# Complexity of finite-variable fragments of propositional modal logics of symmetric frames* 

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#### Abstract

While finite-variable fragments of the propositional modal logic S5-complete with respect to reflexive, symmetric, and transitive frames - are polynomialtime decidable, the restriction to finite-variable formulas for logics of reflexive and transitive frames yields fragments that remain "intractable." The role of the symmetry condition in this context has not been investigated. We show that symmetry either by itself or in combination with reflexivity produces logics that behave just like logics of reflexive and transitive frames, i.e., their finite-variable fragments remain intractable, namely PSPACE-hard. This raises the question of where exactly the borderline lies between modal logics whose finite-variable fragments are tractable and the rest.


KEYWORDS: propositional modal logic, symmetric frames, finite-variable fragments, computational complexity

## 1 Introduction

While the propositional modal logic S5, which has Kripke-style semantics in terms of reflexive, transitive, and symmetric frames, is "computationally intractable"namely, its satisfiability problem is NP-complete - its $n$-variable fragments, for every

[^0]$n \in \mathbb{N}$, are polynomial-time decidable. In contrast, as originally shown in 5] and further elaborated in [2] (see also [7] and [13]), most "natural" modal logics whose Kripke frames are reflexive, transitive, or both, remain intractable even if the number of propositional variables in their languages is restricted to one (for all logics in the intervals $[\mathbf{K}, \mathbf{G L}]$ and $[\mathbf{K}, \mathbf{G r z}]$ ) or zero (for all logics in the interval $[\mathbf{K}, \mathbf{K} 4]$ ). It is, thus, interesting to see if symmetry plays any role in making $\mathbf{S} 5$ stand apart from other propositional modal logics in this regard. More generally, does the addition of the axiom of symmetry to a logic make its finite-variable fragments easier to decide than the entire logic? The role of symmetry in this context has not been investigated in the literature.

In this paper, we answer this question in the negative by showing that all logics in the interval $[\mathbf{K}, \mathbf{K T B}]$, where $\mathbf{K T B}$ is the propositional modal logic of reflexive and symmetric frames, have PSPACE-hard single-variable fragments. As a byproduct, we prove that logics KTB and KB, which is the propositional modal logic of symmetric frames, can be embedded into their single-variable fragments, which are, thus, as semantically expressive - from the point of view of validity and (local) satisfiability - as the entire logics.

The paper is organized as follows. In section 2, we briefly recall the syntax and semantics of the logics we consider in the present paper and establish that all the logics in the interval $[\mathbf{K}, \mathbf{K T B}]$ are PSPACE-hard. Then, in section 3, we present our main results concerning single-variable fragments of logics in $[\mathbf{K}, \mathbf{K T B}]$. We conclude in section 4.

## 2 Preliminaries

The propositional modal language contains countably many propositional variables $p_{1}, p_{2}, \ldots$, the Boolean constant $\perp$ ("falsehood"), the Boolean connective $\rightarrow$, and the modal connective $\square$. Other connectives, as well as formulas, are defined as usual. We also use the following abbreviations:

$$
\begin{aligned}
\square^{0} \varphi & =\varphi, & \square^{\leqslant 0} \varphi & =\varphi, \\
\square^{n+1} \varphi & =\square \square^{n} \varphi, & \square^{\leqslant n+1} \varphi & =\square \leqslant n \varphi \wedge \square^{n+1} \varphi, \\
\square^{+} \varphi & =\varphi \wedge \square \varphi, & \diamond^{n} \varphi & =\neg \square^{n} \neg \varphi .
\end{aligned}
$$

A (Kripke) frame is a pair $\mathfrak{F}=\langle W, R\rangle$, where $W$ is a non-empty set (of worlds) and $R$ is a binary (accessibility) relation on $W$. A (Kripke) model is a pair $\mathfrak{M}=\langle\mathfrak{F}, V\rangle$, where $\mathfrak{F}$ is a frame and $V$ is a valuation function assigning to every propositional variable a subset of $W$; if $\mathfrak{M}$ has the form $\langle\mathfrak{F}, V\rangle$, we say that it is based on $\mathfrak{F}$.

The satisfaction relation between models $\mathfrak{M}$, worlds $w$, and formulas $\varphi$ is defined as follows:

- $\mathfrak{M}, w \models p_{i} \leftrightharpoons w \in V\left(p_{i}\right) ;$
- $\mathfrak{M}, w \models \perp$ never holds;
- $\mathfrak{M}, w \models \varphi_{1} \rightarrow \varphi_{2} \leftrightharpoons \mathfrak{M}, w \models \varphi_{1}$ implies $\mathfrak{M}, w \models \varphi_{2}$;
- $\mathfrak{M}, w \models \square \varphi_{1} \leftrightharpoons \mathfrak{M}, w^{\prime} \models \varphi_{1}$ whenever $w R w^{\prime}$.

Let $\mathfrak{C}$ be a class of frames. A formula is valid on $\mathfrak{C}$ if it is satisfied at every world of every model based on a frame from $\mathfrak{C}$. A formula is satisfiable in $\mathfrak{C}$ if it is satisfied at some world of some model based on a frame from $\mathfrak{C}$.

A propositional modal logic is a set of formulas containing all classical tautologies as well as the formula $\square(p \rightarrow q) \rightarrow(\square p \rightarrow \square q)$ and closed under uniform substitution, modus ponens, and necessitation. Of particular interest to us are the logics $\mathbf{K}$, which is the set of formulas valid on all frames; $\mathbf{K B}$, which is the set of formulas valid on all frames whose accessibility relation is symmetric; and KTB, which is the set of formulas valid on all frames whose accessibility relation is reflexive and symmetric. More generally, we will be concerned with the interval [K, KTB] of logics $L$ such that $\mathbf{K} \subseteq L \subseteq \mathbf{K T B}$. This interval contains a number of logics that have been, for various reasons, of interest to logicians; examples include $\mathbf{T}$, which is the set of formulas valid on reflexive frames; $\mathbf{K B}$; $\mathbf{K D B}$, which is the set of formulas valid on symmetric and serial frames; and Hughes's logic 8]. If a logic $L$ is Kripke-complete, i.e., coincides with the set of formulas valid on some class $\mathfrak{C}$ of frames, we say that a formula $\varphi$ is $L$-satisfiable if $\varphi$ is satisfiable in $\mathfrak{C}$.

We say that a logic is PSPACE-hard (PSPACE-complete) if the problem of membership in it is PSPACE-hard (PSPACE-complete); analogously for fragments of logics. In what follows, we rely on the statement as well as the proof of the following:

Theorem 2.1 Let $L$ be a logic such that $\mathbf{K} \subseteq L \subseteq \mathbf{K T B}$. Then, $L$ is PSPACE-hard.
Proof. The proof is a slight modification of Ladner's proof for logics between $\mathbf{K}$ and S4 (Theorem 3.1 in [9]; see also [1], Section 6.7), which proceeds by reduction from the set TQBF of true quantified Boolean formulas, known to be PSPACE-hard [12]. Note that as PSPACE is closed under complementation, the complement of TQBF is also PSPACE-hard. Since every quantified Boolean formula can be polynomially reduced to one in the prenex normal form, we may assume without a loss of generality that both TQBF and its complement only contain formulas in the prenex normal form.

First, we define a polynomial-time computable translation $f$ from the set of quantified Boolean formulas in the prenex normal form to the set of modal formulas such that

- if $\theta \in \mathrm{TQBF}$, then $f(\theta)$ is KTB-satisfiable;
- if $\theta \notin \mathrm{TQBF}$, then $f(\theta)$ is not $\mathbf{K}$-satisfiable.

Let $\theta=\mathrm{Q}_{1} p_{1} \ldots \mathrm{Q}_{m} p_{m} \varphi\left(p_{1}, \ldots, p_{m}\right)$, where $\mathrm{Q}_{1}, \ldots, \mathrm{Q}_{m} \in\{\exists, \forall\}$ and $\varphi\left(p_{1}, \ldots, p_{m}\right)$ is a propositional formula containing no variables other than $p_{1}, \ldots, p_{n}$. Let $q_{0}, q_{1}, \ldots, q_{m}$ be propositional variables not in $\theta$. Then, $f(\theta)$ is a conjunction of the following formulas:

- $q_{0}$;
- $\square \leqslant m \bigwedge_{i=0}^{m}\left(q_{i} \rightarrow \bigwedge_{j \neq i} \neg q_{j}\right)$;
- $\square^{\leqslant m-1} \bigwedge_{\left\{i: \mathbb{Q}_{i}=\exists\right\}}\left(q_{i-1} \rightarrow \diamond q_{i}\right) ;$
- $\square \leqslant m-1 \bigwedge_{\left\{i: \mathbb{Q}_{i}=\forall\right\}}\left(q_{i-1} \rightarrow \diamond\left(q_{i} \wedge p_{i}\right) \wedge \diamond\left(q_{i} \wedge \neg p_{i}\right)\right)$;
- $\square \leqslant m-1 \bigwedge_{i=1}^{m-1}\left(q_{i} \rightarrow \bigwedge_{j \leqslant i}\left(p_{j} \rightarrow \square\left(q_{i+1} \rightarrow p_{j}\right)\right) \wedge \bigwedge_{j \leqslant i}\left(\neg p_{j} \rightarrow \square\left(q_{i+1} \rightarrow \neg p_{j}\right)\right)\right)$;
- $\square^{m}\left(q_{m} \rightarrow \varphi\right)$.

Note that only the second-to-last formula is substantively different from the formulas used in [9]. Suppose that $\theta$ is true, and thus, there exists a quantifier tree $T$ witnessing its truth. We use $T$ to define a KTB-model satisfying $f(\theta)$. Let $W$ be the set of nodes of $T$ and $R$ be the symmetric and reflexive closure of the "daughter-of" relation of $T$. Thus, $\langle W, R\rangle$ is a KTB-frame. It remains to define the valuation. Let $q_{i}$ be true precisely at the nodes of level $i$ (where the root is a node of level 0 ), let $p_{i}$ be true at a node of level $j \geqslant i$ if, and only if, the substitution of truth values for a variable of $\theta$ connected to that node, or to the node of level $i$ on the same branch of $T$, returns "true" for $p_{i}$, and let $p_{i}$ be false at all nodes of levels $j<i$. It is then straightforward to check that $f(\theta)$ is satisfied at the root of $T$. That falsehood of $\theta$ implies that $f(\theta)$ is not $\mathbf{K}$-satisfiable is argued exactly as in Ladner's proof 9 .

Now, let $L$ be a logic such that $\mathbf{K} \subseteq L \subseteq \mathbf{K T B}$. If $\theta \notin \mathrm{TQBF}$, then $\neg f(\theta) \in \mathbf{K}$ and, hence, $\neg f(\theta) \in L$. Conversely, if $\theta \in \mathrm{TQBF}$, then $\neg f(\theta) \notin \mathbf{K T B}$ and, hence,
$\neg f(\theta) \notin L$. Thus, the translation $t(\theta)=\neg f(\theta)$ reduces the complement of TQBF, which is PSPACE-hard, to $L$. Therefore, $L$ is PSPACE-hard.

As there exist polynomial-space algorithms for deciding satisfiability, and thus validity, for KB, KDB, and KTB (see, e.g., [4]), these logics are PSPACE-complete.

## 3 Complexity of finite-variable fragments

We now show, using a suitable modification of Halpern's technique [5] (see also [11]), that single-variable fragments of all logics in the interval [K,KTB] are PSPACEhard. In the course of the proof we establish that logics KB and KTB can be effectively embedded into their single-variable fragments.

Let $\varphi$ be an arbitrary modal formula. Assume that $\varphi$ only contains propositional variables $p_{1}, \ldots, p_{n}$. First, recursively define the translation ${ }^{\prime}$ as follows:

$$
\begin{array}{ll}
p_{i}{ }^{\prime} & =p_{i}, \quad \text { where } i \in\{1, \ldots, n\} ; \\
\perp^{\prime} & =\perp ; \\
(\phi \rightarrow \psi)^{\prime} & =\phi^{\prime} \rightarrow \psi^{\prime} ; \\
(\square \phi)^{\prime} & =\square\left(p_{n+1} \rightarrow \phi^{\prime}\right) .
\end{array}
$$

Second, put

$$
\widehat{\varphi}=p_{n+1} \wedge \varphi^{\prime}
$$

Notice that $\varphi$ is equivalent to $\widehat{\varphi}\left(p_{n+1} / T\right)$ in $\mathbf{K}$ and, hence, in $\mathbf{K T B}$.
Lemma 3.1 Let $L \in\{\mathbf{K}, \mathbf{K T B}\}$. If $\widehat{\varphi}$ is L-satisfiable, then it is satisfiable in a model based on a frame for $L$ where $p_{n+1}$ is true at every world.

Proof. Suppose that $\mathfrak{M}, w_{0} \models \widehat{\varphi}$ for some model $\mathfrak{M}$ and some world $w_{0}$. Consider the submodel $\mathfrak{M}^{\prime}$ of $\mathfrak{M}$ that consists of worlds where $p_{n+1}$ is true. As $\mathfrak{M}, w_{0} \models p_{n+1}$, the set of worlds of $\mathfrak{M}^{\prime}$ is non-empty. It is straightforward to check both that $\mathfrak{M}^{\prime}$ is based on a frame for $L$ and that $\mathfrak{M}^{\prime}, w_{0} \models \widehat{\varphi}$.

Lemma 3.2 Let $L \in\{\mathbf{K}, \mathbf{K T B}\}$. Then, $\varphi$ is L-satisfiable if, and only if, $\hat{\varphi}$ is L-satisfiable.

Proof. Suppose that $\mathfrak{M}, w_{0} \models \varphi$. To obtain a model satisfying $\widehat{\varphi}$, make $p_{n+1}$ true at every world of $\mathfrak{M}$. Conversely, suppose that $\mathfrak{M}$, $w_{0} \models \widehat{\varphi}$. In view of Lemma 3.1, we may assume that $p_{n+1}$ is universally true in $\mathfrak{M}$. As $\varphi$ is equivalent to $\widehat{\varphi}\left(p_{n+1} / T\right)$,


Figure 1: Model $\mathfrak{M}_{k}$
it follows that $\mathfrak{M}, w_{0} \models \varphi$.
Now, consider the following class M of finite models. For every $k \in\{1, \ldots, n+1\}$, the class M contains a model $\mathfrak{M}_{k}$, depicted in Figure [1, that looks as follows. For brevity, we call some worlds $p$-worlds; if a world is not a $p$-world, we call it a $\bar{p}$-world. The model $\mathfrak{M}_{k}$ is a chain of worlds whose root, $r_{k}$, is a $p$-world. The root is part of a pattern of worlds, described below, which is succeeded by three final $p$-worlds. The pattern looks as follows: a single $p$-world is followed by $2 i+1 \bar{p}$-worlds, for $1 \leqslant i \leqslant k$. Thus, the chain looks as follows: the root (a $p$-world), then three $\bar{p}$ worlds, then a $p$-world, then five $\bar{p}$-worlds, then a $p$-world, ..., then a $p$-world, then $2 k+1 \bar{p}$-worlds, then three $p$-worlds. The accessibility relation $R_{k}$ between the worlds of $\mathfrak{M}_{k}$ is both reflexive and symmetric. To complete the definition of $\mathfrak{M}_{k}$, we define the propositional variable $p$ to be true at exactly the $p$-worlds.

Before proceeding, we prove a lemma about the models in M. Given a model $\mathfrak{M}_{k}$, denote by $c_{i}^{k}$, for $i \in\{1, \ldots, k\}$, the "middle" world of the chain of $2 i+1 \bar{p}$-worlds preceded and succeeded by $p$-worlds; see Figure 1. Also, let

$$
\varepsilon_{i}=\quad \square^{\leqslant i} \neg p \wedge \diamond^{i+1} p \text {, where } i \in \mathbb{N} .
$$

Lemma 3.3 Let $x$ be a world of $\mathfrak{M}_{k}$ that lies between $r_{k}$ and $c_{i}^{k}$, for some $i \leqslant k$ (i.e., $c_{i}^{k}$ cannot be reached from $r_{k}$ by consecutive steps along $R_{k}$ without passing through $x$ ). Then, $\mathfrak{M}_{k}, x \models \varepsilon_{i}$ if, and only if, $x=c_{i}^{k}$.

Proof. Straightforward.
We now define formulas we use to simulate the propositional variables of $\widehat{\varphi}$. First, inductively define, for every $k \in\{1, \ldots, n+1\}$, the following sequence of formulas:

$$
\begin{aligned}
\delta & =\square^{+} p \\
\delta_{k}^{k} & =\varepsilon_{k} \wedge \diamond^{k+2} \delta ; \\
\delta_{i}^{k} & =\varepsilon_{i} \wedge \diamond^{2 i+3} \delta_{i+1}^{k}, \text { where } 1 \leqslant i<k
\end{aligned}
$$

Next, let, for every $k \in\{1, \ldots, n+1\}$,

$$
\alpha_{k}=p \wedge \diamond^{2} \delta_{1}^{k}
$$

and

$$
\beta_{k}=\neg p \wedge \diamond \alpha_{k}
$$

Let $\sigma$ be a (substitution) function that, given a formula $\psi$, replaces all occurrences of $p_{i}$ in $\psi$ by $\beta_{i}$, where $1 \leqslant i \leqslant n+1$. Finally, define

$$
\varphi^{*}=\sigma(\widehat{\varphi})
$$

to produce a single-variable formula $\varphi^{*}$.
Lemma 3.4 Let $L \in\{\mathbf{K}, \mathbf{K T B}\}$. Then, $\varphi$ is L-satisfiable if, and only if, $\varphi^{*}$ is L-satisfiable.

Proof. Suppose that $\varphi$ is not $L$-satisfiable. Then, in view of Lemma 3.2, $\widehat{\varphi}$ is not $L$-satisfiable; hence, $\neg \widehat{\varphi} \in L$. Since $L$ is closed under substitution, $\neg \varphi^{*} \in L$, and so $\varphi^{*}$ is not $L$-satisfiable.

Suppose that $\varphi$ is $L$-satisfiable. Then, in view of Lemmas 3.1] and 3.2, $\mathfrak{M}, w_{0} \models \widehat{\varphi}$ for some $\mathfrak{M}=\langle W, R, V\rangle$, such that $\langle W, R\rangle$ is a frame for $L$ and $p_{n+1}$ is true at every $w \in W$, and for some $w_{0} \in W$. (Recall that $\widehat{\varphi}$ only contains variables $p_{1}, \ldots, p_{n+1}$.) Define model $\mathfrak{M}^{\prime}$ as follows. Attach to $\mathfrak{M}$ all the models from M ; then, for every $x$ in $\mathfrak{M}$, put $x R^{\prime} r_{m}$ and $r_{m} R^{\prime} x$, where $r_{m}$ is the root of $\mathfrak{M}_{m} \in \mathrm{M}$, exactly when $\mathfrak{M}, x \models p_{m}$. Notice that $r_{n+1}$ is accessible in $\mathfrak{M}^{\prime}$ from every $x \in W$. Finally, make $p$ true at exactly those worlds of the attached models where it was true, and make it false at every world in $W$. Notice that $\mathfrak{M}^{\prime}$ is based on a frame for $L$.

To conclude the proof, it suffices to show that $\mathfrak{M}^{\prime}, w_{0} \models \varphi^{*}$. To that end, we first prove two auxiliary Sublemmas:

Sublemma 3.5 Let $x$ be a world of $\mathfrak{M}^{\prime}$ that lies between the root $r_{k}$ of the attached model $\mathfrak{M}_{k}$ and the world $c_{i}^{k}$ of $\mathfrak{M}_{k}$, for some $i \leqslant k$ (i.e., $c_{i}^{k}$ cannot be reached from $r_{k}$ by consecutive steps along $R^{\prime}$ without passing through $\left.x\right)$. Then, $\mathfrak{M}^{\prime}, x \models \varepsilon_{i}$ if, and only if, $x=c_{i}^{k}$.

Proof. Straightforward, using Lemma 3.3.

Sublemma 3.6 Let $x \in W$ and let $\mathfrak{M}^{\prime}, x \models \diamond \alpha_{k}$. Then, $x R^{\prime} r_{k}$.
Proof. Since $\mathfrak{M}^{\prime}, x \models \diamond \alpha_{k}$, so $x R^{\prime} y$ and $\mathfrak{M}^{\prime}, y \models \alpha_{k}$, for some $y$ in $\mathfrak{M}^{\prime}$. We show that $y=r_{k}$. Since $\mathfrak{M}^{\prime}, y \models p$, clearly $y \notin W$, and thus $y$ is the root $r_{m}$ of some $\mathfrak{M}_{m}$. As $\mathfrak{M}^{\prime}, y \models \diamond^{2} \delta_{1}^{k}$, we can reach from $y$ in two $R^{\prime}$-steps a world $y_{1}$ such that $\mathfrak{M}^{\prime}, y_{1} \models \varepsilon_{1}$. Since $w R^{\prime} r_{n+1}$ holds for every $w \in W$, and thus $\mathfrak{M}^{\prime}, w \not \vDash \square \neg p$ for every
$w \in W$, we know that $y_{1} \notin W$, so $y_{1}$ belongs to one of the attached models $\mathfrak{M}_{j}$. In two $R^{\prime}$-steps, we cannot go past $c_{1}^{j}$ for any $j$ and can only reach $c_{1}^{j}$ if $j=m$; hence, due to Sublemma [3.5, $y_{1}=c_{1}^{m}$. Since $\mathfrak{M}^{\prime}, c_{1}^{m} \models \diamond^{5}\left(\varepsilon_{2} \wedge \delta_{2}^{k}\right)$, we can reach from $c_{1}^{m}$ in five $R^{\prime}$-steps a world $y_{2}$ such that $\mathfrak{M}^{\prime}, y_{2} \models \varepsilon_{2}$. As $\mathfrak{M}^{\prime}$, $w \not \models \square \neg p$ for every $w \in W$, we know that $y_{2} \notin W$. In five $R^{\prime}$-steps, we cannot go past $c_{2}^{j}$ for any $j$ and can only reach $c_{2}^{j}$ if $j=m$; hence, due to Sublemma 3.5, $y_{2}=c_{2}^{m}$, and so $\mathfrak{M}^{\prime}, c_{2}^{m} \models \diamond^{7}\left(\varepsilon_{3} \wedge \delta_{3}^{k}\right)$. We can now repeat the argument without worrying about the possibility of satisfying further formulas due to the presence in $\mathfrak{M}^{\prime}$ of the worlds outside of $\mathfrak{M}_{m}$, as we cannot step outside of $\mathfrak{M}_{m}$, starting from $c_{2}^{m}$, in seven steps. By inductively repeating the argument $m$ times, we arrive at the world $c_{m}^{m}$ such that $\mathfrak{M}^{\prime}, c_{m}^{m} \models \diamond^{k+2} \delta$, which can only happen if $m=k$. Thus, all along we have been evaluating the formulas in $\mathfrak{M}_{k}$, and thus $y=r_{k}$, as required.

Now, we proceed with the proof of the main Lemma.
Recall that $\varphi^{*}=\sigma(\widehat{\varphi})=\sigma\left(p_{n+1} \wedge \varphi^{\prime}\right)=\beta_{n+1} \wedge \sigma\left(\varphi^{\prime}\right)$. It is easy to check that $\mathfrak{M}^{\prime}, w_{0} \models \beta_{n+1}$. It then suffices to show that $\mathfrak{M}, x \models \psi^{\prime}$ if, and only if, $\mathfrak{M}^{\prime}, x \models \sigma\left(\psi^{\prime}\right)$, for every subformula $\psi$ of $\varphi$ and every $x \in W$. This can be done by induction on $\psi$.

For the base case, assume that $\mathfrak{M}^{\prime}, x \models \beta_{i}$; in particular, $\mathfrak{M}^{\prime}, x \models \diamond \alpha_{i}$. Then, due to Sublemma 3.6, $x R^{\prime} r_{i}$, and therefore $\mathfrak{M}, x \models p_{i}$ by definition of $\mathfrak{M}^{\prime}$. The other direction is straightforward. The Boolean cases are also straightforward.

Let $\psi=\square \chi$. Assume that $\mathfrak{M}^{\prime}, x \not \vDash \square\left(\beta_{n+1} \rightarrow \sigma\left(\chi^{\prime}\right)\right)$. Then, $x R^{\prime} y$, as well as $\mathfrak{M}^{\prime}, y \models \beta_{n+1}$ and $\mathfrak{M}^{\prime}, y \not \models \sigma\left(\chi^{\prime}\right)$, for some $y$ in $\mathfrak{M}^{\prime}$. In particular, $\mathfrak{M}^{\prime}, y \models \neg p$; thus, $y$ cannot be the root of any of the attached models. Therefore, $y \in W$ and the inductive hypothesis is applicable; this gives us $\mathfrak{M}, x \not \vDash \square\left(p_{n+1} \rightarrow \chi^{\prime}\right)$, as desired. The other direction is straightforward, using the converse of Sublemma 3.6.

Given a formula $\varphi$, let

$$
e(\varphi)=\neg\left((\neg \varphi)^{*}\right) .
$$

Theorem 3.7 Let $L \in\{\mathbf{K}, \mathbf{K T B}\}$. Then, there exists a polynomial-time mapping that embeds $L$ into its single-variable fragment.

Proof. Take the mapping $e$ defined above.

Remark 3.8 Notice that Lemma 3.4 and Theorem 3.7 apply to the logic KB, as well. We did not mention KB in their statements as this is not required for the proof of our main result, Theorem 3.9.

Theorem 3.9 Let L be a logic in the interval $[\mathbf{K}, \mathbf{K T B}]$. Then, the single-variable fragment of $L$ is PSPACE-hard.

Proof. We reduce the PSPACE-hard complement of the set TQBF of true quantified Boolean formulas to the single-variable fragment of $L$. Let $\theta \notin$ TQBF; then, $t(\theta) \in \mathbf{K}$, where $t$ is the translation defined in the proof of Theorem [2.1, hence, in view of Theorem 3.7, $e(t(\theta)) \in \mathbf{K}$, and thus $e(t(\theta)) \in L$. Let, on the other hand, $\theta \in \mathrm{TQBF} ;$ then, as shown in the proof of Theorem [2.1, $t(\theta) \notin \mathbf{K T B} ;$ hence, in view of Theorem 3.7, $e(t(\theta)) \notin \mathbf{K T B}$, and thus $e(t(\theta)) \notin L$. Thus, the polynomialtime computable translation $g(\theta)=e(t(\theta))$ reduces the complement of TQBF to the single-variable fragment of $L$; the statement of the Theorem follows.

Theorem 3.9 implies that the single-variable fragments of all PSPACE-complete logics in $[\mathbf{K}, \mathbf{K T B}]$ are PSPACE-complete; in particular, we have the following:

Corollary 3.10 The single-variable fragments of $\mathbf{T}$, KB, KDB, and KTB are PSPACE-complete.

Note that the PSPACE-completeness of the single-variable fragment of $\mathbf{T}$ has been established in 5].

## 4 Conclusion

We have shown that when it comes to their computational properties, the modal logics of symmetric, as well as of reflexive and symmetric frames, behave in the same way as the logics of transitive, as well as of reflexive and transitive, frames - they remain intractable, namely, PSPACE-hard, when their languages are restricted to only one propositional variable.

Adding the axiom of symmetry to the logic of reflexive and transitive frames, i.e., $\mathbf{S 4}$, is not the only way of arriving at $\mathbf{S 5}$,-it can also be obtained, inter alia, by adding the axiom of Euclideanness, $\neg \square p \rightarrow \square \neg \square p$, to $\mathbf{T}$. The role of Euclideanness is well understood in the context of the present inquiry,- it has been shown in [10] that every extension of the logic of Euclidean frames, K5, is locally tabular; therefore, any finite-variable fragment of such a logic is polynomial-time decidable.

Thus, we have a good understanding of the role played by various properties of Kripke frames of most interest to "traditional" logicians (reflexivity, seriality, symmetry, transitivity, and Euclideanness) - represented by logics included in the "cube


Figure 2: Cube of modal logics
of modal logics" [3] see Figure 2] in the computational behaviour of the finitevariable fragments of the corresponding logics: while Euclideanness - as well as symmetry combined with transitivity, which imply Euclideanness - make such fragments "tractable", reflexivity, symmetry, transitivity, and seriality by themselves-as well as transitivity and symmetry combined either with reflexivity or seriality-do not have this effect (seriality, along with reflexivity and transitivity, has been considered in [2]).

This raises the more general question of where the borderline lies between, on the one hand, logics that behave like those described in [5], [7, [13], [2], and in this paper, i.e., whose finite-variable fragments remain intractable, and on the other, those that behave like $\mathbf{S 5}$, i.e., whose finite-variable fragments are simpler than entire logics (in all the cases known in the literature, this amounts to having polynomialtime decidable finite-variable fragments). It is clear that the answer is not directly linked to the complexity of the logic in question-as shown in [2], satisfiability for single- and two-variable fragments of such logics as S4.3, GL.3, and Grz.3, whose satisfiability problem is NP-complete, is also NP-complete. While the borderline between the NP-hard and the PSPACE-hard in modal logic has received attention in the literature (see, for example, [6]), this question has not, as far as we know, been so far addressed.

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