

A MANY-SORTED VARIANT OF JAPARIDZE'S POLYMODAL PROVABILITY LOGIC

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ABSTRACT. We consider a many-sorted variant of Japaridze's polymodal provability logic **GLP**. In this variant, which is denoted **GLP***, propositional variables are assigned sorts $\alpha \leq \omega$, where variables of finite sort $n < \omega$ are interpreted as Π_{n+1} -sentences of the arithmetical hierarchy, while those of sort ω range over arbitrary ones. We prove that **GLP*** is arithmetically complete with respect to this interpretation. Moreover, we relate **GLP*** to its one-sorted counterpart **GLP** and prove that the former inherits some well-known properties of the latter, like Craig interpolation and PSPACE decidability. We also study a positive variant of **GLP*** which allows for an even richer arithmetical interpretation—variables are permitted to range over theories rather than single sentences. This interpretation in turn allows the introduction of a modality that corresponds to the full uniform reflection principle. We show that our positive variant of **GLP*** is arithmetically complete.

Keywords: provability logics, mathematical logic, modal logic, formal arithmetic, arithmetical completeness

1. INTRODUCTION

The polymodal provability logic **GLP**, due to [17], has received considerable interest in the mathematical logic community. The language of **GLP** features modalities $\langle n \rangle$, for every $n \geq 0$, that can be arithmetically interpreted as *n-consistency*, i.e., the modal formula $\langle n \rangle \varphi$ expresses under this interpretation that φ is consistent with the set of all true Π_n -sentences. This particular interpretation steered interest in **GLP** in mainstream proof theory: in [3], the second author of this

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paper showed how GLP can act as a framework in order to canonically recover an ordinal notation system for Peano arithmetic (PA) and its fragments. Moreover, based on these notions, he obtained a rather abstract version of Gentzen's consistency proof for PA by transfinite induction up to ε_0 and he formulated a combinatorial statement independent from PA [5].

This proof-theoretic analysis is based on the notion of *graded provability algebra*. Let T be an extension of PA. Recall the concept of *Lindenbaum algebra of T* : its elements are equivalence classes of the relation

$$\varphi \sim \psi \iff_{df} T \vdash \varphi \leftrightarrow \psi.$$

Let $[\varphi]$ denote the equivalence class of φ with respect to \sim . The algebra \mathcal{L}_T is equipped with the standard Boolean connectives and the relation

$$[\varphi] \leq [\psi] \iff_{df} T \vdash \varphi \rightarrow \psi.$$

This turns \mathcal{L}_T into a Boolean algebra, the *Lindenbaum algebra of T* . Thus, logical notions are brought into an algebraic setting. The maximal element \top and the minimal element \perp of this algebra are, respectively, the classes of all provable and all refutable sentences of T and deductively closed extensions of T correspond to *filters* of \mathcal{L}_T (see [4] for details).

Let $\langle n \rangle_T$ be a Π_{n+1} -formula that formalizes the notion of n -consistency in arithmetic (see, e.g., [4] for a definition of $\langle n \rangle_T$). The *graded provability algebra \mathcal{M}_T of T* is the algebra \mathcal{L}_T extended by operators $\langle n \rangle_T$ defined on the elements of \mathcal{L}_T by

$$\langle n \rangle_T: [\varphi] \mapsto [\langle n \rangle_T \varphi], \quad \text{for } n \geq 0.$$

Terms in the language of \mathcal{M}_T can be identified with polymodal formulas. Furthermore, for each sound and axiomatizable extension T of PA, Japaridze's arithmetical completeness theorem for GLP states that

$$\text{GLP} \vdash \varphi(\vec{p}) \iff \mathcal{M}_T \models \forall \vec{p} (\varphi(\vec{p}) = \top),$$

where \vec{p} are all the propositional variables from $\varphi(\vec{p})$. The algebra \mathcal{M}_T carries an additional structure in the form of a distinguished family of subsets

$$P_0 \subset P_1 \subset \dots \subset \mathcal{M}_T,$$

where P_n is defined by the class of Π_{n+1} -sentences of the arithmetical hierarchy. This family of subsets is called a *stratification of \mathcal{M}_T* [3]. Since $\langle n \rangle_T$ is a Π_{n+1} -formula, the operator $\langle n \rangle_T$ maps \mathcal{M}_T to P_n . The presence of a stratification thus admits to turn \mathcal{M}_T into a many-sorted algebra in which variables of sort n range over arithmetical Π_{n+1} -sentences. The notion of sort can be readily extended to capture all polymodal terms. It is the goal of this paper to investigate a modal-logical counterpart to this many-sorted algebra.

Let us briefly comment on the general motivations for this study. One of the (global and ambitious) goals of relating provability algebras to the ordinal analysis of theories was to shed more light on the well-known and basic conceptual problem of "natural ordinal notations" in proof theory (see, e.g., [19, 22]). We would like to understand general criteria distinguishing well-behaved ordinal notation systems suitable for proof-theoretic analysis from the "wild" ones, as in Kreisel's counterexamples [22].

The approach of provability algebras is an attempt to recast core proof-theoretic results in a more abstract, essentially algebraic, language. This amounts to introducing structures that are, on the one hand, directly related to strong, computationally universal formal systems, such as Peano arithmetic and its extensions. On the other hand, from these structures one should be able to recover ordinal notation systems in a canonical way. In other words, we consider the natural ordinal notations problem as the question of what kind of information is required for us to be able to speak about proof-theoretic ordinal notation systems in a canonical way.

Within such a project it seems necessary to “pack” all relevant proof-theoretic information into a suitable algebraic framework—and the simpler this framework is the better. Basic results in the proof theory of arithmetic can be viewed as either proofs of reflection schemas restricted to arithmetical complexity classes Π_n , or as Π_n -conservativity relationships between certain systems. Thus, the stratification of the provability algebra into levels of the arithmetical hierarchy of formulas seems to be part of the data that necessarily has to be represented within the sought algebraic framework. (Let us stress that, for example, introducing quantifiers in the style of cylindric algebras would be an overkill, as we would obtain structures that are not “tame”.)

For example, the so-called *reduction property* of provability algebras is a key result needed for the proof-theoretic analysis of Peano arithmetic. The most natural statement of this property in [3] becomes purely algebraic only if the stratification is part of the considered algebraic structure.¹

The present paper considers the most direct approach to incorporating the stratification into the syntactic framework where the propositional variables are assigned “rigid” sorts (types), for every $n \geq 0$, and are understood as ranging over the classes of arithmetical Π_{n+1} -sentences. The corresponding many-sorted variant of GLP will be denoted by GLP^* . Substitution in this logic is required to respect the sorts of variables.² Our main result is a Solovay-style arithmetical completeness theorem for GLP^* , i.e., for any sound and axiomatizable extension T of PA we have

$$\text{GLP}^* \vdash \varphi(\vec{p}) \iff \mathcal{M}_T \models \forall \vec{p} (\varphi(\vec{p}) = \top),$$

where \vec{p} are all propositional variables from $\varphi(\vec{p})$ and a quantifier binding such a variable of sort n only ranges over elements from P_n . In particular, we show that the principle of Σ_{n+1} -completeness,

$$\langle n \rangle_T p \rightarrow p$$

(where p is of sort n), in addition to the postulates of GLP, suffices to obtain arithmetical completeness. We observe that most of the arguments in the proof of arithmetical completeness of GLP also work for the sorted language. Thus, having sorted variables does not really lead to a more complicated arithmetically complete system than GLP itself.

A similar system has been studied by Visser [25, 26] who introduced a Σ_1 -provability logic of PA, i.e., in his logic, variables are arithmetically interpreted as Σ_1 -sentences (see also [1, 15]). The interpretation of propositional variables as Σ_1 -sentences also plays an important role in the study of intuitionistic provability logic and its variable-free fragment; see [27].

In [18] a more flexible, yet more complicated approach is considered, where types corresponding to Σ_n - and to Π_n -sentences, for all $n \geq 1$, are not rigid but can be defined using the modalities $\Sigma_n \varphi$ arithmetizing the predicate “ φ is PA-equivalent to a Σ_n -sentence”. This logic, however, lacks the necessary modalities $\langle n \rangle$, for all $n > 0$, representing the higher reflection principles. It might be interesting to consider the extension of GLP by modalities Σ_n —however, at this point, it is not clear whether this system has substantial advantages compared to the one with rigid types.

The remainder of the paper is organized as follows. After this introductory section, we define basic notions in Section 2. In Section 3, we prove the arithmetical completeness theorem for GLP^* . We continue our exposition on GLP^* in Section 4 by proving that deciding provability in GLP^* is complete for PSPACE and providing a natural many-sorted truth-provability logic. Moreover, we show that GLP^* admits Craig interpolation and study variants of GLP^* that restrict the sorts and modalities we are allowed to use. In Section 5, we study a positive variant $\text{RC}^*\omega$ of GLP^*

¹See also [11] for some generalizations of the reduction property that can be stated without references to sorts.

²Thus, strictly speaking, our treatment does not yield a logic in the usual sense, since it is not closed under unrestricted substitutions. However, we shall use this term without further concern.

whose corresponding one-sorted counterpart has recently gained attraction in the provability logic community. In this fragment, we restrict ourselves to certain positive formulas which allow us to focus on more general arithmetical interpretations—variables are permitted to range over arithmetical theories rather than single sentences. This in turn allows the introduction of an additional modality $\langle \omega \rangle$ that corresponds to the full uniform reflection principle which has no finite, yet recursive axiomatization. We prove that $\text{RC}^*\omega$ is arithmetically complete for this interpretation. We conclude the paper in Section 6.

2. PRELIMINARIES

2.1. The Logics GLP , GLP^* , and J^* . The polymodal provability logic GLP is formulated in the language of the propositional calculus (using the connectives \top , \neg , and \wedge as primitives), enriched by unary connectives $\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \dots$, called *modalities*. Using these connectives, formulas are built inductively in the usual way. The dual connectives $[n]$, for every $n \geq 0$, are abbreviations where $[n]\varphi$ stands for $\neg\langle n \rangle\neg\varphi$. Moreover, we abbreviate the standard Boolean connectives using \top , \neg , and \wedge in the usual manner.

The logic GLP is axiomatized by the following axiom schemas and rules:³

- (i) all tautologies of classical propositional logic;
- (ii) $\langle n \rangle(\varphi \vee \psi) \rightarrow \langle n \rangle\varphi \vee \langle n \rangle\psi$; $[n]\top$;
- (iii) $\langle n \rangle\varphi \rightarrow \langle n \rangle(\varphi \wedge \langle n \rangle\neg\varphi)$ (*Löb's axiom*);
- (iv) $\langle m \rangle\varphi \rightarrow [n]\langle m \rangle\varphi$, for $m < n$;
- (v) $\langle n \rangle\varphi \rightarrow \langle m \rangle\varphi$, for $m < n$ (*monotonicity*); and
- (vi) modus ponens and $\varphi \rightarrow \psi / \langle n \rangle\varphi \rightarrow \langle n \rangle\psi$.

GLP^* is formulated over a propositional language that contains variables each being assigned a unique *sort* α , where $0 \leq \alpha \leq \omega$. Let us formalize this notion more carefully. Let \mathbb{P} denote a fixed, countably infinite set of *propositional variables*. We fix a function $|\cdot|: \mathbb{P} \rightarrow \omega \cup \{\omega\}$ that assigns a *sort* α ($0 \leq \alpha \leq \omega$) to each propositional variable $p \in \mathbb{P}$ in such a way that \mathbb{P} is partitioned into disjoint, countably infinite sets $\mathbb{P}_0, \mathbb{P}_1, \dots, \mathbb{P}_k, \dots, \mathbb{P}_\omega$, where

$$p \in \mathbb{P}_\alpha \iff_{df} |p| = \alpha, \quad 0 \leq \alpha \leq \omega.$$

Formulas in this sorted language (i.e., formulas over variables from \mathbb{P}) are called *many-sorted formulas*. When it is clear from context that we are dealing with many-sorted formulas, we shall, however, often refer to them as “formulas”.

The notion of sort is recursively extended to the set of all polymodal formulas as follows:

- \top and \perp have sort 0;
- $\varphi \wedge \psi$ has sort $\max\{\alpha, \beta\}$ if φ and ψ have the respective sorts α and β ;
- $\neg\varphi$ has sort $1 + \alpha$ if φ has sort α ; and
- $\langle n \rangle\varphi$ has sort n , for $n < \omega$ and any choice of φ .

It is easy to see that the sort of a formula is uniquely determined by the sorts of its constituent propositional variables and we denote by $|\varphi|$ the sort of φ . The sort ω is included to provide

³Usually, GLP is axiomatized by using $[n]$ instead of $\langle n \rangle$. However, it is more convenient for our purposes to use $\langle n \rangle$, since we focus on Π_{n+1} -axiomatized concepts. Note that GLP is closed under the *necessitation rule*: if $\text{GLP} \vdash \varphi$ then $\text{GLP} \vdash [n]\varphi$.

variables that can explicitly be assigned an arbitrary arithmetical sentence in an arithmetical realization. In contrast, variables of finite sort $n < \omega$ can be assigned arithmetical Π_{n+1} -sentences only. Note that if $|\varphi| = \omega$, then also $|\neg\varphi| = \omega$. Moreover, notice that even formulas equivalent in propositional logic may have different sorts, e.g., if p has sort $n \in \omega$, then $\neg p$ has sort $n + 2$.

DEFINITION 2.1. The logic GLP^* is axiomatized by the schemas (i), (ii), (iii), and (v) of GLP , as well as the following axiom schema:

$$(vii) \langle n \rangle \varphi \rightarrow \varphi, \text{ if } |\varphi| \leq n \text{ } (\Sigma_{n+1}\text{-completeness}).$$

Furthermore, GLP^* is closed under modus ponens and $\varphi \rightarrow \psi / \langle n \rangle \varphi \rightarrow \langle n \rangle \psi$, while GLP^* is not closed under arbitrary substitutions of formulas, but one must rather respect the sorts of the propositional variables and formulas involved. That is, one can only substitute formulas of sort at most α for propositional variables of sort α .

Regarding the omission of axiom schema (iv), note that, for $m < n$, $\text{GLP}^* \vdash \langle n \rangle \neg \langle m \rangle \varphi \rightarrow \neg \langle m \rangle \varphi$, whence $\text{GLP}^* \vdash \langle m \rangle \varphi \rightarrow [n] \langle m \rangle \varphi$ follows by propositional logic. Hence, GLP^* extends GLP in the sense that, for any formula φ in the language of GLP , if $\text{GLP} \vdash \varphi$, then $\text{GLP}^* \vdash \varphi'$, where φ' is obtained from φ by arbitrarily assigning sorts to propositional variables.

The logic GLP is not complete for any class of Kripke frames [21]. Therefore, the second author of this paper considered in [7] a weaker logic J that is complete with respect to a natural class of Kripke frames and to which GLP is reducible.⁴ We do so as well and define a many-sorted counterpart J^* of J which arises from GLP^* by dropping the monotonicity axiom schema (v) and adding the schema

$$(vii) \langle m \rangle \langle n \rangle \varphi \rightarrow \langle m \rangle \varphi, \text{ for } m < n.$$

Using monotonicity (schema (v)), we infer $\text{GLP}^* \vdash \langle m \rangle \langle n \rangle \varphi \rightarrow \langle m \rangle \langle m \rangle \varphi$, whence by $\text{GLP}^* \vdash \langle m \rangle \langle m \rangle \varphi \rightarrow \langle m \rangle \varphi$, we see that schema (vii) above is provable in GLP^* , i.e., GLP^* extends J^* . We remark that the definition of J in [7] also comprises the axiom schema

$$(viii) \langle n \rangle \langle m \rangle \varphi \rightarrow \langle m \rangle \varphi, \text{ for } n > m.$$

This schema is readily proved in J^* using one instance of (vii)—notice that $|\langle m \rangle \varphi| = m$.

Remark. We would like to emphasize that formulas in the language of GLP^* and J^* are formulated in a different language than in their respective one-sorted versions GLP and J . Hence, formally, the many-sorted logics and their one-sorted counterparts talk about different objects. However, if we claim that a one-sorted logic proves a many-sorted formula, we mean that the one-sorted logic proves the formula which results from the many-sorted one if we simply disregard the sorts and treat it as a one-sorted formula in the usual sense.

2.2. Kripke Models. A (*Kripke frame*) is a structure $\mathfrak{F} = (W, \{R_n\}_{n \geq 0})$, where W is a non-empty set of *worlds* and each R_k , for $k \geq 0$, is a binary relation on W . The frame \mathfrak{F} is called *finite* if W is finite and $R_k = \emptyset$ for all but finitely many $k \geq 0$.

A *valuation* $\llbracket \cdot \rrbracket$ on a frame \mathfrak{F} maps every propositional variable p to a subset $\llbracket p \rrbracket \subseteq W$. A (*Kripke model*) $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ is a triple such that $\mathfrak{F} := (W, \{R_n\}_{n \geq 0})$ is a Kripke frame and $\llbracket \cdot \rrbracket$ a valuation on \mathfrak{F} . We say that \mathfrak{A} is *based on* \mathfrak{F} .

⁴Ignatiev [21] also considered a weaker logic than GLP that is complete for a class of Kripke models and provided a reduction of GLP to that logic in order to establish arithmetical completeness. However, the arithmetical completeness proof in [8], where J is used, seems to be more convenient for our purposes.

Given any Kripke model $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$, we extend the valuation $\llbracket \cdot \rrbracket$ recursively to the class of all polymodal formulas:

- $\llbracket \top \rrbracket = W$; $\llbracket \perp \rrbracket = \emptyset$;
- $\llbracket \psi \wedge \chi \rrbracket = \llbracket \psi \rrbracket \cap \llbracket \chi \rrbracket$;
- $\llbracket \neg \psi \rrbracket = W \setminus \llbracket \psi \rrbracket$, and
- $\llbracket \langle n \rangle \psi \rrbracket = \{x \in W \mid \exists y (xR_n y \ \& \ y \in \llbracket \psi \rrbracket)\}$.

We often write $\mathfrak{A}, x \models \varphi$ instead of $x \in \llbracket \varphi \rrbracket$. We say that φ is *valid in* \mathfrak{A} , denoted by $\mathfrak{A} \models \varphi$, if $\mathfrak{A}, x \models \varphi$, for every $x \in W$. Moreover, for a frame \mathfrak{F} , we say that φ is *valid in* \mathfrak{F} , if φ is valid in every model based on \mathfrak{F} .

A binary relation R on W is *conversely well-founded* if there is no infinite chain of elements of W of the form $x_0 R x_1 R x_2 \dots$. It is easy to see that, for finite W , this condition is equivalent to R being irreflexive. A Kripke frame $\mathfrak{A} = (W, \{R_n\}_{n \geq 0})$ is called a *J-frame* [7] if

- (a) R_k is conversely well-founded and transitive, for all $k \geq 0$;
- (b) $\forall x, y (xR_n y \Rightarrow \forall z (xR_m z \Leftrightarrow yR_m z))$, for $m < n$; and
- (c) $\forall x, y, z (xR_m y \ \& \ yR_n z \Rightarrow xR_m z)$, for $m < n$.

A *J-model* is a Kripke model that is based on a J-frame. The fact that the R_k must be conversely well-founded and transitive is a classical property required to validate all instances of Löb's axiom (schema (iii)). Frame condition (b) corresponds to the schemas (viii) and (iv), while frame condition (c) corresponds to schema (vii).

THEOREM 2.2 ([7]). *For any polymodal formula φ , $\mathsf{J} \vdash \varphi$ iff φ is valid in all J-frames.*

We call a J-model $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ a *J*-model*, if it is *strongly persistent*, that is, if it satisfies the following two conditions:

- (1) if $|p| \leq n$ and $\mathfrak{A}, y \models p$, then $\mathfrak{A}, x \models p$ whenever $xR_n y$; and
- (2) if $|p| < n$ and $\mathfrak{A}, y \not\models p$, then $\mathfrak{A}, x \not\models p$ whenever $xR_n y$.

Note that, up to now, the notion of strong persistence is the first semantic notion that refers to sorts of variables at all. Sorts thus have no realization on the frame level, but are rather present through the notion of strong persistence on the level of models. Condition (1) states that truth of propositional variables of sort at most n must be propagated downwards along R_n -arcs. Likewise, condition (2) states that falsehood of propositional variables having sort (strictly) less than n must be propagated downwards along R_n -arcs.

Having both conditions in place allows us to extend (1) and (2) to all sorted formulas. This relationship between strong persistence and satisfaction of sorted formulas is the content of the following lemma.

LEMMA 2.3. *Let $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ be a J-model. Then \mathfrak{A} is strongly persistent iff for all formulas φ and all $n \geq 0$ we have*

- if $|\varphi| \leq n$, then $xR_n y$ and $\mathfrak{A}, y \models \varphi$ imply $\mathfrak{A}, x \models \varphi$; and
- if $|\varphi| < n$, then $xR_n y$ and $\mathfrak{A}, y \not\models \varphi$ imply $\mathfrak{A}, x \not\models \varphi$.

Proof. The proof is by induction on the structure of φ . The base case follows immediately by the definition of strong persistence. Assume $\varphi = \neg\psi$ for some ψ . Suppose first that $|\varphi| \leq n$ and xR_ny such that $\mathfrak{A}, y \models \neg\psi$. It follows that $\mathfrak{A}, y \not\models \psi$, and by $|\psi| < |\varphi| \leq n$ and the induction hypothesis, we infer that $\mathfrak{A}, x \not\models \psi$, whence $\mathfrak{A}, x \models \neg\psi$ follows as required. The case where $|\varphi| < n$ is handled similarly.

Suppose now that $\varphi = \langle k \rangle \psi$, for some $k \in \omega$. Assume that $|\langle k \rangle \psi| \leq n$ and let $x, y \in W$ be such that xR_ny and $\mathfrak{A}, y \models \langle k \rangle \psi$. We know that $|\langle k \rangle \psi| = k$, whence $k \leq n$ follows. Let $z \in W$ be such that yR_kz and $\mathfrak{A}, z \models \psi$. Now frame condition (b) and the fact that R_k is transitive (for the case $k = n$) give us xR_kz , whence $\mathfrak{A}, x \models \langle k \rangle \psi$ follows as desired. Suppose now that $|\langle k \rangle \psi| < n$ and let $x, y \in W$ be such that xR_ny and $\mathfrak{A}, y \not\models \langle k \rangle \psi$. Suppose to the contrary that $\mathfrak{A}, x \models \langle k \rangle \psi$. Let $z \in W$ be such that xR_kz and $\mathfrak{A}, z \models \psi$. We know that $k < n$ and by frame condition (b) we infer that yR_kz . Therefore, $\mathfrak{A}, y \models \langle k \rangle \psi$, a contradiction. Hence, $\mathfrak{A}, x \not\models \langle k \rangle \psi$ as required.

Suppose that $\varphi = \psi \wedge \chi$ for some formulas χ, ψ . If $|\varphi| \leq n$, xR_ny , and $\mathfrak{A}, y \models \varphi$, then $|\psi|, |\chi| \leq |\varphi|$, whence by $\mathfrak{A}, y \models \psi$, $\mathfrak{A}, y \models \chi$ and the induction hypothesis it follows that $\mathfrak{A}, x \models \varphi$, as required. Suppose that $|\varphi| < n$. Then also $|\psi|, |\chi| < n$, and if $\mathfrak{A}, y \not\models \varphi$, then $\mathfrak{A}, y \not\models \psi$ or $\mathfrak{A}, y \not\models \chi$. In both cases, the induction hypothesis yields $\mathfrak{A}, x \not\models \varphi$. This finishes the case of conjunction. \square

Note that, in the proof above, it is of importance that \mathfrak{A} is indeed a J-model. In particular, we require that \mathfrak{A} satisfies frame condition (b) and the fact that all R_n are transitive.

LEMMA 2.4. *The axiom schema $\langle n \rangle \varphi \rightarrow \varphi$ is valid in a J-model \mathfrak{A} for all φ such that $|\varphi| \leq n$ iff \mathfrak{A} is strongly persistent.*

Proof. Assuming $\mathfrak{A}, x \models \langle n \rangle \varphi$ gives us $\mathfrak{A}, y \models \varphi$ for some y such that xR_ny , whence $\mathfrak{A}, x \models \varphi$ follows by $|\varphi| \leq n$ and one application of Lemma 2.3.

Conversely, if \mathfrak{A} satisfies all instances of $\langle n \rangle \varphi \rightarrow \varphi$ ($|\varphi| \leq n$), it satisfies these instances for all appropriate propositional variables and their negations (respecting their sorts). Hence, if $|p| \leq n$, $\mathfrak{A}, y \models p$, and xR_ny , then by $\mathfrak{A}, x \models \langle n \rangle p \rightarrow p$ also $\mathfrak{A}, x \models p$. Likewise, if $|p| < n$, $\mathfrak{A}, y \not\models p$, and xR_ny , then $\mathfrak{A}, y \models \neg p$, whence by $\mathfrak{A}, x \models \langle n \rangle \neg p \rightarrow \neg p$ it follows that $\mathfrak{A}, x \not\models p$ as needed. \square

Our goal is now to show that \mathbf{J}^* is sound and complete for the class of all strongly persistent \mathbf{J}^* -models. Soundness follows by a straightforward induction on the length of a derivation invoking Lemma 2.4. For proving completeness, we aim at a reduction of \mathbf{J}^* to \mathbf{J} as detailed in the following.⁵

2.3. Completeness of \mathbf{J}^* . Let φ be a many-sorted formula and let p_1, \dots, p_k exhaust all variables from φ and let $\alpha_1, \dots, \alpha_k$ be their respective sorts. Furthermore, let $\Theta \subset \omega$ be a finite set of natural numbers. Define

$$P_\Theta(\varphi) := \bigwedge_{i=1}^k \bigwedge (\{ \langle j \rangle p_i \rightarrow p_i \mid j \in \Theta, j \geq \alpha_i \} \cup \{ \langle j \rangle \neg p_i \rightarrow \neg p_i \mid j \in \Theta, j > \alpha_i \})$$

and

$$P_\Theta^+(\varphi) := P_\Theta(\varphi) \wedge \bigwedge_{j \in \Theta} [j] P_\Theta(\varphi).$$

If Θ consists of exactly those n such that $\langle n \rangle$ occurs as a modality in φ , then we omit the subscript “ Θ ” in the expression $P_\Theta^+(\varphi)$ and write $P^+(\varphi)$ instead. A similar convention is applied

⁵The authors are thankful to one of the anonymous referees who pointed out a simplification of the completeness proof for \mathbf{J}^* .

to $P_\Theta(\varphi)$. Intuitively, the formula $P^+(\varphi)$ should ensure, when valid in a model, that the model at hand is strongly persistent:

LEMMA 2.5. *Suppose $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ is a finite model such that $\mathfrak{A} \models P_\Theta(\varphi)$, where Θ is chosen such that $R_n \neq \emptyset$ implies $n \in \Theta$. Then \mathfrak{A} is strongly persistent.*

Proof. Let $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ be a model and suppose $\mathfrak{A} \models P_\Theta(\varphi)$. Consider any variable p such that $|p| \leq n$ and some $x, y \in W$ such that $\mathfrak{A}, y \models p$ and $xR_n y$. By the construction of $P_\Theta(\varphi)$, we know that $\mathfrak{A}, x \models \langle n \rangle p \rightarrow p$ and so $\mathfrak{A}, x \models p$ as required. Likewise, if $|p| < n$, $\mathfrak{A}, y \not\models p$, and $xR_n y$, then $P_\Theta(\varphi)$ contains the conjunct $\langle n \rangle \neg p \rightarrow \neg p$, whence $\mathfrak{A}, x \models \neg p$ and thus $\mathfrak{A}, x \not\models p$ follows. \square

Note that the finiteness of the model in Lemma 2.5 is essential, since otherwise $P_\Theta^+(\varphi)$ may not be finite.

Let $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ be a J-model. A *root* of \mathfrak{A} is a world $r \in W$ such that for all $x \in W$, there is a $k \geq 0$ such that $rR_k x$ or $r = x$. A model which has a root is called *rooted*.

LEMMA 2.6 ([8]). *If $J \not\models \varphi$, then there is a finite J-model \mathfrak{A} with root r such that $\mathfrak{A}, r \not\models \varphi$. Moreover, one can choose \mathfrak{A} such that $R_n \neq \emptyset$ implies that $\langle n \rangle$ occurs in φ .*

COROLLARY 2.7. *If $J^* \not\models \varphi$, then there is a finite J^* -model \mathfrak{A} with root r such that $\mathfrak{A}, r \not\models \varphi$.*

Proof. Suppose $J^* \not\models \varphi$. Then also $J \not\models P^+(\varphi) \rightarrow \varphi$, since $J^* \vdash P^+(\varphi)$ and J^* extends J . Using again Lemma 2.6, we know that there is a J-model $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ with root r such that $\mathfrak{A}, r \not\models P^+(\varphi) \rightarrow \varphi$. Furthermore, $R_n \neq \emptyset$ implies that $\langle n \rangle$ occurs in $P^+(\varphi) \rightarrow \varphi$. Hence, $R_n \neq \emptyset$ also implies that $\langle n \rangle$ occurs as modality in φ , since $P^+(\varphi)$ and φ contain the same modalities. Since $\mathfrak{A}, r \models P^+(\varphi)$ and r is the root of \mathfrak{A} , we can infer that $\mathfrak{A}, x \models P(\varphi)$, for all $x \in W$. By Lemma 2.5 it follows that \mathfrak{A} is strongly persistent, i.e., \mathfrak{A} is a J^* -model having root r such that $\mathfrak{A}, r \not\models \varphi$. This proves the claim. \square

From this, the completeness of J^* for the class of J^* -models follows immediately:

COROLLARY 2.8. *$J^* \vdash \varphi$ iff φ is valid in all J^* -models.*

Proof. Soundness is an easy induction on the length of a derivation. Completeness follows immediately by Corollary 2.7. \square

2.4. Formal Arithmetic. We consider first-order theories in the language of arithmetic. The theories we consider are extensions of Peano arithmetic (PA). The class of Δ_0 -formulas are all formulas where each occurrence of a quantifier is of one of the forms

$$\begin{aligned} \forall x \leq t \varphi &:= \forall x (x \leq t \rightarrow \varphi) \text{ or} \\ \exists x \leq t \varphi &:= \exists x (x \leq t \wedge \varphi), \end{aligned}$$

where t is a term that has no occurrence of the variable x . Occurrences of such quantifiers are called *bounded*, and we often call Δ_0 -formulas simply *bounded formulas*. The classes of Σ_n - and Π_n -formulas are defined inductively as follows: Σ_0 - and Π_0 -formulas are the same as Δ_0 -formulas. If $\varphi(\vec{x}, y)$ is a Π_n -formula, then $\exists y \varphi(\vec{x}, y)$ is a Σ_{n+1} -formula. Accordingly, if $\varphi(\vec{x}, y)$ is a Σ_n -formula, then $\forall y \varphi(\vec{x}, y)$ is a Π_{n+1} -formula. A formula is in Δ_{n+1} iff it is both in Σ_{n+1} and Π_{n+1} . When an arithmetical theory T is given, we often identify these classes modulo provable equivalence in T . In this context, we say that a formula is Σ_n in T (Π_n , Δ_n , respectively), if it is provably equivalent to a Σ_n -formula (Π_n -formula, Δ_n -formula, respectively) in T .

We denote by \bar{n} the n -th numeral that represents the number n in our arithmetical language (when reasoning in an arithmetical theory, we shall often write simply n instead of \bar{n}). We assume a standard global assignment $\ulcorner \cdot \urcorner$ of expressions (terms, formulas, etc.) to natural numbers, called the *codes* of the respective expressions. When presenting formulas in the arithmetical language, we usually write $\ulcorner \tau \urcorner$ instead of $\overline{\ulcorner \tau \urcorner}$. We often consider primitive recursive families of formulas φ_n that depend on a parameter $n \in \omega$. In this context, $\ulcorner \varphi_x \urcorner$ denotes a primitive recursive definable term with free variable x whose value for a given n is the Gödel number of φ_n . In particular, the expression $\ulcorner \varphi(\dot{x}) \urcorner$ denotes a primitive recursive definable term whose value given any n is the Gödel number of $\varphi(\bar{n})$, i.e., the Gödel number of the formula resulting from φ when substituting the term \bar{n} for x .

A theory T is *sound* if $T \vdash \varphi$ implies $\mathbb{N} \models \varphi$, for every arithmetical sentence φ . A theory T is *axiomatizable* if T has a recursive set of axioms. For an axiomatizable extension T of PA, we denote by $\Box_T(\alpha)$ the formula that formalizes the notion of provability in T in the usual sense.⁶ We write $\Box_T\varphi$ instead of $\Box_T(\ulcorner \varphi \urcorner)$. The formula \Box_T defines the standard Gödelian provability predicate for T . More generally, given a formula $\text{Prv}(\alpha)$ with one free variable α , we say that Prv is a *provability predicate of level n over T* [21], if for all arithmetical sentences φ, ψ :

- (a) Prv is a Σ_{n+1} -formula;
- (b) $T \vdash \varphi$ implies $\text{PA} \vdash \text{Prv}(\ulcorner \varphi \urcorner)$;
- (c) $\text{PA} \vdash \text{Prv}(\ulcorner \varphi \rightarrow \psi \urcorner) \rightarrow (\text{Prv}(\ulcorner \varphi \urcorner) \rightarrow \text{Prv}(\ulcorner \psi \urcorner))$; and
- (d) if φ is a Σ_{n+1} -sentence, then $\text{PA} \vdash \varphi \rightarrow \text{Prv}(\ulcorner \varphi \urcorner)$ (*provable Σ_{n+1} -completeness*).

It is well-known that \Box_T , in its standard formulation, is a provability predicate of level 0. A provability predicate Prv is *sound* if $\mathbb{N} \models \text{Prv}(\ulcorner \varphi \urcorner)$ implies $\mathbb{N} \models \varphi$, for every arithmetical sentence φ . A sequence π of formulas $\text{Prv}_0, \text{Prv}_1, \dots$ is a *strong sequence of provability predicates over T* , if there is a sequence $r_0 < r_1 < \dots$ of natural numbers such that, for all $n \geq 0$,

- Prv_n is a provability predicate of level r_n over T ; and
- $T \vdash \text{Prv}_n(\ulcorner \varphi \urcorner) \rightarrow \text{Prv}_{n+1}(\ulcorner \varphi \urcorner)$, for any arithmetical sentence φ .

We write $[n]_\pi\varphi$ for $\text{Prv}_n(\ulcorner \varphi \urcorner)$. Moreover, the dual of $[n]_\pi$ is defined by $\langle n \rangle_\pi\varphi := \neg[n]_\pi\neg\varphi$. Given such a sequence π , we denote by $|\pi_n|$ the level of the n -th provability predicate of π .

Since the provability predicate $[n]_\pi$ from π is a Σ_k -sentence for some $k > 0$, we can associate (in analogy to the standard Gödelian provability predicate) a predicate $\text{Prf}_n^\pi(\alpha, y)$ which expresses the statement “ y codes a proof of α ” and

$$T \vdash \text{Prv}_n(\alpha) \leftrightarrow \exists y \text{Prf}_n^\pi(\alpha, y).$$

We assume that Prf_n^π is chosen in such a way such that every number y codes a proof of at most one formula and that every provable formula has arbitrarily long proofs.

We denote by $\text{True}_{\Pi_n}(\alpha)$ the well-known truth-definition for the class of all Π_n -sentences, i.e., $\text{True}_{\Pi_n}(\alpha)$ expresses the statement “ α is the Gödel number of a true arithmetical Π_n -sentence”. The truth-definition for Π_n -sentences serves as a basis for a natural strong sequence of provability predicates. Let $[0]_T := \Box_T$ and

$$[n+1]_T(\alpha) := \exists \beta (\text{True}_{\Pi_n}(\beta) \wedge \Box_T(\beta \rightarrow \alpha)), \quad \text{for } n \geq 0.$$

The formula $[n]_T$ is a provability predicate of level n . It formalizes the notion of being provable in the theory $T + \text{Th}_{\Pi_n}(\mathbb{N})$, where $\text{Th}_{\Pi_n}(\mathbb{N})$ is the set of all true Π_n -sentences.

⁶We assume that Greek letters α, β, \dots range over codes of arithmetical formulas.

Another strong sequence of provability predicates is defined by $[0]_\omega := \Box_{\text{PA}}$ and

$$[n+1]_\omega := \exists\beta (\forall x [n]_\omega \beta(\dot{x}) \wedge [n]_\omega (\forall x \beta(x) \rightarrow \alpha)), \quad \text{for } n \geq 0.$$

The predicate $[n]_\omega$ is of level $2n$ and formalizes the notion of “*provability by n applications of the ω -rule*”. Japaridze originally showed arithmetical completeness of GLP for this interpretation, while completeness with respect to the broader class of interpretations, defined by strong sequences of provability predicates, was later established in [21].⁷

2.4.1. Arithmetical Interpretation. An (*arithmetical*) *realization* is a function f that maps propositional variables to arithmetical sentences. Let π be a strong sequence of provability predicates over T . The realization f is *typed for π* , if, for every propositional variable p , $f(p)$ is an arithmetical $\Pi_{|\pi_n|+1}$ -sentence, provided $n = |p| < \omega$. (We shall simply say that f is *typed* if π is clear from context.) Any realization f can be uniquely extended to a map f_π that is defined for all polymodal formulas as follows:

- $f_\pi(\perp) = \perp$; $f_\pi(\top) = \top$, where \perp (resp., \top) is a convenient contradictory (resp., tautological) statement in the language of arithmetic;
- $f_\pi(p) = f(p)$, for any propositional variable p ;
- $f_\pi(\cdot)$ commutes with the propositional connectives; and
- $f_\pi(\langle n \rangle \varphi) = \langle n \rangle_\pi f_\pi(\varphi)$, for all $n \geq 0$.

(Notice that we include the subscript π in f_π since f_π depends on the choice of π due to the fourth item above.) By some simple closure properties of the class of Π_n -sentences, it follows that $|\varphi| = n$ implies that $f_\pi(\varphi)$ is provably equivalent to a $\Pi_{|\pi_n|+1}$ -sentence in T .

GLP* is arithmetically sound for this semantics:

LEMMA 2.9. *Let T be an axiomatizable extension of PA. If $\text{GLP}^* \vdash \varphi$, then $T \vdash f_\pi(\varphi)$ for all arithmetical realizations f that are typed for π .*

Proof. The lemma is shown by induction on the length of a proof of φ in GLP*. Most of the axioms are clear. In particular, the provability of the instances of Löb’s axiom (axiom schema (iii)) is well-known, although not trivial at all; see, e.g., [3, 15] for a proof. For the schema of Σ_{n+1} -completeness (schema (vii)), notice that $\langle n \rangle \varphi \rightarrow \varphi$ is equivalent to $\neg\varphi \rightarrow [n]\neg\varphi$ in GLP*. The sentence $\neg f_\pi(\varphi)$ is now provably equivalent in T to a $\Sigma_{|\pi_n|+1}$ -sentence, whence $T \vdash \neg f_\pi(\varphi) \rightarrow [n]_\pi \neg f_\pi(\varphi)$ and thus $T \vdash \langle n \rangle_\pi f_\pi(\varphi) \rightarrow f_\pi(\varphi)$ follows by provable $\Sigma_{|\pi_n|+1}$ -completeness (property (d) above). The induction step, i.e., closure under the rules of inference, is easy to establish. We leave the details to the reader. \square

Arithmetical completeness holds under the additional assumption of soundness of the provability predicates involved. As already mentioned, arithmetical completeness for GLP has first been established in [17] and has been significantly extended and simplified in [21]. In fact, arithmetical interpretations for variants of GLP have been pushed even further: in [20], a transfinite version GLP_Λ (for Λ a recursive ordinal) of GLP is considered, where one has a modal operator $[\xi]$ for each $\xi < \Lambda$. The authors of [20] show that GLP_Λ is sound and complete for some suitable theories of second-order arithmetic (see [20] for details), where $[\xi]\varphi$ is interpreted as “ φ is provable using an ω -rule of depth ξ ”.

Regarding our intended arithmetical interpretation of GLP, in [8], the second author of this paper provided yet another simplified proof for the arithmetical completeness theorem for GLP

⁷See [2] for a brief historical background.

that is close to Solovay's original construction for the logic GL [24]. We are going to work along the lines of the construction presented in [8], since it seems to be the most convenient for our purpose. This is due to the fact that, essentially, when redoing the construction for GLP carried out in [8] in the setting of GLP*, we can observe that the arithmetical realization f one extracts from the fact that $\text{GLP}^* \not\vdash \varphi$ is actually typed (for a previously chosen strong sequence of provability predicates). Thus, in the next section, we are first going to present the essentials of the arithmetical completeness proof provided in [8] and afterwards observe that we can restrict ourselves to typed arithmetical realizations.

3. ARITHMETICAL COMPLETENESS

Arithmetical completeness proofs usually rely on reasonable Kripke semantics, since those proofs usually establish the following fact: if φ is a formula that has a Kripke model falsifying φ in a certain world, one can find an arithmetical realization such that the arithmetical theory under consideration does not prove φ under this realization. Since GLP is, however, not complete for any class of Kripke frames, in [8], GLP is reduced to J and then one relies on the Kripke semantics of J in order to prove arithmetical completeness. Our strategy towards obtaining an arithmetical completeness for GLP* is now as follows:

- We revisit the construction of [8] and present the essentials contained in there. We will review all of the necessary information from [8] needed to follow the new parts of the proof. For thorough details, we refer the interested reader to [8].
- We observe that when this construction is carried out using J*-models rather than J-models, we can extract an arithmetical realization that is actually typed (for a previously chosen strong sequence of provability predicates).

3.1. Preliminary Preparations. Before presenting the essentials of the construction in [8], we shall introduce some additional notions.

Let φ be a polymodal formula. Following [8], we define auxiliary formulas $M(\varphi)$ and $M^+(\varphi)$ as follows. Consider an enumeration $\langle m_1 \rangle \varphi_1, \langle m_2 \rangle \varphi_2, \dots, \langle m_s \rangle \varphi_s$ of all subformulas of φ of the form $\langle k \rangle \psi$ and let $n := \max_{i \leq s} m_i$. Define

$$M(\varphi) := \bigwedge_{\substack{1 \leq i \leq s \\ m_i < j \leq n}} (\langle j \rangle \varphi_i \rightarrow \langle m_i \rangle \varphi_i),$$

and, furthermore,

$$M^+(\varphi) := M(\varphi) \wedge \bigwedge_{i \leq n} [i]M(\varphi).$$

Notice that $\text{GLP}^* \vdash M^+(\varphi)$ by the use of the monotonicity axiom schema (v).

The arithmetical completeness theorem we are going to establish reads as follows:

THEOREM 3.1. *Let T be an axiomatizable extension of PA and π a strong sequence of provability predicates over T whose predicates are all sound. Then, for all formulas φ , the following statements are equivalent:*

- (1) $\text{GLP}^* \vdash \varphi$;
- (2) $\text{J}^* \vdash M^+(\varphi) \rightarrow \varphi$;
- (3) $T \vdash f_\pi(\varphi)$, for all arithmetical realizations f that are typed for π .

It is clear that item (2) implies (1) since $\text{GLP}^* \vdash M^+(\varphi)$ and GLP^* extends J^* . Moreover, we have already established that (1) implies (3) in Lemma 2.9. It thus remains to show that (3) implies (2).

Throughout the proof presented in this section, we fix an axiomatizable extension T of PA and a strong sequence of provability predicates π of which every provability predicate is sound. For a proof of the arithmetical completeness theorem for GLP^* , we are going to argue by contraposition and show that $\text{J}^* \not\vdash M^+(\varphi) \rightarrow \varphi$ entails that there is a typed realization f for π such that $T \not\vdash f_\pi(\varphi)$.

3.2. Essentials of the Construction for GLP. We fix a polymodal formula φ and assume that $\text{J}^* \not\vdash M^+(\varphi) \rightarrow \varphi$. Our goal here is to present the essentials of the construction in [8] in order to obtain a realization f such that $T \not\vdash f_\pi(\varphi)$. Afterwards, we are going to show that f is actually typed for π .

By Corollary 2.7, we know that there is a finite J^* -model $\mathfrak{A} = (W, \{R'_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$ with root r such that $\mathfrak{A}, r \models M^+(\varphi)$ and $\mathfrak{A}, r \not\models \varphi$. For technical clarity, assume that $W = \{1, 2, \dots, N\}$ for some $N \geq 1$ and $r = 1$. Construct a new model $\mathfrak{A}_0 = (W_0, \{R_n\}_{n \geq 0}, \llbracket \cdot \rrbracket)$, where

- $W_0 = \{0\} \cup W$;
- $R_0 = \{(0, x) \mid x \in W\} \cup R'_0$;
- $R_k = R'_k$, for $k > 0$; and
- $\mathfrak{A}_0, 0 \models p \iff_{df} \mathfrak{A}, 1 \models p$, for all variables p .

Notice that \mathfrak{A}_0 is still a finite J^* -model such that $\mathfrak{A}_0, r \not\models M^+(\varphi) \rightarrow \varphi$ (r is, however, not the root of \mathfrak{A}_0 anymore). In particular, \mathfrak{A}_0 is still strongly persistent. Throughout the proof, let m be the only number such that $R_m \neq \emptyset$ and $R_k = \emptyset$, for all $k > m$.

As in [8], we define the following auxiliary notions:

$$\begin{aligned} R_k(x) &:= \{y \mid xR_k y\}, \\ R_k^*(x) &:= \{y \mid y \in R_i(x), \text{ for some } i \geq k\}, \text{ and} \\ \tilde{R}_k(x) &:= R_k^*(x) \cup \bigcup \{R_k^*(z) \mid x \in R_{k+1}^*(z), z \in W_0\}. \end{aligned}$$

Note that $R_k(x) \subseteq R_k^*(x) \subseteq \tilde{R}_k(x)$. The set $\tilde{R}_k(x)$ consists of (1) all y that are R_k^* -reachable from x , and (2) all y that are R_k^* -reachable from some $z \in W_0$ such that x is R_{k+1}^* -reachable from z .

The proof now proceeds by defining, for each $x \in W_0$, an arithmetical sentence S_x which expresses that a certain function reaches a limit. More formally, suppose $g: \omega \rightarrow W_0$ is a function that is coded by an arithmetical formula $G(x, y)$ in T . We write $\ell^G = x$ for the formula $\exists N_0 \forall n \geq N_0 G(n, x)$, i.e., the formula that expresses that g reaches a *limit* at point x . The proof in [8] relies on the construction of a sequence h_1, h_2, \dots, h_m of functions that provably satisfy certain properties stated in the lemma below.

Before proceeding with the statement of that lemma, let us clarify some notation first. Given an arithmetical formula $\psi(x)$ and some $A \subseteq W_0$, we use quantifier expressions of the form $\exists x \in A \psi(x)$, $\forall x \in A \psi(x)$, etc., to respectively abbreviate finite disjunctions $\bigvee_{x \in A} \psi(\bar{x})$ and finite conjunctions $\bigwedge_{x \in A} \psi(\bar{x})$ over the elements of A ; similar conventions are employed for $\exists! x \in A \psi(x)$ (“*there exists exactly one* $x \in A$ such that $\psi(x)$ ”). When we know that $F(\vec{x}, y)$ defines a provably total function in T , we shall furthermore often use expressions like $f(\vec{x}) \in A$ to abbreviate a formula of the form $\bigvee_{y \in A} f(\vec{x}) = \bar{y}$, where f is an abbreviation in the metalanguage for the function defined by $F(\vec{x}, y)$.

LEMMA 3.2 ([8]). *There is a sequence of functions $h_0, h_1, \dots, h_m: \omega \rightarrow W_0$ respectively defined by formulas H_0, H_1, \dots, H_m in T , i.e.,*

- (1) $T \vdash \forall x \exists! w \in W_0 H_k(x, w)$,
- (2) $T \vdash \forall x, y (h_k(x) = y \leftrightarrow H_k(x, y))$,

such that the functions h_0, h_1, \dots, h_k , provably in T , satisfy the following properties:

$$h_0(0) = 0 \text{ and } h_k(0) = \ell^{H_{k-1}}, \text{ for } k = 1, \dots, m;$$

$$h_k(n+1) = \begin{cases} z, & \text{if } h_k(n)R_k z \text{ and } \text{Prf}_k(\ulcorner \neg S_z \urcorner, n), \\ h_k(n), & \text{otherwise.} \end{cases}$$

Moreover, for $k = 0, 1, \dots, m$, H_k is in $\Sigma_{|\pi_k|+1}$ and the following properties hold:

- (3) $T \vdash \forall i, j \forall z \in W_0 (i < j \wedge h_k(i) = z \rightarrow h_k(j) \in R_k(z) \cup \{z\})$,
- (4) $T \vdash \exists! x \in W_0 \ell^{H_k} = x$,
- (5) $T \vdash \forall z \in W_0 (\exists n h_k(n) = z \rightarrow \ell^{H_m} \in R_k^*(z) \cup \{z\})$.

In the following, we fix a sequence of functions h_0, h_1, \dots, h_m respectively defined by formulas H_0, H_1, \dots, H_m with the properties as stated in Lemma 3.2. We let S_x be an abbreviation for $\ell^{H_m} = \bar{x}$.

Notice the self-referential character of the definition of the h_k due to their reference to the sentences S_z . Item (3) of Lemma 3.2 above states that h_k is weakly increasing along R_k (i.e., $h_k(n+1)$ either has the value $h_k(n)$ or increases with respect to R_k), item (4) states that h_k reaches a unique limit, while item (5) means that, knowing that $h_k(n) = z$ for some value n , we can conclude that the last function h_m reaches its limit either at z or at some $x \in R_k^*(z)$ (this becomes intuitively clear if we consider the fact that h_{k+1} starts where h_k reaches its limit).

We give an intuitive explanation for the concepts introduced so far using a metaphor.⁸ Think of the domains of the functions h_k as being points in time, expressed via natural numbers. Moreover, imagine that we have m travelers who travel around in our model such that the fact $h_k(n) = x$ expresses that traveler k is at world $x \in W_0$ at time instant n . The limit ℓ^{H_k} of h_k can be seen as a world where the k -th traveler stays indefinitely. Using this metaphor, h_k satisfies the following properties (justified by Lemma 3.2):

- Traveler 0 starts at world 0. Moreover, traveler $k+1$ starts where the k -th traveler stays indefinitely.
- Traveler k can only travel at time instant $n+1$ to the world z such that $h_k(n)R_k z$, if n codes a proof that the *last* traveler (i.e., traveler m) does not stay at world z indefinitely. Otherwise, she must stay at world $h_k(n)$.

Now if we consider the implicit constraints that our model under consideration is finite and that the travelers cannot travel backwards in our model, we would expect that honest travelers all stay at home (i.e., at world 0)—formally, we in particular expect that S_0 is true in the standard model.

Having the notions from Lemma 3.2 in place, the use of the relation \tilde{R}_k can be explained as follows. Assume (in T) that $\ell^{H_m} = x$, where $x \neq 0$. That is, the last traveler m stays in world x indefinitely and x is different from 0. What can we say about the set of worlds at which the last

⁸The metaphor using travelers that follows is inspired by a similar one for the Gödel-Löb logic GL; see, e.g., [2].

traveler can $[k]_\pi$ -provably stay indefinitely? Since $\ell^{H^m} \neq 0$, one traveler has traveled at least one time from one world to another. We certainly know that all the travelers $k, k+1, \dots, m$ respectively travel along the relations R_k, R_{k+1}, \dots, R_m . Furthermore, the $(n+1)$ -st traveler starts where the n -th stays indefinitely. Assuming that the k -th traveler stays at z indefinitely (i.e., $\ell^{H^k} = z$), we must thus have that $\ell^{H^m} \in R_{k+1}^*(z) \cup \{z\}$, i.e., the last traveler stays at some element from $R_{k+1}^*(z) \cup \{z\}$. This then entails $R_k^*(z) \subseteq \tilde{R}_k(x)$. Moreover, $\ell^{H^k} = z$ implies $h(n) = z$ for some $n \geq 0$, whence $\ell^{H^m} \in R_k^*(z) \cup \{z\}$ (item (5) of Lemma 3.2). But we know that $\ell^{H^m} \neq z$, since otherwise h_k could not attain the value z (which is not equal to 0 since $x \neq 0$) at all. Therefore, $\ell^{H^m} \in R_k^*(z) \subseteq \tilde{R}_k(x)$. Essentially, a formalization of this argument constitutes a proof of item (S3) of Lemma 3.3 below. Thus, $\tilde{R}_k(x)$ consists of all the worlds which could be ($[k]_\pi$ -provably) possible candidates for the last traveler to stay indefinitely, provided we assume $\ell^{H^m} = x$ (i.e., S_x) for $x \neq 0$ in T .

The following lemma makes the notions discussed informally using our metaphor more precise.

LEMMA 3.3 ([8]). *The sentences S_x satisfy the following properties:*

- (S1) $T \vdash \bigvee_{x \in W_0} S_x$ and $T \vdash \neg(S_x \wedge S_y)$, for $x \neq y$;
- (S2) $T \vdash S_x \rightarrow \langle k \rangle_\pi S_y$, for all y such that $xR_k y$;
- (S3) $T \vdash S_x \rightarrow [k]_\pi (\bigvee_{y \in \tilde{R}_k(x)} S_y)$, for all $x \neq 0$; and
- (S4) $\mathbb{N} \models S_0$.

Proof. For the sake of clarity, let us repeat some parts of the proof from [8]. Item (S1) states that h_m reaches its limit at one and only one world in the model \mathfrak{A}_0 . Notice that (S1) follows immediately by item (1) of Lemma 3.2.

Item (S2) expresses the fact that, assuming S_x in T , for all y such that $xR_k y$, one can consistently assume (regarding the k -th provability predicate of π) that h_m converges to y . One can prove this item by formalizing the following argument in T :

Assume S_x and $[k]_\pi \neg S_y$ for some y such that $xR_k y$. Then either $\ell^{H^k} = x$ or $\ell^{H^k} = z$, for some $z \in R_{k+1}^*(x)$. In both cases, since \mathfrak{A}_0 is a \mathbf{J}^* -model, we have that $R_k(x) = R_k(\ell^{H^k})$. Pick a number n_0 such that $\forall n \geq n_0$ $h_k(n) = \ell^{H^k}$. Since $[k]_\pi \neg S_y$, there is an $n_1 \geq n_0$ such that $\text{Prf}_k^\pi(\ulcorner \neg S_y \urcorner, n_1)$. But $\ell^{H^k} R_k y$ and $h_k(n_1) = \ell^{H^k}$, so by definition of h_k we obtain $h_k(n_1 + 1) = y \neq \ell^{H^k}$, a contradiction. Thus, $\neg[k]_\pi \neg S_y$, which is equivalent to $\langle k \rangle_\pi S_y$.

For item (S3), we formalize the following argument in T :

Assume S_x , where $x \neq 0$, and assume $\ell^{H^k} = z$. By the construction of the functions h_k , we know that $x \in R_{k+1}(z) \cup \{z\}$. By the definition of \tilde{R}_k , this implies $R_k^*(z) \subseteq \tilde{R}_k(x)$. Since we can define this property by a Δ_0 -formula, we know $[k]_\pi (R_k^*(z) \subseteq \tilde{R}_k(x))$. Hence, $[k]_\pi (\bigvee_{y \in R_k^*(z)} S_y)$ implies $[k]_\pi (\bigvee_{y \in \tilde{R}_k(x)} S_y)$. Moreover, since $\ell^{H^k} = z$, we must have $\exists n$ $h(n) = z$. The latter statement is definable by a $\Sigma_{|\pi_k|+1}$ -formula, whence $[k]_\pi (\exists n$ $h(n) = z)$. By item (5) of Lemma 3.2, we know that, for any $u \in W_0$,

$$\exists n$$
 $h_k(n) = u \implies \ell^{H^m} \in R_k^*(z) \cup \{z\},$

whence

$$[k]_\pi (\exists n$$
 $h_k(n) = u) \implies [k]_\pi (\ell^{H^m} \in R_k^*(z) \cup \{z\}).$

For $u = z$, we thus obtain $[k]_\pi(\ell^{H^m} \in R_k^*(z) \cup \{z\})$. Now we observe that $x \neq 0$ implies $z \neq 0$ and, by construction of h_k , we infer $[k]_\pi \neg S_z$. Therefore, $[k]_\pi(\ell^{H^m} \in R_k^*(z))$, i.e., $[k]_\pi(\bigvee_{y \in R_k^*(z)} S_y)$. We observed above that this implies $[k]_\pi(\bigvee_{y \in \tilde{R}_k(x)} S_y)$, and thus the proof is finished.

Item (S4) can be proved by showing, using an external induction on k , that $\mathbb{N} \models \ell^{H^k} = 0$ for all $k \geq 0$. There, one uses the soundness of $[k]_\pi$: if $\ell^{H^k} = z \neq 0$, then $[k]_\pi \neg S_z$, since by induction hypothesis we have $h_k(0) = \ell^{H^{k-1}} = 0$. Since $[k]_\pi$ is sound, it follows that $\ell^{H^k} \neq z$. Hence, $\ell^{H^k} = 0$. \square

Now we define an arithmetical realization f by

$$f: p \mapsto \bigvee_{\mathfrak{A}_0, x \models p} S_x.$$

In [8], the following ‘‘commutation lemma’’ is shown—recall that we fixed φ in the beginning of our proof:

LEMMA 3.4. *For every subformula θ of φ and each $x \in W_0 \setminus \{0\}$:*

- $\mathfrak{A}_0, x \models \theta$ implies $T \vdash S_x \rightarrow f_\pi(\theta)$;
- $\mathfrak{A}_0, x \not\models \theta$ implies $T \vdash S_x \rightarrow \neg f_\pi(\theta)$.

Using this lemma, we can conclude $T \not\vdash f_\pi(\varphi)$ as follows. If we had $T \vdash f_\pi(\varphi)$, then, since $\mathfrak{A}_0, r \not\models \varphi$, we obtain $T \vdash \neg S_1$. Thus, $T \vdash [0]_\pi S_1$ and since $0R_0r$, using (S2), we obtain $T \vdash \neg S_0$. By the soundness of T , this implies $\mathbb{N} \models \neg S_0$, contradicting (S4). Therefore, $T \not\vdash f_\pi(\varphi)$ as required.

3.3. The Realization f is Typed for π . We now prove, using the assumption that \mathfrak{A}_0 is strongly persistent, that f is actually typed for π which will then conclude the arithmetical completeness proof for GLP*. When reasoning in T , we shall often treat ℓ^{H^i} ($i = 0, 1, \dots, m$) as a world and write $\mathfrak{A}_0, \ell^{H^i} \models p$ as an abbreviation for the fact that, provably in T , $\ell^{H^i} = \bar{u}$ holds for some u such that $\mathfrak{A}_0, u \models p$.

LEMMA 3.5. *For all $k < m$, provably in T , if $k < n \leq m$, then either $\ell^{H^k} = \ell^{H^n}$ or $\ell^{H^k} R_j \ell^{H^n}$, for some $j \in (k, n]$.*

Proof (Sketch). We can easily conclude from Lemma 3.2 that, for $k \geq 0$, either $\ell^{H^k} = \ell^{H^{k+1}}$, or $\ell^{H^k} R_{k+1} \ell^{H^{k+1}}$. Using this property, the claim now follows easily by an external induction on k . \square

LEMMA 3.6. *For any variable p of sort $k \leq m$, provably in T ,*

$$f(p) \iff \forall w \in W_0 \setminus \llbracket p \rrbracket \forall x \neg H_k(x, w).$$

Proof. For the direction from left to right, we reason in T as follows. Assume $f(p)$ and, towards a contradiction, suppose that $\exists x h_k(x) = w$ for some $w \in W_0$ such that $\mathfrak{A}_0, w \not\models p$. By item (5) of Lemma 3.2, we know that, provably in T , $\exists x h_k(x) = w$ implies

$$S_w \vee \bigvee_{u \in R_k^*(w)} S_u.$$

Since \mathfrak{A}_0 is strongly persistent and $\mathfrak{A}_0, w \not\models p$, we know that $\mathfrak{A}_0, u \not\models p$ for all $u \in R_k^*(w)$. This contradicts $f(p)$ by item (S1) of Lemma 3.3.

For the other direction, we reason in T as follows. Assume the right-hand side of the equivalence. We certainly know that $\ell^{H_k} \neq u$ for all $u \in W_0$ such that $\mathfrak{A}_0, u \not\models p$. Now, if $\ell^{H_k} = \ell^{H_m}$, then, by (S1), S_x holds for some $x \in W_0$ such that $\mathfrak{A}_0, x \models p$ and we are thus finished. So suppose that $\ell^{H_k} \neq \ell^{H_m}$. We know that $\mathfrak{A}_0, \ell^{H_k} \models p$, since $\forall x h_k(x) \neq w$ for all $w \in W_0$ such that $\mathfrak{A}_0, w \not\models p$. Assume now that $\mathfrak{A}_0, \ell^{H_m} \not\models p$. By Lemma 3.5 there must be a $j \in (k, m]$ such that $\ell^{H_k} R_j \ell^{H_m}$. By strong persistence, for any $x, y \in W_0$ such that $x R_j y$, it holds that $\mathfrak{A}_0, y \not\models p$ implies that $\mathfrak{A}_0, x \not\models p$. Thus, $\mathfrak{A}_0, \ell^{H_m} \not\models p$ is impossible and therefore $\mathfrak{A}_0, \ell^{H_m} \models p$ by item (S1) of Lemma 3.3. \square

LEMMA 3.7. *For every variable p of sort $k < \omega$, $f(p)$ is $\Pi_{|\pi_k|+1}$ in T .*

Proof. Recall that, according to Lemma 3.2, $H_k(x, y)$ is $\Sigma_{|\pi_k|+1}$ in T . We remind the reader that $f(p)$ is the disjunction of all S_x such that $\mathfrak{A}_0, x \models p$. Observe that S_x is by construction $\Sigma_{|\pi_m|+2}$ in T and hence so is $f(p)$. Moreover, recall that m is the only number such that $R_m \neq \emptyset$ and $R_k = \emptyset$ for all $k > m$.

Suppose first that $k > m$. Then $|\pi_m|+2 \leq |\pi_k|+1$ and so $f(p)$ is also $\Sigma_{|\pi_k|+1}$ in T . Moreover, using item (1) of Lemma 3.2, we observe that, provably in T ,

$$f(p) \iff \bigvee_{\mathfrak{A}_0, x \models p} S_x \iff \bigwedge_{\mathfrak{A}_0, x \not\models p} \neg S_x.$$

The sentences $\neg S_x$ are $\Pi_{|\pi_k|+1}$ in T and thus $f(p)$ is $\Pi_{|\pi_k|+1}$ in T as well.

Suppose now that $k \leq m$. Recall that $H_k(x, y)$ is $\Sigma_{|\pi_k|+1}$ in T and therefore $\neg H_k(x, y)$ is $\Pi_{|\pi_k|+1}$ in T . By Lemma 3.6 we know that, provably in T ,

$$f(p) \iff \forall w \in W_0 \setminus \llbracket p \rrbracket \forall x \neg H_k(x, w).$$

Since $\neg H_k(x, y)$ is $\Pi_{|\pi_k|+1}$ in T and since $\Pi_{|\pi_k|+1}$ -formulas are closed under universal quantification, $f(p)$ is $\Pi_{|\pi_k|+1}$ in T . \square

Now Lemma 3.7 implies that the realization f is actually typed for π . This concludes the proof of the arithmetical completeness theorem (Theorem 3.1) for GLP^* .

4. SOME FURTHER RESULTS ON GLP^*

In this section, we briefly establish some further results on GLP^* that mostly rely on results previously obtained for GLP .

4.1. Truth Provability Logic. Let GLPS denote the extension of the set of theorems of GLP by the schema $\varphi \rightarrow \langle n \rangle \varphi$, for all formulas φ and all $n < \omega$, and with modus ponens as a sole rule of inference. It turns out that the theorems of GLPS are exactly those modal formulas that are true in the standard model of arithmetic under every arithmetical realisation (see [8]). The methods above can be easily extended to characterize a many-sorted analogue of GLPS , which we denote by GLPS^* . More precisely, let GLPS^* denote the logic consisting of the set of theorems of GLP^* extended by the schema $\varphi \rightarrow \langle n \rangle \varphi$ ($n \geq 0$) and with modus ponens as its sole rule of inference.

Let $\langle n_1 \rangle \varphi_1, \dots, \langle n_s \rangle \varphi_s$ be an enumeration of all subformulas from φ of the form $\langle k \rangle \psi$. Furthermore, let

$$U(\varphi) := \bigwedge_{i=1}^s (\varphi_i \rightarrow \langle n_i \rangle \varphi_i).$$

Then the following is a straightforward adaption of a similar result from [8] for GLPS :

THEOREM 4.1. *Let T be a sound axiomatizable extension of PA and π a strong sequence of provability predicates over T of which every provability predicate is sound. Then, for all many-sorted formulas φ , the following statements are equivalent:*

- (1) $\text{GLPS}^* \vdash \varphi$;
- (2) $\text{GLP}^* \vdash U(\varphi) \rightarrow \varphi$; and
- (3) $\mathbb{N} \models f_\pi(\varphi)$, for all realizations f that are typed for π .

Proof (Sketch). The implications from (1) to (3) and from (2) to (1) are easy to establish—observe that $\text{GLPS}^* \vdash U(\varphi)$. We sketch the direction from (3) to (2) again by citing results from [8]. Suppose $\text{GLP}^* \not\vdash U(\varphi) \rightarrow \varphi$. As in the arithmetical completeness proof for GLP^* , we can construct a finite rooted J^* -model \mathfrak{A}_0 with root 0 such that $\mathfrak{A}_0, 0 \models M^+(\varphi) \wedge U(\varphi)$ and $\mathfrak{A}_0, 0 \not\models \varphi$, i.e., \mathfrak{A}_0 is constructed as in the arithmetical completeness proof for GLP^* , with the only difference that $U(\varphi)$ is satisfied at world 0. We can construct the functions h_k based on \mathfrak{A}_0 and the sentences S_x in a similar spirit as in the arithmetical completeness proof of GLP^* . Lemma 3.3 then holds without any changes.

We can then strengthen Lemma 3.4 and prove that, for every subformula θ of φ ,

- $\mathfrak{A}_0, 0 \models \theta$ implies $T \vdash S_0 \rightarrow f_\pi(\theta)$;
- $\mathfrak{A}_0, 0 \not\models \theta$ implies $T \vdash S_0 \rightarrow \neg f_\pi(\theta)$.

(For a proof of this result, we refer the reader to [8].) The proof of the fact that the realization f is actually typed also holds without any changes. Now $\mathbb{N} \models S_0$ (item (S4) of Lemma 3.3) gives us $\mathbb{N} \models f_\pi(\varphi)$. \square

4.2. Reducing GLP^* to GLP . For the results contained in the remainder of this section, we will rely on a reduction of GLP^* to GLP which we shall present here.

We first borrow some notions from [13] used to reduce GLP to J . Let φ be a polymodal formula and let $\langle m_1 \rangle \varphi_1, \langle m_2 \rangle \varphi_2, \dots, \langle m_s \rangle \varphi_s$ be an enumeration of all subformulas of φ of the form $\langle k \rangle \psi$ such that $i < j$ implies $m_i \leq m_j$. Define

$$N(\varphi) := \bigwedge_{\substack{1 \leq i \leq s \\ i < j \leq s}} (\langle m_j \rangle \varphi_j \rightarrow \langle m_i \rangle \varphi_i).$$

Furthermore, let

$$N^+(\varphi) := N(\varphi) \wedge \bigwedge_{1 \leq i \leq s} [m_i] \varphi.$$

Notice that, if ψ is a subformula of φ , then $N^+(\varphi)$ implies $N^+(\psi)$ (in any of our logics under consideration); likewise, in this case, $N(\varphi)$ also implies $N(\psi)$.

Remark. The formula $N^+(\varphi)$ is reminiscent of the formula $M^+(\varphi)$ presented during the arithmetical completeness proof for GLP^* . However, notice that $N^+(\varphi)$ contains only modalities that already occur in φ which may not be the case for $M^+(\varphi)$. This property will be used below.

LEMMA 4.2 ([13]). *For any ψ , $\text{GLP} \vdash \psi \iff \text{J} \vdash N^+(\psi) \rightarrow \psi$.*

LEMMA 4.3. *The following are equivalent for all φ :*

- (1) $\text{GLP}^* \vdash \varphi$;

- (2) $\text{GLP} \vdash P^+(\varphi) \rightarrow \varphi$;
- (3) $\text{J} \vdash N^+(P^+(\varphi) \rightarrow \varphi) \rightarrow (P^+(\varphi) \rightarrow \varphi)$;
- (4) $\text{J}^* \vdash N^+(\varphi) \rightarrow \varphi$.

Proof. It is clear that (2) implies (1) since $\text{GLP}^* \vdash P^+(\varphi)$ and GLP^* extends GLP . Likewise, it is clear that (4) implies (1) since $\text{GLP}^* \vdash N^+(\varphi)$. The equivalence between items (2) and (3) is the content of Lemma 4.2.

We are first going to show that (1) implies (4). Assume $\text{J}^* \not\vdash N^+(\varphi) \rightarrow \varphi$. By Corollary 2.7, we know there is a finite J^* -model $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, [\cdot])$ with root r such that $\mathfrak{A}, r \not\models N^+(\varphi) \rightarrow \varphi$. Moreover, $R_n = \emptyset$ for all n such that $\langle n \rangle$ does not occur in $N^+(\varphi) \rightarrow \varphi$. Hence, $R_n \neq \emptyset$ implies that $\langle n \rangle$ occurs in φ , since $N^+(\varphi)$ and φ contain exactly the same modalities. Our aim is to show that $\mathfrak{A}, r \models M^+(\varphi)$. Consider an enumeration $\langle m_1 \rangle \varphi_1, \langle m_2 \rangle \varphi_2, \dots, \langle m_s \rangle \varphi_s$ of all subformulas of φ of the form $\langle k \rangle \psi$ and let $n := \max_{i \leq s} m_i$. Recall that

$$M(\varphi) := \bigwedge_{\substack{1 \leq i \leq s \\ m_i < j \leq n}} (\langle j \rangle \varphi_i \rightarrow \langle m_i \rangle \varphi_i),$$

and, furthermore, $M^+(\varphi) := M(\varphi) \wedge \bigwedge_{i \leq n} [i]M(\varphi)$. Let $i \in \{1, \dots, s\}$ and consider any j such that $m_i < j \leq n$. Now $\mathfrak{A}, r \models \langle j \rangle \varphi_i$ only if $j = m_k$ for some $k = 1, \dots, s$. In this case, $\mathfrak{A}, r \models \langle m_i \rangle \varphi$ since $\mathfrak{A}, r \models N^+(\varphi)$. Otherwise, if $j \neq m_k$ for all $k = 1, \dots, s$, then trivially $\mathfrak{A}, r \models \langle j \rangle \varphi_i \rightarrow \langle m_i \rangle \varphi_i$, since $\mathfrak{A}, r \not\models \langle j \rangle \varphi_i$ due to the fact that $R_j = \emptyset$. Let $n := \max_{i \leq s} m_i$ and consider any $i \leq n$. A similar line of reasoning as before shows that $\mathfrak{A}, r \models [i]M(\varphi)$. Hence, $\mathfrak{A}, r \models M^+(\varphi)$ and so $\text{J}^* \not\vdash M^+(\varphi) \rightarrow \varphi$, whence $\text{GLP}^* \not\vdash \varphi$ follows by Theorem 3.1.

To complete our proof, it remains to be shown that (4) implies (3). Assume $\text{J} \not\vdash N^+(P^+(\varphi) \rightarrow \varphi) \rightarrow (P^+(\varphi) \rightarrow \varphi)$. By Lemma 2.6, there is a J -model $\mathfrak{A} = (W, \{R_n\}_{n \geq 0}, [\cdot])$ having root r such that $\mathfrak{A}, r \models N^+(P^+(\varphi) \rightarrow \varphi)$, $\mathfrak{A}, r \models P^+(\varphi)$, and $\mathfrak{A}, r \not\models \varphi$. Moreover, \mathfrak{A} is such that $R_n \neq \emptyset$ implies that $\langle n \rangle$ occurs as a modality in $N^+(P^+(\varphi) \rightarrow \varphi) \rightarrow (P^+(\varphi) \rightarrow \varphi)$, and hence in φ . Since $\mathfrak{A}, r \models P^+(\varphi)$, we conclude that $\mathfrak{A} \models P(\varphi)$, whence by Lemma 2.5 it follows that \mathfrak{A} is strongly persistent and thus a J^* -model. Now $\mathfrak{A}, r \models N^+(P^+(\varphi) \rightarrow \varphi)$ entails that $\mathfrak{A}, r \models N^+(\varphi)$ (since φ is a subformula of $P^+(\varphi) \rightarrow \varphi$) and so $\mathfrak{A}, r \not\models N^+(\varphi) \rightarrow \varphi$. Thus, $\text{J}^* \not\vdash N^+(\varphi) \rightarrow \varphi$ by the soundness of J^* for the class of J^* -models. \square

Lemma 4.3 in particular establishes that $\text{GLP}^* \vdash \varphi$ iff $\text{GLP} \vdash P^+(\varphi) \rightarrow \varphi$. In the following, we shall use this reduction of GLP^* to GLP in order to transfer some results known for GLP to GLP^* .

4.3. Craig Interpolation. We say that a logic \mathcal{L} enjoys the *Craig interpolation property* if, whenever $\mathcal{L} \vdash \varphi \rightarrow \psi$, then there is a formula η such that $\mathcal{L} \vdash \varphi \rightarrow \eta$ and $\mathcal{L} \vdash \eta \rightarrow \psi$, and the following conditions hold:

- (i) η contains only variables which are present in both φ and ψ , and
- (ii) η has only modalities that appear in φ or ψ .

The formula η is called *interpolant* for $\varphi \rightarrow \psi$.

THEOREM 4.4 ([6, 21]). *GLP enjoys the Craig interpolation property.*

Remark. Notice we state a rather weak form of Craig interpolation, since we do not demand that the modalities of η occur in both φ and ψ . Indeed, for GLP , one cannot demand that property, as the example $\langle 1 \rangle p \rightarrow \langle 0 \rangle p$ shows (cf. [6]). However, as stated in the theorem above, we can

demand that each modality from η is contained in φ or ψ . We shall use this property below, when we discuss variants of GLP^* that restrict the use of sorts and modalities.

COROLLARY 4.5. *GLP^* enjoys the Craig interpolation property.*

Proof. Suppose $\text{GLP}^* \vdash \varphi \rightarrow \psi$. Let Θ be the set of all modalities from $\varphi \rightarrow \psi$. We have

$$\text{GLP} \vdash P_{\Theta}^+(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi).$$

Note that $P_{\Theta}^+(\varphi \rightarrow \psi)$ is equivalent in GLP to $P_{\Theta}^+(\varphi) \wedge P_{\Theta}^+(\psi)$. Hence,

$$\text{GLP} \vdash P_{\Theta}^+(\varphi) \wedge P_{\Theta}^+(\psi) \rightarrow (\varphi \rightarrow \psi),$$

whence by propositional logic

$$\text{GLP} \vdash P_{\Theta}^+(\varphi) \wedge \varphi \rightarrow (P_{\Theta}^+(\psi) \rightarrow \psi).$$

Since GLP enjoys the Craig interpolation property, there is an interpolant η containing only variables which occur in $P_{\Theta}^+(\varphi) \wedge \varphi$ and $P_{\Theta}^+(\psi) \rightarrow \psi$ such that

$$\text{GLP} \vdash P_{\Theta}^+(\varphi) \wedge \varphi \rightarrow \eta \quad \text{and} \quad \text{GLP} \vdash \eta \rightarrow (P_{\Theta}^+(\psi) \rightarrow \psi).$$

But $\text{GLP}^* \vdash P_{\Theta}^+(\varphi)$ and $\text{GLP}^* \vdash P_{\Theta}^+(\psi)$. Therefore, $\text{GLP}^* \vdash \varphi \rightarrow \eta$ and $\text{GLP}^* \vdash \eta \rightarrow \psi$. Note that η only contains variables which occur in φ and ψ , since $P_{\Theta}^+(\chi)$ contains exactly the variables from χ , for any formula χ . \square

4.4. Complexity. We can also exploit the reduction of GLP^* to GLP to establish a PSPACE-completeness result for GLP^* .

THEOREM 4.6 ([23]). *Deciding whether $\text{GLP} \vdash \varphi$ is complete for PSPACE.*

COROLLARY 4.7. *Deciding whether $\text{GLP}^* \vdash \varphi$ is complete for PSPACE.*

Proof. For membership, in order to check whether $\text{GLP}^* \vdash \varphi$, it suffices to check whether $\text{GLP} \vdash P^+(\varphi) \rightarrow \varphi$. Note that $P^+(\varphi)$ is polynomial in the size of φ . Indeed, let m be the number of different modalities occurring in φ . Then the formula $P^+(\varphi)$ contains for each propositional variable occurring in φ at most m conjuncts of the form $\langle j \rangle p \rightarrow p$ and at most m conjuncts of the form $\langle j \rangle \neg p \rightarrow \neg p$. Both m and the number of variables in φ are clearly bounded by the size of φ . Hence, the size of $P^+(\varphi)$ is at most quadratic in the size of φ .

For hardness, we reduce the task of checking whether $\text{GLP} \vdash \varphi$ to our problem as follows. Let us consider φ as a many-sorted formula whose propositional variables all have sort ω . Now we observe that $\text{GLP} \vdash P^+(\varphi) \rightarrow \varphi$ iff $\text{GLP}^* \vdash \varphi$ and, since φ contains only variables of sort ω , we see that $P^+(\varphi)$ is actually \top (the empty conjunction), i.e., $\text{GLP} \vdash \varphi$ iff $\text{GLP}^* \vdash \varphi$. \square

4.5. Omitting the Sort ω . An interesting question is to consider a variant of GLP^* that is formulated over a language where propositional variables only have finite sorts, that is, only sorts $n \in \omega$. We briefly treat this case here.

We actually work in a slightly more general setting here: let $\alpha \in \omega \cup \{\omega\}$ and let GLP_{α}^* denote the logic that arises from GLP^* when we only allow the use of variables of sort less than α and modalities $\langle \beta \rangle$ with $\beta < \alpha$. Notice that formulas in the language of GLP_{α}^* all have finite sort. Moreover, note that GLP^* extends GLP_{α}^* in the sense that if $\text{GLP}_{\alpha}^* \vdash \varphi$, then also $\text{GLP}^* \vdash \varphi$. Furthermore, if $\beta \leq \alpha$, then GLP_{α}^* extends GLP_{β}^* . Likewise, we can also define a variant J_{α}^* of J^* that enforces similar restrictions on the language as GLP_{α}^* does and it can be easily checked that all the results obtained for J^* carry over to the case of J_{α}^* .

The notion of an arithmetical realization over a strong sequence of provability predicates immediately captures the case of formulas that contain only variables of finite sort. The arithmetical completeness theorem for GLP_α^* then reads:

THEOREM 4.8. *Let T be an axiomatizable extension of PA and π a strong sequence of provability predicates over T whose predicates are all sound. Let φ be a formula in the language of GLP_α^* . The following statements are equivalent:*

- (1) $\text{GLP}_\alpha^* \vdash \varphi$;
- (2) $T \vdash f_\pi(\varphi)$, for all arithmetical realizations f that are typed for π .

Proof (Idea). The direction from (1) to (2) is immediate by the arithmetical completeness theorem for GLP^* (Theorem 3.1) and the fact that GLP^* extends GLP_α^* . For the other direction, the same construction as for GLP^* can be carried out. Essentially, one can just ignore the case of variables of sort $\geq \alpha$ in the construction presented for GLP^* . \square

An easy consequence of this fact is that GLP^* a conservative extension of GLP_α^* :

COROLLARY 4.9. *GLP^* conservatively extends GLP_α^* , i.e., if φ is in the language of GLP_α^* and $\text{GLP}^* \vdash \varphi$, then also $\text{GLP}_\alpha^* \vdash \varphi$. Moreover, GLP_β^* conservatively extends GLP_α^* , for $\beta > \alpha$.*

Proof. If φ is in the language of GLP_α^* , it contains no variables of sort $\geq \alpha$, hence, if $\text{GLP}^* \vdash \varphi$, then $\text{PA} \vdash f_\pi(\varphi)$ for all realizations f (where π is a strong sequence of provability predicates over PA) that are typed for π . The result now follows immediately from Theorem 4.8. \square

Having the above result in place, Craig interpolation for GLP_α^* follows immediately:

COROLLARY 4.10. *GLP_α^* enjoys the Craig interpolation property.*

Proof. If $\text{GLP}_\alpha^* \vdash \varphi \rightarrow \psi$, where φ and ψ contain only variables of finite sort, then $\text{GLP}^* \vdash \varphi \rightarrow \psi$ and hence, by Corollary 4.5, there is an interpolant η such that $\text{GLP}^* \vdash \varphi \rightarrow \eta$ and $\text{GLP}^* \vdash \eta \rightarrow \psi$. The interpolant η contains only variables that jointly appear in φ and ψ . Moreover, each modality from the η is contained in φ or ψ . Hence, η is in the language of GLP_α^* . Since GLP^* conservatively extends GLP_α^* (Corollary 4.9), we obtain $\text{GLP}_\alpha^* \vdash \varphi \rightarrow \eta$ and $\text{GLP}_\alpha^* \vdash \eta \rightarrow \psi$, as desired. \square

For the PSPACE-hardness proof of GLP^* , the use of variables of sort ω become vital, and the proof thus does not immediately carry over to the case of GLP_α^* . We thus aim at a different proof in the following, essentially exploiting a reduction of the intuitionistic propositional calculus (henceforth denoted IPC) to the standard Gödel-Löb logic GL .⁹ For details on IPC and its translation to GL we refer the interested reader to [14].

The translation \cdot^* of formulas from IPC to formulas of GL is defined as follows:

- $\perp^* := \perp$;
- $p^* := \Box p \wedge p$, where p is a propositional variable;
- $(\varphi \rightarrow \psi)^* := \Box(\varphi^* \rightarrow \psi^*) \wedge (\varphi^* \rightarrow \psi^*)$;
- $(\varphi \wedge \psi)^* := \varphi^* \wedge \psi^*$;
- $(\varphi \vee \psi)^* := \varphi^* \vee \psi^*$.

⁹ GL can be axiomatized by axiom schemas (i) to (iii) of GLP (with $\langle n \rangle$ replaced by \diamond) and is closed under modus ponens and $\varphi \rightarrow \psi / \diamond \varphi \rightarrow \diamond \psi$; see, e.g., [15] for an extensive treatment of GL .

Let $\mathfrak{A} = (W, R, \llbracket \cdot \rrbracket)$ be a Kripke model. We say that \mathfrak{A} is *reversely persistent*, if for any variable p and all $x, y \in W$, it holds that $\mathfrak{A}, x \models p$ and xRy imply $\mathfrak{A}, y \models p$.¹⁰ We say that \mathfrak{A} is an *intuitionistic* Kripke model, if it is reversely persistent and R is reflexive and transitive. The *irreflexive version* of \mathfrak{A} is the model $\mathfrak{A}^* := (W^*, R^*, \llbracket \cdot \rrbracket)$ with $W^* := W$, $xR^*y \iff_{df} xRy$ and $x \neq y$, and $\mathfrak{A}^*, x \models p \iff_{df} \mathfrak{A}, x \models p$, for all variables p .

LEMMA 4.11 ([14]). *IPC $\vdash \varphi$ iff GL $\vdash \varphi^*$. Moreover, for any finite intuitionistic Kripke model \mathfrak{A} , it holds that $\mathfrak{A}, x \models \varphi$ iff $\mathfrak{A}^*, x \models \varphi^*$, where \mathfrak{A}^* is the irreflexive version of \mathfrak{A} .*

We are going to use the rather well-known result that deciding whether $\text{IPC} \vdash \varphi$ is complete for PSPACE:

THEOREM 4.12 ([14]). *Deciding whether $\text{IPC} \vdash \varphi$ is complete for PSPACE.*

Recall that GLP_1^* is the fragment of GLP^* that is formulated over variables of sort 0 and only uses the modality $\langle 0 \rangle$, which we abbreviate by \diamond in the following (likewise we write \square for $[0]$). We aim to show that deciding whether $\text{GLP}_1^* \vdash \varphi$ is already hard for PSPACE.

Towards this end, we are going to take an intermediate step and prove that Visser's Σ_1 -logic (see [15, 25]) GLV is PSPACE-complete. The logic GLV consists of all theorems of GL plus the axioms $p \rightarrow \square p$, where p is a propositional variable. GLV is closely related to GLP_1^* with the difference that GLV is arithmetically complete for the interpretation that assigns Σ_1 -sentences to propositional variables rather than Π_1 -sentences as in the case of GLP_1^* (and \square being interpreted as the standard Gödelian provability predicate); cf. [15, 25]. Notice that any reversely persistent model validates the axioms of the form $p \rightarrow \square p$. Moreover:

THEOREM 4.13 ([25]). *GLV is sound and complete for the class of finite, irreflexive, transitive, and reversely persistent Kripke models.*

LEMMA 4.14. *Deciding whether $\text{GLV} \vdash \varphi$ is hard for PSPACE.*

Proof. Consider a formula φ in the language of IPC and its translation φ^* . We claim that $\text{IPC} \vdash \varphi$ iff $\text{GLV} \vdash \varphi^*$ which will then prove the claim of the lemma by virtue of Theorem 4.12.

Indeed, if $\text{IPC} \vdash \varphi$ then $\text{GL} \vdash \varphi^*$, whence $\text{GLV} \vdash \varphi^*$ since GLV clearly extends GL . On the other hand, if $\text{IPC} \not\vdash \varphi$, then there is a finite intuitionistic Kripke model $\mathfrak{A} = (W, R, \llbracket \cdot \rrbracket)$ such that $\mathfrak{A}, x \not\models \varphi$ for some $x \in W$; see [14]. Let \mathfrak{A}^* be the irreflexive version of \mathfrak{A} . By Lemma 4.11 we also have $\mathfrak{A}^*, x \not\models \varphi^*$. Since \mathfrak{A}^* is an irreflexive and transitive model, it validates all theorems of GL . Moreover, since it is reversely persistent, it also satisfies the axioms $p \rightarrow \square p$. By Theorem 4.13 we thus obtain $\text{GLV} \not\vdash \varphi^*$ as required. \square

LEMMA 4.15. *Deciding whether $\text{GLP}_1^* \vdash \varphi$ is hard for PSPACE.*

Proof. Let \neg denote the translation from formulas in the language of GLP_1^* to formulas of GLV that replaces each propositional variable p by its negation $\neg p$. We claim that $\text{GLP}_1^* \vdash \varphi$ iff $\text{GLV} \vdash \varphi^\neg$. As mentioned above, in [25] it is shown that GLV is arithmetically complete for Σ_1 -realizations, i.e., arithmetical realizations that assign Σ_1 -sentences to propositional variables. By Theorem 4.8, we know that GLP_1^* is arithmetically complete for arithmetical realizations that assign Π_1 -sentences to propositional variables. (We choose a strong sequence of provability predicates π that has the standard Gödelian predicate \square_{PA} as its 0-th predicate.) Now every Π_1 -sentence (Σ_1 -sentence, respectively) is equivalent to the negation of a Σ_1 -sentence (Π_1 -sentence,

¹⁰We call this property reversely persistence here, since “persistence” in this paper refers to propagation of truth-values in the other direction. However, we remark that “reverse persistence” is usually called “persistence” (also in [14]).

respectively). Hence, the result follows immediately by Lemma 4.14 and by applying the respective arithmetical completeness theorems for GLP_1^* and GLV . \square

THEOREM 4.16. *For any $\alpha \in \omega \cup \{\omega\}$, deciding whether $\text{GLP}_\alpha^* \vdash \varphi$ is complete for PSPACE.*

Proof. For membership, we observe that deciding $\text{GLP}_\alpha^* \vdash \varphi$ amounts to deciding $\text{GLP}^* \vdash \varphi$, since GLP^* conservatively extends GLP_α^* by Corollary 4.9. For hardness, notice that checking $\text{GLP}_1^* \vdash \varphi$ (where φ is in the language of GLP_1^*) can be reduced to $\text{GLP}_\alpha^* \vdash \varphi$, again by Corollary 4.9. The problem of deciding whether $\text{GLP}_1^* \vdash \varphi$ is hard for PSPACE by Lemma 4.15, whence the claim follows. \square

5. A POSITIVE VARIANT OF GLP^*

In this section we are going to study a positive variant of GLP^* whose one-sorted counterpart has been studied recently in [10]. It was noticed in [9] that the proof-theoretic analysis of Peano arithmetic in the framework of GLP only relies on certain positive formulas. This fragment, denoted by RC , is much simpler than GLP , yet expressive enough for major proof-theoretic applications of GLP as carried out in [3, 4]. In particular, RC allows one to define a system of ordinal notations up to ε_0 .

Formulas of RC are implications of the form $A \Rightarrow B$ (called *sequents*), where A and B are *positive formulas* constructed using \wedge , \top , diamond modalities $\langle n \rangle$, and propositional variables only. RC and fragments thereof were axiomatized in [16], where it is also proved that, in contrast to GLP , RC is complete for a natural class of finite Kripke frames and that theoremhood for RC is decidable in polynomial time.

Apart from its convenient computational properties, RC also allows for a more general arithmetical interpretation than that of standard GLP . In [10], the second author of this paper considers an arithmetical interpretation of positive formulas where propositional variables are interpreted as (primitive recursive enumerations of) arithmetical theories rather than single sentences. This allows one to interpret the diamond modalities as *reflection schemas*, which are generalizations of consistency assertions and are not necessarily finitely axiomatizable (therefore, in [10], positive fragments of GLP are coined *reflection calculi*). In particular, the full uniform reflection principle is realized in [10] as a modality $\langle \omega \rangle$ that is part of the calculus $\text{RC}\omega$ which essentially extends RC to capture this modality.

Apart from the richer interpretation of the standard diamond modalities, the fact that variables can be interpreted as arithmetical theories allows the introduction of additional modalities that have no counterpart in standard GLP . To wit, RC has recently been extended in [12] in order to capture modalities that express partial conservativity operators.

Considering our introduction of many-sorted GLP , it is natural to ask whether many-sorted logics make sense in the positive setting as well. Therefore, in this section, we introduce a many-sorted variant of the reflection calculus $\text{RC}\omega$ presented in [10] and prove that our calculus is arithmetically complete. In the arithmetical completeness proof, we rely on the construction presented in [10] for the one-sorted setting.

5.1. Basics. We shall consider (many-sorted) *positive formulas* that are formed using propositional variables (again having sorts up to ω as in the setting of GLP^*), conjunction (\wedge), the truth constant \top , and the diamond modalities $\langle \alpha \rangle$, where α is either a natural number or ω . We shall write αA instead of $\langle \alpha \rangle A$ in the following. A *sequent* is an expression of the form $A \Rightarrow B$, where A and B are positive formulas—the sequent $A \Rightarrow B$ stands for the formula $A \rightarrow B$. The notion of *sort* is defined in the positive setting in exactly the same way as it is defined for the more general GLP . As before, the sort of A is denoted by $|A|$.

The following axiom schemas and rules of inference are propositional ones and serve as a basis for the calculi to be presented:

- (i) $A \Rightarrow A$; $A \Rightarrow \top$;
- (ii) $A \wedge B \Rightarrow A$; $A \wedge B \Rightarrow B$;
- (iii) if $A \Rightarrow B$ and $B \Rightarrow C$, then infer $A \Rightarrow C$;
- (iv) if $A \Rightarrow B$ and $A \Rightarrow C$, then infer $A \Rightarrow B \wedge C$.

Apart from these propositional axiom schemas and rules, our calculi will all be closed under the following rule that essentially amounts to the necessitation rule for standard modal logics:

- (v) if $A \Rightarrow B$ then infer $\alpha A \Rightarrow \alpha B$, for any $\alpha \leq \omega$.

The positive logic $\text{RC}^*\omega$ is axiomatized by the schemas and rules (i) to (v) as well as the following axiom schemas:

- (vi) $\alpha A \Rightarrow A$, whenever $|A| \leq \alpha$ (α -persistence);
- (vii) $\alpha A \Rightarrow \beta A$, for $\beta < \alpha$ (monotonicity);
- (viii) $\alpha A \wedge B \Rightarrow \alpha(A \wedge B)$, where $|B| < \alpha$.

Remark. It is worth commenting briefly on the axiomatization of $\text{RC}^*\omega$. The calculus presented here is a many-sorted version of the calculus $\text{RC}\omega$ from [10]. Essentially, in $\text{RC}\omega$, the axiom schema of α -persistence can only be applied to the case $\alpha = \omega$. Moreover, in $\text{RC}\omega$, the axiom schema (viii) is replaced by

- $\alpha A \wedge \beta B \Rightarrow \alpha(A \wedge \beta B)$, for $\beta < \alpha$.

It is immediate that $\text{RC}^*\omega$ extends $\text{RC}\omega$ in the same sense as GLP^* extends GLP .

The axiom schema of α -persistence (schema (vi)) is essentially $\Sigma_{\alpha+1}$ -completeness in the setting of GLP^* . Unlike GLP^* and J^* , $\text{RC}^*\omega$ has another axiom schema that refers to the notion of sorts, namely schema (viii). This is due to the lack of negation in the positive calculi. Indeed, suppose $|B| < n < \omega$. Then $\text{J}^* \vdash \neg B \rightarrow \langle n \rangle \neg B$, whence $\text{J}^* \vdash B \rightarrow [n]B$, and so

$$\begin{aligned} \text{J}^* \vdash \langle n \rangle A \wedge B &\rightarrow [n]B \\ &\rightarrow \langle n \rangle (A \wedge B), \end{aligned}$$

by standard modal reasoning. That is, modulo the modality $\langle \omega \rangle$, axiom schema (viii) is readily derived in J^* and thus in GLP^* .

The notion of a proof in $\text{RC}^*\omega$ is defined in the expected manner and theoremhood is denoted by $\text{RC}^*\omega \vdash A \Rightarrow B$. For a set Γ of positive formulas, we shall write $\text{RC}^*\omega \vdash \Gamma \Rightarrow A$ if there are $B_1, \dots, B_n \in \Gamma$ such that $\text{RC}^*\omega \vdash B_1 \wedge \dots \wedge B_n \Rightarrow A$. We denote by $B(p/A)$ the result of substituting the variable p by the positive formula A in B . Substitutions in this logic must again respect the sorts of variables. We then have:

LEMMA 5.1. *Suppose $\text{RC}^*\omega \vdash A \Rightarrow B$ and $|A|, |B| \leq \alpha$. Then $\text{RC}^*\omega \vdash C(p/A) \Rightarrow C(p/B)$ for any C , where $|p| \geq \alpha$.*

Proof. By an easy induction on the structure of C . □

5.2. Arithmetical Interpretation. The arithmetical interpretation for the positive calculi presented in [10] assigns *primitive recursive numerations* of theories extending PA to propositional variables. We shall adapt this interpretation to the many-sorted setting in the following.

Recall that, in the setting of GLP^* , one admissible interpretation of the modality $\langle n \rangle$ is that of *n-consistency*, i.e., consistency in PA plus the set of all true Π_{n+1} -sentences. Also recall that we denote by $[n]_{\text{PA}}(x)$ and $\langle n \rangle_{\text{PA}}(x)$ arithmetical formulas that respectively express *n-provability* and *n-consistency* in PA; cf. Section 2. The arithmetical interpretation of positive formulas in the language of $\text{RC}^*\omega$ is generalized in two ways:

- (1) Propositional variables are interpreted as arithmetical theories extending PA rather than sentences. These theories are formally presented by a bounded formula $\sigma := \sigma(x)$ that arithmetically defines the set of axioms of the theory at hand.
- (2) Diamond modalities are interpreted as generalized consistency assertions, namely, reflection principles for theories extending PA. The modality $\langle \omega \rangle$ is interpreted as the full uniform reflection principle that has no finite axiomatization.

We are going to formalize these two notions in the following.

A (*primitive recursive*) *numeration* is a bounded formula $\sigma := \sigma(x)$ which defines the Gödel numbers of the axioms of an extension S of PA. We say that σ *numerates* S . Furthermore, we say that σ *numerates a Π_{n+1} -axiomatized extension of PA* if

$$\text{PA} \vdash \forall \alpha (\sigma(\alpha) \rightarrow \text{Ax}_{\text{PA}}(\alpha) \vee \alpha \in \Pi_{n+1}),$$

where the expression “ $\alpha \in \Pi_{n+1}$ ” denotes a natural bounded formula which expresses that α is the Gödel number of a Π_{n+1} -sentence (possibly using n as an additional parameter) and $\text{Ax}_{\text{PA}}(\alpha)$ is a formula defining the Gödel numbers of the axioms of PA.¹¹ Thus, in case σ numerates a Π_{n+1} -axiomatized extension of PA, $\sigma(x)$ provably defines the set of axioms of a theory that is an extension of PA by a set of Π_{n+1} -sentences.

For a numeration σ , we denote by $\Box_{\sigma}(\alpha)$ the formula which defines the standard provability predicate of the theory numerated by σ . For numerations σ and τ , we write $\sigma \Rightarrow_{\text{PA}} \tau$ if

$$\text{PA} \vdash \forall \alpha (\Box_{\tau}(\alpha) \rightarrow \Box_{\sigma}(\alpha)),$$

and we write $\sigma \Rightarrow \tau$ if

$$\mathbb{N} \models \forall \alpha (\Box_{\tau}(\alpha) \rightarrow \Box_{\sigma}(\alpha)).$$

We assume that every numeration, provably in PA, numerates an extension of PA, that is, $\tau \Rightarrow_{\text{PA}} \text{Ax}_{\text{PA}}$, for any τ . As usual, we write $\Box_{\sigma}\varphi$ instead of $\Box_{\sigma}(\ulcorner \varphi \urcorner)$ if no confusion arises. We denote by $\text{Con}(\sigma)$ the sentence $\neg \Box_{\sigma} \perp$.

The formula $\text{Con}_n(\sigma)$ expresses that the theory numerated by σ is *n-consistent*. We often regard $\ulcorner \text{Con}_n(\sigma) \urcorner$ as a definable term which depends on n and use that fact without adhering to any special notation.

Now let σ numerate S . The formula $\text{Con}_n(\sigma)$ is another way of expressing to the so-called *global Π_{n+1} -reflection principle for S* ; see, e.g., [4]. When proving statements about $\text{Con}_n(\sigma)$, we shall in the following often use the following equivalent characterization without any further comment:

LEMMA 5.2 ([4]). *For all $n \in \omega$, $\text{Con}_n(\sigma)$ is provably equivalent in PA to*

$$\forall \alpha \in \Pi_{n+1} (\Box_{\sigma}(\alpha) \rightarrow \text{True}_{\Pi_{n+1}}(\alpha)).$$

¹¹Recall our convention that Greek letters α, β, \dots occurring in arithmetical formulas range over codes of formulas.

Given any arithmetical sentence φ , we denote by $\underline{\varphi}$ the numeration

$$\text{Ax}_{\text{PA}}(\alpha) \vee \alpha = \ulcorner \varphi \urcorner,$$

which numerates the theory $\text{PA} + \varphi$. In this setting, for any numeration σ , $\underline{\text{Con}}_n(\sigma)$ numerates the theory $\text{PA} + \text{Con}_n(\sigma)$. The schema

$$\text{Con}_\omega(\sigma): \quad \{\text{Con}_n(\sigma) \mid n \in \omega\}$$

is well-known to be equivalent over PA to the *full uniform reflection principle for S* , see, e.g., [4]. We shall denote by $\underline{\text{Con}}_\omega(\sigma)$ a numeration which numerates the theory $\text{PA} + \text{Con}_\omega(\sigma)$.

We are now ready to formally specify the intended arithmetical interpretation of RC^*_ω :

DEFINITION 5.3. An *arithmetical realization* is a function f from positive formulas to numerations such that the following conditions are satisfied:

- $f(\top) = \text{Ax}_{\text{PA}}$;
- $f(A \wedge B) = f(A) \vee f(B)$;
- $f(\alpha A) = \underline{\text{Con}}_\alpha(f(A))$, for $\alpha \leq \omega$.

We say that f is *typed*, if the following condition is satisfied:

- for every propositional variable p of sort α , $f(p)$ is a numeration which numerates (1) a $\Pi_{\alpha+1}$ -axiomatized extension of PA in case $\alpha < \omega$ and (2) an arbitrary extension of PA in case $\alpha = \omega$;

LEMMA 5.4. *Let f be a typed arithmetical realization and A a formula such that $|A| < \omega$. Then $f(A)$ numerates a $\Pi_{|A|+1}$ -axiomatized extension of PA .*

Proof. By an easy induction on A . The cases for propositional variables and \top are clear. For the induction step, notice that for $n < \omega$, $\text{Con}_n(\sigma)$ provably belongs to Π_{n+1} , for any numeration σ . Furthermore, provably in PA , if φ belongs to Π_m , then also to Π_n , for $n > m$. Using these facts, the claim easily follows. \square

LEMMA 5.5 ([10]). *Let σ numerate S and φ be a Π_{n+1} -sentence. If $S \vdash \varphi$ then $\text{PA} + \text{Con}_n(\sigma) \vdash \varphi$. Moreover, this statement is formalizable uniformly in n in PA , i.e.,*

$$\text{PA} \vdash \forall n \forall \alpha \in \Pi_{n+1} (\Box_\sigma(\alpha) \rightarrow \Box_{\underline{\text{Con}}_n(\sigma)}(\alpha)).$$

LEMMA 5.6. *Let σ be a numeration and $n < \omega$. Then $\underline{\text{Con}}_n(\sigma) \Rightarrow_{\text{PA}} \sigma$, whenever σ numerates a Π_{n+1} -axiomatized extension of PA .*

Proof. We reason in PA as follows. Suppose $\Box_\sigma(\varphi)$ and reason by induction on proof length of φ . The only interesting case is when $\varphi \in \Pi_{n+1}$ is an axiom. By Lemma 5.5, we obtain $\Box_{\underline{\text{Con}}_n(\sigma)}(\varphi)$. Hence, $\underline{\text{Con}}_n(\sigma) \Rightarrow_{\text{PA}} \sigma$ as required. \square

LEMMA 5.7 ([10]). *For any numeration σ , $\underline{\text{Con}}_\omega(\sigma) \Rightarrow_{\text{PA}} \sigma$.*

LEMMA 5.8. *Let φ be a Π_{m+1} -sentence and σ a numeration. For $m < n < \omega$ it holds that*

$$\text{PA} \vdash \text{Con}_n(\sigma) \wedge \varphi \rightarrow \text{Con}_n(\sigma \vee \underline{\varphi}).$$

Proof. We reason in PA as follows. Suppose $\Box_{\sigma \vee \underline{\varphi}}(\psi)$ for $\psi \in \Pi_{n+1}$. Then $\Box_\sigma(\varphi \rightarrow \psi)$ by a formalized version of the standard deduction theorem. We know that $\varphi \rightarrow \psi$ is a Π_{n+1} -sentence since $m < n$. Thus, if $\text{Con}_n(\sigma)$ then also $\text{True}_{\Pi_{n+1}}(\varphi \rightarrow \psi)$ and so $\text{True}_{\Pi_{n+1}}(\varphi) \rightarrow \text{True}_{\Pi_{n+1}}(\psi)$.

Now if φ holds, then, since $\varphi \in \Pi_{n+1}$, we obtain $\text{True}_{\Pi_{n+1}}(\varphi)$ whence $\text{True}_{\Pi_{n+1}}(\psi)$ follows as required. \square

LEMMA 5.9. *Suppose τ numerates a Π_{m+1} -axiomatized extension of PA. Then, for any numeration σ ,*

$$\underline{\text{Con}}_{\omega}(\sigma) \vee \tau \Rightarrow_{\text{PA}} \underline{\text{Con}}_{\omega}(\sigma \vee \tau).$$

Proof (Sketch). We show an informal version of this statement by an argument formalizable in PA. That is, we must show that for each n ,

$$\text{PA} + \underline{\text{Con}}_{\omega}(\sigma) + \tau \vdash \underline{\text{Con}}_n(\sigma \vee \tau).$$

We may assume $n > m$ and use the previous lemma. A formalization of the corresponding argument yields the proof. \square

PROPOSITION 5.10. *$\text{RC}^*\omega$ is arithmetically sound, i.e., if $\text{RC}^*\omega \vdash A \Rightarrow B$ then $f(A) \Rightarrow_{\text{PA}} f(B)$ for every typed arithmetical realization f .*

Proof. By induction on the length of a derivation of $A \Rightarrow B$. The soundness of the propositional rules and axioms (i.e., (i) to (iv)) are immediate. The soundness of the modal axiom schemas (vi), (vii), and (viii) follows from the previous lemmas and corollaries. For the monotonicity axiom schema (vii), it is clear that $\underline{\text{Con}}_{\alpha}(\sigma) \Rightarrow_{\text{PA}} \underline{\text{Con}}_{\beta}(\sigma)$, for $\alpha > \beta$, since the strength of $\underline{\text{Con}}_{\alpha}(\sigma)$ increases with α .

It remains to be shown that the necessitation rule (v) is sound. Suppose $f(A) \Rightarrow_{\text{PA}} f(B)$ and let $n < \omega$. We claim that $\text{PA} + \underline{\text{Con}}_n(f(A)) \vdash \underline{\text{Con}}_n(f(B))$. Indeed, reasoning in $\text{PA} + \underline{\text{Con}}_n(f(A))$, we see that if $\varphi \in \Pi_{n+1}$ and $\Box_{f(B)}(\varphi)$ holds, then also $\Box_{f(A)}(\varphi)$ (since $f(A) \Rightarrow_{\text{PA}} f(B)$) and thus also $\text{True}_{\Pi_{n+1}}(\varphi)$. By Lemma 5.2, we thus obtain $\text{PA} + \underline{\text{Con}}_n(f(A)) \vdash \underline{\text{Con}}_n(f(B))$, i.e., $\underline{\text{Con}}_n(f(A)) \Rightarrow_{\text{PA}} \underline{\text{Con}}_n(f(B))$.

Formalizing this argument also establishes that if $f(A) \Rightarrow_{\text{PA}} f(B)$, then $\underline{\text{Con}}_{\omega}(f(A)) \Rightarrow_{\text{PA}} \underline{\text{Con}}_{\omega}(f(B))$. \square

5.3. Arithmetical Completeness. The arithmetical completeness for $\text{RC}^*\omega$ is obtained in a similar fashion as the results for GLP^* are obtained from the arithmetical completeness proof of GLP . To obtain arithmetical completeness for $\text{RC}^*\omega$, one follows the proof for $\text{RC}\omega$ as given in [10].

Arithmetical completeness for $\text{RC}^*\omega$ can thus be roughly obtained as follows:

- One identifies a class of Kripke models for which $\text{RC}^*\omega$ is sound and complete and which reflects the notion of sort in an appropriate way. It turns out that, as in the case of GLP^* , the notion of strong persistence is appropriate for this purpose.
- The arithmetical completeness of $\text{RC}^*\omega$ is established following the completeness proof for $\text{RC}\omega$ as presented in [10]. One exploits the fact that sequents that are non-provable in $\text{RC}^*\omega$ have Kripke counterexamples that are strongly persistent and observes that redoing the construction of [10] admits the extraction of an arithmetical counterexample that is actually typed. Notice that this is in the same spirit as we conducted the arithmetical completeness proof for GLP^* —after all, it was enough to observe that the assumption of having a strongly persistent counterexample at hand allows one to conclude that the arithmetical realization constructed in the proof for standard GLP is already typed.

In the following, we shall elaborate on the arithmetical completeness proof for $\text{RC}^*\omega$.

5.3.1. *Kripke Models.* We require an appropriate class of Kripke models for which $\text{RC}^*\omega$ is complete. Let Φ be a set of positive formulas and

$$\ell(\Phi) := \{\alpha \leq \omega \mid \alpha \text{ occurs in some } A \in \Phi\}.$$

We say that Φ is *adequate*, if it is closed under subformulas, $\top \in \Phi$, and

- (i) if $\beta A \in \Phi$ and $\beta < \alpha \in \ell(\Phi)$, then $\alpha A \in \Phi$;
- (ii) for any variable p of sort α , if $p \in \Phi$, then $\beta p \in \Phi$, for all $\beta \leq \alpha$.

An $\text{RC}^*\omega$ -theory in Φ is a set $\Gamma \subseteq \Phi$ such that $\text{RC}^*\omega \vdash \Gamma \Rightarrow A$ and $A \in \Phi$ implies $A \in \Gamma$.

The notion of a Kripke model immediately extends to positive formulas as well once we include an accessibility relation R_ω , i.e., Kripke models are structures of the form $\mathfrak{A} = (W, \{R_\alpha\}_{\alpha \leq \omega}, \llbracket \cdot \rrbracket)$. Recall that $A \Rightarrow B$ stands for $A \rightarrow B$ and hence we specify $\mathfrak{A}, x \models A \Rightarrow B$ iff $\mathfrak{A}, x \models A \rightarrow B$. The notion of validity in a model thus immediately extends to sequents as well. Moreover, the notions of J-model and J*-model then carry over to the positive case by additionally considering the relation R_ω . Recall that a J*-model is a strongly persistent J-model, and that a strongly persistent model \mathfrak{A} satisfies the following conditions, for all $0 \leq \alpha \leq \omega$:

- (1) if $|p| \leq \alpha$ and $\mathfrak{A}, y \models p$, then $\mathfrak{A}, x \models p$ whenever $xR_\alpha y$; and
- (2) if $|p| < \alpha$ and $\mathfrak{A}, y \not\models p$, then $\mathfrak{A}, x \not\models p$ whenever $xR_\alpha y$.

In particular, for the case $\alpha = \omega$, the first condition states that the satisfaction of *any* variable is propagated downwards along R_ω -arcs, since all variables have sort at most ω .

Let Φ be an adequate set. We say that a model \mathfrak{A} is Φ -monotone, if for any $\alpha A \in \Phi$ and $\beta \in \ell(\Phi)$ such that $\alpha < \beta$, $\mathfrak{A}, x \models \beta A$ implies $\mathfrak{A}, x \models \alpha A$. The following completeness result for $\text{RC}^*\omega$ is an almost literal repetition of a similar result for $\text{RC}\omega$ proven in [10]. We omit a proof of this theorem, since it can be proved by a straightforward adaption of the according result in [10].

THEOREM 5.11. *Let Φ be a finite adequate set. Then there is a finite model $\mathfrak{A} = (W, \{R_\alpha\}_{\alpha \leq \omega}, \llbracket \cdot \rrbracket)$ such that*

- (1) \mathfrak{A} is an irreflexive J*-model, i.e., a J*-model in which all R_α are irreflexive;
- (2) $R_\alpha = \emptyset$, for all $\alpha \notin \ell(\Phi)$;
- (3) \mathfrak{A} is Φ -monotone;
- (4) for any $\text{RC}^*\omega$ -theory Γ in Φ , there is a node $x \in W$ such that, for any formula A , $A \in \Gamma$ iff $\mathfrak{A}, x \models A$.

5.3.2. *Arithmetical Completeness for $\text{RC}^*\omega$.* We are now going to prove the arithmetical completeness theorem for $\text{RC}^*\omega$, relying on the construction for $\text{RC}\omega$ presented in [10]:

THEOREM 5.12. *The following are equivalent:*

- (1) $\text{RC}^*\omega \vdash A \Rightarrow B$;
- (2) $f(A) \Rightarrow_{\text{PA}} f(B)$, for every typed arithmetical realization f ;
- (3) $f(A) \Rightarrow f(B)$, for every typed arithmetical realization f .

Note that the implication from (1) to (2) was proved in Proposition 5.10 and statement (2) clearly implies (3). In what follows, we establish that (3) implies (1). We do so by proving its contrapositive.

Assume $\text{RC}^*\omega \not\vdash A \Rightarrow B$. Consider a finite adequate set Φ containing $\{A, B\}$. Let $\mathfrak{A} = (W, \{R_\alpha\}_{\alpha \leq \omega}, \llbracket \cdot \rrbracket)$ be a Kripke model satisfying the conditions of Theorem 5.11 such that, for some node $x \in W$, $\mathfrak{A}, x \models A$, yet $\mathfrak{A}, x \not\models B$.

As in the case of GLP^* , one can again assume that \mathfrak{A} is rooted (see [10]). Now one proceeds with the Solovay-type construction similarly as for GLP^* . That is, one identifies the set W with a finite set of natural numbers $\{1, \dots, N\}$ so that 1 is the root. One attaches a new root 0 to \mathfrak{A} by stipulating that $0R_0x$, for all $x \in W$. The valuation of the variables at the new root 0 will be the same as in node 1; abusing notation, let us call the resulting model \mathfrak{A} as well. It is easy to check that \mathfrak{A} still satisfies the properties of Theorem 5.11 and that $\mathfrak{A}, 0 \models A$, but $\mathfrak{A}, 0 \not\models B$. We assume that the relation $x \in \llbracket C \rrbracket$ (where $C \in \Phi$ is a positive formula) and the relations R_α are naturally arithmetized by bounded formulas.

In the following, we shall denote by $\text{Prf}_n(\alpha, y)$ an arithmetical formula (of arithmetical complexity Δ_{n+1}) expressing that “ y is a proof of a formula α from the axioms of PA and all true Π_n -sentences”—recall that $\text{PA} \vdash [n]_{\text{PA}}(\alpha) \leftrightarrow \exists y \text{Prf}_n(\alpha, y)$. We again assume that each provable formula has arbitrarily long proofs and that this holds provably in PA.

Recall from the arithmetical completeness proof of GLP^* that, if $G(x, y)$ codes a function $g: \omega \rightarrow W$ in PA, then the formula $\ell^G = x$ is an abbreviation of the formula $\exists N_0 \forall n \geq N_0 G(n, x)$, i.e., the formula which expresses the fact that g reaches a limit at x .¹²

There is a striking difference in the arithmetical completeness proof of $\text{RC}\omega$ that makes it substantially different from that of GLP : since the arithmetical complexity of the uniform reflection principle is unbounded, finitely many Solovay-style functions do not suffice for obtaining completeness. Instead, in [10], infinitely many such functions of increasing arithmetical complexity are employed.

We are now going to state the major technical lemmas from [10] which will allow us to deduce an arithmetical completeness theorem for $\text{RC}^*\omega$. First, the following lemma states basic properties of the used Solovay-style functions:

LEMMA 5.13 ([10]). *Let M denote the maximum modality $m < \omega$ occurring in Φ , and 0 if there is no such m . There is an infinite sequence h_0, h_1, \dots of functions of type $\omega \rightarrow W$ that satisfy the following properties:*

- (1) *Each h_k is defined by a respective formula H_k in PA which is Δ_{k+1} in PA;*
- (2) *the function $\varphi: k \mapsto \ulcorner H_k \urcorner$ is primitive recursive;*
- (3) *for each h_k , we have that $h_k(x) = y$ if and only if, either*
 - $x = y = 0$, or
 - $h_i(n) \neq h_i(n+1) = y$, for some $i < k$, or
 - $\exists m \geq \max\{M, k\} \text{Prf}_k(\ulcorner \ell^{H_m} \neq y \urcorner, n)$ and $h_k(n)R_k y$ or $h_k(n)R_\omega y$, or
 - $y = h_k(n)$.

In the following, we fix such a sequence h_0, h_1, \dots of functions with the properties as stated in Lemma 5.13 above. Informally speaking, the behavior of the functions h_k in comparison to those employed for GLP^* can be described as follows (see [10]):

¹²We will reuse here most of the notation from the arithmetical completeness proof of GLP^* without further comment.

- The functions with lower index have higher priority in the sense that, whenever h_m makes a move (i.e., if it changes its position to a new world from W), then h_n will make the same move, for any $n > m$;
- h_k also reacts to proofs of limit statements of functions of lower priority, not only to those of itself;
- h_k is also allowed to move along R_ω -edges.

LEMMA 5.14 ([10]). *For each n, m , provably in PA,*

- (1) $\exists! z \in W \ell^{H_n} = z$;
- (2) $\ell^{H_n} R_{n+1} \ell^{H_{n+1}}$ or $\ell^{H_n} R_\omega \ell^{H_{n+1}}$ or $\ell^{H_n} = \ell^{H_{n+1}}$;
- (3) if $m < n$ then $\ell^{H_m} = \ell^{H_n}$ or $\ell^{H_m} R_\alpha \ell^{H_n}$, for some $\alpha \in (m, n) \cup \{\omega\}$.

The first item of Lemma 5.14 states that every function provably reaches a unique limit. The second item states that the limit of h_{n+1} is (provably) reachable from the limit of h_n either via an R_{n+1} -arc or an R_ω -arc. The third item can be obtained from the second one via an (external) induction on n .

For all $n < \omega$, we define an arithmetical formula $L_n(a)$ as follows:

$$L_n(a) := \begin{cases} \exists x h_n(x) = a, & \text{if } n = 0, \\ \exists x (h_n(x) = a \wedge \forall z \geq x h_{n-1}(z) = h_{n-1}(x)), & \text{otherwise.} \end{cases}$$

Notice that $L_n(a)$ is expressible by a Σ_{n+1} -formula. As in the arithmetical completeness proof for GLP^* , let $R_k^*(x)$ denote the set $\{y \in W \mid \exists \alpha \geq k: x R_\alpha y\}$.

LEMMA 5.15 ([10]). *Let $k \geq n$ and $a := \ell^{H_n}$. Then, provably in PA, $L_n(a)$ implies that $\ell^{H_k} \in R_n^*(a) \cup \{a\}$.*

Intuitively, Lemma 5.15 states that, assuming $L_n(a)$ where a is the limit of h_n , the limit of the function h_k is (provably) either a or some point that is reachable via a path from a that consists of arcs R_α , where $n \leq \alpha \leq \omega$. This is because, due to the assumption $L_n(a)$, h_n can move only along such edges from a onward.

The formulas $L_n(a)$ will be important for us to extract an arithmetical realization that is typed. This is due to the following lemma:

LEMMA 5.16. *For all $n < \omega$ and all variables p of sort $k \leq n$, provably in PA,*

$$\ell^{H_n} \in \llbracket p \rrbracket \iff \forall w \in W \setminus \llbracket p \rrbracket \neg L_k(w).$$

Proof. We reason in PA as follows. For the direction from left to right, suppose $\ell^{H_n} \in \llbracket p \rrbracket$ and suppose to the contrary that there is a $w \in W$ and an x such that $w \notin \llbracket p \rrbracket$ and $L_k(x, w)$. By strong persistence, we know that $v \notin \llbracket p \rrbracket$ for all $v \in R_k^*(w)$. Since $k \leq n$, Lemma 5.15 gives us $\ell^{H_n} \in R_k^*(w) \cup \{w\}$, whence $\ell^{H_n} \in W \setminus \llbracket p \rrbracket$. This contradicts the uniqueness of ℓ^{H_n} (that is, item (1) of Lemma 5.14).

For the other direction, suppose (in PA) that $\forall w \in W \setminus \llbracket p \rrbracket \neg L_k(w)$ and assume $\ell^{H_n} \neq x$ for all $x \in \llbracket p \rrbracket$. By item (1) of Lemma 5.14, it follows that $\ell^{H_n} \in W \setminus \llbracket p \rrbracket$. Let $w \in W \setminus \llbracket p \rrbracket$; we first prove that $\ell^{H_k} \neq w$. In case $k = 0$, by $\neg L_k(w)$, we infer $\forall x h_k(x) \neq w$ and thus $\ell^{H_k} \neq w$. Suppose now that $k > 0$. Then $\neg L_k(w)$ is equivalent to $\forall x (h_k(x) \neq w \vee \exists z \geq x h_{k-1}(z) \neq h_{k-1}(x))$. We claim that there are arbitrarily large x such that $h_k(x) \neq w$. Indeed, suppose there is an x_0 such that $\forall y \geq x_0 h_k(y) = w$. By $\neg L_k(w)$, we infer that $\exists z \geq x_0 h_{k-1}(z) \neq h_{k-1}(x_0)$, whence it follows

that there is a $y_0 \geq x_0$ such that $h_{k-1}(x_0) = h_{k-1}(y_0) \neq h_{k-1}(y_0 + 1)$. By the definition of h_k , this implies $w = h_k(y_0 + 1) = h_{k-1}(y_0 + 1)$. Using the assumption $\neg L_k(w)$ again, we infer that $\exists z \geq y_0 + 1$ $h_{k-1}(z) \neq h_{k-1}(y_0 + 1)$. Thus, there is a $y_1 \geq y_0 + 1$ such that $w = h_{k-1}(y_0 + 1) = h_{k-1}(y_1) \neq h_{k-1}(y_1 + 1)$. By the definition of h_k , this again implies $h_k(y_1 + 1) = h_{k-1}(y_1 + 1)$. Notice that $y_1 + 1 > y_1 \geq y_0 + 1 > y_0 \geq x_0$ and $h_k(x_0) = h_k(y_0 + 1) = h_k(y_1) = w$, but certainly $h_k(y_1 + 1) \neq w$. This contradicts the fact that $\forall y \geq x_0$ $h_k(y) = w$. Thus, ℓ^{H_k} cannot reach its limit at w . It remains to observe that this entails $\ell^{H_k} \in \llbracket p \rrbracket$ by item (1) of Lemma 5.14 and thus we infer $\ell^{H_n} \neq \ell^{H_k}$ (recall that we have $\ell^{H_n} \in W \setminus \llbracket p \rrbracket$). However, this means that $n > k$ and so, by item (3) of Lemma 5.14, this implies that $\ell^{H_k} R_\alpha \ell^{H_n}$, for some $\alpha \in (k, n] \cup \{\omega\}$. This contradicts the property of \mathfrak{A} being strongly persistent, since $\ell^{H_k} \in \llbracket p \rrbracket$ but $\ell^{H_n} \in W \setminus \llbracket p \rrbracket$. \square

We shall now define an appropriate arithmetical realization. Let $\{\varphi_i : i \in I\}$ be a primitive recursive set of formulas. We will denote by $[\varphi_i : i \in I]$ a numeration that numerates the theory $\text{PA} + \{\varphi_i : i \in I\}$. Using this notation, we now define an arithmetical realization f as follows:

$$f(p) := [\ell^{H_n} \in \llbracket p \rrbracket : n \geq M].$$

Notice that the formula $\ell^{H_n} \in \llbracket p \rrbracket$ can indeed be constructed primitive recursively from the parameter n , since the function $\varphi : k \mapsto \ulcorner H_k \urcorner$ is primitive recursive according to Lemma 5.13.

The following lemma states that, for $|p| = k < \omega$, the numeration $f(p)$ is indeed a Π_{k+1} -axiomatized extension of PA.

LEMMA 5.17. *For each variable p of sort $k < \omega$, $f(p)$ numerates a Π_{k+1} -axiomatized extension of PA.*

Proof. Let $n \geq M$ and consider the sentence $\ell^{H_n} \in \llbracket p \rrbracket$. If $k \leq n$, then by Lemma 5.16, provably in PA,

$$\ell^{H_n} \in \llbracket p \rrbracket \iff \forall w \in W \setminus \llbracket p \rrbracket \neg L_k(w) \iff \bigwedge_{w \in W \setminus \llbracket p \rrbracket} \neg L_k(\bar{w}).$$

Recall that $L_k(x)$ is Σ_{k+1} in PA, whence it follows that $\neg L_k(\bar{w})$ is Π_{k+1} in PA and thus so is $\ell^{H_n} \in \llbracket p \rrbracket$.

For the case $k > n$, recall the very definition of $\ell^{H_n} \in \llbracket p \rrbracket$ is $\bigvee_{x \in \llbracket p \rrbracket} \ell^{H_n} = \bar{x}$ and the definition of $\ell^{H_n} = \bar{x}$ reads $\exists N_0 \forall z > N_0 H_n(z, \bar{x})$. By virtue of Lemma 5.13, $H_n(x, y)$ is Δ_{n+1} in PA, whence it follows that $\ell^{H_n} = \bar{x}$ is Σ_{n+2} in PA and thus $\ell^{H_n} \neq \bar{x}$ is Π_{n+2} in PA. Observe that, by item (1) of Lemma 5.14, provably in PA,

$$\ell^{H_n} \in \llbracket p \rrbracket \iff \bigwedge \{\ell \neq \bar{x} \mid x \in W \setminus \llbracket p \rrbracket\}.$$

Thus, $\ell^{H_n} \in \llbracket p \rrbracket$ is Π_{k+1} in PA, since $k + 1 \geq n + 2$ by assumption. \square

It follows that f is actually a typed arithmetical realization as desired. We can now proceed along the lines of [10] and quote some more technical lemmas that will allow us to conclude the arithmetical completeness proof for $\text{RC}^*\omega$:

LEMMA 5.18 ([10]). *For any formula $C \in \Phi$,*

- (1) $[\ell^{H_n} \in \llbracket C \rrbracket : n \geq M] \Rightarrow_{\text{PA}} f(C)$;
- (2) $\underline{\ell^{H_0} \neq 0} \vee f(C) \Rightarrow_{\text{PA}} [\ell^{H_n} \in \llbracket C \rrbracket : n \geq M]$.

LEMMA 5.19 ([10]). *For all $n \geq 0$, $\mathbb{N} \models \ell^{H_n} = 0$.*

Intuitively, Lemma 5.18 can be seen as a counterpart to the “commutation lemma” in the arithmetical completeness proof of GLP^* (Lemma 3.4), while Lemma 5.19 simply states that, in the standard model, all functions reach their limit at 0.

Now we can conclude the proof of Theorem 5.12 in accordance with [10] as follows. Recall that we have $\mathfrak{A}, 1 \models A$ but $\mathfrak{A}, 1 \not\models B$. Let σ be the numeration $[\ell^{H_n} = \bar{1} : n \geq M]$ and let S be the theory numerated by σ . By Lemma 5.18, we know that

$$\begin{aligned} \sigma &\Rightarrow_{\text{PA}} [\ell^{H_n} \in \llbracket A \rrbracket : n \geq M] \\ &\Rightarrow_{\text{PA}} f(A). \end{aligned}$$

By Lemma 5.18, we also have

$$\begin{aligned} \underline{\ell^{H_0} \neq 0} \vee f(B) &\Rightarrow_{\text{PA}} [\ell^{H_n} \in \llbracket B \rrbracket : n \geq M] \\ &\Rightarrow_{\text{PA}} [\ell^{H_n} \neq \bar{1} : n \geq M]. \end{aligned}$$

Now if we had $f(A) \Rightarrow f(B)$, then $S \vdash \ell^{H_M} \neq \bar{1}$ and so S would be inconsistent. One can easily show that $\text{PA} \vdash \ell^{H_n} = \bar{1} \rightarrow \ell^{H_m} = \bar{1}$, for all $m \leq n$. Thus, there is a PA -proof of $\ell^{H_n} \neq \bar{1}$, for some $n \geq M$ (otherwise, $\text{PA} \vdash S$ and so PA would be inconsistent too). But this means that h_0 must eventually take a value different from 0 by its definition. This is, however, impossible due to Lemma 5.19.

6. CONCLUSION

We have studied a many-sorted fragment of GLP^* where propositional variables are assigned sorts $\alpha \leq \omega$. The logic GLP^* admits a more fine-grained arithmetical interpretation than standard GLP : variables of finite sort $n < \omega$ range over Π_{n+1} -sentences of the arithmetical hierarchy, while those of sort ω range over arbitrary sentences. The inclusion of sorts in the modal languages naturally corresponds, in the realm of modal logics, to the notion of stratification of graded provability algebras in the algebraic world. We showed that GLP^* is arithmetically complete by exploiting an existing construction for GLP . Moreover, we reduced GLP^* to GLP and thereby transferred results from GLP to GLP^* like Craig interpolation and PSPACE decidability. We studied variants of GLP^* that restrict the use of sorts. A positive variant of GLP^* , denoted $\text{RC}^*\omega$, was introduced which allows for an even richer arithmetical interpretation due to the fact that variables are permitted to range over arithmetical theories rather than single sentences. This arithmetical interpretation allows the introduction of an additional modality $\langle \omega \rangle$ which is not present in GLP^* , and which corresponds to the full uniform reflection principle. We showed that $\text{RC}^*\omega$ is arithmetically complete by again relying on an existing construction for its one-sorted counterpart.

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REFERENCES

- [1] Mohammad Ardeshir and S. Mojtaba Mojtahedi. Reduction of provability logics to Σ_1 -provability logics. *Logic Journal of the IGPL*, 23(5):842–847, 2015.
- [2] Sergei N. Artemov and Lev D. Beklemishev. Provability Logic. In *Handbook of Philosophical Logic, 2nd ed.*, pages 229–403. Kluwer, 2004.
- [3] Lev D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. *Annals of Pure and Applied Logic*, 128(1-3):103–123, 2004.

- [4] Lev D. Beklemishev. Reflection principles and provability algebras in formal arithmetic. *Russian Mathematical Surveys*, 60(2):197–268, 2005. Russian original: *Uspekhi Matematicheskikh Nauk*, 60(2): 3–78, 2005.
- [5] Lev D. Beklemishev. The Worm Principle. In Z. Chatzidakis, P. Koepke, and W. Pohlers, editors, *Lecture Notes in Logic 27. Logic Colloquium '02*, pages 75–95. AK Peters, 2006. Preprint: Logic Group Preprint Series 219, Utrecht Univ., March 2003.
- [6] Lev D. Beklemishev. On the Craig interpolation and the fixed point properties of GLP. In S. Feferman et al., editor, *Proofs, Categories and Computations. Essays in honor of G. Mints*, Tributes, pages 49–60. College Publications, London, 2010.
- [7] Lev D. Beklemishev. Kripke semantics for provability logic GLP. *Annals of Pure and Applied Logic*, 161(6):756–774, 2010.
- [8] Lev D. Beklemishev. A simplified proof of arithmetical completeness theorem for provability logic GLP. *Proceedings of the Steklov Institute of Mathematics*, 274(1):25–33, 2011.
- [9] Lev D. Beklemishev. Calibrating provability logic: from modal logic to reflection calculus. In T. Bolander, T. Braüner, S. Ghilardi, and L. Moss, editors, *Advances in Modal Logic*, v. 9, pages 89–94. College Publications, London, 2012.
- [10] Lev D. Beklemishev. Positive provability logic for uniform reflection principles. *Annals of Pure and Applied Logic*, 165(1):82–105, 2014.
- [11] Lev D. Beklemishev. On the reduction property for GLP-algebras. *Doklady: Mathematics*, 95(1):50–54, 2017.
- [12] Lev D. Beklemishev. On the reflection calculus with partial conservativity operators. In *Logic, Language, Information, and Computation - 24th International Workshop, WoLLIC 2017, London, UK, July 18-21, 2017, Proceedings*, pages 48–67, 2017.
- [13] Lev D. Beklemishev, David Fernández-Duque, and Joost J. Joosten. On provability logics with linearly ordered modalities. *Studia Logica*, 102(3):541–566, 2014.
- [14] Nick Bezhanishvili and Dick de Jongh. Intuitionistic logic. Technical report, Institute for Logic, Language and Computation, University of Amsterdam, 2006.
- [15] George S. Boolos. *The Logic of Provability*. Cambridge University Press, 1993.
- [16] Evgeny V. Dashkov. On the positive fragment of the polymodal provability logic GLP. *Mathematical Notes*, 91(3):318–333, 2012. Original Russian text in: *Matematicheskie Zametki*, 91(3):331–336, 2012.
- [17] Giorgie Dzhaparidze. The polymodal logic of provability. In *Intensional Logics and Logical Structure of Theories: Material from the fourth Soviet-Finnish Symposium on Logic, Telavi, May 20-24, 1985, Metsniereba, Tbilisi*, pages 16–48, 1988. In Russian.
- [18] Giorgie Dzhaparidze. The logic of arithmetical hierarchy. *Annals of Pure and Applied Logic*, 66(2):89–112, 1994.
- [19] Solomon Feferman. Three conceptual problems that bug me. Lecture text for 7-th Scandinavian Logic Symposium, <https://math.stanford.edu/~feferman/papers/conceptualprobs.pdf>, 1996.
- [20] David Fernández-Duque and Joost J. Joosten. The omega-rule interpretation of transfinite provability logic. *CoRR*, abs/1302.5393, 2013.
- [21] Konstantin N. Ignatiev. On Strong Provability Predicates and the Associated Modal Logics. *The Journal of Symbolic Logic*, 58(1):249–290, 03 1993.
- [22] Georg Kreisel. Wie die Beweistheorie zu ihren Ordinalzahlen kam und kommt. *Jahresbericht der DMV*, 78(4):177–223, 1977.
- [23] Ilya Shapировsky. PSPACE-decidability of Japaridze's polymodal logic. In *Advances in Modal Logic*, volume 7, pages 289–304, 2008.
- [24] Robert M. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, 25(3-4):287–304, 1976.

- [25] Albert Visser. *Aspects of diagonalization and provability*. PhD thesis, Utrecht University, 1981.
- [26] Albert Visser. The provability logics of recursively enumerable theories extending Peano Arithmetic at arbitrary theories extending Peano Arithmetic. *Journal of Philosophical Logic*, 13:97–113, 1984.
- [27] Albert Visser. Substitutions of Σ_1^0 -sentences: Explorations between intuitionistic propositional logic and intuitionistic arithmetic. *Annals of Pure and Applied Logic*, 114(1–3): 227–271, 2002.