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Strict paraconsistency of truth-degree preserving intuitionistic logic with dual negation

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Abstract

In this article we provide some results concerning a logic that results from propositional intuitionistic logic when dual negation is added in certain way, producing a paraconsistent logic that has been called da Costa Logic. In particular, we prove the finite model property and strict paraconsistency of this logic.

Keywords: Intuitionistic logic, dual negation, paraconsistency, finite model property.

1 Introduction

Paraconsistent logics have been around for some time. An early example, though the author was not looking for a paraconsistent logic, is Kolmogorov's Minimal Logic (*KML*) (see [4]). However, in *KML* from a contradiction every negation follows. In this respect, *KML* is similar to some paraconsistent logics. In 1990, Urbas (see [9]) took heed of this phenomenon and considered it undesirable for a paraconsistent logic to have the property that from a contradiction every formula of certain form follows. Thus, the concept of a strictly paraconsistent logic emerges, for which we give the precise definition in Section 5.

This article deals with a logic that results from intuitionistic logic by adding the dual of intuitionistic negation. It has been called da Costa Logic by Priest in [6]. However, we think it is more appropriate to call it Truth-Degree Preserving Intuitionistic Logic with Dual Negation, which we abbreviate with *ID*. Our terminology stems from the fact that the algebraic consequence relation of *ID* is defined making use of truth degrees as in [1]. Priest provided natural deduction, Kripke semantics (already given by Rauszer in [7]), tableaux and topological and algebraic semantics. It is easy to see that *ID* is paraconsistent (we give a proof in Section 4).

In this article, using a Frege–Hilbert style presentation of *ID*, we show that *ID* has the finite model property (FMP) and related properties such as the finite satisfiability property. Finally, we prove that *ID* is strictly paraconsistent.

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2 An intuitionistic logic with dual negation

The language of *ID* is given by the set $\mathfrak{F}$ of formulas resulting as usual from propositional letters, the binary connectives $\wedge$, $\vee$ and $\rightarrow$ and the unary connective D, that will behave as dual of intuitionistic negation, in the sense that in the algebraic semantics given in Section 3 the corresponding operation, i.e. the join complement, is the dual of the meet complement, which is the corresponding operation of intuitionistic negation in the usual Heyting algebras semantics.

The logic *ID* has any set of axiom schemas for propositional positive logic (*PL*), e.g.

$$\varphi \rightarrow (\psi \rightarrow \varphi),$$
$$\varphi \rightarrow (\psi \rightarrow \chi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)),$$
$$\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)),$$
$$(\varphi \wedge \psi) \rightarrow \varphi,$$
$$(\varphi \wedge \psi) \rightarrow \psi,$$
$$\varphi \rightarrow (\varphi \vee \psi),$$
$$\psi \rightarrow (\varphi \vee \psi),$$
$$(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi)),$$

plus the axiom schema

$$DI: \varphi \vee D\varphi.$$

The rules of *ID* are *modus ponens* (*MP*) and the rule DE: $\varphi \vee \psi / D\varphi \rightarrow \psi$, but in what follows we will see that *DE* will only be applied in certain cases.

To be precise, we say that the formula φ is *derivable* (using notation $\vdash \varphi$) iff there exists a finite sequence of formulas, the last being equal to φ, such that for any formula ψ in the sequence it holds that either ψ is an axiom or ψ comes from previous formulas in the sequence by *MP* or *DE*.

Now, we say that the formula φ is a *consequence* of the set Γ of formulas (using notation $\Gamma \vdash \varphi$) iff there exists a finite sequence of formulas, the last equal to φ, such that for any formula ψ in the sequence it holds that either $\psi \in \Gamma$ or $\vdash \psi$ or ψ comes from previous formulas in the sequence by *MP*.

Note that it follows that $\emptyset \vdash \varphi$ iff $\vdash \varphi$.

We will use notation $\varphi \vdash \psi$, for any formulas φ and ψ, to mean that ψ is a consequence of the set $\{\varphi\}$. We will use notation $\psi \vdash$, for any formula ψ, to mean that every formula is a consequence of the set $\{\psi\}$.

This presentation of *ID* has the same language (excepting for the use of D instead of $\neg$) and consequences as the natural deduction presentation given in [6], where the rule corresponding to *DE* is the disjunctive syllogism restricted to derivable disjunctions.

The following properties follow: (i) $D(\varphi \vee D\varphi) \vdash$; (ii) *ID* enjoys the Deduction Theorem: if $\varphi \vdash \psi$, then $\vdash \varphi \rightarrow \psi$; and (iii) if $\varphi \dashv\vdash \psi$, then $D\varphi \dashv\vdash D\psi$.

To see (i), note that, using DI, $\vdash (\varphi \vee D\varphi) \vee \psi$. So, we also have that $\vdash D(\varphi \vee D\varphi) \rightarrow \psi$. It follows that $D(\varphi \vee D\varphi) \vdash \psi$.

To see (ii), just proceed by a straightforward induction.

To see (iii), suppose that $\varphi \vdash \psi$. Then, by (ii), $\vdash \varphi \rightarrow \psi$. As $\vdash \varphi \vee D\varphi$, then $\vdash \psi \vee D\varphi$. Then, using DE, $\vdash D\psi \rightarrow D\varphi$ and so $D\psi \vdash D\varphi$. The other case is analogous. Note that some paraconsistent logics do not enjoy this property (see [5, 4.3]).

3 Algebraic semantics

An algebraic semantics for *ID* was given by Priest in [6]. He called the corresponding class of algebras da Costa algebras. These algebras are term equivalent to the Heyting algebras with dual pseudocomplement studied by Sankappanavar in [8]. In this article these algebras will be called *HD* algebras, where *H* stands for Heyting and *D* for dual.

A *HD* algebra $\mathbf{A} = (A, \wedge, \vee, \rightarrow, D)$ is an algebra of type $(2,2,2,1)$ such that $(A, \wedge, \vee, \rightarrow)$ is a generalized Heyting algebra (*gH* algebra) and for any $x \in A$, the join complement $Dx = min\{y\colon \text{for all } z, z \leq x \vee y\}$ exists. A *gH* algebra is an algebra $(A, \wedge, \vee, \rightarrow)$ of type $(2,2,2)$ such that $(A, \wedge, \vee)$ is a lattice and for any $x, y \in A$, the relative meet complement $x \rightarrow y = max\{z : x \wedge z \leq y\}$ exists.

The class of *HD* algebras forms a variety with equations as in a *gH* algebra, i.e.

identities defining lattices,

$x \wedge (x \rightarrow y) = x \wedge y$,

$x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$

and

$z \wedge ((x \wedge y) \rightarrow x) = z$,

plus the equations corresponding in the usual way to the inequalities

$y \leq x \vee Dx$,

$D(x \vee Dx) \leq y$

and

$Dy \leq x \vee D(x \vee y)$.

It is easily seen that D exists in every finite *gH* algebra: just check that in a finite *gH* algebra, for any x, $\bigwedge\{y\colon \text{for all } z,\ z \leq x \vee y\}$ exists and is equal to Dx.

Every *gH* algebra (as a consequence also every *HD* algebra) has a top element $1 = x \rightarrow x$, for any x. It also follows that, for the obvious translations of the axioms of *PL* given in Section 2 into *gH* terms t (which will also be *HD* terms), we have that $t = 1$. We also have, for any x, y in the universe of a *gH* algebra, that if $x \rightarrow y = 1$ and $x = 1$, then $y = 1$. Moreover, we have that $x \vee Dx = 1$ and that if $x \vee y = 1$, then $Dx \rightarrow y = 1$, for any x, y in the universe of a *HD*.

Now, let us define an *algebraic consequence* relation in the following way: $\Gamma \models \varphi$ iff for every *HD* algebra A, valuation (homomorphism) v and $a \in A$, we have that if $a \leq v(\psi)$, for all $\psi \in \Gamma$, then $a \leq v(\varphi)$. This way of defining the algebraic consequence relation was extensively studied in [1]. Note that it follows that for every *HD* algebra A and valuation v, we have that if $v(\psi) = 1$, for all $\psi \in \Gamma$, then $v(\varphi) = 1$.

It may be seen that *ID* is sound and complete w.r.t. the given algebraic consequence relation.

For soundness let us first state as a lemma that for all φ, if $\vdash \varphi$, then $v(\varphi) = 1$, for every *HD* algebra and valuation v. This may be proved by a straightforward induction. The general case, i.e. if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$, can also be seen by a straightforward induction noting, for the case of *MP*, that it is immediate to see that if $a \leq x \rightarrow y$ and $a \leq x$, then $a \leq y$, for any a, x, y in the universe of a *gH* algebra.

To see completeness, first define $\theta_\Gamma = \{(\alpha, \beta) : \Gamma, \alpha \vdash \beta \text{ and } \Gamma, \beta \vdash \alpha\}$, where $\Gamma \cup \{\alpha\} \cup \{\beta\} \subseteq \mathfrak{F}$. Then $\mathfrak{F}/\theta$ is a *gH* algebra. Let $f_D([\alpha]) = [D\alpha]$, which is well defined because we have that if $\alpha \dashv\vdash \beta$, then $D\alpha \dashv\vdash D\beta$. Then $L = (\mathfrak{F}/\theta, f_D)$ is a *HD* algebra. Let v be the valuation such that $v(p) = [p]$, for every propositional letter p. Then, it is easily seen that $v(\varphi) = 1$ iff $\Gamma \vdash \varphi$. Now, suppose that $\Gamma \models \varphi$. Then it holds that, for every *HD* algebra A and valuation w, if $w(\psi)=1$, for all $\psi\in\Gamma$, then $w(\varphi)=1$. However, in the *HD* algebra L we have that $v(\psi)=1$, for all $\psi\in\Gamma$, because every formula $\psi\in\Gamma$ is such that $\Gamma\vdash\psi$. Then, it follows that $v(\varphi)=1$, and so $\Gamma\vdash\varphi$.

4 Some results

In this section, using the results of the previous one, we will see that (i) *ID* is *D*-paraconsistent; (ii) *ID* and classical logic (*CL*) share the same derivable formulas in the $\{\wedge,\vee,D\}$-fragment; (iii) *ID* is a conservative extension of *PL*; and (iv) *ID* has the FMP.

Using soundness it is easily seen that *ID* is *D*-paraconsistent. Just consider the three element algebra H_3 (the middle element noted m) and the valuation given by $v(p)=m$ and $v(q)=0$. Then $v(p\wedge Dp)=m$ and so, $v(p\wedge Dp)\nleq v(q)$. So, $p\wedge Dp\nvdash q$.

To prove (ii) reason as follows. It is immediate that if $\vdash\varphi$, then $\vdash_{CL}\varphi$. Now suppose there is a formula φ in the $\{\wedge,\vee,D\}$-fragment such that $\nvdash\varphi$. Then, using completeness, there is a *HD* algebra A and valuation v such that $v(\varphi)\neq 1$. Thus, using Zorn's Lemma, there is a maximal lattice ideal I such that $v(\varphi)\in I$. Now, the function $f:A\to\{0,1\}$ defined by $f(x)=0$ iff $x\in I$ is a $\{\wedge,\vee,D\}$-homomorphism. Then, in the Boolean algebra $\{0,1\}$ and valuation v' induced by fv we have that $v'(\varphi)=0$. So, using soundness of *CL*, it follows that $\nvdash_{CL}\varphi$.

In order to see that *ID* is a conservative extension of *PL*, let us reason in the following way. Let us suppose that, for φ in the $\{\wedge,\vee,\to\}$-fragment, we have $\vdash\varphi$. Then, using soundness, it follows that for every *HD* algebra and valuation v, $v(\varphi)=1$, in particular, for every finite *HD* algebra and valuation v, $v(\varphi)=1$. From this it follows, using that D exists in every finite *gH* algebra, that for every finite *gH* algebra and valuation v, $v(\varphi)=1$. Then, using the FMP and completeness of *PL*, it follows that $\vdash_{PL}\varphi$.

In order to see that *ID* has the *FMP*, let us reason as in a book by Dunn and Hardegree (see [3, Thm. 13.9.3]). First we have the following:

Lemma 4.1
Let $H=(A,\wedge,\vee,\to,D)$ be a *HD* algebra. Let $H'=(A',\wedge,\vee,0,1)$ be a finite sublattice of H. Then there exists a binary operation $\to'$ and a unary operation D' in H' such that $(A',\wedge,\vee,\to',D',0,1)$ is a *HD* algebra such that (i) for all $x,y\in A'$, if $x\to y\in A'$, then $x\to' y=x\to y$; and (ii) for all $x\in A'$, if $Dx\in A'$, then $D'x=Dx$.

Proof. Take $x\to' y$ to be $\bigvee\{z\in A':x\wedge z\leq y\}$ and $D'x$ to be $\bigwedge\{y\in A'$: for all z, $z\leq x\vee y\}$. ∎

Let us write $Sub(\varphi)$ and $lg(\varphi)$, for the set of subformulas and the set of propositional letters of the formula φ, respectively. Now let us prove the following:

Proposition 4.2
For every formula $\varphi\in\mathfrak{F}$, *HD* algebra H and valuation v, the algebra H' whose underlying lattice is the sublattice of H generated by the elements $v(\psi)$, for $\psi\in Sub(\varphi)$, is a finite *HD* algebra and, for any valuation $v':\mathfrak{F}\to H'$ such that $v'(p)=v(p)$, for every $p\in lg(\varphi)$, it holds that $v'(\psi)=v(\psi)$, for every $\psi\in Sub(\varphi)$.

Proof. Let φ, H and v be, respectively, a formula, a *HD* algebra and a valuation. Since $Sub(\varphi)$ is a finite set and the variety of bounded lattices is locally finite, it follows that H' is a finite lattice. By the previous lemma it is also a *HD* algebra. Let v' be a valuation such that $v'(p)=v(p)$, for all $p\in lg(\varphi)$. Let us see by induction that $v'(\varphi)=v(\varphi)$, for every $\psi\in Sub(\varphi)$. The base is trivial and the cases corresponding to $\circ\in\{\wedge,\vee,\to\}$ are easy to see. For $\to$, suppose that $\psi=\alpha\to\beta$. We have that $v'(\alpha \to \beta) = v'(\alpha) \to' v'(\beta)$, since v' is a valuation; $v'(\alpha) \to' v'(\beta) = v(\alpha) \to' v(\beta)$, by inductive hypothesis; $v(\alpha) \to' v(\beta) = v(\alpha) \to v(\beta)$, by the previous lemma, and the latter is $v(\alpha \to \beta)$, since v is a valuation. The case $\psi = D\alpha$ is similar. ■

We have the following:

COROLLARY 4.3
For any $\varphi \in \mathfrak{F}$, if there is a *HD* algebra H and a valuation v such that

FMP. $v(\varphi) \neq 1$, then there is a finite *HD* algebra H' and valuation v' such that $v'(\varphi) \neq 1$;
FSP. $v(\varphi) = 1$, then there is a finite *HD* algebra H' and valuation v' such that $v'(\varphi) = 1$; and
FP. $v(\varphi) \neq 0$, then there is a finite *HD* algebra H' and valuation v' such that $v'(\varphi) \neq 0$.

Note that *FSP* says that *ID* has the finite satisfiability property.
From *FMP* it follows that *ID* is decidable.

5 Strict paraconsistency

We will say that a logic L is strictly paraconsistent with respect to a connective $\neg$ iff for every formula $\varphi(p_0, \ldots, p_n)$ of L, if $\nvdash_L \varphi(\beta_0, \ldots, \beta_n)$ for some formulas $\beta_0, \ldots, \beta_n$ of L, then there exist a set Γ of formulas of L and formulas $\beta_0, \ldots, \beta_n, \alpha$ of L such that $\Gamma, \alpha, \neg\alpha \nvdash_L \varphi(\beta_0, \ldots, \beta_n)$. It is easily checked that this definition is equivalent, e.g., to the one in Carnielli *et al.* (see [2, p. 14]), where the authors use the word 'boldly' instead of 'strictly'.

In order to see that *ID* is strictly paraconsistent with respect to D, it is enough to prove that

(C) If $\nvdash \varphi$ and $p \notin lg(\varphi)$, then $p \wedge Dp \nvdash \varphi$.

In order to see that (C) is enough, reason as follows. First, suppose that (C) holds. Now, let $\varphi(p_0, \ldots, p_n) \in \mathfrak{F}$. Suppose that there exist formulas $\beta_0, \ldots, \beta_n \in \mathfrak{F}$ such that $\nvdash \varphi(\beta_0, \ldots, \beta_n)$. Then, by structurality, $\nvdash \varphi(p_0, \ldots, p_n)$. Now, let $p \notin lg(\varphi(p_0, \ldots, p_n))$. Then, by our first supposition, $p \wedge Dp \nvdash \varphi(p_0, \ldots, p_n)$. Then, there exist $\Gamma, \beta_0, \ldots, \beta n$ and α such that $\Gamma, \alpha, D\alpha \nvdash \varphi(\beta_0, \ldots, \beta_n)$, to wit, $\Gamma = \emptyset, \beta_0 = p_0, \ldots \beta_n = p_n$ and $\alpha = p$.

Now, (C) remains to be proved. It follows from the next:

THEOREM 5.1
If $\varphi \vdash \psi$ and $lg(\varphi) \cap lg(\psi) = \emptyset$, then $\varphi \vdash$ or $\vdash \psi$.

PROOF. Our proof is based on the FMP for *ID* and a version of Birkhoff's representation theorem for finite distributive lattices.

Let us suppose that $\varphi \vdash \psi$, $lg(\varphi) \cap lg(\psi) = \emptyset$, $\varphi \nvdash$ and $\nvdash \psi$.

Since we have *FP* and *FMP*, there are finite *HD* algebras H_1 and H_2 with valuations v_1 and v_2 such that $v_1(\varphi) \neq 0$ and $v_2(\psi) \neq 1$.

Let us consider the sets P_i of prime filters of H_i, for $i = 1, 2$, with inclusion as order, and then take the product $P_1 \times P_2$ with the usual product order.

Consider the finite algebra H with the set of increasing sets of $P_1 \times P_2$ as universe and the operations given by

$A \wedge B := A \cap B$;
$A \vee B := A \cup B$;
$A \to B := (\downarrow (A - B))^c = \{x \in P_1 \times P_2 : x \nleq y$, for every $y \in (A - B)\}$; and
$DA := \uparrow (A^c) = \{x \in P_1 \times P_2 : y \leq x$, for some $y \in A^c\}$.

It is easily seen that the *definientia* are all increasing sets and that H is a *HD* algebra.

Next, let us construct an appropriate valuation on H.

First, let us define embeddings of H_1 and H_2 into H:

$f_1 : H_1 \to H$, given by $f_1(x_1) = \eta_1(x_1) \times P_2$,
$f_2 : H_2 \to H$, given by $f_2(x_2) = P_1 \times \eta_2(x_2)$,

where η_1 and η_2 are the isomorphisms from H_1 and H_2 onto the increasing sets of P_1 and P_2 given by Birkhoff's representation theorem for finite distributive lattices.

It is easily seen that f_1 and f_2 are injective, that $f_1(0) = f_2(0) = \emptyset$ and that $f_1(1) = f_2(1) = P_1 \times P_2$.

Moreover, f_1 and f_2 are morphisms of *HD* algebras, i.e. we have that $f_i(x_i \circ y_i) = f_i(x_i) \circ f_i(y_i)$, for $i = 1, 2, x_i, y_i \in H_i$ and $\circ \in \{\wedge, \vee, \to\}$ and $f_i D(x_i) = D f_i(x_i)$, for $i = 1, 2$ and $x_i \in H_i$.

Observation: it can be seen that if $x_1 \neq 0, 1$ and $x_2 \neq 0, 1$, then $f_1(x_1)$ and $f_2(x_2)$ are incomparable.

Finally, let us take any valuation such that $v(p) = f_1(v_1(p))$ if $p \in \varphi$ and $v(p) = f_2(v_2(p))$ if $p \in \psi$. Such valuations exist because φ and ψ do not share propositional letters.

Then $v(\varphi) = f_1(v_1(\varphi))$ and $v(\psi) = f_2(v_2(\psi))$.

Now, we have three cases: (i) $v(\varphi) = 1$; (ii) $v(\psi) = 0$; and (iii) neither.

In case (i), as $v(\psi) \neq 1$, it follows that $v(\varphi) \nleq v(\psi)$.

In case (ii), as $v(\varphi) \neq 0$, it also follows that $v(\varphi) \nleq v(\psi)$.

In case (iii), using the observation above, it also follows that $v(\varphi) \nleq v(\psi)$.

So, in any case we have that $v(\varphi) \nleq v(\psi)$. Then, using soundness, it follows that $\varphi \nvdash \psi$, a contradiction. ∎

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References

[1] F. Bou, F. Esteva, J. M. Font, A. J. Gil, L. Godo, A. Torrens and V. Verdú. Logics preserving degrees of truth from varieties of residuated lattices. *Journal of Logic and Computation*, **19**, 1031–1069, 2009.

[2] W. A. Carnielli, M. E. Coniglio, and J. Marcos. *Logics of Formal Inconsistency*, 1st edn. Kluwer Academic Publishers, 2003.

[3] J. M. Dunn and G. Hardegree. *Algebraic Methods in Philosophical Logic*, 1st edn. Oxford University Press, 2001.

[4] A. N. Kolmogorov. Sur le principe de tertium non datur. *Matematicheskii Sbornik*, **32**, 646–667, 1925.

[5] G. Priest. Paraconsistency and dialetheism. *Handbook of the History of Logic*, **8**, 129–204, 2007.

[6] G. Priest. Dualising intuitionistic negation. *Principia*, **13**, 165–184, 2009.

[7] C. Rauszer. Applications of Kripke models to Heyting-Brouwer logic. *Studia Logica*, **36**, 61–71, 1977.

[8] H. P. Sankappanavar. Heyting algebras with dual pseudocomplementation. *Pacific Journal of Mathematics*, **117**, 405–415, 1985.

[9] I. Urbas. Paraconsistency. *Studies in Soviet Thought*, **39**, 343–354, 1990.

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