

A GENERAL FRAMEWORK FOR PRODUCT REPRESENTATIONS: BILATTICES AND BEYOND

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ABSTRACT. This paper studies algebras arising as algebraic semantics for logics used to model reasoning with incomplete or inconsistent information. In particular we study, in a uniform way, varieties of bilattices equipped with additional logic-related operations and their product representations. Our principal result is a very general product representation theorem. Specifically, we present a syntactic procedure (called *duplication*) for building a product algebra out of a given base algebra and a given set of terms. The procedure lifts functorially to the generated varieties and leads, under specified sufficient conditions, to a categorical equivalence between these varieties. When these conditions are satisfied, a very tight algebraic relationship exists between the base variety and the enriched variety. Moreover varieties arising as duplicates of a common base variety are automatically categorically equivalent to each other. Two further product representation constructions are also presented; these are in the same spirit as our main theorem and extend the scope of our analysis.

Our catalogue of applications selects varieties for which product representations have previously been obtained one by one, or which are new. We also reveal that certain varieties arising from the modelling of quite different operations are categorically equivalent. Among the range of examples presented, we draw attention in particular to our systematic treatment of trilattices.

1. INTRODUCTION

The notion of product representation plays a central role in the study of interlaced bilattices, with and without any or all of bounds, negation and additional operations (see *inter alia* [4, 28, 30, 7, 9, 25, 14]). Such algebraic structures have been identified by researchers in artificial intelligence and in philosophical logic as of value for analysing scenarios in which information may be incomplete or inconsistent. The literature in the area is now very extensive. Following the introduction of bilattices by Ginsberg [21], various associated logical systems were proposed and studied, *inter alia* by Belnap [6], Fitting [15, 16, 18], Avron and Arieli [2] and, more recently, by Riviuccio, alone and in collaboration with Bou and Jansana [30, 8, 31]; note also the survey by Gargov [20]. Moreover, much research has been done on algebraic structures having bilattice reducts (for example bilattices with an additional operation such as a modality or an implication [22, 3, 7, 9, 32]) and also trilattices [36, 34, 35]. A bewildering proliferation of examples has resulted, with most of the analysis done on a case-by-case basis.

Our objective in this paper is to develop an abstract framework for product representations. Our principal result is Theorem 3.1. Our treatment scores over the traditional one in three ways. Firstly, product representation theorems have traditionally been obtained on a case-by-case basis, whereas our theorem applies in a uniform way to many varieties, as we shall see in Sections 5–8. Secondly, the theorem splits the construction of a product representation for a variety \mathcal{A} into two parts. First we identify a set \mathcal{M} of algebras (frequently a single algebra) that generates \mathcal{A} . We then set up the product representation just for the members of \mathcal{M} . Then Theorem 3.1 automatically proves that each element of \mathcal{A} admits a product representation. Thirdly, the theorem supplies a categorical equivalence from the outset; in the literature product representation theorems have often been given only at the object level and, where such representations were upgraded to categorical equivalences, considerable effort had to be expended for each individual class.

We now present in a little more detail the idea underlying our approach. Consider two classes of algebras: \mathcal{A} , a variety we wish to analyse, and a base variety \mathcal{B} , which we assume to be of the

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form $\mathcal{B} = \mathbb{V}(\mathbf{N})$, the variety generated by some algebra \mathbf{N} . (The single algebra \mathbf{N} above could be replaced by a class \mathcal{N} of algebras of common type.) Then, when suitable conditions are satisfied, we can ‘duplicate’ \mathbf{N} to construct an algebra $\mathbf{M} := \mathbb{P}_\Gamma(\mathbf{N})$ in \mathcal{A} . Here the universe of \mathbf{M} is $N \times N$, where N is the universe of \mathbf{N} . The operations in the product are built from Γ , a set of pairs of algebraic terms in the base language (that of \mathcal{B}), used to define certain operations coordinatewise, and are combined with coordinate manipulation to link the factors. The set Γ is called a *duplicator* (for \mathcal{B}). Moreover the duplication construction lifts to a category equivalence between the base variety $\mathcal{B} = \mathbb{V}(\mathbf{N})$ and the variety $\mathbb{V}(\mathbf{M})$. In practice, the latter is likely to be the variety $\mathbb{V}(\mathbb{P}_\Gamma(\mathcal{N}))$ we are interested in. The mechanism of duplication is rooted in the manipulation of terms in an abstract algebraic language. Indeed, from this perspective product representations can be seen to arise just from a glorified form of term-equivalence (see the discussion before Theorem 9.1). We stress that the construction does not depend on the specific algebraic language of the base class nor that of the duplicated one but only on the relation between their two languages. We shall follow the literature on product representations in confining our examples to varieties of bilattice-based algebras. However the scope of Theorem 3.1 is not restricted to such classes.

As we shall demonstrate in Sections 5–8, distributive lattices, Boolean algebras, Heyting algebras, distributive bilattices, and De Morgan algebras will serve as base varieties in this way, as do their unbounded analogues. The duplicated varieties carry, besides operations from the base language, operations which are order-preserving or order-reversing unary involutions; implication-like operations; assorted other logic-driven unary and binary operations; further pairs of lattice operations. We stress that the duplication formalism helps guide us to the product representations we seek. To illustrate the point, we contrast our treatment of distributive bilattices with conflation in Section 5 with Fitting’s account in [17] and note our remarks on implicative bilattices (Example 8.3).

The generalised form of product representation given in Section 9 takes its cue from two varieties: pre-bilattices (not covered by Theorem 3.1) and interlaced trilattices (covered, but only by carrying out a two-stage duplication). In an appendix we bring our multitude of examples together in two tables. Table 1 lists bilattice-based varieties and the base varieties they duplicate, and so highlights the categorical equivalences revealed by our analysis. Table 2 systematises the product representations available for interlaced bilattices, for interlaced trilattices and for interlaced trilattices augmented with one, two or three involutory operations.

This work has grown out of our study of natural dualities for bilattices and their connection with product representations [11, 12]. In [13] we return to the duality theme and set up an automatic procedure to obtain natural dualities for classes of algebras that fit into the general framework for product representations presented in this paper.

2. PRELIMINARIES ON BILATTICES AND PRODUCT REPRESENTATION

Our investigations involve classes of algebras. Accordingly we shall draw on some of the basic formalism of universal algebra, specifically regarding algebras, terms and varieties (*alias* equational classes); a standard reference is [10]; see also [5, Chapter I] for a categorical perspective. We write $\mathbb{V}(\mathcal{N})$ to denote the variety generated by a family \mathcal{N} of algebras having a common language. Equivalently $\mathbb{V}(\mathcal{N})$ is the class $\mathbb{HISP}(\mathcal{N})$ of homomorphic images of subalgebras of products of algebras in \mathcal{N} . We often encounter classes such that $\mathbb{HISP}(\mathcal{N}) = \mathbb{ISP}(\mathcal{N})$, the class of isomorphic images of subalgebras of products of algebras in \mathcal{N} . We note the elementary but useful fact that an algebra \mathbf{A} belongs to $\mathbb{ISP}(\mathcal{N})$ if and only if the family of homomorphisms from \mathbf{A} into the algebras in \mathcal{N} separates the elements of \mathbf{A} . Most often in our investigations \mathcal{N} will contain a single algebra \mathbf{N} . When this is the case, to simplify the notation, we write \mathbf{N} instead of $\{\mathbf{N}\}$. A class of algebras of common language will be regarded as a category in the usual way: we take morphisms all homomorphisms.

The algebras we consider as examples will be lattice-based, that is, they have reducts in the variety \mathcal{L}_u of all lattices, with basic operations \vee and \wedge . Here the subscript u indicates that the lattices are *unbounded* in the sense that bottom and top elements for the underlying order, even when these exist, are not included in the language. We write \mathcal{L} for the variety of bounded

lattices, *viz.* algebras $(L; \vee, \wedge, 0, 1)$, where $(L; \vee, \wedge) \in \mathcal{L}_u$, and $0, 1$ are respectively, bottom and top elements for the underlying order on L . For any lattice \mathbf{L} , unbounded or bounded, we write \mathbf{L}^∂ to denote the lattice on the same underlying set, but with the order and bounds (when present) reversed.

We now turn to bilattices. We shall assume that readers are familiar with the basic notions; summaries can be found, for example, in [30, 7]. Here we establish notation and terminology, and make only a few comments to set the scene for our study. An (*unbounded*) *pre-bilattice* $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k)$ is an algebra for which $(A; \vee_t, \wedge_t)$ and $(A; \vee_k, \wedge_k)$ belong to \mathcal{L}_u . Here the subscripts $_t$ and $_k$ have the connotation of ‘truth’ and ‘knowledge’ and refer to the associated lattices \mathbf{A}_t and \mathbf{A}_k as the truth and knowledge lattices of \mathbf{A} ; the corresponding lattice orders are denoted by \leq_t and \leq_k . Analogous definitions can be formulated in the bounded case. Here we follow the notation we used in [12] and choose to deviate from that adopted in recent bilattice literature, in which the truth operations are denoted \vee and \wedge and the knowledge operations by \oplus and \otimes .

Here, as in [30, 7] and elsewhere, the term *bilattice* is reserved for an algebra \mathbf{A} which is a pre-bilattice enriched with a negation operation \neg , which is required to be an involution that preserves \leq_k and reverses \leq_t . We shall normally assume that a negation operator is present, and delay until Section 9 the adaptation of our approach to encompass also the product representation for pre-bilattices. Unlike negation, whose inclusion or omission leads to significantly different outcomes, whether or not the algebraic language includes nullary operations interpreted as lattice bounds is largely a matter of choice, governed for example by the logic being modelled. Thus we are ambivalent about constants, sometimes including them and sometimes not; the adaptations required for the other case are generally minor.

An interaction between the lattice operations \vee_t, \wedge_t and \vee_k, \wedge_k of a bilattice is needed for a good structure theory. At a minimum, we need to impose the condition of *interlacing*, asserting that the operations in $\{\vee_t, \wedge_t\}$ and in $\{\vee_k, \wedge_k\}$ are monotonic with respect to \leq_k and \leq_t , respectively. Interlacing is both necessary and sufficient for the existence of a product representation (see [30] and also [14]). We write \mathcal{BL}_u and \mathcal{BL} for the varieties of unbounded and bounded interlaced bilattices, respectively. We recall the product representation for interlaced unbounded bilattices. Given a lattice $\mathbf{L} = (L; \vee, \wedge)$, then $\mathbf{L} \odot \mathbf{L}$ denotes the bilattice with universe $L \times L$ and lattice operations given by

$$\begin{aligned} (a_1, a_2) \vee_t (b_1, b_2) &= (a_1 \vee b_1, a_2 \wedge b_2), & (a_1, a_2) \vee_k (b_1, b_2) &= (a_1 \vee b_1, a_2 \vee b_2), \\ (a_1, a_2) \wedge_t (b_1, b_2) &= (a_1 \wedge b_1, a_2 \vee b_2), & (a_1, a_2) \wedge_k (b_1, b_2) &= (a_1 \wedge b_1, a_2 \wedge b_2); \end{aligned}$$

negation is given by $\neg(a, b) = (b, a)$. The Product Representation Theorem for unbounded interlaced bilattices states that, given $\mathbf{A} \in \mathcal{BL}_u$, there exists $\mathbf{L} = (L; \vee, \wedge) \in \mathcal{L}_u$ such that $\mathbf{A} \cong \mathbf{L} \odot \mathbf{L}$.

We can see that the operations of $\mathbf{L} \odot \mathbf{L}$ are constructed from the operations of \mathbf{L} just by manipulating coordinates and applying to them the operations in \mathbf{L} . This simple observation is the starting point for the results of this paper, as outlined in Section 1.

3. ALGEBRAIC FRAMEWORK FOR PRODUCT REPRESENTATIONS

In this section we set up our general algebraic-categorical framework. We assume given a variety \mathcal{A} of algebras for which we desire a product representation theorem, and that $\mathcal{B} = \mathbb{V}(\mathcal{N})$ is a well-behaved and well-understood variety on which we want to base our representation for \mathcal{A} . We aim to realise \mathcal{A} as a variety $\mathbb{V}(\mathcal{M})$, where \mathcal{M} is obtained from \mathcal{N} , in the manner outlined in Section 1, by means of a set Γ of pairs of terms in the language of \mathcal{B} , except that now do not restrict to singly-generated varieties.

The set Γ is used to build a product structure $\mathbf{M} \cong P_\Gamma(\mathbf{N})$ of each algebra $\mathbf{N} \in \mathcal{N}$. We then seek to show that $\mathcal{B} := \mathbb{V}(\mathcal{N})$ and $\mathbb{V}(P_\Gamma(\mathcal{N}))$ are categorically equivalent, with the second variety being what we call a duplicate of the first (the formal definition is given below). Two extreme cases naturally arise here: \mathcal{N} is already our base variety \mathcal{B} or \mathcal{N} may contain a single algebra \mathbf{N} . The former case will arise in practice when \mathcal{B} is not finitely generated, as occurs for example when \mathcal{B} is \mathcal{L} or \mathcal{L}_u . Our programme will, however, yield the most powerful results in the latter case

and when, better still, we can show that \mathcal{A} is generated by $P_\Gamma(\mathbf{N})$, for some choice of Γ . In these circumstances Theorem 3.1 tells us that a product representation of a generator for \mathcal{A} lifts to a product representation applicable to the entire equational class \mathcal{A} , and that this lifting operates functorially. We then have a very tight relationship between $\mathcal{B} = \mathbb{V}(\mathbf{N})$ and $\mathcal{A} = \mathbb{V}(P_\Gamma(\mathbf{N}))$; indeed these varieties are equivalent as categories. This is exactly what happens, as we shall demonstrate later, for many much-studied varieties, and it retrospectively vindicates the emphasis in much of the literature (see for example [21, 22, 15, 16, 2, 33, 3]) on individual bilattice-based algebras as opposed to the classes they generate: algebraic information not visible at the level of the generator becomes instantly accessible, leading to a much richer theory.

Let \mathcal{N} be a class of Σ -algebras, where Σ is some algebraic language and let $\mathbb{V}(\mathcal{N})$ be the variety generated by \mathcal{N} . Let Γ be a set of pairs of Σ -terms such that, for each $(t_1, t_2) \in \Gamma$, there exists $n_{(t_1, t_2)} \in \{0, 1, \dots\}$ such that t_1 and t_2 are terms on $2n_{(t_1, t_2)}$ variables. We shall view Γ as playing the role of an algebraic language for a family of algebras $P_\Gamma(\mathbf{A})$ ($\mathbf{A} \in \mathbb{V}(\mathcal{N})$), where the arity of $(t_1, t_2) \in \Gamma$ is $n_{(t_1, t_2)}$. We write $[t_1, t_2]$ when we are viewing (t_1, t_2) as belonging to Γ , *qua* language, rather than as a pair of terms from the original language. Specifically we define, for $\mathbf{A} \in \mathbb{V}(\mathcal{N})$,

$$P_\Gamma(\mathbf{A}) = (A \times A; \{[t_1, t_2]^{P_\Gamma(\mathbf{A})} \mid (t_1, t_2) \in \Gamma\}),$$

where, writing $n = n_{(t_1, t_2)}$, the operation $[t_1, t_2]^{P_\Gamma(\mathbf{A})}: (A \times A)^n \rightarrow A \times A$ is given by

$$[t_1, t_2]^{P_\Gamma(\mathbf{A})}((a_1, b_1), \dots, (a_n, b_n)) = (t_1^{\mathbf{A}}(a_1, b_1, \dots, a_n, b_n), t_2^{\mathbf{A}}(a_1, b_1, \dots, a_n, b_n)),$$

for $(a_1, b_1), \dots, (a_n, b_n) \in A \times A$.

We let $P_\Gamma(\mathcal{N})$ denote the class of algebras of the form $P_\Gamma(\mathbf{N})$, for $\mathbf{N} \in \mathcal{N}$. It is straightforward to check that $P_\Gamma(\mathbb{V}(\mathcal{N}))$ is contained in $\mathbb{V}(P_\Gamma(\mathcal{N}))$. We claim that the assignment $\mathbf{A} \mapsto P_\Gamma(\mathbf{A})$ (on objects) and $h \mapsto h \times h$ (on morphisms) defines a functor $P_\Gamma: \mathbb{V}(\mathcal{N}) \rightarrow \mathbb{V}(P_\Gamma(\mathcal{N}))$. We need to confirm that P_Γ is well defined on morphisms. Take $\mathbf{A}, \mathbf{B} \in \mathbb{V}(\mathcal{N})$ and $h: \mathbf{A} \rightarrow \mathbf{B}$ a homomorphism. Since the operations in $P_\Gamma(\mathbf{A})$ and $P_\Gamma(\mathbf{B})$ are constructed using Σ -terms $h \times h: A \times A \rightarrow B \times B$ is indeed a homomorphism from $P_\Gamma(\mathbf{A})$ to $P_\Gamma(\mathbf{B})$. It is routine to check that P_Γ is a functor and is faithful.

We introduce the following notation. Given a set X we let $\delta^X: X \rightarrow X \times X$ be the diagonal map given by $\delta^X(x) = (x, x)$ and let $\pi_1^X, \pi_2^X: X \times X \rightarrow X$ be the projection maps; we suppress the label when no ambiguity would arise.

We are now ready to give an important definition. Fix a class \mathcal{N} of Σ -algebras that generates a variety \mathcal{B} and let Γ be a set of pairs of terms as specified above. We say that the variety $\mathcal{A} = \mathbb{V}(P_\Gamma(\mathcal{B}))$ is a *duplicate* of \mathcal{B} (in symbols $\mathcal{B} \ll \mathcal{A}$) if Γ *duplicates* \mathcal{N} . By the latter we mean that the following conditions on \mathcal{N} and Γ are satisfied:

- (L) for each n -ary operation symbol $f \in \Sigma$ and $i \in \{1, 2\}$ there exists an n -ary Γ -term t such that $\pi_i^N \circ t^{P_\Gamma(\mathbf{N})} \circ (\delta^N)^n = f^{\mathbf{N}}$ for each $\mathbf{N} \in \mathcal{N}$;
- (M) there exists a binary Γ -term v such that

$$v^{P_\Gamma(\mathbf{N})}((a, b), (c, d)) = (a, d) \quad \text{for } \mathbf{N} \in \mathcal{N} \text{ and } a, b \in N;$$

- (P) there exists a unary Γ -term s such that

$$s^{P_\Gamma(\mathbf{N})}(a, b) = (b, a) \quad \text{for } \mathbf{N} \in \mathcal{N} \text{ and } a, b \in N;$$

Here L, M and P have the connotations of language, merging and permutation. The role of the term v in (M) is to merge pairs and that of term s in (P) is to permute the coordinates. Therefore, if $\mathbf{N} \in \mathcal{N}$ and S is a subset of $P_\Gamma(\mathbf{N})$ that is closed under v , then $\pi_1^N(S) = \pi_2^N(S)$. If S is closed under s , then $S = \pi_1^N(S) \times \pi_2^N(S)$. It is worth observing that, if Γ satisfies (P), then (L) is equivalent to the weaker condition

- (L') for each n -ary operation symbol $f \in \Sigma$ there exist an n -ary Γ -term t and $i \in \{1, 2\}$ such that $\pi_i^N \circ t^{P_\Gamma(\mathbf{N})} \circ (\delta^N)^n = f^{\mathbf{N}}$ for each $\mathbf{N} \in \mathcal{N}$.

The algebraic language determined by Γ is obtained by means of the pairs of terms in Σ . Condition (L) works in the reverse direction, as a method to obtain Σ from terms in Γ . In Section 9 we elucidate the connection between product representation and term-equivalence.

Illustrations of the duplication mechanism, for various base varieties and with a variety of duplicators Γ , are given in succeeding sections. We shall thereby bring many varieties within the scope of our main result, Theorem 3.1. Whether or not an algebra \mathbf{M} on a universe $N \times N$ can be obtained as a duplicate of some \mathbf{N} with universe N will of course depend on whether Γ , satisfying (L), (M) and (P), can be found so that the operations of \mathbf{M} and $\mathbf{N} \odot_{\Gamma} \mathbf{N}$ match up. See Example 5.1 for an illustration of obstacles to duplication.

Theorem 3.1. *Assume that Γ duplicates a class \mathcal{N} and let $\mathcal{B} = \mathbb{V}(\mathcal{N})$. Then the functor $P_{\Gamma}: \mathcal{B} \rightarrow \mathcal{A}$ sets up a categorical equivalence between \mathcal{B} and its duplicate $\mathcal{A} = \mathbb{V}(P_{\Gamma}(\mathcal{N}))$.*

Proof. As we observed above, P_{Γ} is a well-defined and faithful functor. We only need to check that it is full and dense on $\mathcal{A} = \mathbb{V}(P_{\Gamma}(\mathcal{N}))$. To simplify notation, during this proof we write P instead of P_{Γ} .

We first show that P is full. Let $\mathbf{A}, \mathbf{B} \in \mathcal{B}$ and let $\psi: P(\mathbf{A}) \rightarrow P(\mathbf{B})$ be a homomorphism. Let $h: \mathbf{A} \rightarrow \mathbf{B}$ be defined by $h = \pi_1^{\mathbf{B}} \circ \psi \circ \delta^{\mathbf{A}}$. We shall show that h is a homomorphism and $P(h) = \psi$. By (P), we also have $h = \pi_2^{\mathbf{B}} \circ \psi \circ \delta^{\mathbf{A}}$. By (M), there is a Γ -term v such that $v^{P(\mathbf{N})}((c, c), (d, d)) = (c, d)$ for each $\mathbf{N} \in \mathcal{N}$ and $c, d \in \mathbf{N}$. Since $\mathbf{A}, \mathbf{B} \in \mathcal{B}$, the same equation is valid in \mathbf{A} and \mathbf{B} . Hence

$$\begin{aligned} \psi(a, b) &= \psi(v^{P(\mathbf{A})}((a, a), (b, b))) = v^{P(\mathbf{B})}(\psi(a, a), \psi(b, b)) \\ &= (\pi_1^{P(\mathbf{B})}(\psi(a, a)), \pi_2^{P(\mathbf{B})}(\psi(b, b))) = (h(a), h(b)), \end{aligned}$$

that is, $\psi = h \times h$.

Now let $f \in \Sigma$ be an n -ary operation symbol. By (L), there exist an n -ary Γ -terms t_1 and t_2 such that $\pi_i^{\mathbf{N}} \circ t_i^{P(\mathbf{N})} \circ (\delta^{\mathbf{N}})^n = f^{\mathbf{N}}$ for $\mathbf{N} \in \mathcal{N}$ and $i \in \{1, 2\}$. Moreover there is a Γ -term w such that

$$w^{P(\mathbf{N})} = v^{P(\mathbf{N})}(t_1^{P(\mathbf{N})}, t_2^{P(\mathbf{N})}) = f^{\mathbf{N}} \times f^{\mathbf{N}}$$

for $\mathbf{N} \in \mathcal{N}$, the corresponding statement holds also for each \mathbf{C} that belongs to \mathcal{B} . Hence, for $a_1, \dots, a_n \in \mathbf{A}$,

$$\begin{aligned} h(f^{\mathbf{A}}(a_1, \dots, a_n)) &= \pi_1^{\mathbf{B}} \circ \psi \circ \delta^{\mathbf{B}}(f^{\mathbf{A}}(a_1, \dots, a_n)) \\ &= \pi_1^{\mathbf{B}}(\psi((f^{\mathbf{A}} \times f^{\mathbf{A}})((a_1, a_1), \dots, (a_n, a_n)))) \\ &= \pi_1^{\mathbf{B}}(\psi(w^{P(\mathbf{B})}((a_1, a_1), \dots, (a_n, a_n)))) \\ &= \pi_1^{\mathbf{B}}(w^{P(\mathbf{B})}(\psi(a_1, a_1), \dots, \psi(a_n, a_n))) \\ &= \pi_1^{\mathbf{B}}((f^{\mathbf{B}} \times f^{\mathbf{B}})(\psi(a_1, a_1), \dots, \psi(a_n, a_n))) \\ &= \pi_1^{\mathbf{B}}((f^{\mathbf{B}} \times f^{\mathbf{B}})((h(a_1), h(a_1)), \dots, (h(a_n), h(a_n)))) \\ &= f^{\mathbf{B}}(h(a_1), \dots, h(a_n)). \end{aligned}$$

This concludes the proof that P is full.

It remains to show that P is dense. For every set of algebras $\mathcal{K} \subseteq \mathcal{B}$, it is easy to see that $\prod P(\mathcal{K})$ is isomorphic to $P(\prod \mathcal{K})$. Now let $\mathbf{C} \in \mathcal{N}$ and let $\mathbf{B} \in \mathcal{A}$ be such that \mathbf{B} is a subalgebra of $\mathbf{A} = P(\mathbf{C})$. By (L), $\pi_1^{\mathbf{A}}(\mathbf{B})$ and $\pi_2^{\mathbf{A}}(\mathbf{B})$ are the universes of subalgebras \mathbf{C}_1 and \mathbf{C}_2 of \mathbf{C} . By (P), $\pi_1^{\mathbf{A}}(\mathbf{B}) = \pi_2^{\mathbf{A}}(\mathbf{B})$, hence $\mathbf{C}_1 = \mathbf{C}_2$. By (M), $\mathbf{B} = \pi_1^{\mathbf{A}}(\mathbf{B}) \times \pi_2^{\mathbf{A}}(\mathbf{B})$. It follows that $\mathbf{B} \cong P(\mathbf{C}_1)$.

Let $\mathbf{C} \in \mathcal{B}$ and $\mathbf{A} \in \mathcal{A}$ and assume that $g: P(\mathbf{C}) \rightarrow \mathbf{A}$ is a surjective homomorphism. Consider $q = g \circ \delta^{\mathbf{C}}: \mathbf{C} \rightarrow \mathbf{A}$. We shall show that $\theta := \ker(q)$ is a congruence of \mathbf{C} . Let $f \in \Sigma$ be an n -ary operation and $(a_1, b_1), \dots, (a_n, b_n) \in \theta$. We have already observed that by (L) and (M) there exists a term w such that $w^{P(\mathbf{C})} = f^{\mathbf{C}} \times f^{\mathbf{C}}$. Hence

$$\begin{aligned} q(f^{\mathbf{C}}(a_1, \dots, a_n)) &= q(f^{\mathbf{C}}(a_1, \dots, a_n), f^{\mathbf{C}}(a_1, \dots, a_n)) \\ &= q((f^{\mathbf{C}} \times f^{\mathbf{C}})((a_1, a_1), \dots, (a_n, a_n))) \\ &= w^{P(\mathbf{C})}(q(a_1, a_1), \dots, q(a_n, a_n)) \\ &= w^{P(\mathbf{C})}(q(b_1, b_1), \dots, q(b_n, b_n)) \end{aligned}$$

$$= q((f^{\mathbf{C}} \times f^{\mathbf{C}})((b_1, b_1), \dots, (b_n, b_n))) = q(f^{\mathbf{C}}(b_1, \dots, b_n)).$$

Therefore $(f^{\mathbf{C}}(a_1, \dots, a_n), f^{\mathbf{C}}(b_1, \dots, b_n)) \in \theta$. We claim that the map $\varphi: \mathbf{P}(\mathbf{C}/\theta) \rightarrow \mathbf{A}$ defined by $\varphi([a]_{\theta}, [b]_{\theta}) = g(a, b)$ is well defined and an isomorphism. First observe that if $q(a_1) = q(a_2)$ and $q(b_1) = q(b_2)$, then, for $i = 1, 2$,

$$g(a_i, b_i) = g(v^{\mathbf{P}(\mathbf{C})}((a_i, a_i), (b_i, b_i))) = v^{\mathbf{A}}(g(a_i, a_i), g(b_i, b_i)) = v^{\mathbf{P}(\mathbf{A})}(q(a_i), q(b_i)).$$

It follows that φ is well defined. The fact that φ is a homomorphism follows from the fact that h and g are homomorphisms and the definition of the operations in $\mathbf{P}(\mathbf{C}/\theta)$. \square

The structural information provided by a product representation for a variety \mathbf{V} is of most value when additional properties of \mathbf{V} follow from it. Here we should distinguish between properties which hold simply because there is a categorical equivalence between \mathbf{A} and the base variety \mathbf{B} and those which rely on the specific algebraic form of the equivalence. Properties of the former type include those expressible in terms of injective homomorphisms, which correspond to monomorphisms [5, Section 14], or surjective homomorphisms, which correspond to regular epimorphisms (note [1, Proposition 7.37 and Definition 7.71], [10, Theorem 6.12]). From this it follows easily that categorically equivalent varieties have isomorphic subvariety lattices—a fact well known to universal algebraists but hard to document explicitly. In particular, assume that Γ duplicates a class of algebras \mathcal{N} , so that the functor \mathbf{P}_{Γ} is a categorical equivalence. Then \mathbf{P}_{Γ} induces an isomorphism between the lattices of subvarieties of $\mathbb{V}(\mathcal{N})$ and of $\mathbb{V}(\mathbf{P}_{\Gamma}(\mathcal{N}))$. Moreover, Γ also duplicates any subvariety \mathcal{K} of $\mathbb{V}(\mathcal{N})$.

We now record as a corollary to Theorem 3.1 further consequences of the existence of a categorical equivalence. In combination with our later results bringing product-representable varieties within the scope of Theorem 3.1, this corollary provides a uniform derivation for results which have been proved piecemeal in the literature in many specific instances [28, 30, 7]; see also [35].

Corollary 3.2. *Assume that Γ duplicates a class of algebras \mathcal{N} . The following statements hold for each $\mathbf{A} \in \mathbb{V}(\mathcal{N})$.*

- (a) $\text{Con}(\mathbf{A}) \cong \text{Con}(\mathbf{P}_{\Gamma}(\mathbf{A}))$, where Con denotes the lattice of congruences of the corresponding algebra.
- (b) \mathbf{A} is subdirectly irreducible if and only if $\mathbf{P}_{\Gamma}(\mathbf{A})$ is.

Proof. (a) follows directly from the relation between congruences and regular epimorphisms, and (b) is a direct consequence of (a). \square

Any functor that determines a categorical equivalence preserves projective objects. Accordingly, if Γ duplicates \mathcal{N} then \mathbf{A} is projective in $\mathbb{V}(\mathcal{N})$ if and only if $\mathbf{P}_{\Gamma}(\mathbf{A})$ is projective in $\mathbb{V}(\mathbf{P}_{\Gamma}(\mathcal{N}))$. However, categorical equivalences do not always preserve free objects. Nonetheless, the following result tells us how to use \mathbf{P}_{Γ} to describe free objects in $\mathbb{V}(\mathbf{P}_{\Gamma}(\mathcal{N}))$ when those in $\mathbb{V}(\mathcal{N})$ are known. Results of this type were obtained for distributive bilattices in [12, Section 8] using natural duality techniques. Here we see that they stem from the product representation, independently of the existence or not of a natural duality.

Proposition 3.3. *Let X be a set, \mathcal{N} a class of algebras with the same language and $\mathbf{B} = \mathbb{V}(\mathcal{N})$ be the variety generated by \mathcal{N} . If Γ duplicates \mathcal{N} and $\mathbf{A} = \mathbb{V}(\mathbf{P}_{\Gamma}(\mathcal{N}))$, then $\mathbf{F}_{\mathbf{A}}(X)$, the \mathbf{A} -free algebra over X , is isomorphic to the algebra $\mathbf{P}_{\Gamma}(\mathbf{F}_{\mathbf{B}}(X \times \{0, 1\}))$ and the isomorphism is obtained by the identification $x \mapsto ((x, 0), (x, 1))$ for $x \in X$, where $\mathbf{F}_{\mathbf{B}}(X \times \{0, 1\})$ is the \mathbf{B} -free algebra over $X \times \{0, 1\}$.*

Proof. It is easy to see that $\{((x, 0), (x, 1)) \mid x \in X\}$ is a set of generators of the algebra $\mathbf{P}_{\Gamma}(\mathbf{F}_{\mathbf{B}}(X \times \{0, 1\}))$.

Now let $\mathbf{B} \in \mathbb{V}(\mathbf{P}_{\Gamma}(\mathcal{N}))$ and consider a map $f: \{((x, 0), (x, 1)) \mid x \in X\} \rightarrow \mathbf{B}$. By Theorem 3.1, there exists $\mathbf{A} \in \mathbb{V}(\mathcal{N})$ such that $\mathbf{B} \cong \mathbf{P}_{\Gamma}(\mathbf{A})$. Let us identify \mathbf{B} with $\mathbf{P}_{\Gamma}(\mathbf{A})$. Let $g: X \times \{0, 1\} \rightarrow \mathbf{A}$ be the map defined by $g(x, i) = \pi_i^{\mathbf{A}}(f((x, 0), (x, 1)))$ for $i = 0, 1$ and $x \in X$. There exists a unique

homomorphism $\bar{g}: \text{Fr}_{\mathcal{N}}(X \times \{0, 1\}) \rightarrow \mathbf{A}$ such that $g(x, i) = \bar{g}(x, i)$ for $(x, i) \in X \times \{0, 1\}$. Let $h = \text{P}(\bar{g}): \text{P}_{\Gamma}(\text{Fr}_{\mathcal{N}}(X \times \{0, 1\})) \rightarrow \text{P}_{\Gamma}(\mathbf{A})$. For $x \in X$,

$$h((x, 0), (x, 1)) = \text{P}(\bar{g})((x, 0), (x, 1)) = (\bar{g}(x, 0), \bar{g}(x, 1)) = (g(x, 0), g(x, 1)) = f((x, 0), (x, 1)).$$

That is, h extends f . \square

4. DUPLICATION IN ACTION: INTERLACED AND DISTRIBUTIVE BILATTICES REVISITED

We fix some notation. Let Σ be a language and f be an n -ary function symbol in Σ , then for each $m \geq n$ and $i_1, \dots, i_n \in \{1, \dots, m\}$, we denote by $f_{i_1 \dots i_n}^m$ the m -ary term

$$f_{i_1 \dots i_n}^m(x_1, \dots, x_m) = f(x_{i_1}, \dots, x_{i_n}).$$

Similarly, x_i^m denotes the m -ary term that selects the i th variable: $x_i^m(x_1, \dots, x_m) = x_i$. For example, let $\Sigma_{\mathcal{L}_u} = \{\vee, \wedge\}$ be the language of lattices. Then \vee_{13}^4 denotes the term $\vee_{13}^4(x_1, x_2, x_3, x_4) = x_1 \vee x_3$.

Now consider the set of $\Sigma_{\mathcal{L}_u}$ -pairs of terms

$$\Gamma_{\mathcal{B}\mathcal{L}_u} = \{(\vee_{13}^4, \wedge_{24}^4), (\wedge_{13}^4, \vee_{24}^4), (\vee_{13}^4, \vee_{24}^4), (\wedge_{13}^4, \wedge_{24}^4), (x_2^2, x_1^2)\}.$$

We name

$$\vee_t = [\vee_{13}^4, \wedge_{24}^4], \wedge_t = [\wedge_{13}^4, \vee_{24}^4], \vee_k = [\vee_{13}^4, \vee_{24}^4], \wedge_k = [\wedge_{13}^4, \wedge_{24}^4], \text{ and } \neg = [x_2^2, x_1^2],$$

to match up our newly-created operations with those in the language of $\mathcal{B}\mathcal{L}_u$. We can clearly see that $\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathbf{L}) = \mathbf{L} \odot \mathbf{L}$. The Product Representation Theorem for unbounded interlaced bilattices implies that every $\mathbf{A} \in \mathcal{B}\mathcal{L}_u$ is isomorphic to $\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathbf{L})$ for some $\mathbf{L} \in \mathcal{L}_u$. Thus $\mathbb{V}(\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathcal{L}_u)) = \mathcal{B}\mathcal{L}_u$. Moreover, it is known that $\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}$ determines a categorical equivalence [8]. This follows directly from $\mathbb{V}(\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathcal{L}_u)) = \mathcal{B}\mathcal{L}_u$ and Theorem 3.1, by simply observing that $\Gamma_{\mathcal{B}\mathcal{L}_u}$ duplicates \mathcal{L}_u . Indeed, it is easy to see that $\Gamma_{\mathcal{B}\mathcal{L}_u}$ satisfies (L) and (P). Observe too that, for $\mathbf{L} \in \mathcal{L}_u$ and $a, b \in L$,

$$((a, b) \wedge_k ((a, b) \vee_t (c, d))) \vee_k ((c, d) \wedge_k ((a, b) \wedge_t (c, d))) = (a, b).$$

Hence the term $v(x, y) = (x \wedge_k (x \vee_t y)) \vee_k (y \wedge_k (x \wedge_t y))$ satisfies (M).

We can easily add bounds: let $\Gamma_{\mathbf{b}} = \{(0, 1), (1, 0), (0, 0), (1, 1)\}$; this is a set of pairs of terms in the language of \mathcal{L} and we may then take $\Gamma_{\mathcal{B}\mathcal{L}} = \Gamma_{\mathcal{B}\mathcal{L}_u} \cup \Gamma_{\mathbf{b}}$. It is straightforward to check that $\Gamma_{\mathcal{B}\mathcal{L}}$ satisfies conditions (L), (M) and (P). Therefore $\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}}}$ determines a categorical equivalence between \mathcal{L} and $\mathbb{V}(\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}}}(\mathcal{L})) = \mathcal{B}\mathcal{L}$.

Lattices are not a finitely generated variety, and our product representation for $\mathcal{B}\mathcal{L}_u$ over \mathcal{L}_u had to take $\mathcal{N} = \mathcal{L}_u$. For the variety $\mathcal{D}\mathcal{B}_u$ distributive bilattices the situation is different: the obvious base variety to use, (unbounded) distributive lattices, is finitely generated. We now fit the product representation for $\mathcal{D}\mathcal{B}_u$ into our general scheme, using Theorem 3.1 as it applies to a singly generated variety.

We denote by \mathcal{D} and \mathcal{D}_u the varieties of bounded distributive lattices and of unbounded distributive lattices, respectively. We let $\mathbf{2}_{\mathcal{D}}$, respectively $\mathbf{2}_{\mathcal{D}_u}$, denote the two-element algebra in \mathcal{D} , respectively \mathcal{D}_u . In both cases we take the underlying set to have elements 0, 1, with $0 < 1$ and denote the corresponding non-strict order by \leq . The following well-known facts will be important later:

$$\mathcal{D}_u = \text{HSP}(\mathbf{2}_{\mathcal{D}_u}) = \text{ISP}(\mathbf{2}_{\mathcal{D}_u}) \quad \text{and} \quad \mathcal{D} = \text{HSP}(\mathbf{2}_{\mathcal{D}}) = \text{ISP}(\mathbf{2}_{\mathcal{D}}).$$

By Theorem 3.1, it follows that $\text{HSP}(\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathbf{2}_{\mathcal{D}_u})) = \text{ISP}(\text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathbf{2}_{\mathcal{D}_u}))$. Letting

$$\mathbf{4}_{\mathcal{D}\mathcal{B}_u} = (\{0, 1\}^2; \vee_t, \wedge_t, \vee_t, \wedge_t, \neg) := \text{P}_{\Gamma_{\mathcal{B}\mathcal{L}_u}}(\mathbf{2}_{\mathcal{D}_u}),$$

we see that \mathcal{D}_u is categorically equivalent to $\text{ISP}(\mathbf{4}_{\mathcal{D}\mathcal{B}_u}) = \text{HSP}(\mathbf{4}_{\mathcal{D}\mathcal{B}_u})$. So it remains to characterise the variety $\text{HSP}(\mathbf{4}_{\mathcal{D}\mathcal{B}_u})$. This is known to be the variety $\mathcal{D}\mathcal{B}_u$ of distributive bilattices, that is, bilattices such that each of the four operations distributes over each of the other three. Moreover, in [12, Proposition 5.1], we presented a proof that $\text{ISP}(\mathbf{4}_{\mathcal{D}\mathcal{B}_u}) = \mathcal{D}\mathcal{B}_u$ that is independent of the product representation. Therefore $\mathcal{D}_u \ll \mathcal{D}\mathcal{B}_u$. Similarly, it follows that $\mathcal{D} \ll \mathcal{D}\mathcal{B}$, where $\mathcal{D}\mathcal{B}$ is the variety of bounded distributive bilattices.

5. BILATTICES WITH CONFLATION

Involutory operations are often added to lattice-based varieties, and hence to bilattice-based varieties too, to provide algebraic models which capture more than just notions of truth and knowledge. We have already built in an involutory operation \neg to model logical negation but wish also, here and in Section 6 too, to allow for involutions which serve to model, for example, what is not known. To fit their intended interpretations, such operations need to act appropriately with respect to the underlying order structures. As we shall see, adding such operations influences our choice of base variety. So we begin this section with a discussion of two finitely generated varieties, De Morgan lattices and De Morgan algebras, we have not encountered previously in this paper. These will prove to be valuable as base varieties in due course. In addition they enable us to provide further illustration of the concept of duplication.

Example 5.1 (De Morgan algebras and De Morgan lattices). In Section 4 we encountered a four-element bounded bilattice, obtained by duplicating the two-element bounded lattice. We shall now compare this with another four-element algebra, that which generates (as a prevariety) the variety \mathbf{DM} of De Morgan algebras (a good reference is [5, Chapter XI]). An algebra $\mathbf{A} = (A; \vee, \wedge, \sim, 0, 1)$ belongs to \mathbf{DM} if $(A; \vee, \wedge, 0, 1) \in \mathbf{D}$ and \sim is an order-reversing involution. The variety is generated, as a prevariety, by the algebra $\mathbf{4}_{\mathbf{DM}}$, the De Morgan algebra whose \mathbf{D} -reduct is $\mathbf{2}_{\mathbf{D}}^2$ and whose negation \sim interchanges the bounds and fixes the other two elements.

We may ask whether $\mathbf{4}_{\mathbf{DM}}$ is a duplicate of a two-element algebra in some naturally related base variety $\mathbb{V}(\mathbf{N})$. It is a consequence of Theorem 3.1 that this could only occur if \mathbf{DM} were categorically equivalent to $\mathbb{V}(\mathbf{N})$. We note that \mathbf{DM} is not categorically equivalent either to \mathbf{D} or to \mathbf{B} , the variety of Boolean algebras (the subvariety lattice of \mathbf{DM} is not isomorphic to that of \mathbf{D} or of \mathbf{B}). It is however quite simple to construct sets Γ of pairs of terms in the languages $\Sigma_{\mathbf{D}} = \{\vee, \wedge, 0, 1\}$ of \mathbf{D} or $\Sigma_{\mathbf{B}} = \{\vee, \wedge, ', 0, 1\}$ of \mathbf{B} such that $\mathbf{4}_{\mathbf{DM}} \cong P_{\Gamma}(\mathbf{2}_{\mathbf{D}})$ or $\mathbf{4}_{\mathbf{DM}} \cong P_{\Gamma}(\mathbf{2}_{\mathbf{B}})$. We might take for example Γ to be Γ_1 or Γ_2 , where

$$\begin{aligned}\Gamma_1 &= \{(\wedge_{13}^2, \wedge_{24}^2), (\vee_{13}^2, \vee_{24}^2), ((')_2^2, (')_1^2), (0, 0), (1, 1)\}; \\ \Gamma_2 &= \{(\wedge_{13}^2, \vee_{24}^2), (\vee_{13}^2, \wedge_{24}^2), (x_2^2, x_1^2), (0, 1), (1, 0)\}.\end{aligned}$$

It is easy to check that $\mathbf{4}_{\mathbf{DM}} \cong P_{\Gamma_1}(\mathbf{2}_{\mathbf{D}}) \cong P_{\Gamma_2}(\mathbf{2}_{\mathbf{B}})$. However Γ_1 satisfies (L') but not (P), and Γ_2 satisfies (P) but not (L'). So neither Γ_1 nor Γ_2 is a duplicator.

The unbounded analogue of \mathbf{DM} is the variety \mathbf{DM}_u of De Morgan lattices, that is, an algebra $\mathbf{A} = (A; \vee, \wedge, \sim) \in \mathbf{DM}_u$ if $(A; \vee, \wedge) \in \mathbf{D}_u$ and \sim is an order-reversing involution. The variety \mathbf{DM}_u coincides with $\mathbb{ISP}(\mathbf{4}_{\mathbf{DM}_u})$, where $\mathbf{4}_{\mathbf{DM}_u} = (\{0, 1\}^2; \vee, \wedge, \sim)$; is the $\{0, 1\}$ -free reduct of $\mathbf{4}_{\mathbf{DM}}$ [27, Theorem 1]. The variety \mathbf{DM}_u does not arise by duplicating either \mathbf{D}_u or the variety of Boolean lattices.

We conclude from the above example that we should regard the varieties \mathbf{DM} and \mathbf{DM}_u as 'atomic': their members are not built from simpler components by duplication. We shall see that they do have an important role to play as base varieties.

We now turn to the main topic of this section. We consider expansions of the varieties \mathbf{DB}_u and \mathbf{DB} of (unbounded and bounded) distributive bilattices obtained by adding a unary operator – called *conflation* and required to act as an endomorphism for the truth lattice structure and a dual endomorphism for the knowledge lattice structure. Customarily it has been assumed that – is an involution and that it commutes with \neg . In this case we denote the resulting expansion of \mathbf{DB}_u by \mathbf{DBC}_u and by \mathbf{DBC} the expansion of \mathbf{DB} .

As indicated above, the variety \mathbf{DBC}_u consists of algebras $(A; \vee_t, \wedge_t, \vee_k, \wedge_k, \neg, -)$ for which the reduct without – belongs to \mathbf{DB}_u and – is an involution preserving \leq_t , reversing \leq_k and commuting with \neg . The class \mathbf{DBC} of bounded distributive bilattices with conflation, where – and \neg commute, is defined in a similar way. The product representation for \mathbf{DBC}_u was first presented in [17, Theorem 8.3]. What we shall do is to demonstrate how this product representation for \mathbf{DBC}_u , and also that for \mathbf{DBC} likewise, is a particular case of our Theorem 3.1. Indeed we shall see that the properties of conflation essentially dictate what the base variety should be.

Until further notice we work with \mathcal{DBC}_u . We first note that we would expect to use a class having a reduct in unbounded distributive lattices, since that will already provide a set $\Gamma_{\mathcal{DB}_u}$ that satisfies (L), (M) and (P), and will allow us to represent the \mathcal{DB}_u -reducts of algebras in \mathcal{DBC}_u . To obtain the conflation operation in a product representation we need a pair of terms (t_1, t_2) such that $[t_1, t_2]$ interprets as an involution that reverses the k -order. This forces $t_1(a \wedge b) = t_1(a) \vee t_1(b)$. This cannot be obtained with $\{\vee, \wedge\}$ -terms since these preserve the order. So it is natural to add an involution to the language of \mathcal{D}_u to obtain the base variety we require. An obvious candidate is to hand, namely the variety \mathcal{DM}_u of De Morgan lattices. It is easy to see that $\Gamma_{\mathcal{DBE}_u} = \Gamma_{\mathcal{BL}_u} \cup \{(\sim_2^2, \sim_1^2)\}$ satisfies (L), since

$$\pi_1^4([\sim_2^2, \sim_1^2]^{P_{\Gamma_{\mathcal{DBE}_u}}(\mathbf{4}_{\mathcal{DM}_u})})(a, a) = \pi_1^4(\sim a, \sim a) = \sim a$$

for every $a \in \mathbf{4}_{\mathcal{DM}_u}$ and $\Gamma_{\mathcal{BL}_u}$ satisfies (L'). Conditions (M) and (P) hold because they hold for $\Gamma_{\mathcal{BL}_u}$. Therefore $\Gamma_{\mathcal{DBE}_u}$ duplicates \mathcal{DM}_u .

To be able to apply Theorem 3.1, it now only remains to prove that the variety \mathcal{DBC}_u coincides with $\mathbb{V}(P_{\Gamma_{\mathcal{DBE}_u}}(\mathcal{DM}_u))$. It is easy to see that $\mathbf{16}_{\mathcal{DBE}_u} := P_{\Gamma_{\mathcal{DBE}_u}}(\mathbf{4}_{\mathcal{DM}_u})$ is a bilattice with conflation and hence that $\mathbb{V}(P_{\Gamma_{\mathcal{DBE}_u}}(\mathcal{DM}_u)) = \mathbb{V}(P_{\Gamma_{\mathcal{DBE}_u}}(\mathbf{4}_{\mathcal{DM}_u})) \subseteq \mathcal{DBC}_u$. The reverse inclusion follows from the following stronger result.

Proposition 5.2. $\mathcal{DBC}_u = \mathbb{ISP}(\mathbf{16}_{\mathcal{DBE}_u})$.

Proof. Let $\mathbf{A} \in \mathcal{DBC}_u$ and take $a \neq b$ in A . By [12, Proposition 5.1], there exists a \mathcal{DB}_u -homomorphism $h: \mathbf{A} \rightarrow \mathbf{4}_{\mathcal{DB}_u}$ such that $h(a) \neq h(b)$. Denote by h_1 and h_2 the unique maps from \mathbf{A} into $\{0, 1\}$ such that $h(c) = (h_1(c), h_2(c))$, for $c \in A$. Define $h': \mathbf{A} \rightarrow \mathbf{16}_{\mathcal{DBE}_u}$ by

$$h'(c) = ((h_1(c), (1 - h_2(-^{\mathbf{A}}c))), (h_2(c), (1 - h_1(-^{\mathbf{A}}c))))$$

for $c \in A$. Clearly $h'(a) \neq h'(b)$. To prove that h' is a \mathcal{DBC}_u -homomorphism, first observe that, since h is a \mathcal{DB}_u -homomorphism,

$$\begin{aligned} h_1(c \vee_t d) &= h_1(c \vee_k d) = h_1(c) \vee h_1(d), & h_1(c \wedge_t d) &= h_1(c \wedge_k d) = h_1(c) \wedge h_1(d); \\ h_2(c \vee_t d) &= h_2(c \wedge_k d) = h_2(c) \wedge h_2(d), & h_2(c \wedge_t d) &= h_2(c \vee_k d) = h_2(c) \vee h_2(d) \end{aligned}$$

and $h_1(c) = h_2(-c)$. It is then easy to see that h' is a \mathcal{DB}_u -homomorphism. Moreover,

$$\begin{aligned} h'(-^{\mathbf{A}}c) &= ((h_1(-^{\mathbf{A}}c), (1 - h_2(c))), (h_2(-^{\mathbf{A}}c), (1 - h_1(c)))) \\ &= (\sim(h_2(c), 1 - h_1(-^{\mathbf{A}}c)), \sim(h_1(c), (1 - h_2(-^{\mathbf{A}}c)))) \\ &= [\sim_2^2, \sim_1^2]^{16_{\mathcal{DBE}_u}}((h_1(c), (1 - h_2(-^{\mathbf{A}}c))), (h_2(c), (1 - h_1(-^{\mathbf{A}}c)))) \\ &= [\sim_2^2, \sim_1^2]^{16_{\mathcal{DBE}_u}}(h(c)). \end{aligned}$$

Hence h' is a \mathcal{DBC}_u -homomorphism. \square

The product representation for \mathcal{DBC} is obtained in a similar way using the variety \mathcal{DM} of De Morgan algebras as a base class and $\Gamma_{\mathcal{DBE}} = \Gamma_{\mathcal{DBE}_u} \cup \Gamma_{\mathbf{b}}$.

We note that neither the requirement that $-$ be an involution nor the assumption that it commute with \vee has been driven by applications. In [13] we relax these restrictions on conflation and provide a product representation and a natural duality for the resulting class.

6. TRILATTICES

Trilattices are, loosely, algebras with three sets of lattice operations, the idea being to model information, truth and falsity. An introduction to the topic from a logical standpoint can be found in [34, 35].

As with bilattices, inclusion of bounds is optional. For illustrative purposes we consider the unbounded case. To simplify notation a little we shall omit $_u$ subscripts from our symbolic names for trilattice and trilattice-based varieties. Thus a *trilattice* is an algebra

$$\mathbf{A} = (A; \vee_t, \wedge_t, \vee_f, \wedge_f, \vee_i, \wedge_i)$$

such that its reducts $\mathbf{A}_t = (A; \vee_t, \wedge_t)$, $\mathbf{A}_f = (A; \vee_f, \wedge_f)$ and $\mathbf{A}_i = (A; \vee_i, \wedge_i)$ are lattices. For any trilattice \mathbf{A} we let $\mathbf{A}_{t,i}$ denote the bilattice reduct of \mathbf{A} obtained by removing the f -operation, and so on.

As with bilattices, at a minimum, an interlacing condition is required in order to obtain a worthwhile structure theory. In Example 9.4 we consider interlaced trilattices. Here we impose the stronger restriction of distributivity, thereby moving into the setting of finitely generated varieties in which a particularly amenable structure theory becomes available. We let \mathcal{DT} denote the variety of (unbounded) distributive trilattices, that is, those trilattices in which all possible distributive laws hold amongst the six lattice operations.

The following examples of trilattices introduce notation we need shortly. $\mathbf{2}^{++}$, $\mathbf{2}^{+-}$, $\mathbf{2}^{-+}$, $\mathbf{2}^{--} \in \mathcal{DT}$ denote the trilattices whose universe is $\{0, 1\}$ and such that

$$\begin{aligned} \mathbf{2}_i^{++} &= \mathbf{2}_i^{+-} = \mathbf{2}_i^{-+} = \mathbf{2}_i^{--} = \mathbf{2}_{\mathcal{D}u}, \\ \mathbf{2}_t^{++} &= \mathbf{2}_t^{+-} = \mathbf{2}_f^{++} = \mathbf{2}_f^{+-} = \mathbf{2}_{\mathcal{D}u}, \text{ and } \mathbf{2}_t^{-+} = \mathbf{2}_t^{--} = \mathbf{2}_f^{+-} = \mathbf{2}_f^{--} = \mathbf{2}_{\mathcal{D}u}^{\partial}. \end{aligned}$$

There are various ways in which one might want involutory operations on trilattices to behave, depending on the desired interpretation. The involutions considered in [34, Definition 5.2] and [31, Sections 3.2–3.4] are dual endomorphisms for one lattice reduct and endomorphisms for the other two reducts. So, a v -involvement (where $v \in \{t, f, i\}$) is an involutory operation on a trilattice that reverses the v -lattice reduct and preserves the other two reducts. Let \mathcal{DT}_t , $\mathcal{DT}_{t,f}$ and $\mathcal{DT}_{t,f,i}$ denote the varieties of trilattices with t -involvement, with t - and f -involvements, and with t -, f - and i -involvements, respectively. Clearly these three varieties cover all the cases we need to consider. We shall assume that all the involutions which we include commute with each other.

As examples of trilattices with a single involutory operation we note that $\mathbf{4}^+$ and $\mathbf{4}^-$ are trilattices with t -involvement $-_t$ having universe $\{0, 1\}^2$ when we define

$$\begin{aligned} \mathbf{4}_t^+ &= \mathbf{4}_t^- = \mathbf{2}_{\mathcal{D}u} \times \mathbf{2}_{\mathcal{D}u}^{\partial}, & \mathbf{4}_i^+ &= \mathbf{4}_i^- = \mathbf{4}_f^+ = \mathbf{2}_{\mathcal{D}u} \times \mathbf{2}_{\mathcal{D}u}, & \mathbf{4}_f^- &= \mathbf{2}_{\mathcal{D}u}^{\partial} \times \mathbf{2}_{\mathcal{D}u}^{\partial}; \\ -_t(a, b) &= (b, a). \end{aligned}$$

Just as a single involutory operation led to the construction of four-element trilattices from two-element ones, sixteen-element trilattices arise naturally from four-element ones when two involutions come into play. We let $\mathbf{16}_{\mathcal{DT}_{t,f}}$ denote the trilattice with t - and f -involvements with universe $(\{0, 1\}^2)^2$ whose operations are defined as follows:

$$\begin{aligned} (\mathbf{16}_{\mathcal{DT}_{t,f}})_t &= (\mathbf{4}_{\mathcal{D}B_u})_t^2, & (\mathbf{16}_{\mathcal{DT}_{t,f}})_f &= (\mathbf{4}_{\mathcal{D}B_u})_k \times (\mathbf{4}_{\mathcal{D}B_u})_k^{\partial}, & (\mathbf{16}_{\mathcal{DT}_{t,f}})_i &= (\mathbf{4}_{\mathcal{D}B_u})_k^2; \\ -_t(a, b) &= (-\mathbf{4}_{\mathcal{D}B_u}(a), -\mathbf{4}_{\mathcal{D}B_u}(b)), \\ -_f(a, b) &= (b, a). \end{aligned}$$

And, finally, we can encompass three involutions. Let $\mathbf{256}$ be the trilattice whose universe is $(\{0, 1\}^4)^2$ with t, f and i -involvements such that

$$\begin{aligned} \mathbf{256}_t &= (\mathbf{16}_{\mathcal{D}B_e_u})_t^2, & \mathbf{256}_f &= (\mathbf{16}_{\mathcal{D}B_e_u})_k^2, & \mathbf{256}_i &= (\mathbf{16}_{\mathcal{D}B_e_u})_k \times (\mathbf{16}_{\mathcal{D}B_e_u})_k^{\partial}; \\ -_t(a, b) &= (-\mathbf{16}_{\mathcal{D}B_e_u}(a), -\mathbf{16}_{\mathcal{D}B_e_u}(b)), & -_f(a, b) &= (-\mathbf{16}_{\mathcal{D}B_e_u}(a), -\mathbf{16}_{\mathcal{D}B_e_u}(b)), \\ -_i(a, b) &= (b, a). \end{aligned}$$

The following lemma is the stepping-off point for further analysis of trilattices by the methods of this paper.

Lemma 6.1.

- (i) $\mathcal{DT} = \text{ISP}(\mathbf{2}^{++}, \mathbf{2}^{+-}, \mathbf{2}^{-+}, \mathbf{2}^{--});$
- (ii) $\mathcal{DT}_t = \text{ISP}(\mathbf{4}^+, \mathbf{4}^-);$
- (iii) $\mathcal{DT}_{t,f} = \text{ISP}(\mathbf{16}_{\mathcal{DT}_{t,f}});$
- (iv) $\mathcal{DT}_{t,f,i} = \text{ISP}(\mathbf{256}).$

Proof. Let $\mathbf{A} \in \mathcal{DT}$ and take $a \neq b$ in A . Then there exists a lattice homomorphism $h: \mathbf{A}_i \rightarrow \mathbf{2}$ such that $h(a) \neq h(b)$. The assumed distributivity of the trilattice operations ensures that, for each $\mathbf{A} \in \mathcal{DT}$, a congruence of \mathbf{A}_i is a congruence of \mathbf{A} (see [8, Proposition 3.13] or [12, Proposition 2.2] for a simple proof). Hence $\ker(h)$ is a congruence of \mathbf{A} and $|\mathbf{A}/\ker(h)| = 2$. Therefore $\mathbf{A}/\ker(h)$ is necessarily isomorphic to $\mathbf{2}^{++}$, $\mathbf{2}^{+-}$, $\mathbf{2}^{-+}$, or $\mathbf{2}^{--}$, and the proof of (i) is complete.

We now prove (ii). Let $\mathbf{B} \in \mathcal{DJ}_t$ and take $a \neq b$ in B . Then $\mathbf{B}_{t,i} \in \mathcal{DB}_u = \mathbb{ISP}(\mathbf{4}_{\mathcal{DB}_u})$. Therefore there exists a homomorphism $h: \mathbf{B}_{t,i} \rightarrow \mathbf{4}_{\mathcal{DB}_u}$ such that $h(a) \neq h(b)$. As before, $\ker(h)$ is also compatible with the f -lattice structure. Then $\mathbf{B}/\ker(h)$ is a trilattice with four elements such that its t, i and t, f reducts are isomorphic to $\mathbf{4}_{\mathcal{DB}_u}$. Therefore $\mathbf{B}/\ker(h)$ is either isomorphic to $\mathbf{4}^+$ or to $\mathbf{4}^-$ and the result follows.

Now let $\mathbf{C} \in \mathcal{DJ}_{t,f}$ and take $a, b \in C$ such that $a \neq b$. Then $\mathbf{C}_{t,i} \in \mathcal{DB}_u = \mathbb{ISP}(\mathbf{4}_{\mathcal{DB}_u})$, so there exists a homomorphism $h: \mathbf{C}_{t,i} \rightarrow \mathbf{4}_{\mathcal{DB}_u}$ such that $h(a) \neq h(b)$. Since $-_f$ preserves the t -order and the i -order, and it commutes with $-_t$, it follows that $h \circ (-_f)$ is also a homomorphism from $\mathbf{C}_{t,i}$ onto $\mathbf{4}_{\mathcal{DB}_u}$. Then the map $g: \mathbf{C} \rightarrow \mathbf{16}_{\mathcal{DJ}_{t,f}}$ defined by $g(a) = (h(a), h(-_f a))$ is a homomorphism from \mathbf{C} to $\mathbf{16}_{\mathcal{DJ}_{t,f}}$ that separates a and b .

The proof of (iv) can be carried out in a similar way to that of (iii). \square

From the definition of $\mathbf{16}_{\mathcal{DJ}_{t,f}}$ it is easy to extract a duplicator $\Gamma_{\mathcal{DJ}_{t,f}}$. Indeed, letting

$$\Gamma_{\mathcal{DJ}_{t,f}} = \left\{ ((\vee_t)_{13}^4, (\vee_t)_{24}^4), ((\wedge_t)_{13}^4, (\wedge_t)_{24}^4), ((\vee_k)_{13}^4, (\vee_k)_{24}^4), ((\wedge_k)_{13}^4, (\wedge_k)_{24}^4), \right. \\ \left. ((\wedge_k)_{13}^4, (\vee_k)_{24}^4), ((\vee_k)_{13}^4, (\wedge_k)_{24}^4), (-_1^2, -_2^2), (x_2^2, x_1^2) \right\}$$

we obtain $\mathbf{16}_{\mathcal{DJ}_{t,f}} = P_{\Gamma_{\mathcal{DJ}_{t,f}}}(\mathbf{4}_{\mathcal{DB}_u})$.

Similarly, from the definition of **256** we can obtain a duplicator $\Gamma_{\mathcal{DJ}_{t,f,i}}$ for $\{\mathbf{16}_{\mathcal{DB}_{e_u}}\}$ and such that **256** = $P_{\Gamma_{\mathcal{DJ}_{t,f,i}}}(\mathbf{16}_{\mathcal{DB}_{e_u}})$. Therefore, Theorem 3.1 and Lemma 6.1 prove that $\mathcal{DB}_u \ll \mathcal{DJ}_{t,f}$ and $\mathcal{DB}_{e_u} \ll \mathcal{DJ}_{t,f,i}$.

Similar results can be obtained for interlaced trilattices without the distributivity condition. Some results on product representations for these more general classes of interlaced trilattices were presented in [31] (see also Example 9.4).

7. BILATTICES WITH IMPLICATION-LIKE OPERATIONS

Bilattices with implication-like operations have been quite extensively considered in the literature (see [9] and the references therein). A natural implication in an algebra with a lattice reduct arises as the adjoint of the meet operation, if this adjoint exists. Given a lattice \mathbf{L} , the operation \rightarrow is the adjoint (or residuum) of \wedge if, for $a, b, c \in \mathbf{L}$,

$$a \wedge b \leq c \iff b \leq a \rightarrow c.$$

An algebra $(A; \vee, \wedge, \rightarrow, 0, 1)$ such that $(A; \vee, \wedge, 0, 1) \in \mathcal{D}$ and \rightarrow is the adjoint of \wedge is a *Heyting algebra* [5, Chapter IX]. We denote the variety of Heyting algebras by \mathcal{H} .

Any bilattice has two lattice reducts, and hence there are two natural candidates for implications: knowledge implication \rightarrow_k , the adjoint of \wedge_k , and truth implication \rightarrow_t , the adjoint of \wedge_t . Despite their definitions being so alike these implications exhibit different behaviour. As we shall see, constants play an important role here.

Bilattices with knowledge implication.

Let $\mathcal{BL}_{\rightarrow_k}$ denote the class of bounded bilattices whose knowledge lattice reduct is a Heyting algebra, with the implication included in the language. More precisely, we consider algebras of the form $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \rightarrow_k, \neg, 0_t, 1_t, 0_k, 1_k)$, where the reduct omitting \rightarrow_k is a bilattice and $(\wedge_k, \rightarrow_k)$ is an adjoint pair. Then $(A; \vee_k, \wedge_k, \rightarrow_k, 0_k, 1_k)$ belongs to \mathcal{H} . We deduce that the class of bilattices with knowledge implication $\mathcal{BL}_{\rightarrow_k}$ is a variety. We shall show that $\mathcal{BL}_{\rightarrow_k}$ is categorically equivalent to \mathcal{H} .

We first show that the class of bilattices with knowledge implication naturally arises as a duplicate of \mathcal{H} . Let $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \rightarrow_k, \neg, 0_t, 1_t, 0_k, 1_k) \in \mathcal{BL}_{\rightarrow_k}$. Then there exists $\mathbf{L} = (L; \vee, \wedge, 0, 1) \in \mathcal{L}$ such that $\mathbf{A}_{\mathcal{BL}}$, the bilattice reduct of \mathbf{A} , is isomorphic to $P_{\Gamma_{\mathcal{BL}}}(\mathbf{L}) = \mathbf{L} \odot \mathbf{L}$. We identify $\mathbf{A}_{\mathcal{BL}}$ with $P_{\Gamma_{\mathcal{BL}}}(\mathbf{L})$. Since (\wedge_k, \rightarrow) is an adjoint pair we have, for $a, b, c \in L$,

$$a \wedge b \leq c \iff (a, 0) \wedge_k (b, 0) \leq_k (c, 0) \iff (b, 0) \leq_k (a, 0) \rightarrow_k (c, 0) \\ \iff b \leq \pi_1((a, 0) \rightarrow_k (c, 0)).$$

Therefore, the operation $\rightarrow^{\mathbf{L}}$, defined by $x \rightarrow^{\mathbf{L}} y = \pi_1((x, 0) \rightarrow_k (y, 0))$, is the adjoint of \wedge and $(L; \vee, \wedge, \rightarrow^{\mathbf{L}}, 0, 1) \in \mathcal{H}$. Moreover, it follows that $(a, b) \rightarrow_k (c, d) = (a \rightarrow^{\mathbf{L}} c, b \rightarrow^{\mathbf{L}} d)$. What we have actually proved is that the set

$$\Gamma_{\mathcal{H}} = \Gamma_{\mathcal{BL}} \cup \{(\rightarrow_{13}^4, \rightarrow_{24}^4)\}$$

satisfies (L), (M) and (P) with respect to the language of \mathcal{H} . Now an application of Theorem 3.1 proves our claim that $\mathcal{BL}_{\rightarrow_k}$ is categorically equivalent to \mathcal{H} .

In [9], the authors introduced Brouwerian bilattices and in [9, Theorem 2.6] they presented a product representation for these. The base class for their product representation is the variety \mathcal{BR} of Brouwerian lattices (also known as generalised Heyting algebras); this is the variety of 0-free reducts of Heyting algebras. The product representation in [9] implicitly relies on a duplicator different from ours, *viz.*

$$\Gamma_{\mathcal{BR}} = \Gamma_{\mathcal{BL}_u} \cup \{(\rightarrow_{13}^4, \wedge_{14}^4)\}.$$

An application of Theorem 3.1 proves that \mathcal{BR} is categorically equivalent to the variety of Brouwerian bilattices. Moreover, if we consider Heyting algebras (bounded Brouwerian lattices) and the duplicator

$$\Gamma'_{\mathcal{H}} = \Gamma_{\mathcal{BL}} \cup \{(\rightarrow_{13}^4, \wedge_{14}^4)\}$$

we can easily see that Heyting algebras are categorically equivalent to bounded Brouwerian bilattices. This leads to a categorical equivalence between bounded Brouwerian bilattices and $\mathcal{BL}_{\rightarrow_k}$ that is actually a term-equivalence.

Bilattices with truth implication.

Here we consider the class $\mathcal{BL}_{\rightarrow_t}$ of bounded bilattices for which \wedge_t admits an adjoint. More precisely, an algebra $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \rightarrow_t, \neg, 0_t, 1_t, 0_k, 1_k)$ belongs to $\mathcal{BL}_{\rightarrow_t}$ if $(A; \vee_t, \wedge_t, \vee_k, \wedge_k, \neg, 0_t, 1_t, 0_k, 1_k)$ is a bilattice and $(\wedge_t, \rightarrow_t)$ is an adjoint pair. Let $b\mathcal{H}$ be the class of bi-Heyting algebras (see [29] and the references therein). We shall prove that the $\mathcal{BL}_{\rightarrow_t}$ is a duplicate of $b\mathcal{H}$.

We let $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \rightarrow_k, \neg, 0_t, 1_t, 0_k, 1_k) \in \mathcal{BL}_{\rightarrow_t}$, and identify $\mathbf{A}_{\mathcal{BL}}$ with identify $\mathbf{A}_{\mathcal{BL}}$ with $\mathbf{L} \odot \mathbf{L}$ for some $\mathbf{L} = (L; \vee, \wedge, 0, 1) \in \mathcal{L}$. Since $(\wedge_t, \rightarrow_t)$ is an adjoint pair, we have, for $a, b, c \in L$,

$$\begin{aligned} a \wedge b \leq c &\iff (a, 1) \wedge_t (b, 1) \leq_t (c, 1) \\ &\iff (b, 1) \leq_t (a, 1) \rightarrow_t (c, 1) \\ &\iff b \leq \pi_1((a, 1) \rightarrow_t (c, 1)) \end{aligned}$$

and

$$\begin{aligned} a \vee b \geq c &\iff (0, a) \wedge_t (0, b) \leq_t (0, c) \\ &\iff (0, b) \leq_t (0, a) \rightarrow_t (0, c) \\ &\iff b \geq \pi_2((0, a) \rightarrow_t (0, c)). \end{aligned}$$

Thus the binary operations $\rightarrow^{\mathbf{L}}$ and $\mapsto^{\mathbf{L}}$ defined by $x \rightarrow^{\mathbf{L}} y = \pi_1((x, 1) \rightarrow_t (y, 1))$ and $x \mapsto^{\mathbf{L}} y = \pi_2((0, x) \rightarrow_t (0, y))$ are the adjoints of \wedge and \vee , respectively. Hence the algebra $(L; \vee, \wedge, \rightarrow^{\mathbf{L}}, \mapsto^{\mathbf{L}}, 0, 1)$ belongs to $b\mathcal{H}$. Moreover, the set

$$\Gamma_{b\mathcal{H}} = \Gamma_{\mathcal{BL}} \cup \{(\rightarrow_{13}^4, \mapsto_{24}^4)\}$$

duplicates $b\mathcal{H}$. Hence an application of Theorem 3.1 proves our claim that $\mathcal{BL}_{\rightarrow_t}$ is categorically equivalent to $b\mathcal{H}$.

Combining the ideas of this section, we observe that if a bilattice is such that \wedge_t has an adjoint, \rightarrow_t , then \wedge_k also admits an adjoint. Moreover, this adjoint can be captured as follows:

$$x \rightarrow_k y = ((x \rightarrow_t y) \wedge_k 1_t) \vee_k (\neg(\neg x \rightarrow_t \neg y) \wedge_k 0_t).$$

An analysis of a third scenario in which an implication is introduced into bilattices is performed in Example 8.3, where we consider implicative bilattices, as these are defined in [2], and show how they fit into a general scheme of Boolean algebra duplicates.

8. FURTHER EXAMPLES

This section brings a non-exhaustive selection of examples within the scope of the general framework for product representations set up in Section 3. The examples concern the adjunction of new operations of different types to different base varieties, and the identification of appropriate duplicates of these varieties. We group the examples according to the variety being duplicated. Thanks to Theorem 3.1, the varieties within any such group are all categorically equivalent to one another, a fact which in many cases has not been recognised before.

Lattice variety duplicates.

We have already mentioned that \mathcal{BL} , \mathcal{BL}_u , \mathcal{DB} and \mathcal{DB}_u are duplicates of \mathcal{L} , \mathcal{L}_u , \mathcal{D} , and \mathcal{D}_u , respectively. We now turn to new examples.

Example 8.1. [Fitting's guard operation] Fitting [17] introduced a binary operation on $\mathbf{4}_{\mathcal{DB}}$, denoted $:$ and given by

$$a : b = \begin{cases} b & \text{if } a \in \{(1, 1), (1, 0)\}, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Observe that $(a_1, a_2) : (b_1, b_2) = ((a_1 \wedge b_1), (a_1 \wedge b_2))$. Let $\mathbf{4}$ be the algebra obtained by adding the operation “ $:$ ” to $\mathbf{4}_{\mathcal{DB}}$. It is easily seen that $\Gamma_{\mathcal{DB}} \cup \{(\wedge_{13}^4, \wedge_{14}^4)\}$ is a duplicator for $\Sigma_{\mathcal{D}}$ on $\mathbf{2}_{\mathcal{D}}$. By Theorem 3.1, $\mathbb{V}(\mathbf{4})$ is categorically equivalent to \mathcal{D} .

As we observed after Theorem 3.1 the equivalence between a variety of algebras and its duplicate determines an isomorphism between the associated lattices of subvarieties. Moreover, we have observed that a duplicator for a variety is also a duplicator for any of its subvarieties. Now we will use this observation to get new base varieties and new duplicates from known duplicators.

We have already used a duplicator of De Morgan lattices to handle unbounded bilattices with conflation, and noted that a similar construction is available in the bounded case using De Morgan algebras. The variety \mathcal{DM} has two non-trivial proper subvarieties: \mathcal{K} (Kleene algebras) and \mathcal{B} (Boolean algebras). The generators of the non-trivial proper subvarieties of \mathcal{DM} also support various additional operations. We show how we can obtain duplicators to capture such operations. These give rise to product representations, old and new, of algebras arising from the addition of various operations related to the De Morgan negation.

Kleene algebra duplicates.

Let $\mathbf{3}_{\mathcal{DM}} = (\{0, u, 1\}; \vee, \wedge, \sim, 0, 1)$ denote the De Morgan algebra whose lattice reduct is the three-element chain $0 < u < 1$. The class $\mathbb{ISP}(\mathbf{3}_{\mathcal{DM}})$ is indeed a subvariety of \mathcal{DM} (that is, $\mathbb{ISP}(\mathbf{3}_{\mathcal{DM}}) = \mathbb{HISP}(\mathbf{3}_{\mathcal{DM}})$). The algebras in $\mathbb{ISP}(\mathbf{3}_{\mathcal{DM}})$ are called Kleene algebras. Let \mathcal{K} denote the variety of Kleene algebras. The categorical equivalence between \mathcal{DM} and \mathcal{DBE} restricts to a categorical equivalence between \mathcal{K} and $\mathbb{ISP}(\mathbb{P}_{\Gamma_{\mathcal{DBE}}}(\mathbf{3}_{\mathcal{DM}})) = \mathbb{V}(\mathbb{P}_{\Gamma_{\mathcal{DBE}}}(\mathbf{3}_{\mathcal{DM}}))$.

Example 8.2 (Negation by failure). In [33] Ruet and Faget introduce an operation called *negation-by-failure* on the bilattice $\mathbf{9}_{\mathcal{DB}} = \mathbb{P}_{\Gamma_{\mathcal{BL}}}(\mathbf{3}_{\mathcal{D}})$ (where $\mathbf{3}_{\mathcal{D}}$ is the three-element lattice whose universe is $\{0, u, 1\}$ and $0 < u < 1$) and the operator $/: \mathbf{9}_{\mathcal{DB}} \rightarrow \mathbf{9}_{\mathcal{DB}}$ is defined by

$$/(a_1, a_2) = \begin{cases} (1 - a_1, a_2) & \text{if } a_1 = 0 \text{ or } 1, \\ (a_1, a_2) & \text{otherwise.} \end{cases}$$

It follows that $/(a_1, a_2) = (\sim a_1, a_2)$.

Let $\mathbf{9}_/$ denote $\mathbf{9}_{\mathcal{DB}}$ with the operation “ $/$ ” added. It follows that $\Gamma_/ = \Gamma_{\mathcal{DB}} \cup \{(\sim_1^2, x_2^2)\}$ duplicates $\mathbf{3}_{\mathcal{DM}}$ and that $\mathbf{9}_/ = \mathbb{P}_{\Gamma_/}(\mathbf{3}_{\mathcal{DM}})$. By Theorem 3.1, $\mathbb{HISP}(\mathbf{9}_/)$ is equivalent to the variety of Kleene algebras.

Boolean algebra duplicates.

The class \mathcal{B} of Boolean algebras equals $\mathbb{ISP}(\mathbf{2}_{\mathcal{B}})$ where $\mathbf{2}_{\mathcal{B}} = (\{0, 1\}; \vee, \wedge, ', 0, 1)$ is the two-element Boolean algebra.

Example 8.3 (Implicative bilattices). In [2], Arieli and Avron considered a special implication operator definable on a logical bilattice (that is, a bilattice together with a prime bifilter). The case of $\mathbf{4}_{\mathcal{D}\mathcal{B}}$ is very special, since $\mathbf{4}_{\mathcal{D}\mathcal{B}}$ only admits one bifilter, *viz.* $\{(1, 1), (1, 0)\}$. In this case the implication is given by

$$a \supset b = \begin{cases} b & \text{if } a \in \{(1, 1), (1, 0)\}, \\ (1, 0) & \text{otherwise.} \end{cases}$$

In other words, $(a_1, a_2) \supset (b_1, b_2) = (a'_1 \vee b_1, a_1 \wedge b_2)$. Let

$$\mathbf{4}_{\supset} = (\{0, 1\}^2; \vee_t, \wedge_t, \vee_k, \wedge_k, \supset, 0_t, 1_t, 0_k, 1_k)$$

be the algebra whose bilattice reduct is $\mathbf{4}_{\mathcal{D}\mathcal{B}}$ and \supset is as defined above. Any algebra in the variety $\mathbb{V}(\mathbf{4}_{\supset})$ is called an *implicative bilattice*. Setting t as the term $t(x_1, x_2, x_3, x_4) = x'_1 \vee x_3$, it follows that the set $\Gamma_{\supset} = \Gamma_{\mathcal{B}\mathcal{L}} \cup \{(t, \wedge_{14}^4)\}$ duplicates $\mathbf{2}_{\mathcal{B}}$ and $\mathbf{4}_{\supset} = \mathbb{P}_{\Gamma_{\supset}}(\mathbf{2}_{\mathcal{B}})$. By Theorem 3.1, the variety $\mathbb{V}(\mathbf{4}_{\supset})$ of implicative bilattices is categorically equivalent to \mathcal{B} .

If we consider the unbounded reduct $\mathbf{4}_{\mathcal{D}\mathcal{B}u, \supset} = (\{0, 1\}^2; \vee_t, \wedge_t, \vee_k, \wedge_k, \supset)$ of $\mathbf{4}_{\supset}$, the set $\Gamma_{\mathcal{B}\mathcal{L}u} \cup \{(t, \wedge_{14}^4)\}$ duplicates $\mathbf{2}_{\mathcal{G}\mathcal{B}}$, where $\mathcal{G}\mathcal{B}$ denotes the class of generalised (lower unbounded) Boolean algebras [5], and hence $\mathbb{V}(\mathbf{4}_{\mathcal{D}\mathcal{B}u, \supset})$ is equivalent to $\mathcal{G}\mathcal{B}$ by Theorem 3.1. This equivalence was already observed in [7] as a consequence of the product representation of Brouwerian bilattices and its application to implicative bilattices.

Example 8.4 (Moore's epistemic operator). Ginsberg's interpretation of Moore's epistemic operator "I know that p " is the operation $L: \mathbf{4}_{\mathcal{D}\mathcal{B}} \rightarrow \mathbf{4}_{\mathcal{D}\mathcal{B}}$ defined by $L(a_1, a_2) = (a_1, a'_1)$.

In [22, Proposition 4.2] it is proved that the algebra

$$\mathbf{4}_L = (\{0, 1\}^2; \vee_t, \wedge_t, \vee_k, \wedge_k, \neg, L)$$

is primal. Therefore $\mathbb{ISP}(\mathbf{4}_L) = \mathbb{V}(\mathbf{4}_L)$. We can obtain the same result independently from the primality of $\mathbf{4}_L$. Consider the language $\Sigma_{\mathcal{B}}$ of Boolean algebras. Trivially

$$\Gamma_L = \Gamma_{\mathcal{B}\mathcal{L}} \cup \{(x_1^2, ('_1)^2)\}$$

duplicates \mathcal{B} . Moreover $\mathbf{4}_L = \mathbb{P}_{\Gamma_L}(\mathbf{2}_{\mathcal{B}})$.

Example 8.5 (Negation-by-failure on $\mathbf{4}_{\mathcal{D}\mathcal{B}}$). In [33], Ruet and Faget consider their negation-by-failure operator restricted to $\mathbf{4}_{\mathcal{D}\mathcal{B}}$, that is, $/: \mathbf{4}_{\mathcal{D}\mathcal{B}} \rightarrow \mathbf{4}_{\mathcal{D}\mathcal{B}}$ is defined by $/(a_1, a_2) = (1 - a_1, a_2)$. Let $\mathbf{4}_/$ be the algebra obtained by enriching the language of $\mathbf{4}_{\mathcal{D}\mathcal{B}}$ with $/$. It is easy to check that $\mathbf{4}_/$ is a subalgebra of $\mathbf{9}_/$. Moreover, by identifying $\mathbf{2}_{\mathcal{B}}$ with the two-element subalgebra of $\mathbf{3}_{\mathcal{D}\mathcal{M}}$, it follows that $\mathbf{4}_/ = \mathbb{P}_{\Gamma_/}(\mathbf{2}_{\mathcal{B}})$, the set $\Gamma_/$ duplicates \mathcal{B} , and the class $\mathbb{ISP}(\mathbf{4}_/) = \mathbb{HSP}(\mathbf{4}_/) = \mathbb{HSP}(\mathbb{P}_{\Gamma_/}(\mathbf{2}_{\mathcal{B}}))$ is categorically equivalent to \mathcal{B} .

Duplicates of residuated lattices.

An algebra $\mathbf{A} = (A; \vee, \wedge, \cdot, \backslash, /)$ is said to be a *residuated lattice* if $(A; \vee, \wedge)$ is a lattice and $a \cdot b \leq c \iff b \leq a \backslash c \iff a \leq c / b$ (see for example [19]). Let us denote the variety of residuated lattices by $\mathcal{R}\mathcal{L}$.

Example 8.6 (Residuated bilattices). In [23], the authors defined the variety $\mathcal{R}\mathcal{B}\mathcal{L}$ of residuated bilattices. Using the notation of the present paper and of [23, Theorem 3.6] it follows that $\mathcal{R}\mathcal{B}\mathcal{L} = \mathbb{V}(\mathbb{P}_{\Gamma_{\mathcal{R}\mathcal{B}\mathcal{L}}}(\mathcal{R}\mathcal{L}))$, where $\Gamma_{\mathcal{R}\mathcal{B}\mathcal{L}} = \Gamma_{\mathcal{B}\mathcal{L}} \cup \{(\wedge_{13}^4, \wedge_{41}^4), (\wedge_{13}^4, \wedge_{32}^4)\}$. Hence, Theorem 3.1 implies that $\mathcal{R}\mathcal{B}\mathcal{L}$ is categorically equivalent to $\mathcal{R}\mathcal{L}$.

Duplicates of modal algebras.

Let $\mathcal{B}\mathcal{M}$ be the variety of bi-modal algebras. An algebra $(A; \vee, \wedge, ', \square_+, \square_-, 0, 1) \in \mathcal{B}\mathcal{M}$ if and only if $(A; \vee, \wedge, ', 0, 1)$ is a Boolean algebra and $\square_+, \square_-: A \rightarrow A$ preserve finite meets.

Example 8.7 (Modal bilattices). In [26], the authors studied a modal expansion of implicative bilattices. They presented a product representation for implicative bilattices with a modal operator. An algebra $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k, \supset, \neg, \square, 0_t, 1_t, 0_k, 1_k)$ is said to be a *modal bilattice* if $(A; \vee_t, \wedge_t, \vee_k, \wedge_k, \supset, \neg, 0_t, 1_t, 0_k, 1_k)$ is an implicative lattice (see Example 8.3) and

$$\square(1_t) = 1_t, \quad \square(a \wedge_t b) = \square(a) \wedge_t \square(b), \quad \square(0_k \supset a) = 0_k \supset \square(a).$$

We denote the variety of modal bilattices by \mathbf{MBL} .

It is easy to see that the set $\Gamma_{\mathbf{MBL}} = \Gamma_{\supset} \cup \{(t_1, t_2)\}$, where $t_1(x_1, x_2) = \square_+(x_1) \wedge \square_-(x_2)$ and $t_2(x_1, x_2) = (\square_+(x_2))'$, duplicates \mathbf{BM} . The result of [26, Theorem 12] proves that $\mathbf{MBL} = \mathbb{V}(\mathbf{P}_{\Gamma_{\mathbf{MBL}}}(\mathbf{BM}))$. Hence, Theorem 3.1 implies that \mathbf{BM} is categorically equivalent to \mathbf{MBL} .

9. BEYOND PRODUCT REPRESENTATION VIA DUPLICATION

Our aim in writing this paper, as its title suggests, is to present a general framework for product representations of classes of algebras. One may ask if Theorem 3.1 is the most general product representation we can obtain. It is not. In this section we indicate how our theorem can be extended in two different directions (and in both simultaneously). Firstly we consider an extension to handle products which are not binary and secondly we show how our duplication mechanism can be modified so that our methodology encompasses product representations which fall outside the scope of duplication, as this appears in Theorem 3.1. Our two variants will be put forward using a similar expository method in each case: we first present a pathfinder example; then we provide a modified version of conditions (L), (M) and (P) to encompass this example; finally, we state the adaptation of Theorem 3.1 associated with the amended conditions.

Let us consider our first modification of the product representation theorem. Our pathfinder example here is a new product representation for distributive trilattices. We have already observed that $\mathcal{D}_u \ll \mathcal{DB}_u$ and $\mathcal{DB}_u \ll \mathcal{DJ}_{t,f}$, and this proves that $\mathcal{DJ}_{t,f}$ is categorically equivalent to \mathcal{D}_u . This equivalence is determined by the composition of the functors $\mathbf{P}_{\Gamma_{\mathcal{DB}_u}}$ and $\mathbf{P}_{\Gamma_{\mathcal{DJ}_{t,f}}}$. Applying these two functors to a distributive lattice \mathbf{L} would yield a trilattice whose universe is L^4 and whose operations are defined as follows:

$$\begin{aligned} (a_1, a_2, a_3, a_4) \vee_t (b_1, b_2, b_3, b_4) &= (a_1 \vee b_1, a_2 \wedge b_2, a_3 \vee b_3, a_4 \wedge b_4), \\ (a_1, a_2, a_3, a_4) \wedge_t (b_1, b_2, b_3, b_4) &= (a_1 \wedge b_1, a_2 \vee b_2, a_3 \wedge b_3, a_4 \vee b_4), \\ (a_1, a_2, a_3, a_4) \vee_f (b_1, b_2, b_3, b_4) &= (a_1 \vee b_1, a_2 \vee b_2, a_3 \wedge b_3, a_4 \wedge b_4), \\ (a_1, a_2, a_3, a_4) \wedge_f (b_1, b_2, b_3, b_4) &= (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \vee b_3, a_4 \vee b_4), \\ (a_1, a_2, a_3, a_4) \vee_i (b_1, b_2, b_3, b_4) &= (a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3, a_4 \vee b_4), \\ (a_1, a_2, a_3, a_4) \wedge_i (b_1, b_2, b_3, b_4) &= (a_1 \wedge b_1, a_2 \wedge b_2, a_3 \wedge b_3, a_4 \wedge b_4), \\ -_t(a_1, a_2, a_3, a_4) &= (a_2, a_1, a_4, a_3), \\ -_f(a_1, a_2, a_3, a_4) &= (a_3, a_4, a_1, a_2). \end{aligned}$$

We shall now describe how to adapt (L), (M) and (P) to yield a multi-factor product representation and thereby to obtain $\mathcal{DJ}_{t,f}$ directly from \mathcal{D}_u without going via \mathcal{DB}_u . Again fix a class \mathcal{N} of Σ -algebras. But now let Γ be a set of m -tuples of terms such that, for each $\mathbf{t} = (t_1, \dots, t_m) \in \Gamma$, there exists $n_{\mathbf{t}} \in \{0, 1, \dots\}$ such that t_1, \dots, t_m are terms on $mn_{\mathbf{t}}$ variables. We define

$$\mathbf{P}_{\Gamma}^m(\mathbf{N}) = (N^m; \{\mathbf{t}^{\mathbf{P}_{\Gamma}^m(\mathbf{N})} \mid \mathbf{t} \in \Gamma\}),$$

where the operation $\mathbf{t}^{\mathbf{P}_{\Gamma}^m(\mathbf{N})}: (N^m)^{n_{\mathbf{t}}} \rightarrow N^m$ is defined by

$$\mathbf{t}^{\mathbf{P}_{\Gamma}^m(\mathbf{N})}(\mathbf{a}_1, \dots, \mathbf{a}_{n_{\mathbf{t}}}) = (t_1^{\mathbf{N}}(\mathbf{a}_1, \dots, \mathbf{a}_{n_{\mathbf{t}}}), \dots, t_m^{\mathbf{N}}(\mathbf{a}_1, \dots, \mathbf{a}_{n_{\mathbf{t}}}), \text{ for } \mathbf{a}_1, \dots, \mathbf{a}_{n_{\mathbf{t}}} \in N^m.$$

We extend our earlier notation in the expected way: given a set X we let $\delta_m^X: X \rightarrow X^m$ be the diagonal map given by $\delta_m^X(x) = (x, x, \dots, x) \in X^m$ and, for $i \in \{1, \dots, m\}$, let $\pi_i: X^m \rightarrow X$ be the projection map onto the i th coordinate.

We consider the following generalisation of conditions (L), (M) and (P):

(L _{m}) for each n -ary operation symbol $f \in \Sigma$ and $i \in \{1, \dots, m\}$ there exists an n -ary Γ -term t such that $\pi_i^{\mathbf{N}} \circ \mathbf{t}^{\mathbf{P}_{\Gamma}^m(\mathbf{N})} \circ (\delta_m^{\mathbf{N}})^n = f^{\mathbf{N}}$ for each $\mathbf{N} \in \mathcal{N}$;

(M _{m}) there exists an m -ary Γ -term v such that

$$\begin{aligned} v^{\mathbf{P}_{\Gamma}^m(\mathbf{N})}((a_1^1, \dots, a_m^1), \dots, (a_1^m, \dots, a_m^m)) &= (a_1^1, a_2^2, \dots, a_m^m) \\ &\text{for } \mathbf{N} \in \mathcal{N} \text{ and } (a_1^1, \dots, a_m^1), \dots, (a_1^m, \dots, a_m^m) \in N^m. \end{aligned}$$

(P_m) for each permutation σ of $\{1, \dots, m\}$ there exists a unary Γ -term s_σ such that

$$s_\sigma^{\mathbf{P}_\Gamma^m(\mathcal{N})}(a_1, \dots, a_n) = (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) \quad \text{for } \mathbf{N} \in \mathcal{N} \text{ and } a_1, \dots, a_n \in N.$$

Observe that, when $m = 1$, the set Γ consists of Σ -terms and conditions (M_1) and (P_1) are trivially satisfied. Moreover, condition (L_1) implies that $\mathbb{V}(\mathbf{P}_\Gamma^1(\mathcal{N}))$ is term-equivalent to $\mathbb{V}(\mathcal{N})$. This justifies our observation that product representation is a generalised form of term-equivalence.

When $m = 2$, conditions (L_m), (M_m) and (P_m) coincide with (L), (M) and (P). Thus Theorem 3.1 is a specialisation of the following theorem, whose proof follows using the same arguments and replacing (L), (M) and (P) with (L_m), (M_m) and (P_m) as appropriate.

Theorem 9.1. *Let \mathcal{N} be a class of Σ -algebras and Γ a set of m -tuples of Σ -terms. If Γ satisfies (L_m), (M_m) and (P_m), then the functor $\mathbf{P}_\Gamma^m: \mathcal{B} \rightarrow \mathcal{A}$ sets up a categorical equivalence between $\mathcal{B} = \mathbb{V}(\mathcal{N})$ and $\mathcal{A} = \mathbb{V}(\mathbf{P}_\Gamma^m(\mathcal{N}))$.*

Example 9.2. It is easy to see that $\Gamma_{\mathcal{D}\mathcal{J}_{t,f,i}}$ given by

$$\begin{aligned} \Gamma_{\mathcal{D}\mathcal{J}_{t,f,i}} = \{ & (\wedge_{15}^8, \vee_{26}^8, \wedge_{37}^8, \vee_{48}^8), (\vee_{15}^8, \wedge_{26}^8, \vee_{37}^8, \wedge_{48}^8), (\wedge_{15}^8, \wedge_{26}^8, \vee_{37}^8, \vee_{48}^8), \\ & (\vee_{15}^8, \vee_{26}^8, \wedge_{37}^8, \wedge_{48}^8), (\wedge_{15}^8, \wedge_{26}^8, \wedge_{37}^8, \wedge_{48}^8), (\vee_{15}^8, \vee_{26}^8, \vee_{37}^8, \vee_{48}^8), \\ & (x_2^4, x_1^4, x_4^4, x_3^4), (\sim_2^4, \sim_1^4, \sim_4^4, \sim_3^4), (x_3^4, x_4^4, x_1^4, x_2^4) \} \end{aligned}$$

satisfies (L_4), (P_4) and (M_4) with respect to \mathcal{DM}_u . Moreover $\mathbf{256} \cong \mathbf{P}_{\Gamma_{\mathcal{D}\mathcal{J}_{t,f,i}}}^4(\mathcal{4}_{\mathcal{DM}_u})$. Combining Theorem 9.1 and Lemma 6.1(iv), it follows that \mathcal{DM}_u is categorically equivalent to $\mathcal{DJ}_{t,f,i}$. The same result can be obtained from $\mathcal{DM}_u \ll \mathcal{DBC}_u$ and $\mathcal{DBC}_u \ll \mathcal{DJ}_{t,f,i}$ and two applications of Theorem 3.1.

Our presentation of our second variant of product representation starts from consideration of the class of interlaced pre-bilattices. An algebra $\mathbf{A} = (A; \vee_t, \wedge_t, \vee_k, \wedge_k)$ is a *pre-bilattice* if both reducts $(A; \vee_t, \wedge_t)$ and $(A; \vee_k, \wedge_k)$ are lattices. Pre-bilattices form a variety, $p\mathcal{BL}_u$; in fact $p\mathcal{BL}_u$ is the variety generated by the \neg -free reducts of (unbounded) bilattices. A pre-bilattice is *interlaced* if each lattice operation is monotonic with respect to the order of the other lattice. There is a product representation for pre-bilattices (see [14] and the references therein). It follows the same lines as that for bilattices, except that, in the absence of \neg , the two factors do not have to have the same universe and the two coordinates operate independently. We now formulate this precisely. Let $\mathbf{P}, \mathbf{Q} \in \mathcal{L}_u$. Then $\mathbf{P} \odot \mathbf{Q}$ is the pre-bilattice whose universe is $P \times Q$ and whose operations are defined by:

$$\begin{aligned} (a_1, a_2) \vee_t (b_1, b_2) &= (a_1 \vee b_1, a_2 \wedge b_2), & (a_1, a_2) \vee_k (b_1, b_2) &= (a_1 \vee b_1, a_2 \vee b_2), \\ (a_1, a_2) \wedge_t (b_1, b_2) &= (a_1 \wedge b_1, a_2 \vee b_2), & (a_1, a_2) \wedge_k (b_1, b_2) &= (a_1 \wedge b_1, a_2 \wedge b_2). \end{aligned}$$

Pre-bilattices of the form $\mathbf{P} \odot \mathbf{Q}$ are necessarily interlaced. The product representation theorem for pre-bilattices states that each interlaced pre-bilattice \mathbf{A} is isomorphic to $\mathbf{P} \odot \mathbf{Q}$ for some $\mathbf{P}, \mathbf{Q} \in \mathcal{L}_u$. Moreover this product representation can be upgraded to a categorical equivalence between $\mathcal{L}_u \times \mathcal{L}_u$ and the variety of interlaced pre-bilattices [7, Section 5.1].

Our next step is to modify the conditions (L), (M) and (P) to be imposed on a set Γ so as to encompass the example of pre-bilattices. Condition (P), on permutation of coordinates, serves to link the factors in a product. We want to dispense with this and to replace by it by a condition, (D), which distinguishes coordinates in such a way that the factors in a product operate independently. We now indicate how this should work.

Let us fix a class \mathcal{N} of Σ -algebras and let Γ be a set of pairs of Σ -terms. Presented with two algebras $\mathbf{P}, \mathbf{Q} \in \mathcal{N}$ we want to use Γ to obtain an algebra $\mathbf{P} \odot_\Gamma \mathbf{Q}$ whose universe is $P \times Q$. Certainly condition (P) cannot be satisfied and the pairs of terms $(t_1, t_2) \in \Gamma$ should not combine elements from different coordinates. More precisely, in order for the operation $[t_1, t_2]^{\mathbf{P} \odot_\Gamma \mathbf{Q}}: (P \times Q)^n \rightarrow P \times Q$, given by

$$\begin{aligned} [t_1, t_2]^{\mathbf{P} \odot_\Gamma \mathbf{Q}}((a_1, b_1), \dots, (a_n, b_n)) &= (t_1^{\mathbf{P}}(a_1, b_1, \dots, a_n, b_n), t_2^{\mathbf{Q}}(a_1, b_1, \dots, a_n, b_n)), \\ &\text{for } (a_1, b_1), \dots, (a_n, b_n) \in P \times Q, \end{aligned}$$

where $n = n_{(t_1, t_2)}$, to be well defined, we need Γ to satisfy a condition that keeps the use of the coordinates disjoint:

(D) for each $(t_1, t_2) \in \Gamma$,

$$t_1(x_1, \dots, x_{2n}) = r_1(x_1, x_3, \dots, x_{2n-1}) \text{ and } t_2(x_1, \dots, x_{2n}) = r_2(x_2, x_4, \dots, x_{2n}),$$

for some n -ary Σ -terms r_1 and r_2 .

Indeed, if Γ satisfies (D) is easy to see that the algebra

$$\mathbf{P} \odot_{\Gamma} \mathbf{Q} = (P \times Q; \{[t_1, t_2]^{\mathbf{P} \odot_{\Gamma} \mathbf{Q}} \mid (t_1, t_2) \in \Gamma\})$$

is well defined whenever $\mathbf{P}, \mathbf{Q} \in \mathbb{V}(\mathcal{N})$. Moreover, the functor $\odot_{\Gamma}: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{A}$, where $\mathcal{B} = \mathbb{V}(\mathcal{N})$ and $\mathcal{A} = \mathbb{V}(\{\mathbf{P} \odot_{\Gamma} \mathbf{Q} \mid \mathbf{P}, \mathbf{Q} \in \mathcal{N}\})$, given by

$$\begin{aligned} \text{on objects:} & \quad (\mathbf{P}, \mathbf{Q}) \mapsto \mathbf{P} \odot_{\Gamma} \mathbf{Q}, \\ \text{on morphisms:} & \quad \odot_{\Gamma} (h_1, h_2)(a, b) = (h_1(a), h_2(b)). \end{aligned}$$

is also well defined.

We now have a candidate set of conditions for a new product decomposition theorem. Its proof is a straightforward modification of that of Theorem 3.1.

Theorem 9.3. *Let \mathcal{N} be a class of Σ -algebras and Γ be a set of pairs of Σ -terms. Assume that Γ satisfies (L), (M) and (D). Then the functor $\odot_{\Gamma}: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{A}$, sets up a categorical equivalence between $\mathcal{B} \times \mathcal{B}$ (where $\mathcal{B} = \mathbb{V}(\mathcal{N})$) and $\mathcal{A} = \mathbb{V}(\{\mathbf{P} \odot_{\Gamma} \mathbf{Q} \mid \mathbf{P}, \mathbf{Q} \in \mathcal{N}\})$.*

Corollary 3.2 gives easy access to algebraic facts about varieties to which Theorem 3.1 applies. A corresponding corollary to Theorem 9.3 can be formulated.

Example 9.4 (Interlaced trilattices). In [31], Riviaccio presented product representations for the varieties of interlaced trilattices and interlaced trilattices with one involution $-_t$. A trilattice is said to be interlaced if the six lattice operations preserve each of the three orders.

For (unbounded) interlaced trilattices, $\mathcal{J}\mathcal{T}$, we take the base variety to be $p\mathcal{B}\mathcal{L}_u$, the variety of pre-bilattices, and define

$$\begin{aligned} \Gamma_{\mathcal{J}\mathcal{T}} = \{ & ((\wedge_t)_{13}^4, (\wedge_t)_{24}^4), ((\vee_t)_{13}^4, (\vee_t)_{24}^4), ((\vee_k)_{13}^4, (\wedge_k)_{24}^4), ((\wedge_k)_{13}^4, (\vee_k)_{24}^4), \\ & ((\wedge_k)_{13}^4, (\wedge_k)_{24}^4), ((\vee_k)_{13}^4, (\vee_k)_{24}^4)\}. \end{aligned}$$

Then the product representation theorem for $\mathcal{J}\mathcal{T}$ [31, Theorem 3.4] can be formulated as the assertion that $\mathcal{J}\mathcal{T} = \mathbb{V}(p\mathcal{B}\mathcal{L}_u \odot_{\Gamma_{\mathcal{J}\mathcal{T}}} p\mathcal{B}\mathcal{L}_u)$. Moreover, since $\Gamma_{\mathcal{J}\mathcal{T}}$ certainly satisfies (L), (M) and (D), Theorem 9.3 implies that $\mathcal{J}\mathcal{T}$ is categorically equivalent to $p\mathcal{B}\mathcal{L}_u \times p\mathcal{B}\mathcal{L}_u$.

Now consider the variety $\mathcal{J}\mathcal{T}_{-t}$ of interlaced trilattices with t -involution. Let $\mathcal{B}\mathcal{L}_u$ be the base variety and let $\Gamma_{\mathcal{J}\mathcal{T}_{-t}}$ be the following set of pairs of terms in the language of $\mathcal{B}\mathcal{L}_u$:

$$\Gamma_{\mathcal{J}\mathcal{T}_{-t}} = \Gamma_{\mathcal{J}\mathcal{T}} \cup \{(-_1^2, -_2^2)\}.$$

Then [31, Theorem 3.6] proves that $\mathcal{J}\mathcal{T}_{-t} = \mathbb{V}(\mathcal{B}\mathcal{L} \odot_{\Gamma_{\mathcal{J}\mathcal{T}_{-t}}} \mathcal{B}\mathcal{L}_u)$. By Theorem 9.3, it follows that $\mathcal{J}\mathcal{T}_{-t}$ is categorically equivalent to $\mathcal{B}\mathcal{L}_u \times \mathcal{B}\mathcal{L}_u$.

Of course we could combine the generalisation to m -factor products and the variant that allows different components in the resulting product. Specifically we could introduce a condition (D_m) and, by applying to (L_m) , (M_m) and (P_m) the same reasoning that we used to replace (M) by (D) in Theorem 9.3, obtain a categorical equivalence between $(\mathbb{V}(\mathcal{N}))^m$ and $\mathbb{V}(\{\mathbf{N}_1 \odot_{\Gamma} \dots \odot_{\Gamma} \mathbf{N}_m \mid \mathbf{N}_1, \dots, \mathbf{N}_m \in \mathcal{N}\})$. We omit the details. By this means we can in particular arrive at a direct proof that $\mathcal{D}\mathcal{T}$ is categorically equivalent to $\mathcal{D}_u \times \mathcal{D}_u \times \mathcal{D}_u \times \mathcal{D}_u$ or that $\mathcal{J}\mathcal{T}$ is categorically equivalent to $\mathcal{L}_u \times \mathcal{L}_u \times \mathcal{L}_u \times \mathcal{L}_u$.

APPENDIX: SUMMARY OF DUPLICATIONS AND EQUIVALENCES

For reference, and to emphasise the uniformity of our approach to product representations across a wide range of varieties we include two tables summarising our results.

The first table covers varieties to which conditions (L), (P) and (M) of Section 3 apply. Any two varieties in the same row are categorically equivalent, and any two duplicates with a common base variety are equivalent to each other. This table may be seen as an amplified version of that given by Jung and Riviaccio [24]. We stress that we are able to view all the examples in our table as being underpinned by a common syntactic mechanism.

Table 2 serves a somewhat different purpose from Table 1. It compares and contrasts the behaviour of (interlaced) trilattices with different numbers of involutions added, from none to three. We have already seen in Section 6 how Theorem 3.1 can be employed to obtain categorical equivalences. Here we focus on the use of the ideas in Section 9.

variety	duplicate of	reference
bilattices \mathcal{BL} (\mathcal{BL}_u)	lattices \mathcal{L} (\mathcal{L}_u)	Section 4
distributive bilattices \mathcal{DB} (\mathcal{DB}_u)	distributive lattices \mathcal{D} (\mathcal{D}_u)	
distributive bilattices with conflation \mathcal{DBC} (\mathcal{DBC}_u)	De Morgan algebras (lattices) \mathcal{DM} (\mathcal{DM}_u)	Section 5
distributive trilattices with t - and f -involution $\mathcal{DT}_{t,f}$	distributive bilattices \mathcal{DB}_u	Section 6
distributive trilattices with t -, f - and i -involution $\mathcal{DT}_{t,f,i}$	distributive bilattices with conflation \mathcal{DBC}_u	
bilattices with knowledge implication $\mathcal{BL}_{\rightarrow_k}$	Heyting algebras \mathcal{H}	Section 7
bilattices with truth implication $\mathcal{BL}_{\rightarrow_t}$	bi-Heyting algebras $b\mathcal{H}$	
bilattices with guard operator $\mathbb{V}(4_{\cdot})$	distributive lattices \mathcal{D}	Example 8.1
bilattices with negation by failure $\mathbb{V}(9_{\cdot})$	Kleene algebras \mathcal{KL}	Example 8.2
implicative bilattices $\mathbb{V}(4_{\supset})$	Boolean algebras \mathcal{B}	Example 8.3
unbounded implicative bilattices $\mathbb{V}(4_{\mathcal{DB}_u, \supset})$	generalised Boolean algebras \mathcal{GB}	
bilattices with Moore's epistemic operator $\mathbb{V}(4_L)$	Boolean algebras \mathcal{B}	Example 8.4
residuated bilattices \mathcal{RL}	residuated lattices \mathcal{RBL}	Example 8.6
modal bilattices \mathcal{MBL}	bi-modal algebras \mathcal{BM}	Example 8.7

TABLE 1. Varieties obtained by duplication

variety	equivalent to	reference
distributive trilattices with t - and f -involution $\mathcal{DT}_{t,f}$	distributive lattices \mathcal{D}_u	Theorem 9.1
distributive trilattices with t - f - and i -involutions $\mathcal{DT}_{t,f,i}$	De Morgan lattices \mathcal{DM}_u	Theorem 9.1
pre-bilattices $p\mathcal{BL}_u$	lattices \times lattices $\mathcal{L}_u \times \mathcal{L}_u$	Theorem 9.3
interlaced trilattices \mathcal{IT}	pre-bilattices \times pre-bilattices $p\mathcal{BL}_u \times p\mathcal{BL}_u$ OR lattices \times lattices \times lattices \times lattices $\mathcal{L}_u \times \mathcal{L}_u \times \mathcal{L}_u \times \mathcal{L}_u$	Theorem 9.3 Theorem 9.3 (4-factor version)
distributive trilattices \mathcal{DT}	$p\mathcal{DB}_u \times p\mathcal{DB}_u$ OR $\mathcal{D}_u \times \mathcal{D}_u \times \mathcal{D}_u \times \mathcal{D}_u$	Theorem 9.3 Theorem 9.3 (4-factor version)
interlaced trilattices with t -involution \mathcal{IT}_{-t}	bilattices \times bilattices $\mathcal{BL}_u \times \mathcal{BL}_u$ OR lattices \times lattices $\mathcal{L}_u \times \mathcal{L}_u$	Theorem 9.3 Theorem 9.3 (& Theorem 3.1)
distributive trilattices with t -involution \mathcal{DT}_{-t}	$\mathcal{DB}_u \times \mathcal{DB}_u$ OR $\mathcal{D}_u \times \mathcal{D}_u$	Theorem 9.3

TABLE 2. Equivalences derived from Theorems 9.1 and 9.3 (no bounds)

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REFERENCES

- [1] Adámek, J., H. Herrlich and Strecker, G.E. *Abstract and Concrete Categories: The Joy of Cats* (online edition). Available at <http://katmat.math.uni-bremen.de/acc>
- [2] Arieli, O. and Avron, A.: Reasoning with logical bilattices. *J. Log. Lang. Inf.* **5** (1996), 25–63
- [3] Arieli, O. and Avron, A.: The value of the four values. *Artificial Intelligence* **102** (1998), 97–141
- [4] Avron, A.: The structure of interlaced bilattices. *Math. Structures Comput. Sci.* **6** (1996), 287–299
- [5] Balbes, R. and Dwinger, Ph.: *Distributive Lattices*. University of Missouri Press, Columbia (1974)
- [6] Belnap, N.D.: A useful four-valued logic: How a computer should think. In: Anderson, A.R. and Belnap, N.D.: *Entailment. The Logic of Relevance and Necessity*, vol. II, pp. 506–541, Princeton University Press (1992)
- [7] Bou, F., Jansana, R. and Riviuccio, U.: Varieties of interlaced bilattices. *Algebra Universalis* **66** (2011), 115–141
- [8] Bou, F. and Riviuccio, U.: The logic of distributive bilattices. *Logic J. IGPL* **19** (2011), 183–216
- [9] Bou, F. and Riviuccio, U.: Bilattices with implications. *Studia Logica* **101** (2013), 651–675
- [10] Burris, S.N. and Sankappanavar, H.P.: *A Course in Universal Algebra*. Graduate Texts in Mathematics, vol. 78. Springer-Verlag (1981) Free download at <http://www.math.waterloo.ca/~snburris>
- [11] Cabrer, L.M., Craig, A.P.K., and Priestley, H.A.: Product representation for default bilattices: an application of natural duality theory. *J. Pure Appl. Alg.* **219** (2015) 2962–2988
- [12] Cabrer, L.M. and Priestley, H.A.: Distributive bilattices from the perspective of natural duality theory. *Algebra Universalis* (to appear, DOI:10.1007/s00012-015-0316-5)
- [13] Cabrer, L.M. and Priestley, H.A.: Natural dualities through product representations: bilattices and beyond. (preprint)

- [14] Davey, B.A.: The product representation theorem for interlaced pre-bilattices: some historical remarks. *Algebra Universalis* **70** (2013), 403–409
- [15] Fitting, M.: Bilattices and the semantics of logic programming. *J. Logic Programming* **11** (1991), 91–116
- [16] Fitting, M.: Kleene’s logic, generalized. *J. Logic Comput.* **1** (1991), 797–810
- [17] Fitting, M.: Kleene’s three-valued logics and their children. *Fund. Inform.* **20** (1994), 113–131
- [18] Fitting, M.: Bilattices are nice things. Self-reference, CSLI Lecture Notes **178**, pp. 53–77, CSLI Publ., Stanford, CA, (2006)
- [19] Galatos N., Jipsen P., Kowalski T. and Ono H.: Residuated lattices: an algebraic glimpse at substructural logics. In: *Stud. Logic Found. Math.*, vol. 151, Elsevier, (2007)
- [20] Gargov, G.: Knowledge, uncertainty and ignorance: bilattices and beyond. *J. Appl. Non-classical Logics* **9** (1999), 195–283
- [21] Ginsberg, M. L.: Multivalued logics: A uniform approach to inference in artificial intelligence. *Comput. Intelligence*, **4** (1988), 265–316
- [22] Ginsberg, M. L.: Bilattices and modal operators. *J. Logic Comput.* **1**(1) (1990), 41–69
- [23] Jansana, R. and Riviuccio, U.: Residuated bilattices, *Soft Comput.* **16** (2012), 493–504
- [24] Jung, A. and Riviuccio U.: Priestley duality for distributive bilattices (extended abstract). 5th Int. Conf. on Topology, Algebra, and Categories in Logic (TACL 2011, Marseilles)
- [25] Jung, A. and Riviuccio, U.: Priestley duality for bilattices, *Studia Logica* **100** (2012), 223–252
- [26] Jung, A. and Riviuccio, U.: Kripke semantics for modal bilattice logic (extended abstract). *Proc. 28th ACM/IEEE Symp. Logic Comp. Sci.* (2013), pp. 438–447
- [27] Kalman, J.: Lattices with involution. *Trans. Amer. Math Soc.* **87** (1958), 485–491
- [28] Mobasher, B., Pigozzi, D., Slutski, V. and Voutsadakis, D.: A duality theory for bilattices. *Algebra Universalis* **43** (2000), 109–125
- [29] Reyes G.E. and Zolfaghari, H.: Bi-Heyting algebras, toposes and modalities, *J. Phil. Log.* **25** (1996), 25–46
- [30] Riviuccio, U.: An Algebraic Study of Bilattice-based Logics. PhD Thesis, University of Barcelona (2010) (available at <http://arxiv.org/abs/1010.2552>)
- [31] Riviuccio, U.: Representation of interlaced trilattices. *J. Appl. Log.* **11** (2013), 174–189
- [32] Riviuccio, U.: Algebraic Semantics for Bilattice Public Announcement Logic. *Studia Logica*, Proc. Trends in Logic XIII, Springer (2014)
- [33] Ruet, P. and Fages, F.: Combining explicit negation and negation by failure via Belnap’s logic. *Theoret. Comput. Sci.* **171** (1997), 61–75
- [34] Shramko, Y., Dunn, J.M. and Takenaka, T.: The trilattice of constructive truth. *J. Logic Computat.* **11** (2001), 761–788
- [35] Shramko, Y. and Wansing, H.: Some useful 16-valued logics: How a computer network should think. *J. Philos. Logic* **34** (2005), 121–153
- [36] Wille, R.: The basic theorem of triadic concept analysis. *Order* **12** (1995), 149–158

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