

A propositional system induced by Japaridze's approach to IF logic*

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Abstract

Cirquent calculus is a new proof-theoretic and semantic approach introduced for the needs of computability logic by G.Japaridze, who also showed that, through cirquent calculus, one can capture, refine and generalize independence-friendly (IF) logic. Specifically, the approach allows us to account for independence from propositional connectives in the same spirit as the traditional IF logic accounts for independence from quantifiers. Japaridze's treatment of IF logic, however, was purely semantical, and no deductive system was proposed. The present paper constructs a formal system sound and complete w.r.t. the propositional fragment of Japaridze's cirquent-based semantics for IF logic. Such a system can thus be considered an axiomatization of purely propositional IF logic in its full generality.

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1 Introduction

Cirquent calculus is a new proof-theoretic and semantic approach introduced by G.Japaridze in [6] and further developed in [7,9–11,17–19]. It was proposed for the needs of his computability logic (CoL) [5,8]. Unlike the more traditional proof theories that manipulate tree-like objects such as formulas, cirquent calculus deals with circuit-style objects termed *cirquents*. The main characteristic feature of the latter is allowing (one or another sort of) *sharing* of subcomponents between different components. Due to sharing, cirquent calculus has higher expressiveness and efficiency. For instance, as shown in [7], the analytic cirquent calculus system CL8 achieves an exponential speedup of proofs over the classical analytic systems.

A qualitative generalization of the concept of cirquents was made in [9], where the idea of *clustering* of propositional connectives was introduced. Intuitively, clusters are switch-style devices that combine tuples of individual disjunctive or conjunctive gates in a parallel way — in a way where the choice (*left* or *right*) of an argument is shared between all members. It was showed semantically in [9] that, through cirquents with clustered connectives (and also quantifiers as generalized connectives), one can capture, refine and generalize the well known *independence-friendly (IF) logic*.¹ The latter, introduced by J. Hintikka and G. Sandu [2] and further developed [3,4,12–16] by a number of authors, is a conservative extension of classical first-order logic, allowing one to express independence relations between quantifiers. The past attempts (cf. [13,14]) to apply the same ideas to propositional connectives and thus develop IF logic at the propositional (as opposed to first-order) level, however, have remained limited only to certain special syntactic fragments of the language.

Japaridze's treatment of IF logic in [9] was purely semantical, and no deductive system was proposed. In this paper, we axiomatically construct a cirquent calculus system called IF_p , with clustered

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¹Clustering only disjunctions is sufficient for the basic IF logic, while the so called *extended IF logic* requires clustering both disjunctions and conjunctions.

Each cirquent C with n atoms can be seen to be an n -ary truth function (even though expressed in a very unusual way) and, for this reason, C can be represented by a disjunctive-normal-form formula F of classical logic obtained in the standard way from the truth table characterizing C . However, the formula F obtained this way from C will generally be exponentially bigger than C . We believe there is no translation from cirquents to equivalent Boolean formulas that only creates polynomial size differences, even though this needs to be proven, of course. Overall, an answer to the question whether our system yields a greater expressive power than classical propositional logic depends on who is asked. A philosopher would probably say “No”, a computer scientist say “Yes”, and a mathematician say that it depends on how “expressive power” is precisely defined.

The rest of this section is not technically relevant to the main results of the present paper and is only meant for those who are familiar with IF logic but not with [9].³ In an attempt to understand what all of the above has to do with IF logic, consider the formula

$$\forall x(\exists y/\forall x)\exists z p(x, y, z) \quad (1)$$

with its standard meaning. According to the latter, given any object x , two objects y and z can be chosen so that $p(x, y, z)$ is true, with the modifier ‘ $\forall x$ ’ attached to ‘ $\exists y$ ’ indicating that here y can be chosen independently from (without any knowledge of) x . Assuming that the universe of discourse is $\{1, 2\}$, (1) can just as well be (re-)written as

$$\left(\left(p(1, 1, 1) \vee^z p(1, 1, 2) \right) \vee^y/\wedge^x \left(p(1, 2, 1) \vee^z p(1, 2, 2) \right) \right) \wedge^x \left(\left(p(2, 1, 1) \vee^z p(2, 1, 2) \right) \vee^y/\wedge^x \left(p(2, 2, 1) \vee^z p(2, 2, 2) \right) \right),$$

which, after further rewriting $p(1, 1, 1), p(1, 1, 2), \dots$ as the more compact p, q, \dots , is the propositional formula

$$\left((p \vee^z q) \vee^y/\wedge^x (r \vee^z s) \right) \wedge^x \left((t \vee^z u) \vee^y/\wedge^x (v \vee^z w) \right). \quad (2)$$

Here we have turned $\forall x$ into \wedge^x , $\exists y$ into \vee^y and $\exists z$ into \vee^z , with the superscript in each case used just to remind us from which quantifier each oconnective was obtained, and \vee^y/\wedge^x indicating that the y -superscripted disjunction is independent of the x -superscripted conjunction. Now, Japaridze’s recipes (see [9], Descriptions 7.4 and 7.5) translate (2) into the following cirquent:

$$\left((p \vee_2 q) \vee_1 (r \vee_3 s) \right) \wedge \left((t \vee_4 u) \vee_1 (v \vee_5 w) \right). \quad (3)$$

Note that cluster 1 contains two disjunctive oconnectives — namely, those originating from $\exists y$, and all other clusters are singletons. It is left as an exercise for the reader to convince himself or herself that, in any given model (interpretation) $*$, (3) is true in our sense if and only if (2) is true in the sense of IF logic. Well, the present case is a “lucky” case because we easily understand what “true in the sense of IF logic” means for (2) — after all, (2) originates from (and will be handled in the same way as) the first-order (1). As an example of an “unlucky” case, consider the cirquent

$$(p \vee_1 q) \wedge ((r \vee_1 s) \wedge q). \quad (4)$$

It is just as meaningful from the point of view of our semantics as any other cirquent, including (3). An attempt to express the same in the traditional formalism of IF logic apparently yields something like

$$(p \vee/\wedge^x q) \wedge^x ((r \vee/\wedge^x s) \wedge^y q). \quad (5)$$

Unlike (2), however, (5) is problematic for the traditional semantical approaches (the ones based on imperfect information games) to IF logic. Namely, because of a problem called signaling, it is far from clear how its truth should be understood.

If the connections and differences between our present semantics and that of IF logic are still not clear, see the first 6 sections of [9] for more explanations, discussions and examples.

³As for those unfamiliar with IF logic, they may want to consult [12] or [15].

3 Main results

3.1 System IF_p introduced

Throughout the rest of this paper, for convenience of descriptions, we will be always omitting the IDs for singleton clusters when writing cirquents. Note that, this way, all classical cirquents mechanically turn into (rather than are just identified with) formulas of classical logic.

As will be seen shortly, the inference rules of our system IF_p modify cirquents at any level rather than only around the root. Thus, IF_p is in fact a *deep inference* system, in the style of [1]. This explains our borrowing some notation from the Calculus of Structures. Namely, we will be using $\Phi\{\}$ or $\Psi\{\}$ to denote any cirquent where a vacancy (“hole”) $\{\}$ appears in the place of a subcirquent. The vacancy $\{\}$ can be filled with any cirquent. For example, if $\Phi\{\} = (p \vee_1 q) \vee_1 (\{\} \wedge q)$, then $\Phi\{\neg p\} = (p \vee_1 q) \vee_1 (\neg p \wedge q)$, $\Phi\{q\} = (p \vee_1 q) \vee_1 (q \wedge q)$, and $\Phi\{p \vee_1 q\} = (p \vee_1 q) \vee_1 ((p \vee_1 q) \wedge q)$. Multiple vacancies are also allowed and they are treated similarly.

The **axioms** of IF_p are all classical cirquents that (seen as formulas) are tautologies of classical logic. We schematically represent the **rules of inference** of IF_p in Figure 1, where $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ stand for any cirquents and \circ is a variable that stands for either \wedge or \vee or \vee_l for some (whatever) particular index l such that cluster l is a non-singleton one. It is important to point out that, in each rule, *all* occurrences of \circ stand for the same object (\wedge , \vee or \vee_l).

$$\begin{array}{c}
 \frac{\Phi\{\Psi\{\mathcal{A}\} \vee_k \mathcal{C}\}}{\Phi\{\Psi\{\mathcal{A} \vee_k \mathcal{B}\} \vee_k \mathcal{C}\}} \text{ Rule I (left)} \qquad \frac{\Phi\{\mathcal{C} \vee_k \Psi\{\mathcal{A}\}\}}{\Phi\{\mathcal{C} \vee_k \Psi\{\mathcal{B} \vee_k \mathcal{A}\}\}} \text{ Rule I (right)} \\
 \\
 \frac{\Phi\{(\mathcal{A} \circ \mathcal{C}) \vee_k (\mathcal{B} \circ \mathcal{C})\}}{\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \circ \mathcal{C}\}} \text{ Rule II (left)} \qquad \frac{\Phi\{(\mathcal{C} \circ \mathcal{A}) \vee_k (\mathcal{C} \circ \mathcal{B})\}}{\Phi\{\mathcal{C} \circ (\mathcal{A} \vee_k \mathcal{B})\}} \text{ Rule II (right)} \\
 \\
 \frac{\Phi\{(\mathcal{A} \circ \mathcal{C}) \vee_k (\mathcal{B} \circ \mathcal{D})\}}{\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \circ (\mathcal{C} \vee_k \mathcal{D})\}} \text{ Rule III}
 \end{array}$$

Figure 1: The rules of IF_p

In each application of these rules, we call the oconnective(s) \vee_k in the premise (resp. conclusion), as shown in Figure 1, the **key oconnective(s)** of this application in the premise (resp. conclusion).

A **proof** of a cirquent \mathcal{A} in IF_p , as expected, is a sequence of cirquents such that the first cirquent in the sequence is an axiom of IF_p , the last cirquent is \mathcal{A} , and every cirquent, except the axiom, follows from the preceding cirquent by one of the rules of IF_p . When such a proof exists, \mathcal{A} is said to be **provable** in IF_p . Below is an example of a proof:

$$\begin{array}{c}
 \frac{}{((q \wedge p) \vee (p \wedge \neg q)) \vee ((q \wedge \neg p) \vee (\neg p \wedge \neg q))} \text{ Axiom} \\
 \frac{}{((q \wedge p) \vee_2 (q \wedge \neg p)) \vee ((p \wedge \neg q) \vee_2 (\neg p \wedge \neg q))} \text{ Rule III} \\
 \frac{}{(q \wedge p) \vee_2 (q \wedge \neg p) \vee ((p \vee_2 \neg p) \wedge \neg q)} \text{ Rule II (left)} \\
 \frac{}{(q \wedge (p \vee_2 \neg p)) \vee ((p \vee_2 \neg p) \wedge \neg q)} \text{ Rule II (right)} \\
 \frac{}{(q \wedge (p \vee_2 \neg p)) \vee_1 ((p \vee_2 \neg p) \wedge (s \vee_1 \neg q))} \text{ Rule I (right)} \\
 \frac{}{((q \vee_1 r) \wedge (p \vee_2 \neg p)) \vee_1 ((p \vee_2 \neg p) \wedge (s \vee_1 \neg q))} \text{ Rule I (left)}
 \end{array}$$

Lemma 3.1 *Given an interpretation $*$ and a metaselection f , a cirquent $\Phi\{\mathcal{A} \vee_k \mathcal{B}\}$ is metatrue w.r.t. $(*, f)$ iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and the cirquent $\Phi\{\mathcal{A}\}$ (resp. $\Phi\{\mathcal{B}\}$) is metatrue w.r.t. $(*, f)$.*

Proof. We prove the proposition by induction on the number of oconnectives of $\Phi\{\}$.

For the basis, assume that the number of oconnectives of $\Phi\{\}$ is 0. Then $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \mathcal{A} \vee_k \mathcal{B}$, $\Phi\{\mathcal{A}\} = \mathcal{A}$ and $\Phi\{\mathcal{B}\} = \mathcal{B}$. By the definition of metatruth, we immediately have $\mathcal{A} \vee_k \mathcal{B}$ is metatrue w.r.t. $(*, f)$ if and only if $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and its resolvent \mathcal{A} (resp. \mathcal{B}) is so.

Now (induction hypothesis) assume that the proposition holds when the number of oconnectives of $\Phi\{\}$ is n . We want to show that the proposition still holds when the number of oconnectives of $\Phi\{\}$ is $n + 1$. The following three cases (i), (ii), (iii) need to be considered:

(i) Assume that the main connective of $\Phi\{\}$ is \wedge , namely, $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \Psi\{\mathcal{A} \vee_k \mathcal{B}\} \wedge \mathcal{C}$ for some cirquent \mathcal{C} (the other possibility $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \mathcal{C} \wedge \Psi\{\mathcal{A} \vee_k \mathcal{B}\}$ is similar). By the definition of metatruth, $\Phi\{\mathcal{A} \vee_k \mathcal{B}\}$ is metatrue w.r.t. $(*, f)$ iff both $\Psi\{\mathcal{A} \vee_k \mathcal{B}\}$ and \mathcal{C} are so. But, by the induction hypothesis, $\Psi\{\mathcal{A} \vee_k \mathcal{B}\}$ is metatrue w.r.t. $(*, f)$ iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Psi\{\mathcal{A}\}$ (resp. $\Psi\{\mathcal{B}\}$) is metatrue w.r.t. $(*, f)$. Therefore, $\Phi\{\mathcal{A} \vee_k \mathcal{B}\}$ is metatrue w.r.t. $(*, f)$ iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Psi\{\mathcal{A}\} \wedge \mathcal{C} = \Phi\{\mathcal{A}\}$ (resp. $\Psi\{\mathcal{B}\} \wedge \mathcal{C} = \Phi\{\mathcal{B}\}$) is metatrue w.r.t. $(*, f)$.

(ii) Assume that the main connective of $\Phi\{\}$ is \vee , namely, $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \Psi\{\mathcal{A} \vee_k \mathcal{B}\} \vee \mathcal{C}$ (the case $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \mathcal{C} \vee \Psi\{\mathcal{A} \vee_k \mathcal{B}\}$ is similar). This means that the displayed (main) occurrence of \vee is in a singleton cluster i for some positive integer $i \neq k$. Two (sub)cases are to be further considered here.

Case (a): $f(i) = \text{right}$. Then $\Psi\{\mathcal{A} \vee_k \mathcal{B}\} \vee \mathcal{C}$ is metatrue w.r.t. $(*, f)$ iff its resolvent \mathcal{C} is so. But exactly the same (and for the same reason) holds for both $\Psi\{\mathcal{A}\} \vee \mathcal{C}$ and $\Psi\{\mathcal{B}\} \vee \mathcal{C}$. Consequently, vacuously adding “ $f(k) = \dots$ ”, we arrive at the desired conclusion that $\Psi\{\mathcal{A} \vee_k \mathcal{B}\} \vee \mathcal{C}$ is metatrue w.r.t. $(*, f)$ iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Psi\{\mathcal{A}\} \vee \mathcal{C}$ (resp. $\Psi\{\mathcal{B}\} \vee \mathcal{C}$) is metatrue w.r.t. $(*, f)$.

Case (b): $f(i) = \text{left}$. Then $\Phi\{\mathcal{A} \vee_k \mathcal{B}\}$ is metatrue w.r.t. $(*, f)$ iff its resolvent $\Psi\{\mathcal{A} \vee_k \mathcal{B}\}$ is so, which, in turn (by the induction hypothesis), is the case iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Psi\{\mathcal{A}\}$ (resp. $\Psi\{\mathcal{B}\}$) is metatrue w.r.t. $(*, f)$. This, in turn, is the case iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Phi\{\mathcal{A}\}$ (resp. $\Phi\{\mathcal{B}\}$) is metatrue w.r.t. $(*, f)$. Hence the desired conclusion holds.

(iii) Assume that the main connective of $\Phi\{\}$ is \vee_l , for some (whatever) particular index l such that cluster l is not a singleton. Namely, assume $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \Psi\{\mathcal{A} \vee_k \mathcal{B}\} \vee_l \mathcal{C}$ (the case $\Phi\{\mathcal{A} \vee_k \mathcal{B}\} = \mathcal{C} \vee_l \Psi\{\mathcal{A} \vee_k \mathcal{B}\}$ is similar). If $l \neq k$, then we can employ an essentially the same argument as the one used in (ii). And if $l = k$, then the case is even simpler, so we leave details to the reader. ■

Lemma 3.2 *All rules of IF_p preserve truth in both top-down and bottom-up directions.*

Proof. Pick an arbitrary interpretation $*$.

Rule I: Here we will only look at Rule I (left), with Rule I (right) being similar. Consider an arbitrary metaselection f . By Lemma 3.1, the premise is metatrue w.r.t. $(*, f)$ iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Phi\{\Psi\{\mathcal{A}\}\}$ (resp. $\Phi\{\mathcal{C}\}$) is metatrue w.r.t. $(*, f)$. But exactly the same is the case for the conclusion as well (only, now Lemma 3.1 needs to be applied twice). Thus, the premise is metatrue w.r.t. $(*, f)$ iff so is the conclusion. This, of course, implies that the premise is true under $*$ iff so is the conclusion, as desired.

Rule II: Again, we will only consider Rule II (left), with Rule II (right) being similar. We want to show that the premise $\Phi\{(\mathcal{A} \circ \mathcal{C}) \vee_k (\mathcal{B} \circ \mathcal{C})\}$ is true under $*$ iff so is the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \circ \mathcal{C}\}$. The following two cases need to be considered here.

Case (a): \circ is \wedge or \vee_l . In this case, by Lemma 3.1 applied to both the premise and the conclusion, we immediately get that, for an arbitrary metaselection f , the premise is metatrue w.r.t. $(*, f)$ iff so is the conclusion. Hence, the premise is true under $*$ iff so is the conclusion.

Case (b): \circ is \vee . Assume that, after restoring the (otherwise always omitted) singleton-cluster IDs, the premise is $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{C})\}$ and the conclusion is $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m \mathcal{C}\}$, where clusters i, j, m are singletons. Let $\{l_1, \dots, l_n\}$ be the collection of all singleton clusters in the subcirquent \mathcal{C} of the conclusion. And let $\{l'_1, \dots, l'_n\}$ (resp. $\{l''_1, \dots, l''_n\}$) be the collection of all singleton clusters in the left (resp. right) occurrence of \mathcal{C} in the premise satisfying the condition that, for any $h \in \{1, \dots, n\}$, l'_h (resp. l''_h) occurs in this \mathcal{C} at the same place as l_h occurs in the \mathcal{C} part of the conclusion.

Suppose that the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m \mathcal{C}\}$ is true under $*$. Then there is a metaselection f such that $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m \mathcal{C}\}$ is metatrue w.r.t. $(*, f)$. This, by Lemma 3.1, implies that $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_m \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_m \mathcal{C}\}$) is metatrue w.r.t. $(*, f)$. Let g be a metaselection

satisfying the conditions that $g(i) = g(j) = f(m)$, $g(l'_h) = g(l''_h) = f(l_h)$ for any $h \in \{1, \dots, n\}$ and g agrees with f on all other clusters. Then we have $g(k) = \text{left}$ (resp. $g(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_i \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_j \mathcal{C}\}$) is metatrue w.r.t. $(*, g)$. This, again by Lemma 3.1, implies that the premise $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{C})\}$ is metatrue w.r.t. $(*, g)$. Therefore, the premise is true under $*$.

Now suppose that the premise $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{C})\}$ is true under $*$, meaning that there is a metaselection u such that $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{C})\}$ is metatrue w.r.t. $(*, u)$. By Lemma 3.1, this implies that $u(k) = \text{left}$ (resp. $u(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_i \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_j \mathcal{C}\}$) is metatrue w.r.t. $(*, u)$. Let v be a metaselection satisfying the following two conditions: if $u(k) = \text{left}$ (resp. $u(k) = \text{right}$), then $v(m) = u(i)$ (resp. $v(m) = u(j)$) and $v(l_h) = u(l'_h)$ (resp. $v(l_h) = u(l''_h)$) for any $h \in \{1, \dots, n\}$; v agrees with u on all other clusters. Then, we have $v(k) = \text{left}$ (resp. $v(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_m \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_m \mathcal{C}\}$) is metatrue w.r.t. $(*, v)$. This, by Lemma 3.1, implies that the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m \mathcal{C}\}$ is metatrue w.r.t. $(*, v)$. Hence, the conclusion is true under $*$.

Rule III: We want to show that the premise $\Phi\{(\mathcal{A} \circ \mathcal{C}) \vee_k (\mathcal{B} \circ \mathcal{D})\}$ is true under $*$ iff so is the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \circ (\mathcal{C} \vee_k \mathcal{D})\}$. Two cases are to be considered.

Case (a): \circ is \wedge or \vee_l . Consider an arbitrary metaselection f . It is not hard to see that, by Lemma 3.1 applied twice, the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \circ (\mathcal{C} \vee_k \mathcal{D})\}$ is metatrue w.r.t. $(*, f)$ iff $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Phi\{\mathcal{A} \circ \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \circ \mathcal{D}\}$) is metatrue w.r.t. $(*, f)$. This, in turn, is the case iff the premise $\Phi\{(\mathcal{A} \circ \mathcal{C}) \vee_k (\mathcal{B} \circ \mathcal{D})\}$ is metatrue w.r.t. $(*, f)$. Therefore, the conclusion is true under $*$ iff so is the premise.

Case (b): \circ is \vee . Assume that (after restoring the singleton-cluster IDs) the premise is $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{D})\}$ and the conclusion is $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m (\mathcal{C} \vee_k \mathcal{D})\}$, where clusters i, j, m are singletons. As can be seen from the following two paragraphs, the present case is very similar to Case (b) of Rule II.

Suppose that the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m (\mathcal{C} \vee_k \mathcal{D})\}$ is true under $*$. Then there is a metaselection f such that $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m (\mathcal{C} \vee_k \mathcal{D})\}$ is metatrue w.r.t. $(*, f)$. This, by Lemma 3.1 applied twice, implies that $f(k) = \text{left}$ (resp. $f(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_m \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_m \mathcal{D}\}$) is metatrue w.r.t. $(*, f)$. Let g be a metaselection satisfying the conditions that $g(i) = g(j) = f(m)$ and g agrees with f on all other clusters. Then we have $g(k) = \text{left}$ (resp. $g(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_i \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_j \mathcal{D}\}$) is metatrue w.r.t. $(*, g)$. This, again by Lemma 3.1, implies that the premise $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{D})\}$ is metatrue w.r.t. $(*, g)$. Therefore, the premise is true under $*$.

Now suppose that the premise $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{D})\}$ is true under $*$. Then there is a metaselection u such that $\Phi\{(\mathcal{A} \vee_i \mathcal{C}) \vee_k (\mathcal{B} \vee_j \mathcal{D})\}$ is metatrue w.r.t. $(*, u)$. By Lemma 3.1, this implies that $u(k) = \text{left}$ (resp. $u(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_i \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_j \mathcal{D}\}$) is metatrue w.r.t. $(*, u)$. Let v be a metaselection satisfying the following two conditions: when $u(k) = \text{left}$ (resp. $u(k) = \text{right}$), $v(m) = u(i)$ (resp. $v(m) = u(j)$); v agrees with u on all other clusters. Then, we have $v(k) = \text{left}$ (resp. $v(k) = \text{right}$) and $\Phi\{\mathcal{A} \vee_m \mathcal{C}\}$ (resp. $\Phi\{\mathcal{B} \vee_m \mathcal{D}\}$) is metatrue w.r.t. $(*, v)$. This, by Lemma 3.1 applied twice, implies that the conclusion $\Phi\{(\mathcal{A} \vee_k \mathcal{B}) \vee_m (\mathcal{C} \vee_k \mathcal{D})\}$ is metatrue w.r.t. $(*, v)$. Hence, the conclusion is true under $*$. ■

3.2 The soundness and completeness of IF_p

Theorem 3.3 *A cirquent is valid if and only if it is provable in IF_p .*

Proof. The soundness part is immediate, because the axioms are obviously valid and, by Lemma 3.2, all rules preserve truth and hence validity. For the completeness part, consider an arbitrary cirquent \mathcal{C} and assume it is valid. We want to show that \mathcal{C} is provable in IF_p .

In the context of a given cirquent \mathcal{D} , we define the **level** of an oconnective a , denoted by $\mathcal{L}^{\mathcal{D}}(a)$, to be the total number of oconnectives b such that a is in the scope of b . An oconnective b is a **child** of an oconnective a and a is the **parent** of b when b is in the scope of a and $\mathcal{L}^{\mathcal{D}}(b) = \mathcal{L}^{\mathcal{D}}(a) + 1$. The relations “descendant” and “ancestor” are the transitive closures of the relations “child” and “parent”, respectively. The **distance** between an oconnective a and one of its descendants b is defined to be the positive integer k such that $k = \mathcal{L}^{\mathcal{D}}(b) - \mathcal{L}^{\mathcal{D}}(a)$; when the distance between a and b is less than the

distance between a and another descendant c of a , we say that b is **nearer** to a (or vice versa) than c is. Next, for any two oconnectives a and b , we denote their nearest common ancestor oconnective by \underline{ab} .

If our cirquent \mathcal{C} is classical (i.e. every cluster of it is a singleton), then the validity of \mathcal{C} can be seen to mean nothing but its validity in the sense of classical logic. So, in this case, \mathcal{C} is an axiom of IF_p and hence is provable.

Now, for the rest of this proof, assume that \mathcal{C} is not classical. We construct, bottom-up, a proof of \mathcal{C} as follows.

First, repeat applying Rule I to the current (topmost in the so far constructed proof) cirquent until no longer possible.

Every time Rule I is applied, the current cirquent loses one pair of disjunctive oconnectives a, b such that a, b are in the same cluster and b is a descendant of a (or vice versa). So, sooner or later, we get a cirquent \mathcal{C}_1 where no descendant-ancestor pair of oconnectives shares the same cluster. Since Rule I preserves truth in the bottom-up direction (Lemma 3.2), \mathcal{C}_1 is valid.

Let k_1, k_2, \dots, k_n ($n \geq 0$) be a list of all non-singleton clusters of \mathcal{C}_1 . Our construction of a proof of \mathcal{C} continues upward from \mathcal{C}_1 as follows.

Pick any cluster k_i from the above list. Repeat the following three steps until cluster k_i becomes a singleton cluster in the current (topmost) cirquent. Below we use \mathcal{D} to denote the current cirquent:

Step 1. Pick any pair a, b of disjunctive oconnectives such that the following two conditions are satisfied: a, b are both in cluster k_i ; $\mathcal{L}^{\mathcal{D}}(\underline{ab}) \geq \mathcal{L}^{\mathcal{D}}(\underline{cd})$ for any pair of oconnectives c, d in cluster k_i . Set $m = 2$ and $l = \mathcal{L}^{\mathcal{D}}(\underline{ab})$.⁴

Step 2. (a) Repeatedly perform the following two actions until $\mathcal{L}^{\mathcal{D}}(a) = l + 1$: (i) Apply (bottom-up) Rule II to \mathcal{D} , with a being the key oconnective of this application (in the conclusion); (ii) Rename the key oconnective of this application (in the premise) into a .

(b) Repeatedly perform the following two actions until $\mathcal{L}^{\mathcal{D}}(b) = l + 1$: (i) Apply (bottom-up) Rule II to \mathcal{D} , with b being the key oconnective of this application (in the conclusion); (ii) Rename the key oconnective of this application (in the premise) into b .

Step 3. Apply (bottom-up) Rule III to \mathcal{D} , with a, b being the key oconnectives of this application (in the conclusion). Set $m = 1$.

During the above three-step procedure, the situation that some descendant together with its ancestor is in the same cluster will not emerge. To see why, note that there are no descendant-ancestor pairs within any given cluster in the current cirquent \mathcal{D} at the beginning of the procedure. If so, step 1 will not give rise to the situation. Next, applying Rule II to \mathcal{D} in step 2 means that $\mathcal{L}^{\mathcal{D}}(a)$ (resp. $\mathcal{L}^{\mathcal{D}}(b)$) is greater than $\mathcal{L}^{\mathcal{D}}(\underline{ab}) + 1$; fourthly, during step 1 and step 2, l 's being maximal ensures that, when applying Rule II as Figure 1 shows to the current cirquent \mathcal{D} , the subcirquent \mathcal{C} does not contain any oconnective e that is also in cluster k_i (otherwise, $\mathcal{L}^{\mathcal{D}}(\underline{ae}) > \mathcal{L}^{\mathcal{D}}(\underline{ab})$ (resp. $\mathcal{L}^{\mathcal{D}}(\underline{be}) > \mathcal{L}^{\mathcal{D}}(\underline{ab})$), which contradicts the conditions satisfied by a, b). Finally, based on the above conditions, applying Rule III in step 3 will not create any descendant-ancestor pairs within any given cluster, either.

Below we verify that the above three-step procedure terminates in a finite number of steps.

For the current cirquent \mathcal{D} at any given stage of the procedure — hencefore we shall use \mathcal{D} as (also) a name of that stage — we define the **state** of \mathcal{D} to be the four-tuple (x, y, z, t) as follows, where $m_{\mathcal{D}}$, $a_{\mathcal{D}}$, $b_{\mathcal{D}}$ are the values of the corresponding three variables of the procedure at the beginning of stage \mathcal{D} :

- x is the number of elements in cluster k_i of \mathcal{D} ;
- $y = x - m_{\mathcal{D}}$;
- $z = \mathcal{L}^{\mathcal{D}}(a_{\mathcal{D}}) + \mathcal{L}^{\mathcal{D}}(b_{\mathcal{D}})$ if $m_{\mathcal{D}} = 2$, and $z = \mathcal{L}^{\mathcal{D}}(a_{\mathcal{D}}) - 1$ if $m_{\mathcal{D}} = 1$;
- t is the total number of elements in all other non-singleton clusters of \mathcal{D} except cluster k_i .

⁴Here m is a variable that records the number of elements of the collection of key oconnectives in the current cirquent. It is introduced into the process mainly for later difinitions and proofs.

Further, we define the relation “ \leq ” on the set of all such tuples as follows. For any two tuples (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) , $(x_1, y_1, z_1, t_1) \leq (x_2, y_2, z_2, t_2)$ if and only if one of the following conditions holds: (i) $x_1 < x_2$; (ii) $x_1 = x_2$ and $y_1 < y_2$; (iii) $x_1 = x_2$, $y_1 = y_2$ and $z_1 < z_2$; (iv) $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ and $t_1 < t_2$; (v) $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ and $t_1 = t_2$. It is easy to see that “ \leq ” well-orders the set of all states, with each tuple (state) denoting an ordinal $< \omega^4$.

Now we show that every step of the process strictly decreases the state of the current cirquent. One can see that the state of the current cirquent depends on cluster k_i and the picked pair of a, b . At the beginning of this procedure, for the picked cluster k_i and the picked pair of a, b in step 1, the state (x, y, z, t) has its original value. In step 2, every time substeps (i),(ii) are performed for a (resp. b), x, y do not change, but z decreases by 1; then in step 3, when Rule III is applied, x is decreased by 1. As long as cluster k_i is not a singleton, the process will come into the next iteration of the loop. During each iteration, a new pair of a, b is picked and the value of m is changed from 1 to 2 in step 1, which makes x unchanged but y is decreased by 1; then every iteration of the inner loop in step 2 leaves x, y unchanged but decreases z by 1; and then step 3 decreases x by 1. Thus, the state keeps decreasing during the procedure, meaning that the latter terminates at some point.

Then after applying the three-step procedure, we get the resulting cirquent \mathcal{C}_2 where the number of non-singleton clusters is $n - 1$, since no new non-singleton clusters are created during the procedure and cluster k_i became a singleton. Pick any cluster k_j ($j \neq i, 1 \leq j \leq n$) in \mathcal{C}_2 and carry out the same steps as we did with \mathcal{C}_1 , then we get the resulting cirquent \mathcal{C}_3 , with $n - 2$ non-singleton clusters. Every time the above steps are performed, the number of non-singleton clusters in the current (topmost) cirquent decreases by 1. Therefore, the outermost procedure will end sooner or later with the resulting cirquent \mathcal{C}_{n+1} having no non-singleton clusters. Since only Rule II and Rule III are applied during this procedure when we get \mathcal{C}_{n+1} from \mathcal{C}_1 and both of these rules preserve truth in the bottom-up direction, all the cirquents $\mathcal{C}_2, \mathcal{C}_3, \dots, \mathcal{C}_{n+1}$ are valid. Hence, our construction of a proof of \mathcal{C} ends up with the top most cirquent \mathcal{C}_{n+1} , which is an axiom of IF_p . ■

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