

## PAPER

## Nonlinear Estimation of Harmonic Signals

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**SUMMARY** A nonlinear harmonic estimator (NHE) is proposed for extracting a harmonic signal and its fundamental frequency in the presence of white noise. This estimator is derived by applying an extended complex Kalman filter (ECKF) to a multiple sinusoidal model with state-representation and then efficiently specializing it for the case of harmonic estimation. The effectiveness of the NHE is verified using computer simulations.

**key words:** nonlinear filter, Kalman filter, harmonic estimation, frequency estimation, noise reduction, singular value decomposition

## 1. Introduction

Estimating a harmonic signal and its fundamental frequency from a finite number of noisy observation data is of great importance in many applications of signal processing, including speech enhancement and harmonic retrieval in power system. Indeed, various estimation methods have been proposed [1]–[6].

The comb filter is one of popular methods, of which pass-bands are adjusted to a discrete spectral structure of harmonic signals. This filter is easily implemented in real time after an initial design of the impulse response. The performance is, however, much dependent on pitch estimation accuracy [2]. The alternative methods are based on the use of the Kalman filter, which dynamically estimates harmonic amplitudes and a fundamental frequency (pitch frequency) [3]–[6]. However, they required precise knowledge for describing the dynamics of phase rotation or amplitude of each harmonic component.

Under the circumstances, a nonlinear multiple sinusoidal estimator (NMSE) [8] has been proposed for extracting multiple complex sinusoids in the presence of white noise. The NMSE is an extended and improved version of the single sinusoidal estimator [7], which was previously derived by the author using an extended complex Kalman filter (ECKF). Fortunately, the stability (convergence) of the NMSE is also theoretically clarified [8]. Also, the singular value decomposition (SVD) [9], [10] is effectively combined with the ECKF as the initial state estimation.

In this paper, the NMSE is efficiently specialized to harmonic estimation so that its computational complexity is adequately reduced. The resulting filter is more applicable than the NMSE for practical harmonic estimation problems, which is referred to as a nonlinear harmonic estimator (NHE). Although the NHE is derived on the basis of a harmonic model with time-invariant parameters, it can easily cope with the time-varying cases by adding a certain process noise in the state equation. The performance of the NHE is indeed evaluated for harmonic signals with time-varying fundamental frequency using computer simulations.

The rest of this paper is organized as follows. Section 2 presents the harmonic estimation problem treated here. Section 3 outlines the NMSE. Based on the results of Sect. 3, the NHE is derived in Sect. 4. In Sect. 5, we present the results of some simulations that were performed to verify the derivation of the NHE and show its effectiveness for harmonic estimation. Finally, the conclusions are given in Sect. 6.

## 2. Problem Formulation

Let an observation signal  $y_k$  be a superposition  $z_k$  of  $M$  harmonics with additive observation noise  $v_k$  as

$$y_k = z_k + v_k, \quad k = 1, 2, 3, \dots, N \quad (1)$$

where

$$z_k = \sum_{m=1}^M a_m \exp(j\omega_m t_k), \quad t_k = k\Delta t$$

$$a_m = |a_m|e^{j\phi_m}, \quad \omega_m = m\omega_1, \quad \omega_1 = 2\pi f_1 \quad (2)$$

in which  $f_1$ ,  $a_m$ , and  $\phi_m$  are the fundamental frequency, the complex amplitude, and the initial phase of the  $m$ th harmonic, respectively, and they are unknown and possibly time-varying. Also,  $\Delta t$  is a sampling period and  $j^2 = -1$ . The observation noise  $v_k$  is a stationary, white complex Gaussian noise with zero mean and variance  $\sigma_v^2$ , where real and imaginary parts of  $v_k$  are mutually independent and have the same variance, i.e.,  $\sigma_v^2 = \sigma_{v_r}^2 + \sigma_{v_i}^2$  and  $\sigma_{v_r}^2 = \sigma_{v_i}^2$ .

The aim in this paper is to estimate a harmonic signal  $\{z_k\}$  and its fundamental frequency  $f_1$  under the assumption that the number  $M$  of harmonics is known but the variance  $\sigma_v^2$  of noise are unknown.

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### 3. Nonlinear Estimation of Multiple Complex Sinusoids

The key points to successfully use the extended Kalman filters (EKF) [11], [12] are accurate modeling and effective initialization. In this section, the single sinusoidal estimator [7] based on the ECKF is effectively extended to the case of multiple complex sinusoids, using the SVD as the initial estimation [8].

#### 3.1 Nonlinear Multiple Sinusoidal Estimator (NMSE)

For a noiseless signal  $\{z_k\}$  which consists of  $M$  complex sinusoids, noting that the difference equation  $z_{k+1} = \alpha(1)z_k + \dots + \alpha(M)z_{k-M+1}$  holds, we obtain

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{w}_k \quad (3)$$

$$y_k = \mathbf{H}\mathbf{x}_k + v_k \quad (4)$$

where

$$\begin{aligned} \mathbf{x}_k &= \begin{bmatrix} \alpha_k^T & \mathbf{z}_k^T \end{bmatrix}^T \in \mathbb{C}^{2M} \\ &= [\alpha_k(1), \dots, \alpha_k(M), z_k, \dots, z_{k-M+1}]^T \\ \mathbf{f}(\mathbf{x}_k) &= [\alpha_k(1), \dots, \alpha_k(M), \\ &\quad \alpha_k(1)z_k + \dots + \alpha_k(M)z_{k-M+1}, \\ &\quad z_k, \dots, z_{k-M+2}]^T \\ \mathbf{H} &= [\mathbf{H}_1 \ \mathbf{H}_2], \mathbf{H}_1 = \mathbf{0}^T, \mathbf{H}_2 = [1, 0, \dots, 0]. \end{aligned} \quad (5)$$

A small amount of system noise  $\mathbf{w}_k$  with zero mean and covariance  $\Sigma_w$ , which is added in (3), enables the NMSE to track statistical variations in sinusoids, where every entry  $w_{m,k}$  of  $\mathbf{w}_k$  is assumed to be mutually independent.

Applying the ECKF to the above state-space model and arranging it, we obtain a nonlinear multiple sinusoidal estimator (NMSE) which sequentially provides the quasi-optimal estimate  $\hat{\mathbf{x}}_{k|k}$  of the state vector  $\mathbf{x}_k$  as

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(y_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}) \\ \hat{\mathbf{x}}_{k+1|k} &= \mathbf{f}(\hat{\mathbf{x}}_{k|k}) \end{aligned} \quad (6)$$

$$\mathbf{K}_k = \hat{\mathbf{P}}_{k|k-1} \mathbf{H}^* \mathbf{H} \hat{\mathbf{P}}_{k|k-1} \mathbf{H}^{*T} + 1)^{-1} \quad (7)$$

$$\begin{aligned} \hat{\mathbf{P}}_{k|k} &= \hat{\mathbf{P}}_{k|k-1} - \mathbf{K}_k \mathbf{H} \hat{\mathbf{P}}_{k|k-1} \\ \hat{\mathbf{P}}_{k+1|k} &= \mathbf{F}_k \hat{\mathbf{P}}_{k|k} \mathbf{F}_k^* + \mathbf{Q}_w \end{aligned} \quad (8)$$

where

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= [\hat{x}_{k|k}(1), \dots, \hat{x}_{k|k}(2M)]^T = [\hat{\alpha}_{k|k}^T \ \hat{\mathbf{z}}_{k|k}^T]^T \\ \mathbf{F}_k &= \frac{\partial \mathbf{f}(\mathbf{x}_k)}{\partial \mathbf{x}_k} \bigg|_{\mathbf{x}_k = \hat{\mathbf{x}}_{k|k}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B}_k & \mathbf{A}_k \end{bmatrix} \in \mathbb{C}^{2M \times 2M} \\ \mathbf{A}_k &= \begin{bmatrix} \hat{\alpha}_{k|k}(1) & \dots & \dots & \dots & \hat{\alpha}_{k|k}(M) \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

$$\mathbf{B}_k = \begin{bmatrix} \hat{z}_{k|k} & \dots & \dots & \hat{z}_{k-M+1|k} \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{Q}_w &= \Sigma_w / \sigma_v^2, \Sigma_w = E\{\mathbf{w}_k \mathbf{w}_k^* T\}, \sigma_v^2 = E\{v_k^2\} \\ \hat{\mathbf{P}}_{k|k} &= \hat{\Sigma}_{k|k} / \sigma_v^2, \hat{\mathbf{P}}_{k+1|k} = \hat{\Sigma}_{k+1|k} / \sigma_v^2 \in \mathbb{C}^{2M \times 2M} \\ \hat{\Sigma}_{k+1|k} &= E\{(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})^* T | \mathcal{Y}_k\} \\ \hat{\Sigma}_{k|k} &= E\{(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k})^* T | \mathcal{Y}_k\} \end{aligned} \quad (9)$$

in which  $*$  and  $T$  denote the complex conjugate and transpose operations, respectively,  $\mathbf{I}$  the identity matrix,  $E\{\cdot\}$  the expectation of  $\cdot$ , and  $\mathcal{Y}_k = \{y_0, \dots, y_k\}$ . It should be noted here that the derived estimator is a nonlinear filter with respect to the state vector, so that the gain  $\mathbf{K}_k$  and covariance matrices  $\hat{\mathbf{P}}_{k|k}$ ,  $\hat{\mathbf{P}}_{k+1|k}$  depend on the estimate  $\hat{\mathbf{x}}_{k|k}$ . However, the nonlinearity is considerably weak. Indeed, the Taylor expansion of  $\mathbf{f}(\mathbf{x}_k)$  does not involve terms higher than the second order.

#### 3.2 Initial Estimation for the NMSE

Frequency estimation of multiple complex sinusoids based on the SVD is described [9], [10]. The resultant estimates are used as the initial state variables in the NMSE, which have a good accuracy when frequency of each sinusoid is time-invariant.

In a linear prediction method, the coefficients  $\{\alpha(i)\}$  approximately satisfy the following linear prediction equations:

$$\begin{bmatrix} y_L & \dots & y_1 \\ y_{L+1} & \dots & y_2 \\ \vdots & \ddots & \vdots \\ y_{N_s-1} & \dots & y_{N_s-L} \end{bmatrix} \begin{bmatrix} \alpha(1) \\ \alpha(2) \\ \vdots \\ \alpha(L) \end{bmatrix} = \begin{bmatrix} y_{L+1} \\ y_{L+2} \\ \vdots \\ y_{N_s} \end{bmatrix}$$

or in a vector-matrix form

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{b} \quad (10)$$

where  $N_s$  is the length of data used in the SVD, and  $L$  must be larger than the number  $M$  of complex sinusoids being sought.

The minimum norm solution to minimize the squared error  $\|\mathbf{b} - \mathbf{A}\boldsymbol{\alpha}\|^2$  is given by

$$\hat{\boldsymbol{\alpha}} = \mathbf{A}^+ \mathbf{b}, \quad \hat{\boldsymbol{\alpha}} = [\hat{\alpha}_{SVD}(1), \dots, \hat{\alpha}_{SVD}(L)]^T \quad (11)$$

where  $\mathbf{A}^+$  is determined as

$$\begin{aligned} \mathbf{A}^+ &= \mathbf{V} \hat{\boldsymbol{\Lambda}}^+ \mathbf{U}^{*T}, \\ \hat{\boldsymbol{\Lambda}}^+ &= \text{diag}\{1/\lambda_1, \dots, 1/\lambda_M, 0, \dots, 0\} \end{aligned} \quad (12)$$

using the SVD of  $\mathbf{A}$  ( $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{*T}$ ). That is, a diagonal matrix  $\hat{\boldsymbol{\Lambda}}^+$  of the same size as  $\mathbf{A}$  is obtained from  $\boldsymbol{\Lambda} = \text{diag}\{\lambda_1, \dots, \lambda_L\}$  by setting to zero all but its  $M(< L)$  largest singular values and by replacing each nonzero diagonal entry with its reciprocal.

By solving  $z^L - \hat{\alpha}_{SVD}(1)z^{L-1} - \dots - \hat{\alpha}_{SVD}(L) = 0$  and employing the  $M$  nearest roots to the unit circle, we can obtain the frequency estimate of each of complex sinusoids ( $\exp(j\omega_m t_k), m = 1, \dots, M$ ).

#### 4. The Nonlinear Harmonic Estimator

There is a possibility of efficiently reducing the NMSE for harmonic estimation since each harmonic has a strong regularity on frequency such as  $\omega_m = m\omega_1$ .

##### 4.1 Specialization of the NMSE to Harmonic Estimation

Similar to sinusoids, a complex signal  $\{z_k\}$  consisting of  $M$  harmonics satisfies

$$z_{k+1} = \alpha(1)z_k + \alpha(2)z_{k-1} + \cdots + \alpha(M)z_{k-M+1} \quad (13)$$

whose coefficients  $\{\alpha(i)\}$  equal to those of the following equation:

$$z^M - (\gamma_1 + \cdots + \gamma_1^M)z^{M-1} - \cdots + (-1)^M \gamma_1^{(1+\cdots+M)} = 0 \quad (14)$$

where  $\gamma_1 = e^{j\omega_1 \Delta t}$ . This fact can be verified by expanding the polynomial

$$(z - \gamma_1)(z - \gamma_1^2) \cdots (z - \gamma_1^M)$$

and then regarding  $z^{-1}$  as a delay operator.

Using (13), we can represent the state-space model of  $M$  harmonics as

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \mathbf{w}_k \quad (15)$$

$$y_k = \mathbf{H}\mathbf{x}_k + v_k \quad (16)$$

where

$$\begin{aligned} \mathbf{x}_k &= [\gamma_1, z_k, \cdots, z_{k-M+1}]^T \in \mathbb{C}^{M+1} \\ \mathbf{f}(\mathbf{x}_k) &= [\gamma_1, \alpha(1)z_k + \cdots + \alpha(M)z_{k-M+1}, \\ &\quad z_k, \cdots, z_{k-M+2}]^T \in \mathbb{C}^{M+1} \\ \mathbf{H} &= \begin{bmatrix} 0 & \mathbf{H}_2 \end{bmatrix} \in \mathbb{R}^{1 \times (M+1)}. \end{aligned} \quad (17)$$

Applying the ECKF to the above-mentioned model provides

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(y_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}) \\ \hat{\mathbf{x}}_{k+1|k} &= \mathbf{f}(\hat{\mathbf{x}}_{k|k}) \end{aligned} \quad (18)$$

$$\mathbf{K}_k = \hat{\mathbf{P}}_{k|k-1} \mathbf{H}^* T (\mathbf{H} \hat{\mathbf{P}}_{k|k-1} \mathbf{H}^* T + 1)^{-1} \quad (19)$$

$$\begin{aligned} \hat{\mathbf{P}}_{k|k} &= \hat{\mathbf{P}}_{k|k-1} - \mathbf{K}_k \mathbf{H} \hat{\mathbf{P}}_{k|k-1} \\ \hat{\mathbf{P}}_{k+1|k} &= \mathbf{F}_k \hat{\mathbf{P}}_{k|k} \mathbf{F}_k^* T + \mathbf{Q}_w \end{aligned} \quad (20)$$

where

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= [\hat{x}_{k|k}(1), \cdots, \hat{x}_{k|k}(M+1)]^T = [\hat{\gamma}_1 | \hat{\mathbf{z}}_{k|k}^T]^T \\ \mathbf{F}_k &= \left. \frac{\partial \mathbf{f}(\mathbf{x}_k)}{\partial \mathbf{x}_k} \right|_{\mathbf{x}_k = \hat{\mathbf{x}}_{k|k}} \\ &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{b}_k & \mathbf{A}_k \end{bmatrix} \in \mathbb{C}^{(M+1) \times (M+1)} \\ \mathbf{b}_k &= [(-1)^2 \hat{\alpha}'_{k|k}(1) \hat{x}_{k|k}(2) + \cdots \\ &\quad + (-1)^{M+1} \hat{\alpha}'_{k|k}(M) \hat{x}_{k|k}(M+1), 0, \cdots, 0]^T. \end{aligned} \quad (21)$$

Note that  $\mathbf{A}_k$  in (21) is identical to  $\mathbf{A}_k$  in (9). This filter is

referred to as a nonlinear harmonic estimator (NHE) hereafter.

Unfortunately, all the estimates  $\{\hat{\alpha}_{k|k}(i)\}_{i=1}^M$  of  $\{\alpha(i)\}_{i=1}^M$  included in  $\mathbf{A}_k$  cannot be directly obtained from the estimated state vector  $\hat{\mathbf{x}}_{k|k}$ . Furthermore, the derivative  $\hat{\alpha}'_{k|k}(i)$  of  $\hat{\alpha}_{k|k}(i)$  with respect to  $\gamma_1$  is required to determine  $\mathbf{b}_k$  in  $\mathbf{F}_k$ .

##### 4.2 Determination of the Coefficients of the Polynomial

Although  $\{\alpha(i)\}_{i=1}^M$  are determined from  $\gamma_1$ , it becomes very complicated for large  $M$ . The following gives an example of the coefficients  $\{\alpha(i)\}_{i=1}^M$  for  $M = 11$ :

$$\begin{aligned} \alpha(1) &= \gamma_1 + \gamma_1^2 + \cdots + \gamma_1^{11} \\ \alpha(5) &= \gamma_1^{15} + \gamma_1^{16} + 2\gamma_1^{17} + 3\gamma_1^{18} \\ &\quad + 5\gamma_1^{19} + 7\gamma_1^{20} + 10\gamma_1^{21} + 12\gamma_1^{22} \\ &\quad + 16\gamma_1^{23} + 19\gamma_1^{24} + 23\gamma_1^{25} + 25\gamma_1^{26} \\ &\quad + 29\gamma_1^{27} + 30\gamma_1^{28} + 32\gamma_1^{29} + 32\gamma_1^{30} \\ &\quad + 32\gamma_1^{31} + 30\gamma_1^{32} + 29\gamma_1^{33} + 25\gamma_1^{34} \\ &\quad + 23\gamma_1^{35} + 19\gamma_1^{36} + 16\gamma_1^{37} + 12\gamma_1^{38} \\ &\quad + 10\gamma_1^{39} + 7\gamma_1^{40} + 5\gamma_1^{41} + 3\gamma_1^{42} \\ &\quad + 2\gamma_1^{43} + \gamma_1^{44} + \gamma_1^{45} \\ \alpha(6) &= -(\gamma_1^{21} + \gamma_1^{22} + 2\gamma_1^{23} + 3\gamma_1^{24} \\ &\quad + 5\gamma_1^{25} + 7\gamma_1^{26} + 10\gamma_1^{27} + 12\gamma_1^{28} \\ &\quad + 16\gamma_1^{29} + 19\gamma_1^{30} + 23\gamma_1^{31} + 25\gamma_1^{32} \\ &\quad + 29\gamma_1^{33} + 30\gamma_1^{34} + 32\gamma_1^{35} + 32\gamma_1^{36} \\ &\quad + 32\gamma_1^{37} + 30\gamma_1^{38} + 29\gamma_1^{39} + 25\gamma_1^{40} \\ &\quad + 23\gamma_1^{41} + 19\gamma_1^{42} + 16\gamma_1^{43} + 12\gamma_1^{44} \\ &\quad + 10\gamma_1^{45} + 7\gamma_1^{46} + 5\gamma_1^{47} + 3\gamma_1^{48} \\ &\quad + 2\gamma_1^{49} + \gamma_1^{50} + \gamma_1^{51}) \\ \alpha(11) &= \gamma_1^{66}. \end{aligned}$$

Through some considerations, we have reached a simple recursive algorithm to efficiently determine  $\{\alpha(i)\}_{i=1}^M$  from  $\gamma_1$  as follows.

**Lemma 1:** For any  $M > 0$ , the coefficients  $\{\alpha(i)\}_{i=1}^M$  is recursively computed using  $\gamma_1$  as

$$\alpha(i) = (-1)^{i+1} \alpha(i; M), \quad 1 \leq i \leq M \quad (22)$$

where

$$\begin{aligned} \alpha(i; M) &= \begin{cases} \sum_{n=1}^M \gamma_1^n, & i = 1 \\ \alpha(i; M-1) + \alpha(i-1; M-1) \gamma_1^M, & 2 \leq i \leq M-1 \\ \sum_{n=1}^M \gamma_1^n, & i = M \end{cases} \end{aligned} \quad (23)$$

The lemma alone is not sufficient to determine  $F_k$ . The derivatives  $\{\hat{\alpha}'_{k|k}(m)\}$  appeared in  $b_k$  are also necessary.

**Lemma 2:** For any  $M > 0$ , the derivatives  $\{\alpha'(i)\}_{i=1}^M$  of  $\{\alpha(i)\}_{i=1}^M$  with respect to  $\gamma_1$  are recursively obtained as

$$\alpha'(i) = (-1)^{i+1} \alpha'(i; M), \quad 1 \leq i \leq M \quad (24)$$

where

$$\alpha'(i; M) = \begin{cases} \sum_{n=1}^M n \gamma_1^{n-1}, & i = 1 \\ \alpha'(i; M-1) + \alpha'(i-1; M-1) \gamma_1^M + \alpha(i-1; M-1) M \gamma_1^{M-1}, & 2 \leq i \leq M-1 \\ \left( \sum_{n=1}^M n \right) \gamma_1^{n-1}, & i = M \end{cases} \quad (25)$$

Note that all the proofs are described in Appendix. ■

#### 4.3 Derivation of the Algorithm

Eventually, the resulting harmonic estimator is summarized as follows.

**Theorem 1: (The Nonlinear Harmonic Estimator: NHE)** For the following state-space model of  $M$  harmonics:

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}(\mathbf{x}_k) + \mathbf{w}_k \\ y_k &= \mathbf{H} \mathbf{x}_k + v_k, \quad \mathbf{H} = \begin{bmatrix} 0 & \mathbf{H}_2 \end{bmatrix} \end{aligned}$$

the ECKF is recursively performed by

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k (y_k - \mathbf{H} \hat{\mathbf{x}}_{k|k-1}) \quad (26)$$

$$\hat{\mathbf{x}}_{k+1|k} = \mathbf{f}(\hat{\mathbf{x}}_{k|k}) \quad (27)$$

$$\mathbf{K}_k = \hat{\mathbf{P}}_{k|k-1} \mathbf{H}^* T (\mathbf{H} \hat{\mathbf{P}}_{k|k-1} \mathbf{H}^* T + 1)^{-1} \quad (28)$$

$$\hat{\mathbf{P}}_{k|k} = \hat{\mathbf{P}}_{k|k-1} - \mathbf{K}_k \mathbf{H} \hat{\mathbf{P}}_{k|k-1} \quad (29)$$

$$\hat{\mathbf{P}}_{k+1|k} = \mathbf{F}_k \hat{\mathbf{P}}_{k|k} \mathbf{F}_k^* T + \mathbf{Q}_w \quad (29)$$

where

$$\begin{aligned} \hat{\mathbf{x}}_{k|k} &= [\hat{x}_{k|k}(1), \dots, \hat{x}_{k|k}(M+1)]^T = [\hat{\gamma}_{1|k} \quad \hat{\mathbf{z}}_{k|k}^T]^T \\ \mathbf{f}(\hat{\mathbf{x}}_{k|k}) &= [\hat{x}_{k|k}(1), \hat{\alpha}_{k|k}(1) \hat{x}_{k|k}(2) + \dots \\ &\quad + \hat{\alpha}_{k|k}(M) \hat{x}_{k|k}(M+1), \hat{x}_{k|k}(2), \dots, \hat{x}_{k|k}(M)]^T \\ \mathbf{F}_k &= \frac{\partial \mathbf{f}(\mathbf{x}_k)}{\partial \mathbf{x}_k} \bigg|_{\mathbf{x}_k = \hat{\mathbf{x}}_{k|k}} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{b}_k & \mathbf{A}_k \end{bmatrix} \\ \mathbf{A}_k &= \begin{bmatrix} \hat{\alpha}_{k|k}(1) & \dots & \dots & \dots & \hat{\alpha}_{k|k}(M) \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \end{aligned} \quad (30)$$

$$\begin{aligned} \mathbf{b}_k &= [(-1)^2 \hat{\alpha}'_{k|k}(1) \hat{x}_{k|k}(2) + \dots \\ &\quad + (-1)^{M+1} \hat{\alpha}'_{k|k}(M) \hat{x}_{k|k}(M+1), 0, \dots, 0]^T \end{aligned}$$

in which  $\hat{\alpha}_{k|k}(i)$  and  $\hat{\alpha}'_{k|k}(i)$  can be recursively computed by

$$\hat{\alpha}_{k|k}(i) = (-1)^{i+1} \hat{\alpha}_{k|k}(i; M), \quad 1 \leq i \leq M \quad (31)$$

$$\hat{\alpha}_{k|k}(i; M) = \begin{cases} \sum_{n=1}^M \hat{\gamma}_{1|k}^n, & i = 1 \\ \hat{\alpha}_{k|k}(i; M-1) + \hat{\alpha}_{k|k}(i-1; M-1) \hat{\gamma}_{1|k}^M, & 2 \leq i \leq M-1 \\ \sum_{n=1}^M n \hat{\gamma}_{1|k}^{n-1}, & i = M \end{cases} \quad (32)$$

and

$$\hat{\alpha}'_{k|k}(i) = (-1)^{i+1} \hat{\alpha}'_{k|k}(i; M), \quad 1 \leq i \leq M \quad (33)$$

$$\hat{\alpha}'_{k|k}(i; M) = \begin{cases} \sum_{n=1}^M n \hat{\gamma}_{1|k}^{n-1}, & i = 1 \\ \hat{\alpha}'_{k|k}(i; M-1) + \hat{\alpha}'_{k|k}(i-1; M-1) \hat{\gamma}_{1|k}^M + \hat{\alpha}_{k|k}(i-1; M-1) M \hat{\gamma}_{1|k}^{M-1}, & 2 \leq i \leq M-1 \\ \left( \sum_{n=1}^M n \right) \hat{\gamma}_{1|k}^{n-1}, & i = M \end{cases} \quad (34)$$

respectively, using the estimate  $\hat{\gamma}_{1|k}$  of  $\gamma_1$ .

**Proof 1:** It is clear from Lemmas 1 and 2. ■

Furthermore, we will find that there is the following relationship between  $\alpha(i)$  and  $\alpha(M-i)$ .

**Corollary 1:** The coefficients of the polynomial of (13) satisfy

$$\begin{aligned} \alpha(C+i) &= \alpha(C-(i-1)) \times \gamma_1^{i(M+1)}, \quad 1 \leq i \leq C-1 \end{aligned} \quad (35)$$

when  $M$  is an even number, which do

$$\begin{aligned} \alpha(C+i) &= -\alpha(C-(i-1)) \gamma_1^{\left(\frac{M+1}{2}\right)(2i-1)}, \quad 1 \leq i \leq C \end{aligned} \quad (36)$$

when  $M$  is an odd number. Here,

$$C = \begin{cases} \frac{M}{2}, & \text{when } M \text{ is an even number} \\ \frac{M-1}{2}, & \text{otherwise} \end{cases}$$

and  $\alpha(0) = 1$ . ■

To determine  $\mathbf{A}_k$ , therefore, it is sufficient to calculate half of the coefficients such as  $\alpha(1), \alpha(2), \dots, \alpha(C), \alpha(M)$ . Similarly, we have a simplification for  $\alpha'(i)$ .

**Corollary 2:** The derivatives of the coefficients satisfy

$$\begin{aligned} \alpha'(C+i) &= i(M+1)\alpha(C-(i-1))\gamma_1^{i(M+1)-1} \\ &\quad + \alpha'(C-(i-1))\gamma_1^{i(M+1)}, \quad 1 \leq i \leq C-1 \end{aligned} \quad (37)$$

when  $M$  is an even number, which do

$$\begin{aligned} \alpha'(C+i) &= -\frac{M+1}{2}(2i-1)\gamma_1^{\frac{M+1}{2}(2i-1)-1}\alpha(C-(i-1)) \\ &\quad - \gamma_1^{\frac{M+1}{2}(2i-1)}\alpha'(C-(i-1)), \quad 1 \leq i \leq C \end{aligned} \quad (38)$$

when  $M$  is an odd number.

Corollaries 1 and 2 will provide further computational reduction in the NHE.

#### 4.4 Computational Complexity

The major computational burden of the NMSE and the NHE lies in the updating of  $\hat{\mathbf{P}}_{k|k-1}$  and  $\hat{\mathbf{P}}_{k|k}$ . For estimation of  $M$  harmonics, the NHE reduces the order of arithmetic operations per iteration from  $\mathcal{O}(16M^3)$  to  $\mathcal{O}(2M^3)$  for large  $M$  in comparison with the NMSE.

Furthermore, multiplying  $\mathbf{K}_k$  with  $\mathbf{H}\hat{\mathbf{P}}_{k|k-1}$  in the updating of  $\hat{\mathbf{P}}_{k|k-1}$  and using the sparseness of the matrices  $\mathbf{A}_k$  and  $\mathbf{B}_k$  for the calculation of  $\mathbf{F}_k\hat{\mathbf{P}}_{k|k}\mathbf{F}_k^T$ , one could reduce the computational complexity from  $\mathcal{O}(16M^2)$  to  $\mathcal{O}(4M^2)$ .

### 5. Simulation Study

In this section, the performance of the NHE is evaluated using computer simulations. Remark that all calculations are carried out with MATLAB.

#### 5.1 Performance for a Synthesized Signal

The proposed NHE is tested on a synthetic harmonic signal, consisting of 11 harmonics ( $M = 11$ ), whose fundamental angular frequency varies in time as

$$\omega_1(k) = \left(1 - 0.2 \cos\left(\frac{\pi(k-1)}{L}\right)\right)\omega_1, \quad L = 180 \quad (39)$$

and complex amplitudes  $a_1, a_2, \dots, a_M$  are set to  $e^{-j\pi/3}$ ,  $1/2e^{j\pi/5}$ ,  $1/3e^{j\pi/2}$ ,  $1/4e^{j\pi/4}$ ,  $1/5e^{-j\pi/3}$ ,  $1/6e^{j\pi/5}$ ,  $1/7e^{-j\pi/3}$ ,  $1/8e^{j\pi/2}$ ,  $1/9e^{j\pi/4}$ ,  $1/10e^{-j\pi/3}$ ,  $1/11e^{j\pi/5}$ , respectively, where  $L$  being 180 is data length. The variance  $\sigma_v^2$  of a stationary observation noise  $v_k$  with zero mean is chosen so as to yield a given signal-to-noise ratio (SNR), which is defined by

$$SNR = 10 \log_{10} \frac{\max\{|a_m|^2\}}{\sigma_v^2} = 10 \log_{10} \frac{1}{\sigma_v^2} [\text{dB}]. \quad (40)$$

Note that  $\sigma_v^2 = \sigma_{v_r}^2 + \sigma_{v_i}^2$  and  $\sigma_{v_r}^2 = \sigma_{v_i}^2$  hold due to the complex Gaussian property of  $v_k$ .

Figure 1 shows an original harmonic signal  $\{z_k\}$  (broken line) with  $f_1 = 2.3/80$ , the observation  $\{y_k\}$  (solid line) at  $SNR = 5[\text{dB}]$ , and Fig. 2 the estimates  $\{\hat{z}_{k|k}\}$ ,  $\hat{\omega}_1(k)$  (solid line) of  $\{z_k\}$  and  $\omega_1(k)$  (broken line) by the NHE. In the initial setting of the NHE,  $\hat{x}_{1|0}(1)$  is put to the estimate  $\hat{\gamma}_{SVD}(1)$  of  $\gamma_1$  obtained by the SVD ( $N_s = 60, L = 20$ ), while  $\hat{x}_{1|0}(i)$ ,  $i = 2, \dots, M+1$  are merely set to the samples of the observed data, i.e.,  $\hat{x}_{1|0}(1) = \hat{\gamma}_{SVD}(1)$ ,  $\hat{x}_{1|0}(2) = y_M$ ,  $\dots$ ,  $\hat{x}_{1|0}(M+1) = y_1$ . Note that a combination of the ECKF and the SVD in the NHE reduces each of their disadvantages since the ECKF is poor at initial estimation whereas the SVD, which is batch-processed, is unsuitable for the time-varying cases.

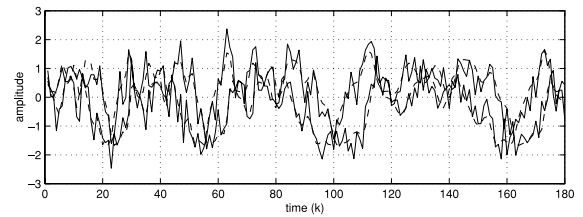
From experience, the error covariance matrix  $\hat{\mathbf{P}}_{k|k-1}$  in the NHE is initialized with

$$\hat{\mathbf{P}}_{1|0} = \begin{bmatrix} \hat{\mathbf{P}}_{1|0}^{11} & 0 \\ 0 & \hat{\mathbf{P}}_{1|0}^{22} \end{bmatrix} \in \mathbb{C}^{M+1 \times M+1} \quad (41)$$

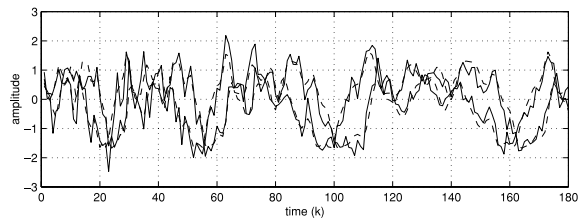
$$\hat{\mathbf{P}}_{1|0}^{11} = 10^{-4}, \quad \hat{\mathbf{P}}_{1|0}^{22} = 10^4 \mathbf{I} \in \mathbb{C}^{M \times M} \quad (42)$$

and the covariance matrix  $\mathbf{Q}_w$  of system noise  $\mathbf{w}_k$  is chosen as  $\mathbf{Q}_w = \text{diag}\{\rho_1, \dots, \rho_{M+1}\}$ ,  $\rho_1 = 1.0 \times 10^{-3}$ ,  $\rho_2 = 1.0 \times 10^{-2}$ ,  $\rho_3 = 1.0 \times 10^{-4}$ ,  $\rho_4 = 1.0 \times 10^{-6}$ , and zero otherwise, where  $\text{diag}\{\cdot\}$  denotes the diagonal matrix.

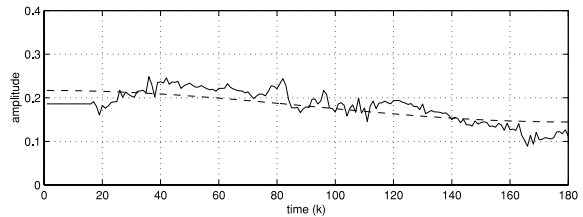
As seen in Fig. 2, the NHE is available for estimating a harmonic signal as well as its time-varying fundamental frequency.



**Fig. 1** Original signal (broken line) and observed one (solid line);  $f_1 = 2.3/80$ ,  $M = 11$ ,  $SNR = 5$  dB.



(a) Estimate of  $z_k$



(b) Estimate of  $\omega_1(k)$

**Fig. 2** Estimation results by the NHE with  $M = 11$  where the broken line denotes the true values.

**Table 1** Performance Evaluation of the NHE for a synthesized harmonic signal with  $M = 11$ .

	$CR$	$NR$ [dB]
ACF	0.66	4.65
NHE	0.86	-3.24

Table 1 shows the performance of the NHE along with an adaptive comb filter (ACF) [2], where the correlation  $CR$  and the noise reduction  $NR$  are defined as

$$CR = \frac{|\sum_{k=1}^L z_k \hat{z}_k|^2}{\sum_{k=1}^L |z_k|^2 \sum_{k=1}^L |\hat{z}_k|^2} \quad (43)$$

$$NR = 10 \log \frac{\sum_{k=1}^L |\hat{z}_k - z_k|^2}{\sum_{k=1}^L |v_k|^2} \text{ [dB]}. \quad (44)$$

The length of the ACF is 3 pitch period, which is experimentally optimized, and the pitch frequency (fundamental frequency) is same as the initial estimate in the NHE that is obtained by the SVD. From the results in Table 1, we see that the ACF cannot track such quick variations in fundamental frequency.

On the other hand, the computation time of the NMSE for the harmonic estimation was more than 2.7 times that of the NHE on SUN SPARC II, where the computation time was measured using the *etime* command in MATLAB. Additionally, the NHE was superior to the NMSE in frequency estimation accuracy and convergence.

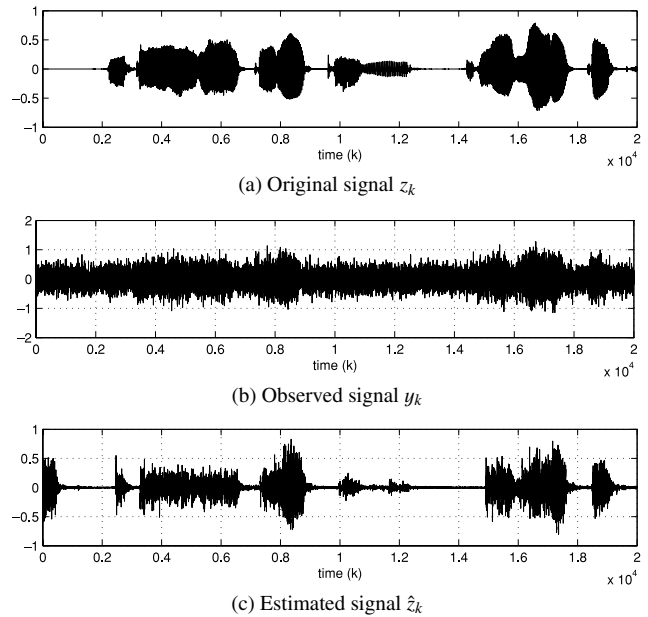
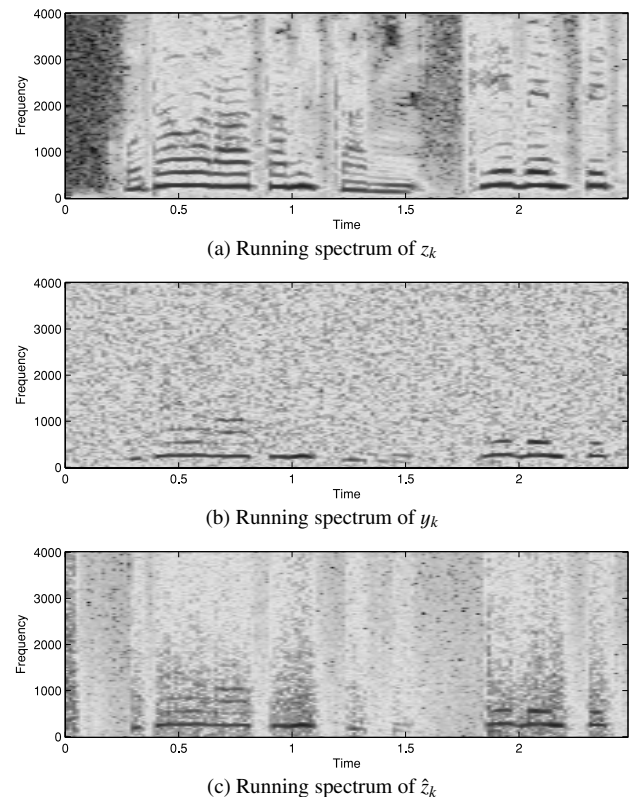
## 5.2 Performance for a Speech Signal

We next tested the performance of the NHE for an actual speech signal corrupted by white noise. Figure 3 shows the original, observed, and estimated signals of a speech in Japanese (*Jodawara atami kan keiben tetsu*), respectively. Here, the speech signal is sampled with 8 kHz, and is transformed into a corresponding complex signal using the Hilbert transform. All the parameters of the NHE are the same as those used in the previous subsection. Unexpectedly, the NHE works well for such a speech signal. The evaluation quantities  $CR$  and  $NR$  of the NHE for the speech signal were 0.62 and  $-17.7$ [dB], respectively. Unfortunately, the ACF was not suitable for estimation of such a noisy speech signal because it is very hard to decide the voiced periods.

For further analysis, Figure 4 shows the sliding-window spectra, which run with time, for real part of each signal of Fig. 3.

## 6. Conclusion

A nonlinear harmonic estimator (NHE) has been proposed for simultaneously extracting a harmonic signal and its fundamental frequency in the presence of white noise. Computer simulations demonstrated that the NHE is effective for estimating a harmonic signal even when its fundamental frequency varies in time. However, the NHE usually needs an additional computational cost due to the Hilbert transform

**Fig. 3** Estimation of a speech signal by the NHE with  $M = 11$ .**Fig. 4** Spectral analysis on the original, observed, and estimated signals.

for a real-valued signal such as speech.

A more detail evaluation of the NHE and its practical applications will be undertaken in the future.

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## Appendix: Proofs of Lemmas and Corollaries

### A.1 Proof of Lemma 1

Expanding the following polynomial :

$$\begin{aligned} & \{z^{M-1} + \alpha(1; M-1)z^{M-2} - \alpha(2; M-1)z^{M-3} \\ & \quad \cdots + (-1)^M \alpha(M-1; M-1)\} \{z - \gamma_1^M\} \end{aligned} \quad (\text{A} \cdot 1)$$

we obtain

$$\begin{aligned} & z^M + \{\alpha(1; M-1) + \gamma_1^M\} z^{M-1} \\ & \quad - \{\alpha(2; M-1) + \gamma_1^M \alpha(1; M-1)\} z^{M-2} \\ & \quad \cdots + (-1)^M \gamma_1^M \alpha(M-1; M-1). \end{aligned} \quad (\text{A} \cdot 2)$$

Defining the coefficients of the polynomial of (A·2) as  $\alpha(i; M)$ , we have

$$\alpha(i; M) = \alpha(i; M-1) + \alpha(i-1; M-1)\gamma_1^M \quad (\text{A} \cdot 3)$$

for  $1 < i < M$ . The coefficients when  $i = 1$  or  $i = M$  are immediately obtained from (A·2).

### A.2 Proof of Lemma 2

Calculating the derivative of (23) with respect to  $\gamma_1$ , we easily obtain the results in Lemma 2.

### A.3 Proof of Corollary 1

The relationship between the coefficients of the polynomial is deduced from the symmetry.

### A.4 Proof of Corollary 2

Differentiating the coefficients in Corollary 1 with respect to  $\gamma_1$ , we can obtain (37) and (38).



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