) = \begin{bmatrix} \Omega_0(x) \\ \sqrt{g_1(x)} \Omega_1(x) \\ \vdots \\ \sqrt{g_k(x)} \Omega_k(x) \end{bmatrix}, \quad x \in X_g.$$

From these observations, we can utilize positive functions that are defined and positive only in such as  $D_q$  and  $G_e^{(j)}$  by choosing appropriate basis, where we can adopt functions involving terms of  $\sqrt{d_{qi}(x)}$  and  $\sqrt{g_{ei}^{(j)}}$  from the expressions of the mode-invariant sets and the guards, respectively.

Now we consider sum-of-squares formulation to obtain a Lyapunov function  $V_q(x)$ . Choose basis  $\omega_q^{(m)}(x): x \in D_q \rightarrow \mathbf{R}, m = 1, \dots, M_q, q \in \mathbf{Q}$  defined on  $D_q$  and derive  $l_q(x)$  and  $\Lambda_q(\cdot)$  as the counter parts of  $l(x)$  and  $\Lambda(\cdot)$  in Sect. 2.3, respectively. Then we set

$$V_q(x) = \langle l_q(x), \Lambda_q^*(Y_q) \rangle, \quad Y_q = Y_q^T \in \mathbf{R}^{M_q \times M_q}.$$

In the same way, let

$$\sigma_q(x) = \langle \tilde{l}_q(x), \tilde{\Lambda}_q^*(\tilde{Y}_q) \rangle, \quad \tilde{Y}_q = \tilde{Y}_q^T \in \mathbf{R}^{\tilde{M}_q \times \tilde{M}_q},$$

where  $\tilde{l}_q(x)$  and  $\tilde{\Lambda}_q(\cdot)$  are generated from basis  $\tilde{\omega}_q^{(m)}(x): x \in D_q \rightarrow \mathbf{R}, m = 1, \dots, \tilde{M}_q, q \in \mathbf{Q}$ . These parameterizations of  $V_q(x)$  and  $\sigma_q(x)$  imply their nonnegativity on  $D_q$  if  $Y_q \geq 0, \tilde{Y}_q \geq 0$ . One can easily modify the development of this subsection to strictly guarantee (14) and  $\sigma_q(x) > 0$ ; the detail is omitted.

Next, consider the condition (15). Schur complement yields the following equivalent matrix inequality:

$$\begin{bmatrix} \frac{\partial V_q(x)}{\partial x} f_q^A(x) + \sigma_q(x) \|f_q^C(x)\|^2 & * \\ \frac{1}{2} \frac{\partial V_q(x)}{\partial x} f_q^B(x) & -\sigma_q(x) I \end{bmatrix} =: W_q(x) \leq 0, \quad (18)$$

where the left-hand side is symmetric and '\*' represents off-diagonal elements abbreviated. Choose basis  $\hat{\omega}_q^{(m)}(x): x \in D_q \rightarrow \mathbf{R}^{n_q+1}, m = 1, \dots, \hat{M}_q, q \in \mathbf{Q}$  and derive  $\hat{l}_q(x)$  and  $\hat{\Lambda}_q(\cdot)$  as in Sect. 2.3. Then consider the following equation

$$W_q(x) = -\langle \hat{l}_q(x), \hat{\Lambda}_q^*(\hat{Y}_q) \rangle, \quad (19)$$

which implies (15), where  $\hat{Y}_q = \hat{Y}_q^T \in \mathbf{R}^{\hat{M}_q \times \hat{M}_q}$ . We derive an equivalent equation to (19) that does not involve the variable  $x$ . Define matrices  $\mathcal{A}_q, \mathcal{B}_{qj}, \mathcal{C}_q, \mathcal{D}_q, \mathcal{E}_q$  with appropriate sizes so that for some vector-valued function  $\phi_q^F(x), x \in D_q, q \in \mathbf{Q}$ , whose elements are linearly independent to each other, for which the following equalities hold

$$\begin{aligned} \frac{\partial l_q(x)}{\partial x} f_q^A(x) &= \mathcal{A}_q \phi_q^F(x), \\ \frac{\partial l_q(x)}{\partial x} [f_q^B(x)]^{*j} &= \mathcal{B}_{qj} \phi_q^F(x), \quad j = 1, \dots, n_q^B, \\ \tilde{l}_q(x) \|f_q^C(x)\|^2 &= \mathcal{C}_q \phi_q^F(x), \quad \tilde{l}_q(x) = \mathcal{D}_q \phi_q^F(x), \\ \hat{l}_q(x) &= \mathcal{E}_q \phi_q^F(x) \end{aligned}$$

for  $x \in D_q, q \in \mathbf{Q}$ . Then it is easy to see that (19) holds if and only if

$$\begin{cases} \mathcal{A}_q^T \Lambda_q^*(Y_q) + \mathcal{C}_q^T \tilde{\Lambda}_q^*(\tilde{Y}_q) + \mathcal{E}_q^T [\hat{\Lambda}_q^*(\hat{Y}_q)]^{11} = 0, \\ \frac{1}{2} \mathcal{B}_{qj}^T \Lambda_q^*(Y_q) + \mathcal{E}_q^T [\hat{\Lambda}_q^*(\hat{Y}_q)]^{1,j+1} = 0, \\ \quad j = 1, \dots, n_q^B, \\ -\mathcal{D}_q^T \tilde{\Lambda}_q^*(\tilde{Y}_q) + \mathcal{E}_q^T [\hat{\Lambda}_q^*(\hat{Y}_q)]^{j+1,j+1} = 0, \\ \quad j = 1, \dots, n_q^B, \\ \mathcal{E}_q^T [\hat{\Lambda}_q^*(\hat{Y}_q)]^{i+1,j+1} = 0, \\ \quad j = 1, \dots, n_q^B, i \neq j. \end{cases} \quad (20)$$

Lastly, consider the condition (16). For  $e \in E$ , let  $\bar{l}_{ej}(\xi)$  and  $\bar{\Lambda}_{ej}(\cdot)$  be a pair of a vector-valued function and a map that provide a sum of squares by  $\mu_{ej}(x) = \langle \bar{l}_{ej}(x), \bar{\Lambda}_{ej}^*(\bar{Y}_{ej}) \rangle$  with a positive semidefinite matrix  $\bar{Y}_{ej}$ . Define matrices  $\mathcal{F}_{ej}, \mathcal{G}_{ej}, \mathcal{H}_{ej}$  and a vector-valued function  $\phi_{ej}^R(\xi)$  with linearly independent elements satisfying  $l_q \circ g_{e0}^{(j)}(\xi) = \mathcal{F}_{ej} \phi_{ej}^R(\xi), l_r \circ R_e^N \circ g_{e0}^{(j)}(\xi) = \mathcal{G}_{ej} \phi_{ej}^R(\xi)$  and  $\bar{l}_{ej}(\xi) = \mathcal{H}_{ej} \phi_{ej}^R(\xi)$  for  $\xi \in G_e^{(j)}, j = 1, \dots, N_{G_e}, e = (q, r) \in E$ . Then  $V_q(x) - V_r(R_e^N(x)) = \langle \bar{l}_{ej}(x), \bar{\Lambda}_{ej}^*(\bar{Y}_{ej}) \rangle$  holds for all  $x \in G_e$  if and only if

$$\mathcal{F}_{ej}^T \Lambda_q^*(Y_q) - \mathcal{G}_{ej}^T \Lambda_r^*(Y_r) = \mathcal{H}_{ej}^T \bar{\Lambda}_{ej}^*(\bar{Y}_{ej}) \quad (21)$$

for  $j = 1, \dots, N_{G_e}, e = (q, r) \in E$ .

**Remark 4:** If  $R_e^N(x)$  is an identity map, i.e., the continuous

variable does not change in discrete evolutions of  $e$ , then it is often natural to set  $\mu_{ej}(\xi) = 0$ , by which the Lyapunov function is continuous at those discrete evolutions.  $\square$

We summarize the result:

**Theorem 1:** Suppose that (20) and (21) hold for positive semidefinite matrices  $Y_q, \tilde{Y}_q, \hat{Y}_q, q \in \mathbf{Q}$  and  $\tilde{Y}_{ej}, j = 1, \dots, N_{G_e}, e = (r, q) \in E$ . Then  $V_q(x) = \langle l_q(x), \Lambda_q^*(Y_q) \rangle \geq 0$  and  $\sigma_q(x) = \langle \tilde{l}_q(x), \tilde{\Lambda}_q^*(\tilde{Y}_q) \rangle \geq 0$  satisfy (15) and (16).  $\square$

Thus the stability analysis problem is reduced to a standard semidefinite programming problem, which enjoys various fast algorithms.

#### 4. Numerical Example

Let us consider the following hybrid automaton:

$$\mathbf{Q} = \mathbf{Q}_0 = \{+, -\}, \quad \mathbf{X}_+ = \mathbf{X}_- = \mathbf{R}^2,$$

$$\text{Init} = \{(+, x) : x \in \mathbf{R}^2, x_1 > 0\},$$

$$f_+^A(x) = \begin{bmatrix} -x_1 - x_2 - x_1^3 - x_1^2 x_2 - x_1 x_2^2 - x_2^3 \\ 2x_1 - x_2 + x_1^3 - x_1^2 x_2 + x_1 x_2^2 - x_2^3 \end{bmatrix},$$

$$f_-^A(x) = \begin{bmatrix} \begin{pmatrix} -0.5x_1 - 4x_2 + 0.5x_1^3 \\ -x_1^2 x_2 - x_1 x_2^2 - 4x_2^3 \end{pmatrix} \\ \begin{pmatrix} x_1 - 4x_2 + 0.5x_1^3 \\ -x_1^2 x_2 + x_1 x_2^2 - 4x_2^3 \end{pmatrix} \end{bmatrix},$$

$$f_+^B(x) = f_-^B(x) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top,$$

$$f_+^C(x) = f_-^C(x) = x_1,$$

$$D_+ = \{x : x_1 \geq 0\}, \quad D_- = \{x : x_1 \leq 0\},$$

$$E = \{e_1 = (-, +), e_2 = (+, -)\},$$

$$G_{e_1} = G_{e_2} = \{x : x_1 = 0\},$$

$$R_{e_1}^N(x) = R_{e_2}^N(x) = x.$$

In this section the  $i$ -th element of  $x$  is denoted by  $x_i$ . We seek a Lyapunov function and find an upper bound of  $k_+$  and  $k_-$  for which **Equi** is stable. We choose basis for  $V_q(x)$ ,  $\sigma_q(x)$  and  $W_q(x)$  as

$$\Omega_q(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \end{bmatrix}$$

$$\tilde{\Omega}_q(x) = \begin{bmatrix} x_1 & x_2 & 1 \end{bmatrix},$$

$$\hat{\Omega}_q(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 & x_1^3 & x_1^2 x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & x_1 x_2^2 & x_2^3 & 0 & 0 & 1 & \\ & 0 & 0 & x_1 & x_2 & 1 \end{bmatrix}$$

for  $q \in \mathbf{Q}$ , respectively, while we set  $\mu_{ej}(\xi) = 0$ . A Lyapunov function is obtained as

$$\begin{aligned} V_+(x) &= x_1^2 - 0.5x_1 x_2 + 0.25x_2^2 + 0.22x_1^4 \\ &\quad - 0.12x_1^3 x_2 + 0.23x_1^2 x_2^2 + 0.040x_1 x_2^3 + 0.12x_2^4, \\ V_-(x) &= 0.30x_1^2 - 0.40x_1 x_2 + 0.25x_2^2 + 0.074x_1^4 \\ &\quad - 0.31x_1^3 x_2 + 0.45x_1^2 x_2^2 - 0.22x_1 x_2^3 + 0.12x_2^4 \end{aligned}$$

with

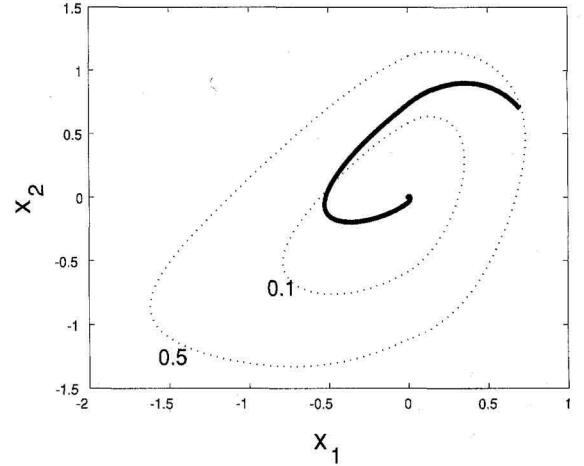


Fig. 1 An execution and level surfaces of the derived Lyapunov function.

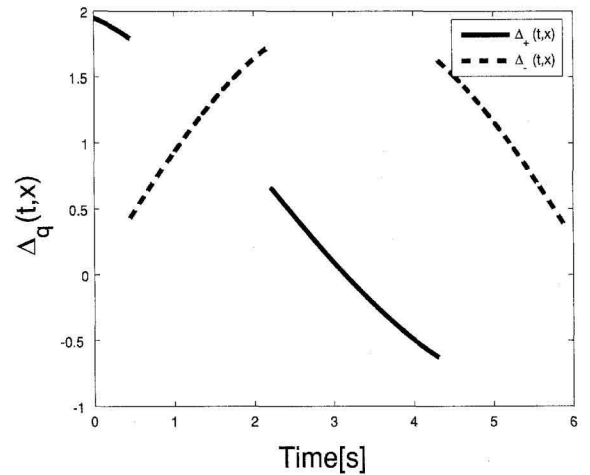


Fig. 2 Uncertainty profile.

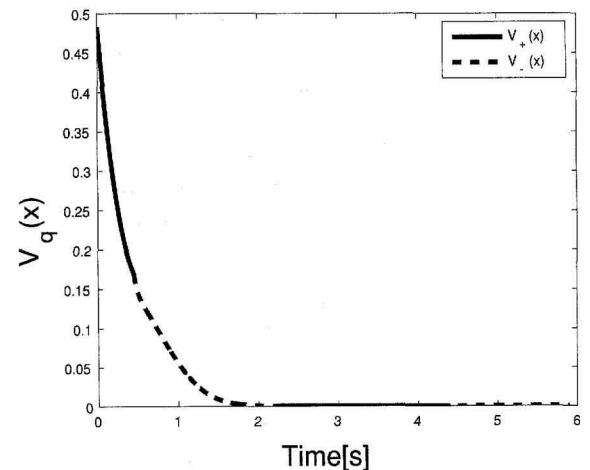


Fig. 3 Response of the derived Lyapunov function.

$$\begin{aligned} \sigma_+(x) &= 0.40x_1^2 + 0.014x_1 x_2 + 0.36x_2^2 + 0.39 \\ \sigma_-(x) &= 0.061x_1^2 - 0.015x_1 x_2 + 0.32x_2^2 + 0.105 \end{aligned}$$

and an upper bound of  $k_q$  is derived as  $k^* = 1.9975$ . We plot  $x_k(t)$  of the execution satisfying (13) in Fig. 1 for  $t_0 = 0$ ,  $q_0 = +$ ,  $x_0(t_0) = [0.7 \ 0.7]^\top$  with setting

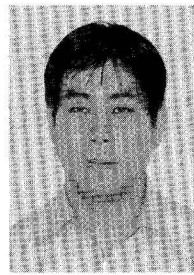
$\Delta_+(t, x) = k^* e^{(-t/8)} \cos(t/2)$  and  $\Delta_-(t, x) = k^* \sin(t/2)$ , shown as in Fig. 2. Level surfaces of the derived Lyapunov function  $V_q(x)$  are shown in Fig. 1, while  $V_{q_k}(x_k(t))$  is plotted in Fig. 3.

## 5. Conclusion

We have shown a method of computing Lyapunov functions for stability analysis of hybrid automata with set-valued vector fields. We generalized the formulation of scalar-valued sums of squares in [6] to matrix-valued ones and applied them to derive an LMI/LME problem whose solution yields a Lyapunov function.

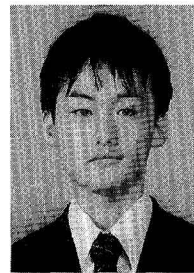
## References

- [1] R. Alur, C. Courcoubetis, T.A. Henzinger, and P.-H. Ho, "Hybrid automata: An algorithmic approach to the specification verification of hybrid systems," in *Hybrid Systems, LNCS 736*, ed. R.L. Grossman, A. Nerode, A.P. Ravn, and H. Rischel, pp.209–229, Springer, 1993.
- [2] R. Alur and D.L. Dill, "A theory of timed automata," *Theor. Comput. Sci.*, vol.126, no.2, pp.183–236, 1994.
- [3] M.S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Autom. Control*, vol.43, no.4, pp.475–482, 1998.
- [4] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Autom. Control*, vol.43, no.4, pp.555–559, 1998.
- [5] I. Masubuchi, S. Yabuki, and T. Tsuji, "Computation of Lyapunov functions for hybrid automata via LMIs," *IEICE Trans. Fundamentals*, vol.E87-A, no.11, pp.2937–2943, Nov. 2004.
- [6] Y. Nesterov, "Squared functional systems and optimization problems," in *High Performance Optimization*, ed. H. Frenk, K. Roos, and T. Terlaky, pp.405–440, Kluwer Academic Pub., 2000.
- [7] S. Prajna and A. Papachristodoulou, "Analysis of switched and hybrid systems: Beyond piecewise quadratic methods," *American Control Conference*, pp.2779–2784, 2003.
- [8] P.A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming Ser. B*, vol.96, no.2, pp.293–320, 2003.
- [9] S. Pettersson and B. Lennartson, "LMI for stability and robustness of hybrid systems," *Proc. Automatic Control Conference 1997*, pp.1714–1718, 1997.
- [10] S. Pettersson and B. Lennartson, "LMI for stability and robustness of hybrid systems," *Technical Report CTH/RT/I-96/005*, Control Engineering Lab., Chalmers University of Technology, 1998.
- [11] H. Ye, A.N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *IEEE Trans. Autom. Control*, vol.43, no.4, pp.461–474, 1998.
- [12] J. Zhang, K.H. Johansson, J. Lygeros, and S. Sastry, "Zeno hybrid systems," *Int. J. Robust and Nonlinear Control*, vol.11, no.5, pp.435–451, 2001.



He is a member of SICE, ISCIE, JSME and IEEE.

**Izumi Masubuchi** received the B.E., M.E. and Dr. Eng. degrees from Osaka University in 1991, 1993 and 1996, respectively. He was a research associate of Japan Advanced Institute of Science and Technology and Kobe University in 1996–1998 and 1998–2001, respectively. He is currently with Graduate School of Mechanical System Engineering, Hiroshima University, as an associate professor. His research interests include systems and control theory, robust control design and theory of hybrid



**Tokihisa Tsuji** received the B.E. and M.Eng. degrees from Hiroshima University in 2004 and 2006, respectively. He is currently with Daihatsu Motor Co., LTD.