Regional boundary controllability of time fractional diffusion processes

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Abstract

In this paper, we are concerned with the regional boundary controllability of the Riemann-Liouville time fractional diffusion systems of order $\alpha \in (0, 1]$. The characterizations of strategic actuators are established when the systems studied are regionally boundary controllable. The determination of control to achieve regional boundary controllability with minimum energy is explored. We also show a connection between the regional internal controllability and regional boundary controllability. Several useful results for the optimal control from an implementation point of view are presented in the end.

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1. Introduction

In the past several decades, a lot of work has been carried out to deal with the problem of steering a system to a target state, especially after the introduction of the notions of actuators and sensors [1, 2]. However, in many real-world applications, we are only concerned with those cases where the target states of the problem studied are defined in a given subregion of the whole space domain. Then the regional idea emerges and we refer the reader to [3, 4, 5] for more information on the concept of regional analysis for the Gaussian diffusion process. Besides, it should be pointed out that not only does the concept of regional analysis make sense closer to real-world problems, it also generalizes the results of existence contributions.

In addition, after the introduction of continuous time random walks (CTRWs) by Montroll and Weiss [6], the anomalous diffusion equation of fractional order has attracted increasing interest and has been proven to be a useful tool in modeling many real-world problems

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[7, 8, 9, 10, 11]. More precisely, the mean squared displacement (MSD) of anomalous diffusion process is described by a power law of fractional exponent, which is smaller (in the case of sub-diffusion) or bigger (in the case of super-diffusion) than that of Brownian motion. It is confirmed that the time fractional diffusion system, where the traditional first order time derivative is replaced by a Riemann-Liouville time fractional derivative of order $\alpha \in (0, 1]$, can be used to well characterize those sub-diffusion process [7, 8]. For example, the flow through porous media microscopic processes [12], or swarm of robots moving through dense forest [13] etc. For the fractional calculus, as we all know, it has shown great potential in science and engineering applications and some phenomena such as self-similarity, nonstationary, non-Gaussian process and short or long memory process are all closely related to fractional calculus [14, 15, 16]. It is now widely believed that, using fractional calculus in modeling can better capture the complex dynamics of natural and man-made systems, and fractional order controls can offer better performance not achievable before using integer order controls [17, 18], which in fact raise important theoretical challenges and open new research opportunities.

Motivated by the argument above, the contribution of this present work is on the regional boundary controllability of the anomalous transport process described by time fractional diffusion systems. More precisely, for an open bounded subset $\Omega \subseteq \mathbf{R}^n$ with smooth boundary $\partial\Omega$, we consider:

- A subregion Γ of $\partial\Omega$ which may be unconnected.
- Various kinds of actuators (zone, pointwise, internal or boundary) acting as controls.

The rest of this paper is organized as follows. The mathematical concept of regional boundary controllability and several preliminaries are presented in the next section, then we present an example which is regional boundary controllability but not globally boundary controllable. Section 3 is focused on the characterizations of $\Gamma-$strategic actuators and our main result on regional boundary controllability with minimum energy problem is given in Section 4. In Section 5, a connection between internal and boundary regional controllability is established and at last, we work out some useful results for the optimal control from an implementation point of view.

2. Regional boundary controllability

2.1. Problem statement

In this paper, we consider the following abstract time fractional diffusion system:

$$\begin{cases} {}_0D_t^\alpha z(t) = Az(t) + Bu(t), \quad t \in [0, b], \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} z(t) = z_0, \end{cases} \tag{2.1}$$

where A generates a strongly continuous semigroup $\{\Phi(t)\}_{t\geq 0}$ on the Hilbert space $Z := H^1(\Omega)$, $z \in L^2(0,b;Z)$ and the initial vector $z_0 \in Z$. It is supposed that $B : \mathbf{R}^p \to Z$ is the control operator and $u \in L^2(0,b;\mathbf{R}^p)$ depends on the number and structure of actuators. Moreover, the Riemann-Liouville fractional derivative ${}_0D_t^\alpha$ and the Riemann-Liouville fractional integral ${}_0I_t^\alpha$ are, respectively, given by [15],[16]

$${}_0D_t^\alpha z(t) = \frac{d}{dt}{}_0I_t^{1-\alpha}z(t),\ \ \alpha \in (0,1] \ \text{ and } \ {}_0I_t^\alpha z(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}z(s)ds,\ \ \alpha > 0. \quad (2.2)$$

Definition 2.1. *[19] For any given $f \in L^2(0,b;Z)$, $\alpha \in (0,1]$, a function $v \in L^2(0,b;Z)$ is said to be a mild solution of the following system*

$$\begin{cases} {}_0D_t^\alpha v(t) = Av(t) + f(t),\ t \in [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha}v(t) = v_0 \in Z, \end{cases} \quad (2.3)$$

if it satisfies

$$z(t) = K_\alpha(t)v_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)f(s)ds, \quad (2.4)$$

where $K_\alpha(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\Phi(t^\alpha\theta)d\theta$, $\{\Phi(t)\}_{t\geq 0}$ is the strongly continuous semigroup generated by A, $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\psi_\alpha(\theta^{-\frac{1}{\alpha}})$ and ψ_α is a probability density function defined by

$$\psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\ \ \theta > 0. \quad (2.5)$$

In addition, we have [20, 21]

$$\int_0^\infty \psi_\alpha(\theta)d\theta = 1 \ \ \textit{and} \ \ \int_0^\infty \theta^\nu\phi_\alpha(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)},\ \nu \geq 0. \quad (2.6)$$

By Lemma 2.1, the mild solution $z(.,u)$ of (2.1) can be given by

$$z(t,u) = K_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1}K_\alpha(t-s)Bu(s)ds. \quad (2.7)$$

Let $H : L^2(0,b;\mathbf{R}^p) \to Z$ be

$$Hu = \int_0^b (b-s)^{\alpha-1}K_\alpha(b-s)Bu(s)ds,\ \forall u \in L^2(0,b;\mathbf{R}^p). \quad (2.8)$$

Suppose that $\{\Phi^*(t)\}_{t\geq 0}$, generated by the adjoint operator of A, is also a strongly continuous semigroup on the space Z. For any $v \in Z$, it follows from $\langle Hu, v\rangle = \langle u, H^*v\rangle$ that

$$H^*v = B^*(b-s)^{\alpha-1}K_\alpha^*(b-s)v, \quad (2.9)$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing of space Z, B^* is the adjoint operator of B and

$$K_\alpha^*(t) = \alpha\int_0^\infty \theta\phi_\alpha(\theta)\Phi^*(t^\alpha\theta)d\theta.$$

Let $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ be the trace operator of order zero, which is linear continuous and surjective, γ^* denotes the adjoint operator. Moreover, if $\Gamma \subseteq \partial\Omega$, $p_\Gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\Gamma)$ defined by

$$p_\Gamma z := z|_\Gamma \tag{2.10}$$

and for any $\bar{z} \in H^{\frac{1}{2}}(\Gamma)$, the adjoint operator p_Γ^* can be given by

$$p_\Gamma^* \bar{z}(x) := \begin{cases} \bar{z}(x), & x \in \Gamma, \\ 0, & x \in \partial\Omega \backslash \Gamma. \end{cases} \tag{2.11}$$

2.2. Definition and characterizations

Let $\omega \subseteq \Omega$ be a given region of positive Lebesgue measure. Denote the projection operator on ω by the restriction map

$$p_\omega : H^1(\Omega) \to H^1(\omega), \tag{2.12}$$

then we are ready to state the following definitions.

Definition 2.2. *The system* (2.1) *is said to be exactly (respectively, approximately) regionally controllable on ω at time b if for any $y_b \in H^1(\omega)$, given $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that*

$$p_\omega z(b, u) = y_b \left(\text{respectively, } \|p_\omega z(b, u) - y_b\|_{H^1(\omega)} \leq \varepsilon\right). \tag{2.13}$$

Definition 2.3. *The system* (2.1) *is said to be exactly (respectively, approximately) regionally boundary controllable on $\Gamma \subseteq \partial\Omega$ at time b if for any $z_b \in H^{\frac{1}{2}}(\Gamma)$, given $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that*

$$p_\Gamma (\gamma z(b, u)) = z_b \left(\text{respectively, } \|p_\Gamma (\gamma z(b, u)) - z_b\|_{H^{\frac{1}{2}}(\Gamma)} \leq \varepsilon\right). \tag{2.14}$$

Proposition 2.1. *The following properties are equivalent:*

(1) *The system* (2.1) *is exactly regionally boundary controllable on Γ at time b;*
(2) $Im(p_\Gamma \gamma H) = H^{\frac{1}{2}}(\Gamma)$;
(3) $Ker(p_\Gamma) + Im(\gamma H) = H^{\frac{1}{2}}(\partial\Omega)$;
(4) *For any $z \in H^{\frac{1}{2}}(\Gamma)$, there exists a positive constant c such that*

$$\|z\|_{H^{\frac{1}{2}}(\Gamma)} \leq c\|H^* \gamma^* p_\Gamma^* z\|_{L^2(0,b;\mathbf{R}^p)}. \tag{2.15}$$

Proof. By Definition 2.3, it is not difficult to see that (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3) : For any $z \in H^{\frac{1}{2}}(\Gamma)$, let $\hat{z}$ be the extension of z to $H^{\frac{1}{2}}(\partial\Omega)$. Since $Im(p_\Gamma \gamma H) = H^{\frac{1}{2}}(\Gamma)$, there exists $u \in L^2(0, b; \mathbf{R}^p)$, $z_1 \in Ker(p_\Gamma)$ such that $\hat{z} = z_1 + \gamma H u$.

(3) $\Rightarrow$ (2) : For any $\tilde{z} \in H^{\frac{1}{2}}(\partial\Omega)$, $\tilde{z} = z_1 + z_2$, where $z_1 \in Ker(p_\Gamma)$ and $z_2 \in Im(\gamma H)$. Then there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that $\gamma H u = z_2$. Hence, it follows from the definition of p_Γ that $Im(p_\Gamma \gamma H) = H^{\frac{1}{2}}(\Gamma)$.

$(1) \Leftrightarrow (4)$: The equivalence between (1) and (4) can be deduced from the following general result [22]: Let E, F, G be reflexive Hilbert spaces and $f \in \mathcal{L}(E, G)$, $g \in \mathcal{L}(F, G)$. Then the following two properties are equivalent

(1) $Im(f) \subseteq Im(g)$,

(2) $\exists\, \gamma > 0$ such that $\|f^* z^*\|_{E^*} \le \gamma \|g^* z^*\|_{F^*}, \quad \forall z^* \in G$.

By choosing $E = G = H^{\frac{1}{2}}(\Gamma)$, $F = L^2(0, b; \mathbf{R}^p)$, $f = Id_{H^{\frac{1}{2}}(\Gamma)}$ and $g = p_\Gamma \gamma H$, then we complete the proof.

Proposition 2.2. *There is an equivalence among the following properties:*

$\langle 1 \rangle$ *The system* (2.1) *is approximately regionally boundary controllable on* Γ *at time* b*;*

$\langle 2 \rangle$ $\overline{Im(p_\Gamma \gamma H)} = H^{\frac{1}{2}}(\Gamma)$;

$\langle 3 \rangle$ $Ker(p_\Gamma) + \overline{Im(\gamma H)} = H^{\frac{1}{2}}(\partial\Omega)$;

$\langle 4 \rangle$ *The operator* $p_\Gamma \gamma H H^* \gamma^* p_\Gamma^*$ *is positive definite.*

Proof. By Proposition 2.1, $\langle 1 \rangle \Leftrightarrow \langle 2 \rangle \Leftrightarrow \langle 3 \rangle$. Finally, we show that $\langle 2 \rangle \Leftrightarrow \langle 4 \rangle$. In fact, since

$$\overline{Im(p_\Gamma \gamma H)} = H^{\frac{1}{2}}(\Gamma) \Leftrightarrow (p_\Gamma \gamma H u, z)_{H^{1/2}(\Gamma)} = 0 \text{ for any } u \in L^2(0, b; \mathbf{R}^p) \text{ implies } z = 0,$$

where $(\cdot, \cdot)_{H^{1/2}(\Gamma)}$ is the inner product of $H^{\frac{1}{2}}(\Gamma)$. Let $u = H^* \gamma^* p_\omega^* z$. Then we see that

$$\overline{Im(p_\Gamma \gamma H)} = H^{\frac{1}{2}}(\Gamma) \Leftrightarrow (p_\Gamma \gamma H H^* \gamma^* p_\Gamma^* z, z)_{H^{1/2}(\Gamma)} = 0 \text{ implies } z = 0,\ z \in H^{\frac{1}{2}}(\Gamma),$$

i.e., the operator $p_\Gamma \gamma H H^* \gamma^* p_\Gamma^*$ is positive definite and the proof is complete.

Remark 2.1. (1) *A system which is boundary controllable on* Γ *is boundary controllable on* Γ_1 *for every* $\Gamma_1 \subseteq \Gamma$.

(2) *The definitions* (2.2) *can be applied to the case where* $\Gamma = \partial\Omega$ *and there exist systems that are not boundary controllable but which are regionally boundary controllable. This is illustrated by the following example*

2.3. An example

Consider the following two dimension time fractional diffusion equation defined on $\Omega = [0, 1] \times [0, 1]$, which is excited by a zone actuator:

$$\begin{cases} {}_0D_t^\alpha z(x, y, t) = \frac{\partial^2}{\partial x^2} z(x, y, t) + \frac{\partial^2}{\partial y^2} z(x, y, t) + p_D u(t) \ \text{ in } \ \Omega \times [0, b], \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} z(x, y, t) = 0 \quad \text{in} \quad \Omega, \\ z(\xi, \eta, t) = 0 \quad \text{on} \quad \partial\Omega \times [0, b], \end{cases} \tag{2.16}$$

where $\alpha \in (0, 1]$, $D = \{0\} \times [d_1, d_2] \subseteq \Omega$, $A = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ with $\lambda_{ij} = -(i^2 + j^2)\pi^2$, $\xi_{ij}(x, y) = 2a_{ij} \cos(i\pi x) \cos(j\pi y)$, $a_{ij} = (1 - \lambda_{ij})^{-\frac{1}{2}}$, $\Phi(t)z = \sum\limits_{i,j=1}^{\infty} \exp(\lambda_{ij} t)(z, \xi_{ij})_Z \xi_{ij}$ and $K_\alpha(t)z(x) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) \Phi(t^\alpha \theta) z(x) d\theta = \sum\limits_{i,j=1}^{\infty} E_{\alpha,\alpha}(\lambda_{ij} t^\alpha)(z, \xi_{ij})_Z \xi_{ij}(x)$. Further, since

$$(H^* \gamma^* z)(t) = (b - t)^{\alpha - 1} \sum_{i,j=1}^{\infty} E_{\alpha,\alpha}(\lambda_{ij}(b - t)^\alpha)(\gamma^* z, \xi_{ij})_Z (p_D, \xi_{ij})_Z$$

and $(p_D,\xi_{ij})_Z=\frac{2a_{ij}}{j\pi}\left[\sin(j\pi d_2)-\sin(j\pi d_1)+j\pi(\cos(j\pi d_2)-\cos(j\pi d_1)\right]$, there exists $d_1,d_2\in[0,1]$ satisfying $Ker(H^*)\neq\{0\}$ ($\overline{Im(p_DH)}\neq L^2(\omega)$), i.e., the system (2.16) is not boundary controllable.

Moreover, let $d_1=0$, $d_2=\frac{1}{2}$, $\Gamma=\{0\}\times[\frac{1}{4},\frac{3}{4}]$ and $z_*=\xi_{ij}(0,y),(i,j=4k,k=1,2,3,\cdots)$. Obviously, z_* is not reachable on $\partial\Omega$. However, since

$$E_{\alpha,\alpha}(t)>0\ (t\geq 0)\text{ and }(p_D,\xi_{ij})_Z=\frac{2a_{ij}}{j\pi}\left[\sin(j\pi/2)+j\pi(\cos(j\pi/2)-1)\right],\ j=1,2,\cdots,$$

we see that

$$\begin{aligned}(H^*\gamma^*p_\Gamma^*z_*)(t)&=\sum_{i,j=1}^{\infty}\tfrac{E_{\alpha,\alpha}(\lambda_{ij}(b-t)^\alpha)}{(b-t)^{1-\alpha}}(\xi_{ij},\gamma^*z_*)_{H^{1/2}(\Gamma)}(p_D,\xi_{ij})_Z\\&=\sum_{i,j=1,j\neq 4k}^{\infty}\tfrac{2a_{ij}E_{\alpha,\alpha}(\lambda_{ij}(b-t)^\alpha)}{j\pi(b-t)^{1-\alpha}}(\xi_{ij},\gamma^*z_*)_{H^{1/2}(\Gamma)}\\&\qquad\times\left[\sin(j\pi/2)+j\pi(\cos(j\pi/2)-1)\right]\\&\neq 0.\end{aligned}\tag{2.17}$$

Hence z_* is regionally boundary controllable on $\Gamma=\{0\}\times[\frac{1}{4},\frac{3}{4}]$.

To end this section, we finally recall a necessary lemma to be used afterwards.

Lemma 2.1. *[23] Let $\Omega\subseteq\mathbf{R}^n$ be an open set and $C_0^\infty(\Omega)$ be the class of infinitely differentiable functions on Ω with compact support in Ω and $u\in L^1_{loc}(\Omega)$ be such that*

$$\int_\Omega u(x)\psi(x)dx=0,\quad\forall\psi\in C_0^\infty(\Omega).\tag{2.18}$$

Then $u=0$ almost everywhere in Ω.

3. Regional strategic actuators

The characteristic of actuators to achieve the regionally approximately boundary controllable of the system (2.1) will be explored in this section.

As cited in [1], a actuator can be expressed by a couple (D,g) where $D\subseteq\Omega$ is the support of the actuator and g is its spatial distribution. To state our main results, it is supposed that the control are made by p actuators $(D_i,g_i)_{1\leq i\leq p}$ and let $Bu=\sum\limits_{i=1}^{p}p_{D_i}g_i(x)u_i(t)$, where $p\in\mathbf{N}$, $g_i(x)\in Z$, $u=(u_1,u_2,\cdots,u_p)$ and $u_i(t)\in L^2(0,b)$. As cited in [24], all these distributed parameter systems with moving sensors and actuators form the so-called cyber-physical systems, which are rich in real world applications. For instance, in the pest spreading process, p is the number the spreading machines and $u_i(\cdot)$ stands for the control input strategic of every spreading machines with respect to time t [25]. Then the system (2.1) can be rewritten as

$$\begin{cases}{}_0D_t^\alpha z(t,x)=Az(t,x)+\sum\limits_{i=1}^{p}p_{D_i}g_i(x)u_i(t),\ (t,x)\in[0,b]\times\Omega,\\ \lim\limits_{t\to 0^+}{}_0I_t^{1-\alpha}z(t,x)=z_0(x).\end{cases}\tag{3.1}$$

Moreover, we suppose that $-A$ is a self-adjoint uniformly elliptic operator, by [26], we get that there exists a sequence $(\lambda_j, \xi_{jk}) : k = 1, 2, \cdots, r_j,\ j = 1, 2, \cdots$ such that

(1) For each $j = 1, 2, \cdots$, λ_j is the eigenvalue of operator A with multiplicities r_j and

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots, \quad \lim_{j\to\infty} \lambda_j = -\infty.$$

(2) For each $j = 1, 2, \cdots$, $\xi_{jk}(k = 1, 2, \cdots, r_j)$ is the orthonormal eigenfunction corresponding to λ_j, i.e.,

$$(\xi_{jk_m}, \xi_{jk_n}) = \begin{cases} 1, & k_m = k_n, \\ 0, & k_m \neq k_n, \end{cases}$$

where $1 \leq k_m, k_n \leq r_j$, $k_m, k_n \in \mathbf{N}$ and $(\cdot, \cdot)$ is the inner product of space Z.

Hence, the sequence $\{\xi_{jk}, k = 1, 2, \cdots, r_j, j = 1, 2, \cdots\}$ is a orthonormal basis in Z, the strongly continuous semigroup $\{\Phi(t)\}_{t\geq 0}$ on Z generated by A is

$$\Phi(t)z(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \exp(\lambda_j t)(z, \xi_{jk})\xi_{jk}(x), \quad x \in \Omega \tag{3.2}$$

and for any $z(x) \in Z$, it can be expressed as $z(x) = \sum\limits_{j=1}^{\infty}\sum\limits_{k=1}^{r_j}(z, \xi_{jk})\xi_{jk}(x)$.

Definition 3.1. *A actuators (suite of actuators) is said to be $\Gamma-$strategic if the system under consideration is regionally approximately boundary controllable on Γ at time b.*

Before to show our main result in this part, by Eq.(3.2), for any $z \in L^2(\Omega)$, we have

$$\begin{aligned} K_\alpha(t)z(x) &= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\Phi(t^\alpha\theta)z(x)d\theta \\ &= \alpha \int_0^\infty \theta\phi_\alpha(\theta)\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \exp(\lambda_j t^\alpha\theta)(z, \xi_{jk})\xi_{jk}(x)d\theta \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{\infty} \frac{\alpha(\lambda_j t^\alpha)^n}{n!}(z, \xi_{jk})\xi_{jk}(x)\int_0^\infty \theta^{n+1}\phi_\alpha d\theta \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{\infty} \frac{\alpha(n+1)!(\lambda_j t^\alpha)^n}{\Gamma(\alpha n+\alpha+1)n!}(z, \xi_{jk})\xi_{jk}(x) \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} \alpha E^2_{\alpha,\alpha+1}(\lambda_j t^\alpha)(z, \xi_{jk})\xi_{jk}(x), \end{aligned}$$

where $E^\mu_{\alpha,\beta}(z) := \sum\limits_{n=0}^{\infty}\frac{(\mu)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}$, $z \in \mathbf{C}$, $\alpha, \beta, \mu \in \mathbf{C}$, $\mathbf{Re}\,\alpha > 0$ is the generalized Mittag-Leffler function in three parameters and here, $(\mu)_n$ is the Pochhammer symbol defined by (see [27], Section 2.1.1)

$$(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1),\ n \in \mathbf{N}. \tag{3.3}$$

If $\alpha, \beta \in \mathbf{C}$ such that $\mathbf{Re}\,\alpha > 0$, $\mathbf{Re}\,\beta > 1$, then (see Section 2.3.4, [28] or Section 5.1.1, [29])

$$\alpha E^2_{\alpha,\beta} = E_{\alpha,\beta-1} - (1+\alpha-\beta)E_{\alpha,\beta}. \tag{3.4}$$

It then follows that

$$K_\alpha(t)z(x) = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j} E_{\alpha,\alpha}(\lambda_j t^\alpha)(z,\xi_{jk})\xi_{jk}(x) \tag{3.5}$$

and

$$\int_0^t \tau^{\alpha-1}K_\alpha(\tau)Bu(t-\tau)d\tau = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{p}\int_0^t g^i_{jk}u_i(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_j\tau^\alpha)d\tau\xi_{jk}(x), \tag{3.6}$$

where $g^i_{jk} = (p_{D_i}g_i, \xi_{jk}),\ j = 1,2,\cdots,\ k = 1,2,\cdots,r_j,\ i = 1,2,\cdots,p$ and $E_{\alpha,\beta}(z) := \sum\limits_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)}$, $\mathbf{Re}\,\alpha > 0,\ \beta, z \in \mathbf{C}$ is known as the generalized Mittag-Leffler function in two parameters.

Theorem 3.1. *For any $j = 1,2,\cdots,$ define $p\times r_j$ matrices G_j as*

$$G_j = \begin{bmatrix} g^1_{j1} & g^1_{j2} & \cdots & g^1_{jr_j} \\ g^2_{j1} & g^2_{j2} & \cdots & g^2_{jr_j} \\ \vdots & \vdots & \vdots & \vdots \\ g^p_{j1} & g^p_{j2} & \cdots & g^p_{jr_j} \end{bmatrix}, \tag{3.7}$$

where $g^i_{jk} = (p_{D_i}g_i, \xi_{jk}),\ j = 1,2,\cdots,\ k = 1,2,\cdots,r_j,\ i = 1,2,\cdots,p$. Then the suite of actuators $(D_i, g_i)_{1\le i\le p}$ is said to be $\Gamma-$strategic if and only if

$$p \ge r = \max\{r_j\} \quad \textit{and} \quad \textit{rank}\ G_j = r_j \ \textit{for}\ j = 1,2,\cdots. \tag{3.8}$$

Proof. For any $z_* \in H^{\frac{1}{2}}(\Gamma)$, denote by $(\cdot,\cdot)_{H^{1/2}(\Gamma)}$ the inner product of space $H^{\frac{1}{2}}(\Gamma)$, we then see that

$$(p_\Gamma\gamma Hu, z_*)_{H^{1/2}(\Gamma)} = \sum_{j=1}^{\infty}\sum_{k=1}^{r_j}\sum_{i=1}^{p}\int_0^b \tau^{\alpha-1}E_{\alpha,\alpha}(\lambda_j\tau^\alpha)u_i(b-\tau)d\tau g^i_{jk}z_{jk} = 0,\ t\in[0,b], \tag{3.9}$$

where $z_{jk} = (p_\Gamma\gamma\xi_{jk}, z_*)_{H^{1/2}(\Gamma)}$, $j = 1,2,\cdots$, $k = 1,2,\cdots,r_j$. Further, Lemma 2.1 gives

$$\sum_{j=1}^{\infty}\sum_{k=1}^{r_j} t^{\alpha-1}E_{\alpha,\alpha}(\lambda_j t^\alpha)g^i_{jk}z_{jk} = \mathbf{0}_p := (0,0,\cdots,0) \in \mathbf{R}^p \text{ for } t > 0, i = 1,2,\cdots,p. \tag{3.10}$$

Then we conclude that the suite of actuators $(D_i, g_i)_{1\le i\le p}$ is $\Gamma-$strategic if and only if

$$\sum_{j=1}^{\infty} b^{\alpha-1}E_{\alpha,\alpha}(\lambda_j b^\alpha)G_j z_j = \mathbf{0}_p \Rightarrow z_* = 0, \tag{3.11}$$

where $z_j = (z_{j1}, z_{j2}, \cdots, z_{jr_j})^T$ is a vector in $\mathbf{R}^{r_j}$ and $j = 1, 2, \cdots$.

(a) If we assume that $p \geq r = \max\{r_j\}$ and $rank\ G_j < r_j$ for some $j = 1, 2, \cdots$, there exists a nonzero element $\tilde{z} \in H^{\frac{1}{2}}(\Gamma)$ with $\tilde{z}_j = \left(\tilde{z}_{j1}, \tilde{z}_{j2}, \cdots, \tilde{z}_{jr_j}\right)^T \in \mathbf{R}^{r_j}$ such that

$$G_j \tilde{z}_j = \mathbf{0}_p. \tag{3.12}$$

It then follows from $E_{\alpha,\alpha}(\lambda_j t^\alpha) > 0$ $(t \geq 0)$ that we can find a nonzero vector $\tilde{z}$ satisfying

$$\sum_{j=1}^{\infty} b^{\alpha-1} E_{\alpha,\alpha}(\lambda_j b^\alpha) G_j \tilde{z}_j = \mathbf{0}_p. \tag{3.13}$$

This means that the actuators $(D_i, f_i)_{1 \leq i \leq p}$ are not $\Gamma-$strategic.

(b) However, on the contrary, if the actuators $(D_i, g_i)_{1 \leq i \leq p}$ are not $\Gamma-$strategic, i.e., $\overline{Im(p_\Gamma \gamma H)} \neq H^{\frac{1}{2}}(\Gamma)$, then there exists a nonzero element $z \neq \mathbf{0}_n$ satisfying

$$(p_\Gamma \gamma H u, z)_{H^{1/2}(\Gamma)} = 0 \text{ for all } u \in L^2(0, b; \mathbf{R}^p). \tag{3.14}$$

Then we can find a nonzero element $z_{j^*} \in \mathbf{R}^{r_j}$ such that

$$G_{j^*} z_{j^*} = \mathbf{0}_p. \tag{3.15}$$

This allows us to complete the conclusion of the theorem.

4. Regional boundary controllability with minimum energy control

In this section, we explore the possibility of finding a minimum energy control when the system (2.1) can be steered from a given initial vector z_0 to a target function z_b on the boundary subregion Γ. The method used here is an extension of those in [1, 2, 3, 4, 5].

Consider the following minimization problem

$$\begin{cases} \inf\limits_u J(u) = \int_0^b \|u(t)\|^2_{\mathbf{R}^p} dt \\ u \in U_b = \{u \in L^2\left(0, b; \mathbf{R}^p\right) : p_\Gamma \gamma z(b, u) = z_b\}, \end{cases} \tag{4.1}$$

where, obviously, U_b is a closed convex set. We then show a direct approach to the solution of the minimum energy problem (4.1).

Theorem 4.1. *If the system* (2.1) *is regionally approximately boundary controllable on* Γ, *then for any* $z_b \in H^{\frac{1}{2}}(\Gamma)$, *the minimum energy problem* (4.1) *has a unique solution given by*

$$u^*(t) = (p_\Gamma \gamma H)^* R_\Gamma^{-1} \left(z_b - p_\Gamma \gamma K_\alpha(b) z_0\right), \tag{4.2}$$

where $R_\Gamma = p_\Gamma \gamma H H^* \gamma^* p_\Gamma^*$ *and* H^* *is defined in Eq.*(2.9).

Proof. To begin with, since the solution of (2.1) excited by the control u^* is given by

$$z(t, u^*) = K_\alpha(t)z_0 + \int_0^t (t-s)^{\alpha-1} K_\alpha(t-s)Bu^*(s)ds, \tag{4.3}$$

we get that

$$\begin{aligned} p_\Gamma \gamma z(b, u^*) &= p_\Gamma \gamma \left[K_\alpha(b)z_0 + \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s)Bu^*(s)ds \right] \\ &= p_\Gamma \gamma K_\alpha(b)z_0 + p_\Gamma \gamma H \left(p_\Gamma \gamma H\right)^* R_\Gamma^{-1} \left(z_b - p_\Gamma \gamma K_\alpha(b)z_0\right) \\ &= z_b. \end{aligned}$$

Next, we show that if the system (2.1) is regionally approximately boundary controllable on Γ at time b, then the operator R_Γ is coercive. In fact, for any $z_1 \in H^{\frac{1}{2}}(\Gamma)$, there exists a control $u \in L^2(0, b, \mathbf{R}^p)$ such that

$$z_1 = p_\Gamma \gamma \left[K_\alpha(b)z_0 + Hu \right] \tag{4.4}$$

and

$$\begin{aligned} \langle R_\Gamma z_1, z_1 \rangle_{H^{1/2}(\Gamma)} &= \left\| H^* \gamma^* p_\Gamma^* z_1 \right\|^2_{L^2(0,b,\mathbf{R}^p)} \\ &= \left\| B^*(b-\cdot)^{\alpha-1} K_\alpha^*(b-\cdot)\gamma^* p_\Gamma^* z_1 \right\|^2_{L^2(0,b,\mathbf{R}^p)} \\ &\geq \left\| z_1 \right\|^2_{H^{1/2}(\Gamma)}. \end{aligned}$$

Moreover, since $R_\Gamma \in \mathcal{L}\left(H^{\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)\right)$, by the Theorem 1.1 in [30], it follows that R_Γ is an isomorphism.

Finally, we prove that u^* solves the minimum energy problem (4.1). For this purpose, since $p_\Gamma \gamma z(b, u^*) = z_b$, for any $u \in L^2(0, b, \mathbf{R}^p)$ with $p_\Gamma \gamma z(b, u) = z_b$, one has

$$p_\Gamma \gamma \left[z(b, u^*) - z(b, u) \right] = 0, \tag{4.5}$$

which follows that

$$0 = p_\Gamma \gamma \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s)B\left[u^*(s) - u(s)\right]ds = p_\Gamma \gamma H \left[u^* - u\right].$$

Thus, by

$$\begin{aligned} J'(u^*)(u^* - u) &= 2\int_0^b \langle u^*(s) - u(s), u^*(s) \rangle ds \\ &= 2\int_0^b \left\langle u^*(s) - u(s), \left(p_\Gamma \gamma H\right)^* R_\Gamma^{-1} \left(z_b - p_\Gamma \gamma K_\alpha(b)z_0\right) \right\rangle ds \\ &= 2\int_0^b \left\langle p_\Gamma \gamma H \left[u^*(s) - u(s)\right], R_\Gamma^{-1} \left(z_b - p_\Gamma \gamma K_\alpha(b)z_0\right) \right\rangle ds \\ &= 0, \end{aligned}$$

it follows that $J(u) \geq J(u^*)$, i.e., u^* solves the minimum energy problem (4.1) and the proof is complete.

5. The connection between internal and boundary regional controllability

Based on an intension of the regional controllability of integer order differential equations developed in [3, 4], we here give a transfer on the internal and boundary regional controllability of fractional order sub-diffusion equations (2.1) and develop two types of controls, i.e., zone or pointwise.

5.1. Internal and boundary regional controllability

In this part, we present a internal and boundary regional controllability transfer of the problem (2.1). To this end, suppose that $z(b,u) \in Z$ and we first define a operator

$$T : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega) \text{ such that } \gamma T g = g. \ \forall g \in H^{\frac{1}{2}}(\partial\Omega), \tag{5.1}$$

which is linear and continuous [31]. Let $z_b \in H^{\frac{1}{2}}(\Gamma)$ with the extension $p_\Gamma^* z_b \in H^{\frac{1}{2}}(\partial\Omega)$ and consider the sets

$$\Omega_1 = \left\{ T p_\Gamma^* z_b \in Z | z_b \in H^{\frac{1}{2}}(\Gamma) \right\} \text{ and } \Omega_2 = \bigcup_{z_b \in H^{1/2}(\Gamma)} \text{Supp } T p_\Gamma^* z_b. \tag{5.2}$$

For any $r > 0$ be arbitrary sufficiently small, consider

$$D_r = \bigcup_{z \in \Gamma} B(z,r) \text{ and let } \omega_r = D_r \cap \Omega_2, \tag{5.3}$$

where $B(z,r)$ is a ball of radius r centred in z.

Theorem 5.1. *If the system* (2.1) *is exactly(respectively, approximately) controllable on* ω_r*, then it is also exactly(respectively, approximately) boundary controllable on* Γ.

Proof. Let $z_b \in H^{\frac{1}{2}}(\Gamma)$ be the target function. By utilizing the trace theorem [33], there exists $T p_\Gamma^* z_b \in Z$ with a bounded support such that $\gamma(T p_\Gamma^* z_b) = p_\Gamma^* z_b$. Then

1) if the system (2.1) is exactly controllable on ω_r, for any $y_b \in H^1(\omega_r)$, there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$p_{\omega_r} z(b,u) = y_b. \tag{5.4}$$

Then $p_{\omega_r} T p_\Gamma^* z_b \in H^1(\omega_r)$ and there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$p_{\omega_r} z(b,u) = p_{\omega_r} T p_\Gamma^* z_b \text{ and } \gamma p_{\omega_r} z(b,u) = p_\Gamma^* z_b. \tag{5.5}$$

Thus $p_\Gamma \gamma p_{\omega_r} z(b,u) = z_b$, i.e., the system (2.1) is exactly boundary controllable on Γ.

2) if the system (2.1) is approximately controllable on ω_r, for and $\varepsilon > 0$ and any $y_b \in H^1(\omega_r)$, there exists a control $u \in L^2(0,b;\mathbf{R}^p)$ such that

$$\|p_{\omega_r} z(b,u) - y_b\|_{H^1(\omega_r)} \le \varepsilon. \tag{5.6}$$

Then for any $\varepsilon > 0$, there exists a control $u \in L^2(0, b; \mathbf{R}^p)$ such that

$$\|p_{\omega_r} z(b, u) - p_{\omega_r} T p_\Gamma^* z_b\|_{H^1(\omega_r)} \leq \varepsilon. \tag{5.7}$$

Moreover, by the continuity of the trace mapping γ on $H^1(\omega_r)$, one has

$$\|\gamma(p_{\omega_r} z(b, u)) - \gamma(p_{\omega_r} T p_\Gamma^* z_b)\|_{H^1(\partial\omega_r)} \leq \varepsilon, \tag{5.8}$$

therefore $\|p_\Gamma \gamma(p_{\omega_r} z(b, u)) - z_b\|_{H^1(\Gamma)} \leq \varepsilon$, Thus (2.1) is approximately boundary controllable on Γ and the proof is complete.

5.2. Regional boundary target control

This part is concerned with the approach for the control which drives the problem (2.1) from z_0 to z_b on Γ. Let $z_b \in H^{\frac{1}{2}}(\Gamma)$ with the extension $p_\Gamma^* z_b \in H^{\frac{1}{2}}(\partial\Omega)$. By Theorem 5.1, the problem may be solved by driving the system (2.1) from z_0 to $y_b \in H^1(\omega_r)$ on ω_r.

The following two sets will be used in our discussion.

$$G = \{g \in H^1(\Omega) : g = 0 \text{ in } \Omega \backslash \omega_r\} \text{ and } E = \{e \in H^1(\Omega) : e = 0 \text{ in } \omega_r\}. \tag{5.9}$$

5.2.1. Case of zone actuator

Let us consider the system (2.1) with a zone actuator (D, f) where $D \subseteq \Omega$ is the support of the actuator and f is its spatial distribution. Then the system can be written in the form

$$\begin{cases} {}_0D_t^\alpha z(x,t) = Az(x,t) + p_D f(x) u(t) \text{ in } \Omega \times [0, b], \\ \lim\limits_{t \to 0^+} {}_0I_t^{1-\alpha} z(x,t) = z_0(x) \text{ in } \Omega, \\ z(x,t) = 0 \text{ on } \partial\Omega \times [0, b]. \end{cases} \tag{5.10}$$

For any $g \in G$, consider the system

$$\begin{cases} Q_t D_b^\alpha \left[(b-t)^{1-\alpha} \varphi(x,t)\right] = A^* Q \left[(b-t)^{1-\alpha} \varphi(x,t)\right] \text{ in } \Omega \times [0, b], \\ \lim\limits_{t \to 0^+} Q_t I_b^{1-\alpha} \left[(b-t)^{1-\alpha} \varphi(x,t)\right] = p_{\omega_r}^* g(x) \text{ in } \Omega, \\ \varphi(x,t) = 0 \text{ on } \partial\Omega \times [0, b]. \end{cases} \tag{5.11}$$

where Q is a reflection operator on interval $[0, b]$ such that

$$Qf(t) := f(b-t). \tag{5.12}$$

By the argument in [32], we see that the following properties on operator Q hold:

$$Q_0 I_t^\alpha f(t) = {}_t I_b^\alpha Q f(t), \quad Q_0 D_t^\alpha f(t) = {}_t D_b^\alpha Q f(t) \tag{5.13}$$

and

$${}_0I_t^\alpha Q f(t) = Q_t I_b^\alpha f(t), \quad {}_0D_t^\alpha Q f(t) = Q_t D_b^\alpha f(t). \tag{5.14}$$

Then system (5.11) can be rewritten as

$$\begin{cases} {}_0D_t^{\alpha} Q\left[(b-t)^{1-\alpha}\varphi(x,t)\right] = A^* Q\left[(b-t)^{1-\alpha}\varphi(x,t)\right] \quad \text{in} \ \ \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} Q\left[(b-t)^{1-\alpha}\varphi(x,t)\right] = p_{\omega_r}^* g(x) \ \ \text{in} \ \ \Omega, \\ \varphi(x,t) = 0 \ \ \text{on} \ \ \partial\Omega \times [0,b] \end{cases} \tag{5.15}$$

and its unique mild solution is $\varphi(x,t) = (b-t)^{\alpha-1} K_\alpha^*(b-t) p_{\omega_r}^* g(x)$. Moreover, we define the semi-norm

$$g \in G \to \|g\|_G^2 = \int_0^b (f, \varphi(\cdot,t))_{L^2(D)}^2 dt \tag{5.16}$$

on G and obtain the following result.

Lemma 5.1. (5.16) *defines a norm on G if the system* (5.10) *is regionally approximately controllable on ω at time b.*

Proof. For any $g \in G$, if the system (5.10) is regionally approximately controllable on ω, we have

$$Ker(H^* p_\omega^*) = Ker\left[(b-s)^{\alpha-1}\left(f, K_\alpha^*(b-s) p_{\omega_r}^* g\right)_{L^2(D)}\right] = Ker\left[(f, \varphi(\cdot,t))_{L^2(D)}\right] = \{0\}.$$

It then follows from

$$\|g\|_G^2 = \int_0^b (f, \varphi(\cdot,t))_{L^2(D)}^2 dt = 0 \Leftrightarrow (f, \varphi(\cdot,t))_{L^2(D)} = 0$$

that $\|\cdot\|_G$ is a norm of space G and the proof is complete.

Moreover, let $u(t) = (f, \varphi(\cdot,t))_{L^2(D)}$ and decomposed the system (5.10) into an autonomous system and a homogeneous initial condition one

$$\begin{cases} {}_0D_t^{\alpha} \psi_1(x,t) = A\psi_1(x,t) + p_D f(x) \left(f, \varphi(\cdot,t)\right)_{L^2(D)} \quad \text{in} \ \ \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi_1(x,t) = 0 \quad \text{in} \ \ \Omega, \\ \psi_1(x,t) = 0 \ \ \text{on} \ \ \partial\Omega \times [0,b] \end{cases} \tag{5.17}$$

and

$$\begin{cases} {}_0D_t^{\alpha} \psi_2(x,t) = A\psi_2(x,t) \ \ \text{in} \ \ \Omega \times [0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi_2(x,t) = z_0(x) \ \ \text{in} \ \ \Omega, \\ \psi_2(x,t) = 0 \ \ \text{on} \ \ \partial\Omega \times [0,b]. \end{cases} \tag{5.18}$$

Let $\wedge$ be the operator $\wedge : G \to E^\perp$ given by

$$\wedge g = p_{\omega_r} \psi_1(\cdot, b), \quad \forall g \in G. \tag{5.19}$$

Then for any $z_b \in H^1(\omega_r)$, the regional control problem on ω_r is equivalent to the resolution of the equation

$$\wedge g = z_b - p_{\omega_r} \psi_2(\cdot, b) \tag{5.20}$$

and we have the following result.

Theorem 5.2. *Assume that the system* (5.10) *is regionally approximately controllable on* ω_r *at time* b*, then* (5.20) *admits a unique solution* $g \in G$ *and the control*

$$u^*(t) = (f, \varphi(\cdot, t))_{L^2(D)} \tag{5.21}$$

steers the problem (5.10) *to* z_b *on* ω_r*. Moreover,* u^* *solves the minimum energy problem*

$$\inf_u J(u) = \int_0^b \|u(t)\|^2_{\mathbf{R}^p} dt. \tag{5.22}$$

Proof. From Lemma 5.1, if the system (5.10) is regionally approximately controllable on ω_r at time b, then $\|\cdot\|_G$ is a norm of space G. Let the completion of G with respect to the norm $\|\cdot\|_G$ again by G.

Next, we show that (5.20) admits a unique solution in G. For any $g \in G$, by Eq. (5.19), it follows that

$$\begin{aligned}
\langle g, \wedge g \rangle &= \langle g, p_{\omega_r}\psi_1(\cdot, b)\rangle \\
&= \left\langle g, p_{\omega_r} \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) p_D f(\cdot) \, (f, \varphi(\cdot, s))_{L^2(D)} ds \right\rangle \\
&= \int_0^b \| \, (f, \varphi(\cdot, t))_{L^2(D)} \, \|^2 ds = \|g\|^2_G.
\end{aligned}$$

Hence, it follows from the Theorem 1.1 in [30] that (5.20) admits a unique solution in G.

Let $u = u^*$ in problem (5.10), then $p_{\omega_r} z(b, u^*) = z_b$. Finally, we show that u^* minimize the const functional (5.22). For any $u_1 \in L^2(0, b, \mathbf{R}^p)$ with $p_{\omega_r} z(b, u_1) = z_b$, we have

$$p_{\omega_r} [z(b, u^*) - z(b, u_1)] = 0. \tag{5.23}$$

Then

$$0 = p_{\omega_r} \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) p_D f(x) \, [u^*(s) - u_1(s)] ds.$$

Moreover, since

$$\begin{aligned}
J'(u^*)(u^* - u_1) &= 2\int_0^b (u^*(s) - u_1(s)) u^*(s) ds \\
&= 2\int_0^b (u^*(s) - u_1(s)) \, (f, \varphi(\cdot, t))_{L^2(D)} ds \\
&= 2\int_0^b \big(p_D f \, [u^*(s) - u_1(s)] \, , (b-t)^{\alpha-1} K^*_\alpha(b-t) p^*_{\omega_r} g\big) ds \\
&= 2\left(p_{\omega_r} \int_0^b (b-s)^{\alpha-1} K_\alpha(b-s) p_D f(x) \, [u^*(s) - u_1(s)] ds, g\right) \\
&= 0,
\end{aligned}$$

one has $J(u) \geq J(u^*)$, i.e., u^* solves the minimum energy problem (5.22) and the proof is complete.

5.2.2. Case of pointwise actuator

Consider the system (2.1) with a pointwise internal actuator, which can be written in the form

$$\begin{cases} {}_0D_t^\alpha z(x,t) = Az(x,t) + \delta(x-\sigma)u(t) \ \text{ in } \ \Omega\times[0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} z(x,t) = z_0 \ \text{ in } \ \Omega, \\ z(x,t) = 0 \ \text{ on } \ \partial\Omega\times[0,b], \end{cases} \tag{5.24}$$

where σ is the actuator support. For any $g\in G$, consider (5.11) and define the semi-norm

$$g \to \|g\|_G^2 = \int_0^b \|\varphi(\sigma,s)\|^2 ds, \tag{5.25}$$

which defines a norm on G if (5.24) is regionally approximately controllable.

Similar to the argument in section 5.2.1, let $u(t) = \varphi(\sigma,t)$ and we consider the following system

$$\begin{cases} {}_0D_t^\alpha \psi_1(x,t) = A\psi_1(x,t) + \delta(x-\sigma)\varphi(\sigma,t) \ \text{ in } \ \Omega\times[0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi_1(x,t) = 0 \ \text{ in } \ \Omega, \\ \psi_1(x,t) = 0 \ \text{ on } \ \partial\Omega\times[0,b] \end{cases} \tag{5.26}$$

and

$$\begin{cases} {}_0D_t^\alpha \psi_2(x,t) = A\psi_2(x,t) \ \text{ in } \ \Omega\times[0,b], \\ \lim\limits_{t\to 0^+} {}_0I_t^{1-\alpha} \psi_2(x,t) = z_0(x) \ \text{ in } \ \Omega, \\ \psi_2(x,t) = 0 \ \text{ on } \ \partial\Omega\times[0,b]. \end{cases} \tag{5.27}$$

Then the regional control problem on ω_r is equivalent to the resolution of the equation

$$\wedge\, g = z_b - p_{\omega_r}\psi_2(\cdot,b) \tag{5.28}$$

and we see the following result.

Theorem 5.3. *Assume that the system* (5.24) *is regionally approximately controllable on* ω_r *at time* b*, then* (5.28) *admits a unique solution* $g\in G$ *and the control*

$$u^*(t) = \varphi(\sigma,t) \tag{5.29}$$

steers (5.10) *to* z_b *on* ω_r*. Moreover, this control minimize the cost functional* (5.22).

5.2.3. Simulation

The resolution of the regional boundary control problem may be seen via the following simplified steps (see the case of pointwise actuator for example).

1) Initial data Ω, Γ, z_b and the actuator;

2) Solve the problem (5.28) ($\to g$);

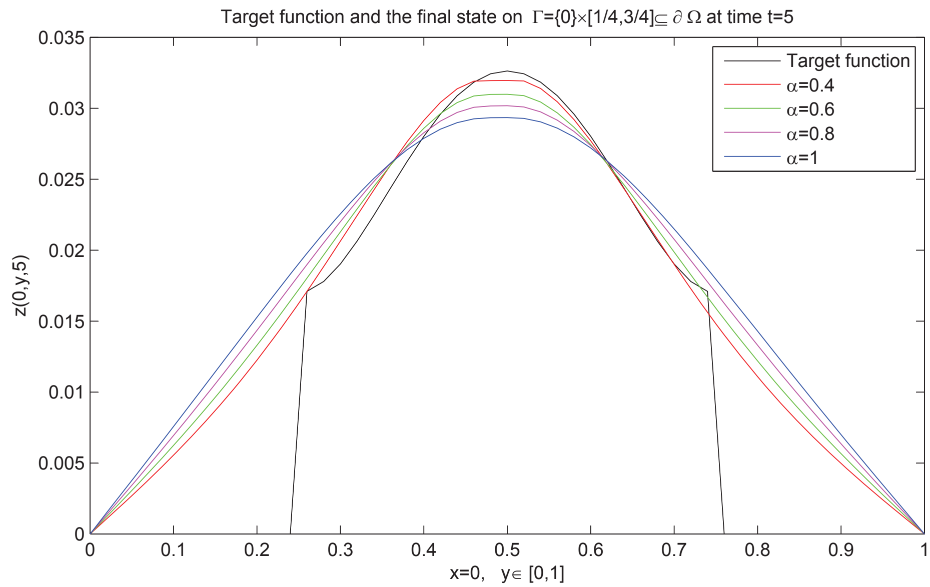

Figure 1: Final reached state and target function on $\Gamma \subseteq \partial\Omega$ at time $t = 5$.

3) Solve the problem (5.11) ($\to \varphi(\sigma, t)$);

4) Apply the control $u^*(t) = \varphi(\sigma, t)$.

For example, consider the system (2.16) and let $\Omega = [0, 1] \times [0, 1]$, $\Gamma = \{0\} \times [1/4, 3/4]$, $b = 5$. For the target function z_b on $\Gamma \subseteq \partial\Omega$, we assume that

$$z_b(0, y) = \begin{cases} 0, & 0 \leq y < 1/4; \\ 0.017 + 4(y - 1/4)^2(y - 3/4)^2, & 1/4 \leq y \leq 3/4; \\ 0, & 3/4 < y \leq 1 \end{cases} \tag{5.30}$$

and the actuator is supposed to be located in $D = \{0\} \times \{0.5\} \subseteq \Omega$.

Figure 1 shows how the final reached state is very close to the target function on $\Gamma \subseteq \partial\Omega$ at time $t = 5$ when $\alpha = 0.4,\ 0.6,\ 0.8,\ 1.0$. This also implies that time fractional diffusion systems can offer better performance compared with those using integer order distributed parameter systems. Moreover, when $\alpha = 0.4$, the corresponding control input, which is calculated by the formula (5.29), is presented at Figure 2.

6. CONCLUSIONS

In this paper, the regional boundary controllability of the Riemann-Liouville time fractional diffusion systems of order $\alpha \in (0, 1]$ is discussed, which is motivated by many realistic situation encountered in various applications. The results here provide some insights into the qualitative analysis of the design of fractional order diffusion equations, which can also be extended to complex fractional order distributed parameter dynamic systems. Various

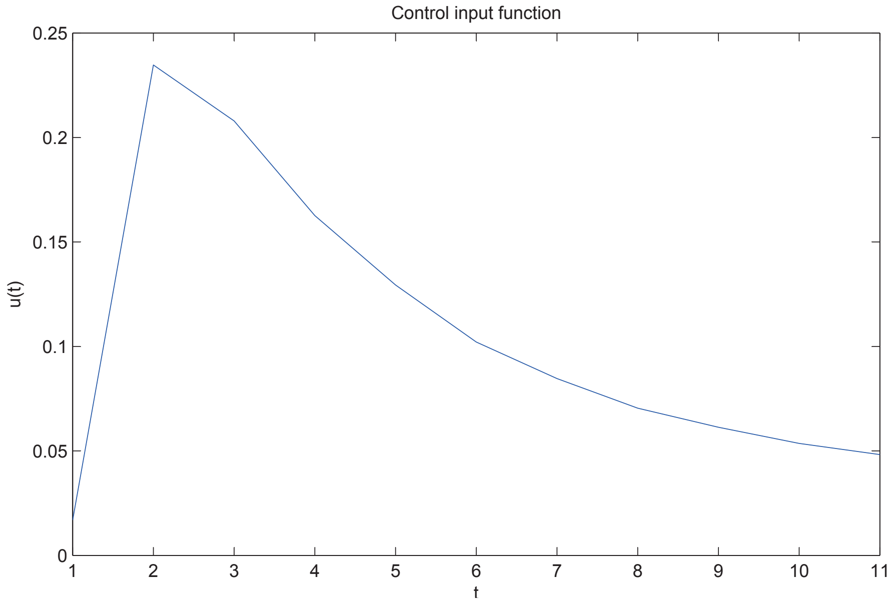

Figure 2: Control input function, which is calculated by the formula (5.29).

open questions are still under consideration. The problem of constrained control as well as the case of fractional order distributed parameter dynamic systems with more complicated regional sensing and actuation configurations are of great interest. For more information on the potential topics related to fractional order distributed parameter systems, we refer the readers to [34] and the references therein.

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